Points to Remember

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1. Linearisation: $L(x) = y = f(a) + f'(a)(x - a)$
2. Bisection Method: Intermediate value theorem to approximate the root
3. Newton Raphson Procedure: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
4. Taylor Theorem with Integral Remainder:
$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + R_n(x)$$
where
$$R_n(x) = \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt$$
5. Taylor's Inequality: The error in using an n^{th}-order polynomial $P_{n,x_0}(x)$ as an approximation to f(x) satisfies the inequality
$$|R_n(x)| \leq K\frac{|x - x_0|^{n+1}}{(n+1)!}$$
where $|f^{n+1}(z)| \leq K$ for all values of z between x and x_0
6. Maclaurin Series
   (a) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, for $|x| \leq 1$
   (b) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, for all x
   (c) $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, for all x
   (d) $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, for all x
   (e) $(1+x)^k = \sum_{n=0}^{\infty} {}^kC_n x^n$, for all x
7. The Alternating Series Estimation Theorem: Suppose the series $\sum_{n=0}^{\infty}(-1)^n a_n$ can be shown to converge using the Alternating Series Test. Let S denote its sum. If $n = 0$ we use the partial sum $S_n = a_0 + a_1 + ... + a_n$ to approximate the value of S, then the error satisfies the inequality $|S - Sn| \leq a_{n+1}$. That is, the truncation error for an alternating series is no greater than the first term omitted.

8. Integrals

   (a) $\int$ sec x tan x $dx =$ sec x $+$ C

   (b) $\int$ sec $xdx =$ ln|sec x + tan x| $+$ C

9. Approximation of Integrals using Taylor Polynomials:

$$\int_0^x f(t)dt = \int_0^x P_{n,l}(t)dt + \int_0^x R_n(t)dt$$

   where l is the center of the approximation defined appropriately. Then we can approximate

$$\Big|\int_0^x R_n(t)dt\Big| \leq \int_0^x |R_n(t)|dt$$

   The last line is only valid if $x > 0$ othewise one has to interchange the limits on the right hand side

10. Big-O: Given two functions f and g we say that "f is an order of g as $x \to x_0$" and write $f(x) = \mathcal{O}(g(x))$ as $x \to x_0$ if there exists a constant A greater than zero such that $|f(x)| \leq A|g(x)|$

11. Multivariate Taylor Series:

$$\begin{gathered} f(x,y) = f(P_0) \\ +f_x(P_0)h + f_y(P_0)k \\ +\frac{1}{2!}[f_{xx}(P_0)h^2 + 2f_xy(P_0)hk + f_{yy}(P_0)k^2] \\ +\frac{1}{3!}[f_{xxx}(P_0)h^3 + 3f_{xxy}(P_0)h^2k + 3f_{xyy}(P_0)hk^2 + f_{yyy}(P_0)k^3] \end{gathered}$$

12. Multivariate Linear Approximation: $\Delta f \approx f_x(P_0)\Delta x + f_y(P_0)\Delta y$

13. Gradient vector: $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$. This changes chain rule to: $\nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$

14. Directional derivatives: $D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}$

15. Un-constrained optimization: To locate the critical point set $\nabla f = \vec{0}$.
    Then to classify the critical points calculate $D(x,y) = f_{xx}f_{yy} - (f_{xy})^2$

    (a) If $D(P_0) > 0$ then f has an extremum at P_0

        i. if $f_{xx}(P_0) < 0$ then maxima

        ii. if $f_{xx}(P_0) > 0$ then minima

    (b) If $D(P_0) < 0$ then f does not have an extremum at P_0 (it is a saddle point instead)

    (c) If $D(P_0) = 0$ then this test gives no conclusion

16. Method of Lagrange: To find the critical points of $f(x,y)$ subject to constraint $g(x,y) = K$ where K is a constant, find the values of x and y for which $\nabla f = \lambda \nabla g$ and $g(x,y) = K$, for some constant K