ECE 205 - Fall 2018

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6th September 2018

Week 1 - Review & Introduction

1 Complex Numbers

Mathematics: "i" $= \sqrt{-1}$
Engineering: "j" $= \sqrt{-1}$

1.1 Euler Formula

$$e^{j\theta} = cos\theta + jsin\theta$$

We will have expression of the form $e^m t$ where m satisfies $am^2 + bm + c = 0$

1.1.1 Real Roots

If $am^2 + bm + c = 0$ has real roots r_1, r_2 then that leads to solutions $e^{r_1 t}, e^{r_2 t}$
We often take linear combinations of those solutions: $a_1 e^{r_1 t} + a_2 e^{r_2 t}$

1.1.2 Complex Roots

If $am^2 + bm + c = 0$ has complex roots $\alpha + j\beta, \alpha - j\beta,\ (\alpha,\ \beta \,\epsilon\, \mathbb{R})$ then that leads to functions $e^{(\alpha+j\beta)t}, e^{(\alpha-j\beta)t}$
We also take linear combinations of those solutions:

$$\frac{1}{2}[e^{(\alpha+j\beta)t} + e^{(\alpha-j\beta)t}] = e^{\alpha t}cos\beta t$$

because the complex component cancels out from Euler's formula.
Similarly taking a complex linear combination,

$$\frac{1}{2j}[e^{(\alpha+j\beta)t} - e^{(\alpha-j\beta)t}] = e^{\alpha t}sin\beta t$$

Taking a real linear combination, we will have a solution of the form

$$Ae^{\alpha t}cos\beta t + Be^{\alpha t}sin\beta t\ (A,\ B \,\epsilon\, \mathbb{R})$$

2 Linear Algebra

Everything is set up to manage and use Vector Spaces as $\mathbb{R}^n$, M_{m+n} P_3

2.1 Linear Equation

$$A\vec{x} = \vec{b}$$

$A \epsilon M_{m+n}$ $[known]$, $b \epsilon \mathbb{R}^n$ $[unknown]$, $b \epsilon \mathbb{R}^m$ $[known]$

2.1.1 Special Cases

Homogeneous Equation:$A\vec{x} = \vec{0}_{\mathbb{R}^m}$
$\vec{0}_{\mathbb{R}^n}$ is a solution
A solution set S is a vector subspace of $\mathbb{R}^n$
If $\vec{x}_1, \vec{x}_2 \epsilon S$ then $\vec{x}_1 + \vec{x}_2 \epsilon S$ and if $\vec{x}_1 \epsilon S$ then $c\vec{x}_1 \epsilon S$

2.1.2 Some other points

Consider In-homogeneous equation

$$A\vec{x} = \vec{b}$$

$\vec{b} \neq \vec{0}$ with solution set $\tilde{S}$

1. Lemma 1: Let $\vec{y_1} \, \epsilon \, \tilde{S}$ and $\vec{y_2} \, \epsilon \, \tilde{S}$
   Then $\vec{y_1} \; - \; \vec{y_2} \, \epsilon \, S$
   Moreover $\vec{y_1} \; - \; \vec{y_2} \; = \; \vec{x_1}$ where $\vec{x_1} \, \epsilon \, S$ then $\vec{y_1} \; = \; \vec{x_1} \; + \; \vec{y_2}$
2. Lemma 2: We can obtain $\tilde{S}$ by finding one solution $\vec{y_2} \, \epsilon \, \tilde{S}$ [a particular solution] and then add it to each and every solution in set S one by one
   If there are no solutions i.e. you cannot obtain $\vec{y_2}$, then $\tilde{S}$ is empty

7th September 2018
Lecture 1

1 Differential Equations

1.1 Introduction

In calculus we deal with functions sometimes the functions are provided explicitly, however often we only have information about derivatives(s) of the function. *Eg:* Newtown's law in 1 dimension, with constant mass:

$$F = \frac{d\vec{p}}{dt}$$

Using momentum as $\vec{p} = m\vec{v} = m\frac{dx}{dt}$

$$F = m\frac{d^2x}{dt^2}$$

Eg: one dimensional Motion under gravity, $g = \frac{d^2x}{dt^2} = 9.81ms^{-2}$

$$F = m\vec{a} = m\vec{g}$$

1.1.1 Definitions and Vocabulary

There are many types and we need some vocabulary to distinguish between them
Def #1: A differential equation [DE] is an equation which involves the derivative(s) of some unknown function(s).
Def #2: When we have a DE we will have one (or more) function(s) which we are trying to determine. This function(s) is(are) called the dependent variable(s). The other variables are called independent variables.
Eg: $\frac{d^2x}{dt^2} = g$, g is constant, x is the dependent variable, t is the independent variable
Eg: $c\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial t^2}$, c is constant, z is the dependent variable, t, x is the independent variable.

These are usually obvious: *Eg:* $\frac{dx}{dy} = 2$ or $\frac{dy}{dx} = \frac{1}{2}$ (which is usually dependent)

Def #3: A DE is called an ordinary differential equation (ODE) to mean that it only contains ordinary derivatives.
A DE is called a partial differential equation (PDE) to mean that it contains partial derivatives.
Eg: ODE

$$\frac{d^2x}{dt^2} + sin(x)\frac{dx}{dt} + x^2 - sin(t^2) = 0$$

Eg: PDE

$$\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^3 z}{\partial t^3} + t^2 = 1$$

Note: ECE 205 has at least 10 weeks of ODE

Def #4: The order of a DE is the highest derivative that appears. In general: higher order, means more difficult
Eg:

$$\frac{d^3x}{dt^3} + (\frac{d^2x}{dt^2})^2 + (\frac{dx}{dt})^{29} + sin(t) = 0$$

Def #5: We say that a DE is linear to mean that all the terms in the dependent variable(s) are linear expressions
Eg: Let $x = x(t)$, x is the dependent variable
Not linear

$$\frac{dx}{dt} + x^2 = 0$$

Linear

$$\frac{d^2x}{dt^2} + xt^2 = 0$$

Not linear (product of terms in x)

$$\frac{d^2x}{dt^2}\frac{dx}{dt} + sin(t) = 0$$

Linear

$$\frac{d^2x}{dt^2} + sin(t)\frac{dx}{dt} + cos(t)x = t^2$$

Not linear

$$\frac{d^2x}{dt^2} + sin(t)\frac{dx}{dt} + cos(tx) = t^2$$

In this course we will examine, 1st order ODE (4 types, one of them is linear), 2nd order linear

7th September 2018
Lecture 2

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Def #6: The most general n^{th} order ODE is of the form

$$a_n(t)\frac{d^n x}{dt^n} + a_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + a_{n-2}(t)\frac{d^{n-2}x}{dt^{n-2}} + ... + a_0(t)\frac{dx}{dt} + f(t) = 0$$

The coefficient function $a_n(t) \neq 0$, meaning it cannot be the 0^{th} function all the time, but it can have occasional zeros, *Eg:* It can be $sin(t)$. If it is always zero then it will not be n^{th} order anymore

Def #7: We say that n^th order ODE is homogeneous to mean that $f(t) \equiv 0$, *Eg:* $a_1(t)\frac{dx}{dt} + a_0(t)x$

Def #8: If the ODE is not homogeneous then we call it in-homogeneous or non-homogeneous

Def #9: Given a linear in-homogeneous ODE,

$$a_n(t)\frac{d^n x}{dt^n} + a_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + a_{n-2}(t)\frac{d^{n-2}x}{dt^{n-2}} + ... + a_0(t)\frac{dx}{dt} + f(t) = 0,\ \ f(t) \not\equiv 0$$

Then we call the ODE,

$$a_n(t)\frac{d^n x}{dt^n} + a_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + a_{n-2}(t)\frac{d^{n-2}x}{dt^{n-2}} + ... + a_0(t)\frac{dx}{dt} + 0 = 0,$$

the associated homogeneous equation

Def #10:
- A *solution* of an ODE is a function that satisfies the ODE
- A *solution set* of a DE is the set of all solutions of the DE
- The *general solution* of a DE is the form of typical solutions

10th September 2018

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$$\int_a^b f(x)dx = N \ \ a,\ b \,\epsilon\, \mathbb{R} \implies \frac{d}{dt}(\int_a^b f(x)dx) = \frac{dN}{dt} = 0$$

$$\int^{\tau} sin(x)dx = g(\tau) \implies g(\tau) = -cos(\tau) + c \implies \frac{d}{d\tau}\int^{\tau} sin(x)dx = sin(\tau)$$

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Week 2

2 First Order Ordinary Differential Equations

- Seperable
- Linear
- Exact
- Integration factor method

2.1 Seperate ODE's

We say that $\frac{dy}{dx} = F(x, y)$ is a seperable ODE to mean that $F(x, y) = A(x).B(y)$ i.e. $F(x, y)$ is the product of a function of "x" only and "y" only

Method of Solution

1. Is $B(y)$ ever zero?
   If $B(y_1) = 0$, then $y = y_1$ is a solution of this ODE
   Find all such functions $y = y_1$, $y = y_2, \ldots$
2. Otherwise when $y \neq y_1$, $y \neq y_2, \ldots$, $B(y) \neq 0$, which means we can divide by it

$$\frac{dy}{dx} = A(x).B(y)$$

$$\implies \frac{1}{B(y)}\frac{dy}{dx} = A(x)$$

$$\implies \int \frac{1}{B(y)}\frac{dy}{dx}dx = \int A(x)dx$$

$$\implies \int \frac{1}{B(y)}dy = \int A(x)dx$$

On integrating if you can,

$$H(y) = K(x)$$

Only need constant on one side of these integrations

Eg:

$$\frac{dy}{dx} = -xy$$

$$\implies \frac{1}{y}\frac{dy}{dx} = -x$$

$$\implies \int \frac{1}{y}\frac{dy}{dx}dx = \int -xdx$$

$$\implies ln|y| = \frac{-x^2}{2} + c$$

$$\implies |y| = e^{\frac{-x^2}{2}+c}$$

$$\implies |y| = e^{\frac{-x^2}{2}}.e^c, \;\; e^c = A\epsilon\mathbb{R}^+$$

$$\implies y = \pm Ae^{\frac{-x^2}{2}}, \;\; A\,\epsilon\,\mathbb{R}^+$$

$$\implies y = Ae^{\frac{-x^2}{2}}, \;\; A\,\epsilon\,\mathbb{R}\,-\,\{0\}$$

But solution set,

$$\{y = \pm Ae^{\frac{-x^2}{2}}, \;\; A\,\epsilon\,\mathbb{R}\}$$

11th September

2.2 Linear

The differential equation $\frac{dy}{dx} = F(x,y)$ is said to be linear to meant that $F(x,y) = p(x)y + q(x)$, for some function $p(x)$ and $q(x)$ of x along

$$\frac{dy}{dx} = p(x)y + q(x) \implies \frac{dy}{dx} - p(x)y = q(x) \tag{1}$$

Homogeneous means $q(x) \equiv 0$ when not homogeneous.

$$\frac{dy}{dx} - p(x)y = 0 \tag{2}$$

This 2 is called the associated homogeneous DE to 1

Important Idea: Given any $H(x)$

$$\int (\frac{d}{dx}H(x))dx = H(x)$$

i.e. the easiest thing to integrate is a derivative CFT(I)

Method of Solution

1. Collect the y terms
$$\frac{dy}{dx} - p(x)y = q(x)$$
2. It would be nice if the LHS was $\frac{d}{dx}(H(x))$, however it is not.
We multiply the differential equation *both sides* by a function $I(x)$ called an Integration Factor (IF)
$$I(x)\frac{dy}{dx} - p(x)I(x)y = q(x)I(x)$$
3. We now insist that the LHS is a pure derivative, i.e. $\frac{d}{dx}H(x)$
We insist that the LHS is $\frac{d}{dx}[I(x)y(x)]$
4. What we got, and what we want:
$$I\frac{dy}{dx} - pIy = I(x)\frac{dy}{dx} + y(x)\frac{dI}{dx} \implies -p(x)I(x)y(x) = y(x)\frac{dI}{dx}$$
for $y(x) \neq 0$
$$\implies -p(x)I(x) = \frac{dI}{dx} \implies -\int^x p(a)da = \int \frac{1}{I(x)}\frac{dI}{dx}dx \text{ [Solving 1st order separable ODE]}$$
But don't forget to find the solutions of $I(x) \equiv 0 \implies I = 0$, but that solution is not useful
$$\implies -\int^x p(a)da = ln(|I(x)|) \implies |I(x)| = e^{-\int^x p(a)da} \implies I(x) = \pm e^{-\int^x p(a)da}$$

5. Choosing the positive integration factor, and moreover we will choose the constant of the integration on the right to be zero so that we get exactly one Integration factor.

6. We now know that the LHS of the DE is $\frac{d}{dx}(I(x)y(x))$. Thus our DE is:

$$\frac{d}{dx}(Iy) = q(x)I(x) \implies \int \frac{d}{dx}(I(x)y(x))dx = \int (q(x)I(x))dx$$

$$\implies I(x)y(x) = \int (q(x)I(x))dx$$

Eg:

$$\frac{dy}{dx} = 2x + 3y \implies \frac{dy}{dx} - 3y = 2x$$

Integrating factor $I(x) = e^{-\int^x p(a)da} \implies I(x) = e^{-\int^x 3da} \implies I(x) = e^{-3x}$

$$\implies e^{-3x}\frac{dy}{dx} - e^{-3x}3y = 2xe^{-3x} \implies e^{-3x}(y) = \int 2xe^{-3x}dx$$

$$\implies e^{-3x}y = 2x\int e^{-3x} - \int \frac{d(2x)}{dx}[\int e^{-3x}dx]dx$$

$$\implies e^{-3x}y = 2x\frac{e^{-3x}}{-3} - \int 2[\frac{e^{-3x}}{-3}]dx$$

$$\implies e^{-3x}y = 2x\frac{e^{-3x}}{-3} - 2\frac{e^{-3x}}{9} + c \qquad c \in \mathbb{R}$$

$$\implies y = -\frac{2x}{3} - \frac{2}{9} + ce^{3x} \qquad c \in \mathbb{R}$$

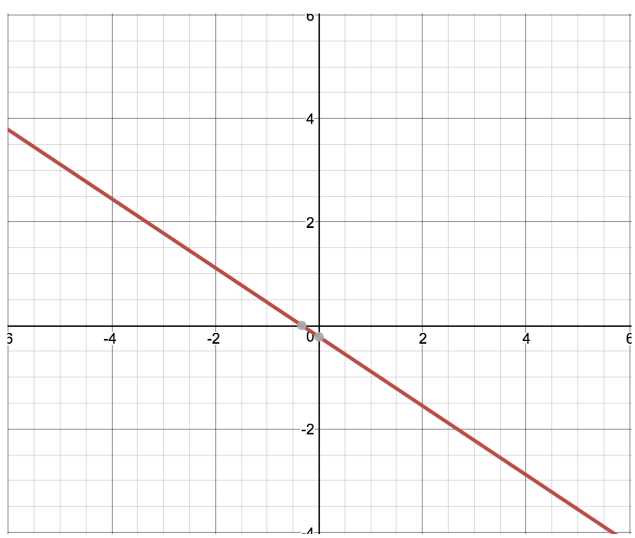


Figure 1: c = 0

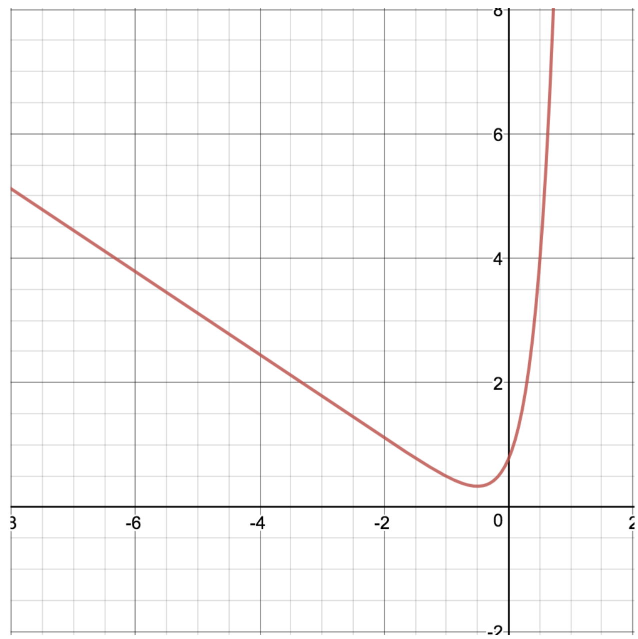


Figure 2: c = 1

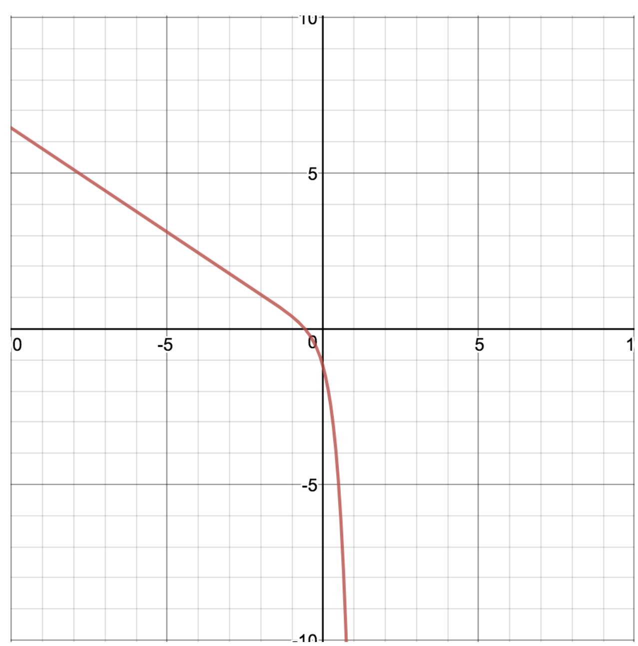


Figure 3: c = -1

14th September Part 1

2.2 Linear ODE Revisited

Consider

$$\frac{dy}{dx} = p(x)y + q(x)$$

$$\frac{dy}{dx} - p(x)y = q(x) \qquad \text{Solution Set } S$$

$$\frac{dy}{dx} - p(x)y = 0 \qquad \text{Solution Set } \widetilde{S}$$

Lemma 1: If $y_1, y_2 \ \epsilon \ \widetilde{S}$ then $c_1y_1 + c_2y_2 \ \epsilon \ \widetilde{S}$, $c_1,\ c_2 \epsilon \mathbb{R}$
Proof: We know $\frac{dy}{dx} - p(x)y_1 = 0$ and $\frac{dy}{dx} - p(x)y_2 = 0$, what about $\frac{d}{dx}(c_1y_1 + c_2y_2) - p(x)(c_1y_1 + c_2y_2)$
On simplifying the last equation we get it's equal to zero

Lemma 2: Let $z_1,\ z_2\ \epsilon S$ then $z_1,\ z_2\ \epsilon \widetilde{S}$
Proof: We know $\frac{dz_1}{dy} - p(x)z_1 = q(x)$ and $\frac{dz_2}{dy} - p(x)z_2 = q(x)$, what about $\frac{d}{dx}(z_1 - z_2) - p(x)(z_1 - z_2) = 0$ On simplifying the last equation we get it's equal to zero

Lemma 3: Let $y_p \ \epsilon \ S$ (a particular solution), then $S = \{y + y_p : \ y \ \epsilon \ \widetilde{S}\}$ *Eg:* $\frac{dy}{dx} - 3y = 3x$
Consider $\frac{dy}{dx} - 3y = 0$, One solution is $y \equiv 0$ otherwise $\int \frac{dy}{y} = \int 3dx$

$$\implies ln|y| = 3x + c \implies y = De^{3x} \qquad D\epsilon\mathbb{R}$$

Solution set of the associated homogeneous ODE is $\{De^3x : D \ \epsilon \ \mathbb{R}\}$ We need a y_p - a special solution of the DE. We will guess this one. Let

$$y_p = ax + b$$

, a and b to be found.

$$\frac{dy_p}{dx} - 3y_p \equiv 2x \implies a - 3(ax + b) \equiv 2x \implies a - 3b - 3ax \equiv 2x$$

Comparing coefficients:

$$-3a = 2 \implies a = \frac{-2}{3}$$

$$a = 3b \implies b = \frac{-2}{9}$$

Thus $y_p = -\frac{2}{3}x - \frac{2}{9}$ and the complete solution set is $\{-\frac{2}{3}x - \frac{2}{9} + De^3x : D \ \epsilon \ \mathbb{R}\}$

2.3 Initial Value Problems

When we solve a first order ODE we do not get a single function. We actually obtain a one parameter (c - int constant) family of functions.
Often in practice we are provided with some extra information which allows us to choose one and only one of this family *i.e.* the unique solution.
Often we have $x = x(t)$ and conditions at $t = 0$ and so the condition(s) of are referred to as initial condition. The differential equation along with the initial condition together form an initial value problem

14th September Part 2

Aside Differentials dx change in x, dy change in y, leads to dz change in z

$$z = H(x, y)$$

$$dz = \frac{\partial H}{\partial x} dx + \frac{\partial H}{\partial y} dy$$

But if we're on $z = \quad constant \implies dz = 0 \implies z = c = H(x,\ y)$

$$0 = \frac{\partial H}{\partial x} dx + \frac{\partial H}{\partial y} dy \text{ [implicit equation for } y]$$

and we can do something called implicit differentiation

$$\frac{\partial H}{\partial x} + \frac{\partial H}{\partial y}\frac{dy}{dx} = 0$$

2.4 Exact 1^{st} order ODEs

Suppose we have our 1^{st} order ODE

$$\frac{dy}{dx} = F(x, y)$$

. When we have to solve the ODE, we can always express the answer in the form

$$G(x, y) = k$$

where k is a constant.

Our solution from earlier becomes:

$$G(x, y) = \frac{y + \frac{2}{3}x + \frac{2}{9}}{e^{3x}}$$

If we differentiate this expression:

$$\frac{\partial G}{\partial x} + \frac{\partial G}{\partial y}\frac{dy}{dx} = 0$$

$$\implies \frac{dy}{dx} = -\frac{\frac{\partial G}{\partial x}}{\frac{\partial G}{\partial y}}$$

Comparing the two

$$\frac{dy}{dx} = F(x, y)$$

and

$$\implies \frac{dy}{dx} = -\frac{\frac{\partial G}{\partial x}}{\frac{\partial G}{\partial y}}$$

We must have,

$$-\frac{\frac{\partial G}{\partial x}}{\frac{\partial G}{\partial y}} = F(x, y)$$

There is some choice here:

$$\frac{\partial G}{\partial x} = m(x, y)F(x, y)$$
$$\frac{\partial G}{\partial y} = -m(x, y)$$

If we can find a $G(x, y)$ with $\frac{\partial G}{\partial x} = m(x, y)F(x, y)$ and $\frac{\partial G}{\partial y} = -m(x, y)$. Then the solution is $G(x, y) = A$.

In practice we consider $\frac{dy}{dx} = F(x, y)$ is when teh function $F(x, y) = \frac{-M(x,y)}{N(x,y)}$, which we re-write as:

$$M(x, y)dx + N(x, y)dy = 0$$

We want a solution of the following form $G(x, y) = c$

$$\frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy = 0$$

We need

$$\frac{\partial G}{\partial x} = M(x, y)$$

and

$$\frac{\partial G}{\partial y} = N(x, y)$$

How do we know whether these partial differential equations can be solved? If

$$\frac{\partial^2 G}{\partial x \partial y} = \frac{\partial^2 G}{\partial y \partial x}$$

Extending the earlier equations:

$$\frac{\partial}{\partial y}\frac{\partial G}{\partial x} = \frac{\partial M(x, y)}{\partial y} \equiv \frac{\partial}{\partial x}\frac{\partial G}{\partial y} = \frac{N(x, y)}{\partial x}$$

Given the ODE $mdx + Ndy = 0$ we say that the equation is exact to mean that the condition above is satisfied $\frac{N(x,y)}{\partial x} = \frac{\partial M(x,y)}{\partial y}$

Eg:

$$\frac{dy}{dx} = \frac{-1}{\frac{-(6y^2-x^2+3}{3x^2-3xy+2}}$$

We re-write that as: $(3x^2 - 3xy + 2)dx + (6y^2 - x^2 + 3)dy = 0$
Is it possible that $\frac{\partial G}{\partial x} = 3x^2 - 2xy + 2 = M$ and $\frac{\partial G}{\partial y} = 6y^2 - x^2 + 3 = N$
We can solve this if $\frac{\partial^2 G}{\partial x \partial y} = \frac{\partial^2 G}{\partial y \partial x}$
$\frac{\partial M}{\partial y} = -2x$ and $\frac{\partial N}{\partial x} = -2x$ There is a solution, since this DE is exact

We want a function G which $\frac{\partial G}{\partial x} = 3x^2 + 2xy + 2$ A solution is $G(x, y) = x^3 - x^2y + 2x$
The most general solution is $G(x, y) = x^3 - x^2y + 2x + A(y)$ where $A(y)$ is a function of y

We also need G to satisfy $\frac{\partial G}{\partial y} = 6y^2 - x^2 + 3$
A solution is $G = 2y^3 - x^2y + 3y$.
The most general solution is $G = 2y^3 - x^2y + 3y + B(x)$ where $B(x)$ for some function $B(x)$

Thus, the general solution of the DE is: $G(x, y) = x^3 + 2y^3 - x^2y + 2x + 3y + c$

18th September

Last time we had the solution $G(x, y) = x^3 + 2y^3 - x^2y + 2x + 3y + k$
Solution is $G = c$, but we don't want two constants so we have to choose either of these:

$x^3 + 2y^3 - x^2y + 2x + 3y + k = 0$ or $x^3 + 2y^3 - x^2y + 2x + 3y = c$

Instead of remembering formulas remember:

$Mdx + Ndy = 0$ and that our solution for that is $\frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial y}dy = 0$ and thus, $M = \frac{\partial G}{\partial x}$ and $N = \frac{\partial G}{\partial y}$.

And for a nice function f

$$\frac{\partial^2 G}{\partial x \partial y} = \frac{\partial^2 G}{\partial y \partial x} \implies \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \text{Clairaut's theorem on equality of mixed partials}$$

2.4.1 Why could the Clairaut's theorem on equality of mixed partials fail for a G

$$xe^{xy} = c$$

Differentiating,

$$(e^{xy} + xye^{xy})dx + x^2e^{xy}dy = 0 \implies \frac{dy}{dx} = -\frac{e^{xy} + xye^{xy}}{x^2e^{xy}} \qquad xy \neq 0$$

And since $e^{xy} \neq 0$ then can we not factor it out of the equation

$$(1 + xy)dx + x^2dy = 0$$

Now the question being whether this is exact? Let

$$M = 1 + xy \qquad N = x^2$$
$$\frac{\partial M}{\partial y} = x \qquad \frac{\partial N}{\partial x} = 2x$$

This means that the equation earlier is not exact since there are not exactly the same functions.

Now we are in a situation that we need to find the missing function, and need to find what it is. It's called the Integration factor method

2.5 Integration Factors

Suppose we have to find a first order ODE that is not exact.
How de we proceed?
Solution: We multiply the ODE by an unknown function $I(x, y)$, called an integration factor. We try to make the new ODE exact

$$\text{i.e.} \quad Mdx + Ndy = 0$$

And $M_y \not\equiv N_x$ which is not exact, and we remedy by multiplying with I $\implies MIdx + NIdy = 0$
We want this to be exact $(MI)_y \equiv (NI)_x$, which now imposes a restriction on I

Sadly we started from an ODE, but we moved to a PDE i.e. we are in a bigger mess now. Differentiating:

$$I\frac{\partial M}{\partial y} + M\frac{\partial I}{\partial y} \equiv I\frac{\partial N}{\partial x} + N\frac{\partial I}{\partial x} \qquad (**)$$

This is very hard to solve in general, however there are two special cases:
Case 1: $I(x, y) = I(x)$ The PDE $(**)$ becomes

$$I\frac{\partial M}{\partial y} \equiv I\frac{\partial N}{\partial x} + N\frac{dI}{dx}$$

Collect all the terms in I together,

$$\frac{1}{I}\frac{dI}{dx} = \frac{1}{N}(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})$$

Since the LHS in the left, is just a function of x only, then the RIHS must also be a function of x only.
The RHS involves terms from the ODE, and we now have a test for an integration factor of the form $I = I(x)$

Eg: $= \frac{dy}{dx} = -\frac{1+xy}{x^2}$, given $x^2 \neq 0$

$$(1 + xy)dx + (x^2)dy$$
$$\implies M = 1 + xy \implies M_y = x$$
$$\implies N = x^2 \implies N_x = 2x$$

Thus, $x \not\equiv 2x$, i.e. they are not identical, but they can be equal at $x = 0$ We examine $\frac{M_y - N_x}{N} = \frac{x-2x}{x^2} = -\frac{1}{x}$
Since is this a function of x only,

- There will be an integration factor for the DE of the form $I(x,y) = I(x)$
- $\frac{dlnI}{dx} = -\frac{1}{x}$, integration $ln(I) = ln|\frac{1}{x}|$ and one solution is $I(x) = x^{-1}$

Solution: $(1+xy)dx + x^2dy = 0$, multiplying by $I(x)$ we get

$$(x^{-1} + y)dx + xdy = 0$$

Find $G(x,y)$ such that $G_x = x^{-1} + y \implies ln|x| + xy + H(y) = G$ and $G_y = x \implies xy + H(x) = G$
Therefore $G(x,y) = xy + ln|x|$ and the solution becomes: $G(x,y) = xy + ln|x| = c$

21st September Part 1

Case 2: $I = I(y)$ and $I_x = 0$

$$I_yM + IM_y - IN_x = 0$$
$$\frac{dI}{dy}\frac{1}{I} = \frac{N_x - M_y}{M}$$

The LHS is only a function of y then the RHS must also be a function of y
We need a combination $\frac{M_y - N_x}{-M}$ to be a function of only y. In this, we will be able to obtain $I(y)$ i.e. an integration factor which is a function of only y

$$\frac{dy}{dx} = \frac{-y^2}{1+xy} \implies y^2dx + (1+xy)dy = 0$$

$M = y^2$ and $N = 1 + xy$
$M_y = 2y$ and $N_x = y$, so this DE is not exact
We multiply by $I(x,y)$ to get

$$Iy^2dx + I(1+xy)dy = 0 \qquad (1)$$

Consider $I = I(x)$ $(Iy^2)_y = I2y$ and $(I(1+xy))_x = \frac{dI}{dx}(1+xy) + I_y$
If $(Iy^2)_y = (I(1+xy))_x$ then $2yI = \frac{dI}{dx}(1+xy) + I_y \implies \frac{dI}{dy}\frac{1}{I} = \frac{y}{1+xy}$, in which sadly RHS is not just a function of x while LHS is
Now considering, $I = I(y)$ $(Iy^2)_y = \frac{dI}{dy}y^2 + 2yI$ and $(I(1+xy))_x = I_y$
If we equate these, $y^2\frac{dI}{dy} + 2yI = I_y \iff y^2\frac{dI}{dy} + yI$
Assuming $y \not\equiv 0$,

$$\frac{dI}{dy}\frac{1}{I} = -\frac{1}{y}$$

Integrating we get,

$$ln|I| = -ln|y| \implies ln|I||y| = 0 \implies |Iy| = e^0 \implies Iy = \pm 1 \implies I = \frac{1}{y} \quad \text{(Choosing the positive one only)}$$

(1) becomes,

$$ydx + \frac{1+xy}{y}dy = 0$$

There exists a $G(x,y)$ such that $G_x = y \implies G = xy + F(y)$ and $G_y = \frac{1}{y} + x \implies G = xy + ln|y| + H(x)$
Thus,

$$G = xy + ln|y| \quad \text{and the Solution is } G = xy + ln|y| = c$$

3 Second Order Linear ODEs

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

We will only deal with those equation when a_2, a_1, a_0 are all constants and not functions
We say that the DE has constant coefficients to mean that $a_2(x) = constant, a_1(x) = constant, a_0(x) = constant$
Dividing by $a_2(x)$, we get

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = f(x)$$

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = f(x) \qquad \text{Solution set } S$$

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0 \qquad \text{Solution set } \tilde{S}$$

The second equation above is the associated homogeneous DE

Lemma 1: $\tilde{S}$ is a vector subspace of function space (It is a 2 dimensional)
Lemma 2: $z_1, z_2 \epsilon S$ then $z_1 - z_2 \epsilon \tilde{S}$
Lemma 3: $S = \{y_p + y : y \epsilon \tilde{S}\}$ where $y_p \epsilon S$

We begin by considering $\tilde{S}$, i.e. solve

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$$

21st September Part 2

Try to e^{mx} and adjust m to make this work
Putting $y = e^{mx}$ into the LHS of the DE with $\frac{dy}{dx} = me^{mx}$ and $\frac{d^2y}{dx^2} = m^2e^{mx}$
We get, $m^2e^{mx} + pme^{mx} + qe^{mx} - (m^2 + pm + q)e^{mx}$ We want to get 0 in RHS, thus We need $(m^2 + pm + q)e^{mx} \equiv 0$
$e^{mx} \neq 0 \implies (m^2 + pm + q) = 0$
There are three cases:

- Two real distinct roots m_1, m_2
  Two solutions e^{m_1x}, e^{m_2x} which are two linearly independent solutions
  General solution: $y = Ae^{m_1x} + Be^{m_2x}$
  Solution set is $\{Ae^{m_1x} + Be^{m_2x}\} = span(\{e^{m_1x}, e^{m_2x}\})$
- Real equal roots $m_1 = m_2$; $y = e^{m_1x}$ is a solution
  This is bad luck (i.e. Conrad Resonance of Type 1)
  Another independent solution is $y_2 = xy_1 = xe^{m_1x}$
  General solution: $Ae^{m_1x} + Bxe^{m_1x}$
  Solution set is $\{Ae^{m_1x} + Bxe^{m_1x}\} = \{span(\{e^{m_1x}, xe^{m_1x}\})$
- Two complex roots $m_1 = \alpha + j\beta,\;\; m_2 = \alpha - j\beta \qquad \alpha \,\epsilon\, \mathbb{R}, \quad \beta \,\epsilon\, \mathbb{R} \,-\, \{0\}$

$$y_1 = e^{m_1x} = e^{(\alpha+j\beta)x} = e^{\alpha x}e^{j\beta x} = e^{\alpha x}(cos\beta x + jsin\beta x)$$
$$y_2 = e^{m_2x} = e^{(\alpha-j\beta)x} = e^{\alpha x}e^{-j\beta x} = e^{\alpha x}(cos\beta x - jsin\beta x)$$
$$\text{sum/2: } e^{\alpha x}cos(\beta x) = y_1 \qquad \text{This is real}$$
$$\text{difference/(2j): } e^{\alpha x}sin(\beta x) = y_2 \qquad \text{This is real too}$$

  General solution: $Ae^{\alpha x}cos(\beta x) + Be^{\alpha x}sin(\beta x)$
  Solution set is $\{Ae^{\alpha x}cos(\beta x) + Be^{\alpha x}sin(\beta x), \quad A,\ B \,\epsilon\, \mathbb{R}\}$

Note: $Acos(\beta t) + Bsin(\beta t) = Rcos(\beta t - \phi)$ then $R = \sqrt{A^2 + B^2}$ and $tan(\phi) = \frac{B}{A}$
Eg: Solve

- $y'' + 2y' - 8y = 0$
  Try e^{mx}, we get $(m^2 + 2m - 8)e^{mx} = 0 \implies (m^2 + 2m - 8) = 0$
  $\implies (m + 4)(m - 2) = 0 \implies m = -4, 2$
  The two linearly independent solutions are $y_1 = e^{-4x},\ y_2 = e^{2x}$
- $y'' - 10y' + 25y = 0$ Try e^{mx}, we get $(m^2 - 10m + 25)e^{mx} = 0 \implies (m^2 - 10m + 25) = 0$
  $\implies (m - 5)(m - 5) = 0 \implies m = 5$
  One solution is $y_1 = e^{5x}$. This is bad luck (i.e. Conrad Resonance of Type 1), and therefore the second solution is $y_2 = xe^{5x}$
  The two linearly independent solutions are $y_1 = e^{5x},\ y_2 = xe^{5x}$
  Solution set is $\{Ae^{5x} + Bxe^{5x},\ A,\ B \,\epsilon\, \mathbb{R}\}$
  Let us check y_2 :, $y_2' = (1 + 5x)e^{5x}, y_2'' = (10 + 25x)e^{5x}$
  Putting this back in the original differential equation results in zero
- $y'' - 2y' + 5y = 0$ Try e^{mx}, we get $(m^2 - 2m + 5)e^{mx} = 0 \implies (m^2 - 2m + 5) = 0$
  $\implies m = 1 \pm 2j$
  We have

24th September

Today we consider a particular solution of the inhomogeneous differential equation.

$$y'' + py' + qy = f(x)$$

We want y_p

3.1 The method of undetermined coefficients

The guiding principle is to choose something "like" $f(x)$

$f(x)$	**Try for** y_p
Polynomial of order n	Polynomial of order n
Exponential $e^{\alpha x}$	$ae^{\alpha x}$
$sin(\beta t)$ or $cos(\beta t)$	$asin(\beta t) + bcos(\beta t)$
Sums and/or products of above	Sums and/or products of above

Eg: $y'' + 2y' - 8y = x^2 + 2x + 3$
When we solved the homogeneous equation $y'' + 2y' - 8y = 0$ the solution was $y = Ae^{2x} - Be^{-4x}$

Let $y_p = ax^2 + bx + x \implies y_p' = 2ax + b$ and $y_p'' = 2a$ Putting this into the LHS of the equation:

$$2a + 4ax + 2b - 8ax^2 - 8bx - 8c$$

while the RHS is $x^2 + 2x + 3$
For $x^2 \implies 8a = 1 \implies a = -\frac{1}{8}$
For $x \implies 4a - 8b = 2 \implies b = \frac{4a-2}{8} \implies b = -\frac{5}{16}$
For constants $\implies 2a + 2b - 8c = 3 \implies c = \frac{2a+2b-3}{8} \implies c = -\frac{31}{64}$
$\implies y_p = -\frac{1}{8}x^2 - \frac{5}{16}x - \frac{31}{64}$, with the general solution being $y = y_p + Ae^{2x} + Be^{-4x}$

Going to be on the test?
Eg: $y'' + 2y' - 8y = xe^{3x}$
Try $y_p = (A + Bx)e^{3x}$, write down the most general polynomial of the RHS
$y_p' = [3A + 3Bx + +B]e^{3x}$ and $y_p'' = [9A + 9Bx + +3B + 3B]e^{3x}$
Put these terms into the LHS of the equation to get:
$(9A + 6B + 9Bx + 6A + 2B + 6Bx - 8a - 8Bx)e^{3x} = ((7A + 8B) + (7B)x)e^{3x}$
We want to get xe^{3x} Comparing coefficients:
For $xe^{3x} \implies 7B = 1 \implies B = \frac{1}{7}$
For $e^{3x} \implies 7A + 8B = 0 \implies A = -\frac{8}{49}$
$y_p = (-\frac{8}{49} + \frac{x}{7})e^{3x}$ with the general solution being $y = y_p + Ce^{2x} + De^{-4x}$, $C,\ D\ \epsilon\mathbb{R}$

Eg: $y'' + 2y' - 8y = sin(2x)$
Try $y_p = asin2x + bcos2x$, $y_p' = 2acos2x - 2bsin2x$ and $y_p'' = -4asin2x - 4bcos2x$

We put y_p into the LHS of the DE:
$-4y_p + 2(acos2x - 2bsin2x) - 8y_p = 4acos2x - 4bsin2x - 12asin2x - 12bcos2x$
We get $(-4b-12a)sin2x + (4a-12b)cos2x$ which we want to equal $sin(2x)$ Comparing coefficients:
For $cos2x \implies 4a - 12b = 0 \implies a = 3b$
For $sin2x \implies -4b - 12a = 1 \implies a(-12 - \frac{4}{3}) = 1 \implies a = -\frac{3}{40}$ and $b = -\frac{1}{40}$

$y_p = -\frac{3}{40}sin2x + -\frac{1}{40}cos(2x)$ with the general solution being $y = y_p + Ce^{2x} + De^{-4x}$, $C,\ D\ \epsilon\mathbb{R}$

Eg: $y'' + 2y' - 8y = e^{2x}$
Try $y_p = ae^{2x}$ We put y_p into the LHS of the DE:
LHS becomes zero, because it is already of the homogeneous solution. We know that this will give zero, i.e. we have bad luck (i.e. Conrad Resonance of Type 2)
Instead we try: $y_p = x(ae^{2x}) \implies y_p' = a(2x+1)e^{2x}$ and $y_p'' = a(4x+2+2)e^{2x}$
Putting this in the LHS of the DE we get $a(4x+4+4x+2-8x)e^{2x} = 6ae^{2x}$
We want this to be equal to RHS, we need $6a = 1 \implies a = \frac{1}{6}$

Eg: $y'' - 10y' + 25y = x^2e^{5x}$
When we solved the homogeneous equation $y'' - 10y' + 25y = 0$ the solution was $y = Ae^{5x} - Bxe^{5x}$
What do we try: $y_p = (ax^2+bx+c)e^{5x}$, this way the terms with b and c are part of the homogeneous and they will get cancelled i.e. we have bad luck (i.e. Conrad Resonance of Type 3)
We use x^2 because it is over 2 Thus we have to try: $y_p = x^2(ax^2 + bx + c)e^{5x}$

25th September

3.2 Initial Conditions

Since the DE is 2^{nd} order we need 2 peices of information for a particular DE

1. Initial Values/conditions
   $x(0) = x_0$ and $\frac{dx}{dt}|_{t=0} = V_0$
2. Boundary Value Problem
   $x(0) = x_0$ and $x(T) = x_1$
3. Mixed conditions
   Either $x(0) = x_0$ and $\frac{dx}{dt}|_{t=T} = V_t$ or $x(T) = x_1$ and $\frac{dx}{dt}|_{t=0} = V_0$

Eg: $y'' + 2y' - 8y = sin(2x)$ when $y(0) = 0, y'(0) = 0$
We already solved this earlier, so using given information: $y(0) = a + b - \frac{1}{40}$ and $y'(0) = 2a - 4b - 6/40 \implies a = \frac{1}{60}, b = \frac{1}{120}$

3.2.1 The Oscillator DE

$\ddot{x} + 2\lambda\dot{(x)} + \omega_0^2 x = F(t)$ i.e. $\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega_0^2 x = F(t)$

3.2.2 LCR Circuits

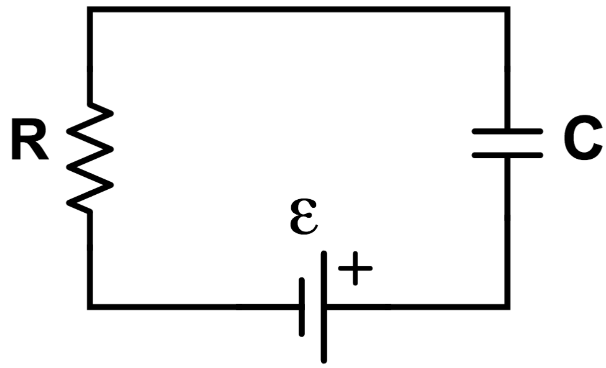
Using Kirchoff's Law's
$IR + \frac{Q}{C} + L\frac{dI}{dt} = V(t)$
Since we know $I = \frac{dQ}{dt} \implies \frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{CL} = \frac{V(t)}{L}$

3.2.3 Mass on Spring

Spring constant k
Hooke's Law $F = -kx$
Air resistance for low velocity $\approx \alpha\frac{dx}{dt}$
$\frac{d}{dt}(mv) = mg - kx - \alpha\frac{dx}{dt}$

27th September

Eg: C-R circuit



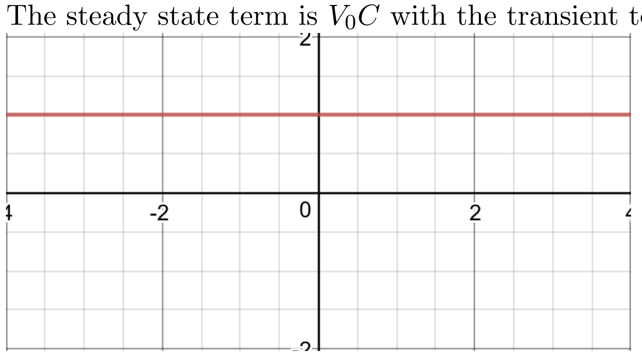
Applying Kitchoff's Law: $IR + \frac{Q}{C} = V(t)$ but $I = \frac{dQ}{dt}$
Linear for V = V(t), separable if $V(t) = K$
Multiply by integration factor $J(t)$

$$J\frac{dQ}{dt} + \frac{JQ}{RC} = J\frac{V}{R}$$

We insist that the LHS is $\frac{d}{dt}(J(t)Q(t)) \implies \frac{dJ}{dt}Q + J\frac{dQ}{dt}$
We need: $\frac{dJ}{dt}Q = \frac{JQ}{RC}$, and for $Q \not\equiv 0$ we have $\frac{dJ}{dt} = J\frac{1}{RC}$.
Solving for J in the second equation we get $J = e^{\frac{t}{RC}}$
At this point we can rewrite teh original equation as:

$$\frac{d}{dt}(Qe^{\frac{t}{RC}}) = \frac{1}{R}e^{\frac{t}{RC}}V(t)$$

1. Let $V(t) = V_0$ be a constant DC supply
   The DE is now
   $$\frac{d}{dt}(Qe^{\frac{t}{RC}}) = \frac{V_0}{R}e^{\frac{t}{RC}}$$
   Integrating:
   $$Qe^{\frac{t}{RC}} = \frac{V_0}{R}RCe^{\frac{t}{RC}} + K \qquad k\epsilon R$$
   $$Q = V_0C + Ke^{-\frac{t}{RC}}$$
   The steady state term is V_0C with the transient term being $Ke^{-\frac{t}{RC}}$



$t = RC$, RC is a time in seconds which provides us with a measure of how long the transient term is significant

2. Let $V(t) = V_0 sin(\omega t)$ a constant AC supply

$$\frac{d}{dt} e^{\frac{t}{RC}} Q = e^{\frac{t}{RC}} \frac{V_0}{R} sin(\omega t)$$

Integrating we get:

$$Q = \frac{V_0 C}{1 + \omega^2 R^2 C^2} [-\omega RC cos(\omega t) + sin(\omega t)] + ke^{\frac{-t}{RC}}$$

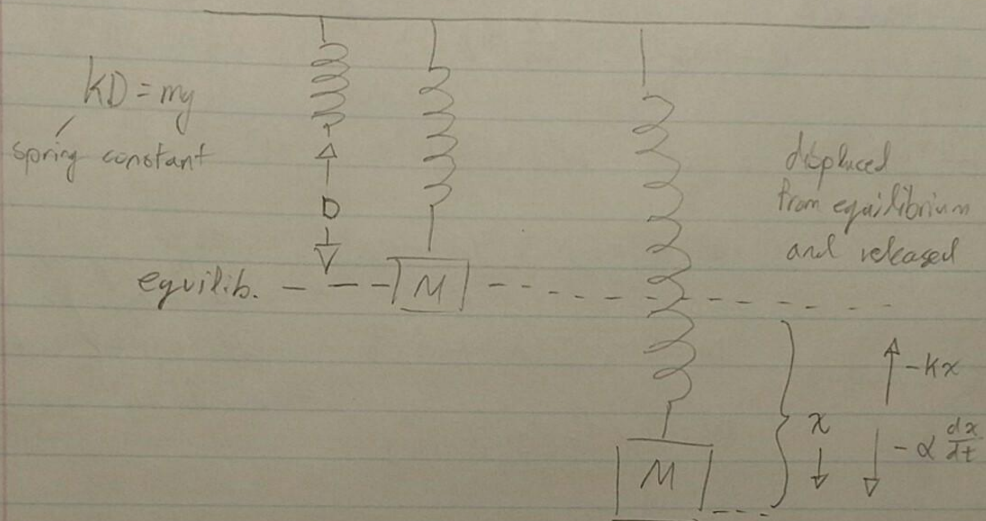
The first part is the steady state term, and other one dies away and is the transient part

28th September

pt 28

ECE 205

Mass on spring:



Forces involved: Newton's 2nd

- w/ const. mass

$$m\frac{d^2x}{dt^2} = -kx - \alpha\frac{dx}{dt} \leftarrow \text{linear air resistance}$$

$$m\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} + kx = f(t)$$

eg. B) $$\frac{d^2x}{dt^2} + \frac{\alpha}{m}\frac{dx}{dt} + \frac{k}{m}x = f(t)$$

* System will have a resonant frequency (a phenomena in all differential systems).

consider: $\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = F(t) \leftarrow$ Forcing terms

eg. A) $$\frac{d^2Q}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{Q}{LC} = \left(\frac{V(t)}{L}\right)$$

A) $Q \to x \quad \left(\frac{R}{L}\right) \to 2\lambda \quad \frac{1}{LC} = \omega_0^2 \qquad \omega_0 > 0$

B) $x \to x \quad \frac{\alpha}{m} \to 2\lambda \quad \left(\frac{k}{m}\right) = \omega_0^2 \qquad // \ \frac{\alpha}{m}$ resistance per unit mass

$$\frac{\lambda}{\omega_0} = \gamma$$

Case 1: Homogeneous $F(t)=0$

//m is unknown constant, not mass here

Try $x = e^{mt}$

$\dot{x} = mx, \quad \ddot{x} = m^2 x \qquad \frac{\lambda}{\omega_0} = \gamma$

put into DE to get: $m^2 x + 2\lambda m + \omega_0^2 = 0$

//λ term is resistance

$\rightarrow \gamma = \frac{\lambda}{\omega_0}$ is resistance

$$m = \frac{-2\lambda \pm (4\lambda^2 - 4\omega_0^2)^{\frac{1}{2}}}{2}$$

this is why we have 2λ, so the 2's cancel out

$$m = -\lambda \pm (\lambda^2 - \omega_0^2)^{\frac{1}{2}}$$

take out ω_0^2

$$m = \omega_0 \left[-\gamma \pm (\gamma^2 - 1)^{\frac{1}{2}}\right]$$

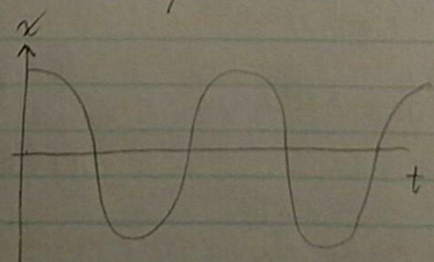
If $\gamma = 0$, no resistance, zero damping:

$m = \pm \omega_0 j$ purely complex roots

Solution: $x = A\cos(\omega_0 t) + B\sin(\omega_0 t) = R\cos(\omega_0 t - \phi)$

Oscillating motion that lasts forever, no transient term

We call this simple harmonic motion (SHM)



Period: $T = \frac{2\pi}{\omega_0}$

Freq: $f = \frac{\omega_0}{2\pi}$

Angular Freq $= \omega_0$

ECE 205

Case 2: $0 < \gamma < 1$ under-damping

- m has complex roots, but with a non-zero real part
- $m_1 = \omega_0\left[-\gamma - \sqrt{\gamma^2-1}\right]$ $m_2 = \omega_0\left[-\gamma + \sqrt{\gamma^2-1}\right]$

$m_1 = -\gamma\omega_0 - \omega_0\sqrt{1-\gamma^2}\,j$; $m_2 = -\gamma\omega_0 + \omega_0\sqrt{1-\gamma^2}\,j$ // we factor out j because $0<\gamma<1$ so sqrt of negative #

Solution:

$$x = e^{-\gamma\omega_0 t}\left[A\sin\left(\omega_0\sqrt{1-\gamma^2}\,t\right) + B\cos\left(\omega_0\sqrt{1-\gamma^2}\,t\right)\right]$$

$$x = Re^{-\gamma\omega_0 t}\cos\left(\omega_0\sqrt{1-\gamma^2}\,t - \phi\right) \qquad // x = e^{mt}$$

time dependent amplitude

The motion is oscillatory, not period (energy is lost)

$\lim_{t\to\infty} x(t) = 0$; $x=0$ is referred to as steady state, equilibrium state

The entire solution is the transient solution

Fri Sept 20

ECE 205 -

SHM: $\frac{T}{2} = \frac{\pi}{\omega_0}$ no damping:

Non periodic Oscillators:

$$T_0 = \frac{\pi}{\omega_0\sqrt{1-\gamma^2}} > \left(\frac{\pi}{\omega_0}\right)$$

Critical Damping, No oscillations:

$\gamma = 1$ $\quad m = \omega_0\left[-\gamma \pm \sqrt{\gamma^2 - 1}\right. \quad m_1 = m_2 = -\gamma\omega_0$

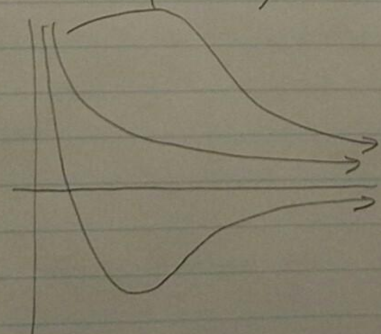
Exponential Decay:

One soln is:

$$x = Ae^{-\gamma\omega_0 t} + Bte^{-\gamma\omega_0 t}$$ A,B determined by initial conditions

function has at most one zero

$$x = (A + Bt)e^{-\gamma\omega_0 t}$$



possible graphs

$\lim_{t\to\infty} x(t) = 0$ soln tends to equilibrium

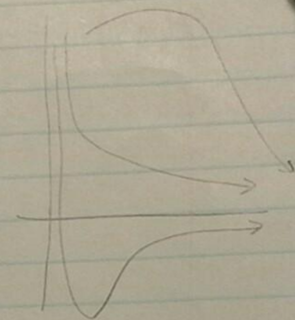
Overdamped $\gamma > 1$

$$m = \omega_0\left[-\gamma \pm \sqrt{\gamma^2 - 1}\right] \quad // \sqrt{\gamma^2-1} < \gamma$$

$$m_1 = \omega_0\left[-\gamma + \sqrt{\gamma^2-1}\right] \quad m_2 = \omega_0\left[-\gamma - \sqrt{\gamma^2-1}\right]$$

both negative

$$x = Ae^{\omega_0\left[-\gamma+\sqrt{\gamma^2-1}\right]t} + Be^{\omega_0\left[-\gamma-\sqrt{\gamma^2-1}\right]t}$$



$\lim_{t\to\infty} x(t) = 0$ solution tends to equilibrium

There are two natural time scales associated with the problem

$t_{osc} = \frac{1}{\omega_0}$ an oscillation time scale

this is a measurement of time, over which we see oscillation

$t_{damping} = \frac{1}{\lambda}$ a damping time scale

this is a time period over which we see some effects of damping

$$\boxed{\gamma = \frac{\lambda}{\omega_0} = \frac{\frac{1}{\omega_0}}{\frac{1}{\lambda}} = \frac{t_{osc}}{t_{damp}}}$$

$\gamma \ll 1 \iff t_{damping} \gg t_{osc}$ see oscillations before damping has a big effect

$\gamma \gg 1 \iff t_{osc} \gg t_{damp}$ see damping first before oscillations (there are none)

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Lemma: Critical Damping is the situation that returns to equillibrium
" faster (in general) than the overdamped situation.

Proof: In the overdamped case the general solution is the sum of two decaying solutions
One of these tends to be slower than the critically damped solution

Critically damping: $e^{-\omega_0 t} = e^{-\gamma \omega_0 t}$ $(\gamma = 1)$

Overdamped: $e^{\omega_0(-\gamma + \sqrt{\gamma^2 - 1})t}$ $(\gamma > 1)$

consider $t(\gamma) = \gamma - (\gamma^2 - 1)^{1/2}$ $t(1) = 1$

$$\frac{dt}{d\gamma} = 1 - \frac{1}{2}(\gamma^2 - 1)^{-1/2} 2\gamma = \frac{(\gamma^2-1)^{1/2} - \gamma}{(\gamma^2-1)^{1/2}} < 0$$

$\therefore t(\gamma)$ is decreasing

$$t(\gamma) = \gamma - (\gamma^2 - 1)^{1/2} < 1 \quad ; (\gamma > 1)$$
$$= -\gamma + (\gamma^2 - 1)^{1/2} > -1$$
$$= (-\gamma + (\gamma^2 - 1)^{1/2})\omega_0 t > -\omega_0 t$$
$$e^{(-\gamma + (\gamma^2 - 1)^{1/2})\omega_0 t}$$

Part 2: Driving term/ Forcing Term

$$\ddot{x} + 2\lambda\dot{x} + \omega_0^2 x = F(t)$$

ω_0 is the natural frequency of the system. We will consider

$$F(t) = cos(\Omega t)$$

Comments:

1. In many circumstances we do have a periodic driving term

2. Other functions can be decomposed into a linear combination of trig functions and we are looking at one piece of this decomposition. *(see Fourier series)*

3.2.4 Case A: No damping

$$\ddot{x} + \omega_0^2 x = F(t) = cos(\Omega t)$$

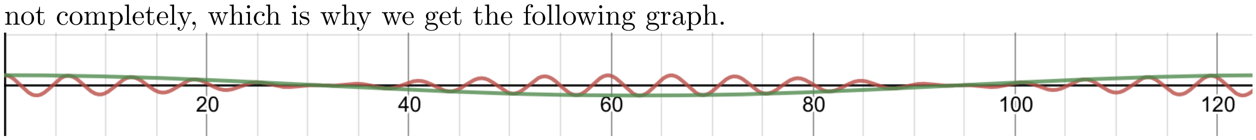
$x = x_H + x_P$ where x_H solves $\ddot{x} + \omega_0^2 x = 0$, $x_H = Rcos(\omega_0 t - \phi)$
Case 1: $\Omega \neq \omega_0$
$x_P = Asin(\Omega t) + Bcos(\Omega t)$
$\dot{x} = \Omega[Acos\Omega t - Bsin\Omega t]$
$\ddot{x} = \Omega^2[-x_p]$
Put into the DE to get

$$-\Omega^2 x_p + \omega_0^2 x_p = cos(\Omega t)$$

Thus

$$(\omega_0^2 - \Omega^2)[Asin(\Omega t) + Bcos(\Omega t)] = 1\dot{c}os(\Omega t)$$

$A = 0, B = \frac{1}{\omega_0^2 - \Omega^2}$
Thus the general solution is $x_p = \frac{1}{\omega_0^2 - \Omega^2} cos(\Omega t)$
Which means $x = x_H + x_p = Rcos(\omega_0 t + \phi) + \frac{1}{\omega_0^2 - \Omega^2} cos(\Omega t)$
Special case, setting $\phi = 0$
At which point $x = x_H + x_p = Rcos(\omega_0 t) + \frac{1}{\omega_0^2 - \Omega^2} cos(\Omega t)$
One special case arises when the two amplitudes are equal (or very close) $\implies R = \frac{1}{|\omega_0^2 - \Omega^2|}$ or $R \approx \frac{1}{|\omega_0^2 - \Omega^2|}$. The two contributions could cancel out for some time.
Adding the two *cos* terms we get $x = R2[cos(\frac{(\omega_0+\Omega)t}{2})cos(\frac{(\omega_0-\Omega)t}{2})]$
This means that the second special case arises when the frequencies are almost the same, but not completely, which is why we get the following graph.



This is called the phenomenon of **Beats**
Case 2: $\Omega = \omega_0$
$x_P = t(Asin(\Omega t) + Bcos(\Omega t))$

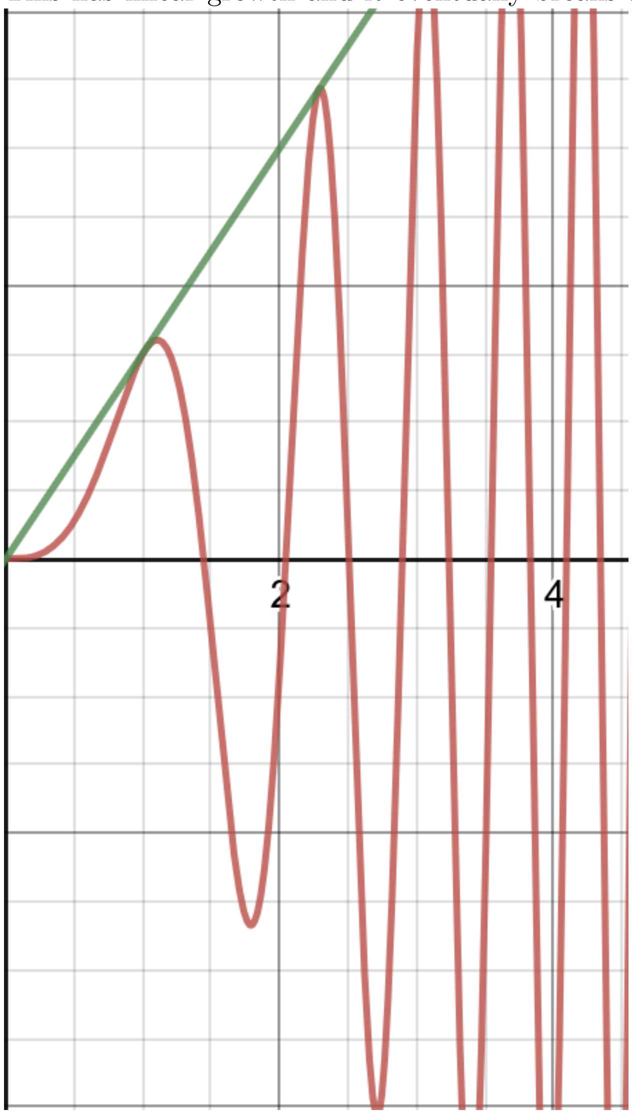
$$\dot{x_p} = Asin(\Omega t) + Bcos(\Omega t) + t\Omega(Acos(\Omega t) - Bsin(\Omega t))$$

$$\ddot{x_p} = 2\omega_o(Acos(\Omega t) - Bsin(\Omega t)) + t\omega_0^2(-Asin(\Omega t) - Bcos(\Omega t))$$

Put into the DE to get:

$$\ddot{x_p} + \omega_0^2 x_p = 2\omega_o(Acos(\Omega t) - Bsin(\Omega t)) \equiv cos(\Omega t)$$

$\implies 2A\omega_0 = -1 \implies A = \frac{1}{2\omega_0}$
$\implies B = 0$
Thus $x_p = \frac{1}{2\omega_0} tsin(\omega_0 t)$
This has linear growth and it eventually breaks as visible from the graph on the next page



Lemma: $L[t^n f(t)] = (-1)^n \frac{d^n F(s)}{ds^n}$, where $L[f(t)] = F(s)$

4 Fourier Series

4.1 Hyper applied linear algebra

We're going to be looking in $C^\infty[-\pi,\pi]$, i.e. infinitely differentiable in $[-\pi,\pi]$ [Which is also infinite dimensional]

Unique Representation Theorem: Any vector in a given space can be represented in exactly one way using the given bases [There is only one way because the basis are all linearly independent]

To get from one base to another, there are 2 ways.

1. Solve the equation
2. Change of basis/co-ordinate matrix
3. Orthogonal / Orthonormal basis

4.1.1 Ortho-normal Basis

$B_1 = \{ \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix} \}$
p and q are orthogonal means $p.q = 0$
An orthogonal basis for $\mathbb{R}^n$ is one in which the vectors in the basis are mutually orthogonal. [The basis B_1 above is not orthogonal, while B_2 below is] $B_2 = \{ \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}5\\-4\\1\end{pmatrix} \}$

Let $z = [v_1, v_2, v_3$ is an orthogonal basis for $\mathbb{R}^3$. Let $v \epsilon \mathbb{R}^3$, $v = a_1v_1 + a_2v_2 + a_3v_3$
$v \cdot v1 = (a_1v_1 + a_2v_2 + a_3v_3) \cdot v$ $v \cdot v_1 = a_1v_1 \cdot v_1$
$v \cdot v_2 = a_2v_2 \cdot v_2$
$v \cdot v_3 = a_3v_3 \cdot v_3$

We say that a basis $B = \{v_1, \ldots v_p\}$ is an orthonormal basis to mean that:

1. B is an orthogonal basis
2. Each vector is a unit vector

4.1.2 Function Space

Let $F(x) \,\epsilon\, C^\infty(-\pi,\ \pi)$
Taylor Series/Theorem: Basis = $\{1, x, x^2, x^3 \ldots x^n\}$
Fourier Basis: $\{1, cos(x), cos(2x), cos(3x) \ldots cos(nx), sin(x), sin(2x), \ldots sin(nx)\}$

$F(x)" = " \frac{1}{2}a_0 + a_1cos(1x) + a_2cos(2x) \cdots + a_ncos(nx)$

Let V be a vector space
A function $<,>: V \times V \to \mathbb{C}$ which satisfies:

$$< v, w >= \overline{< w, v >} \text{ Conjugate symmetry}$$

$$< a_1 v_1 + a_2 v_2, w >= a_1 < v_1, w > +a_2 < v_2, w > \text{ Linearity in 1st "slot"}$$

$$< v, v > \ \epsilon\, \mathbb{R}, < v, v > \geq 0 \text{ and } < v, v >= 0 \text{ iff } v = 0 \text{ Non negativity}$$

We call $(V, <, >)$ is called an inner product sapce with $<, >$ is the inner product
The length of v is $||v|| = \sqrt{< v, v >}$
eg: $\mathbb{R}^n$ vector space $v \cdot w = v_1 w_1 + ...v_n w_n$ i.e the "dot product" is an inner product

Now considering $C^\infty[-\pi, \pi]$ and $f(x), g(x)\ \epsilon\ C^\infty(-\pi, \pi)$.
We define $< f(x), g(x) >= \int_{-\pi}^{\pi} f(x)g(x)dx$

Consider the basis $B = \{1, cosx, cos2x, ...sin(x), sin(2x)....sin(mx)\}$ and these "vectors" are orthogonal.

$$< 1, cosnx >= \int_{-\pi}^{\pi} 1cos(nx)dx = \frac{sin(nx)}{n}|_{-\pi}^{\pi} = 0$$

$$< 1, sinmx >= \int_{-\pi}^{\pi} 1sin(mx)dx = \frac{cos(mx)}{-m}|_{-\pi}^{\pi} = 0$$

If $m \neq n$,

$$< cos(mx), cos(nx) >= \int_{-\pi}^{\pi} cos(mx)cos(nx)dx = 0$$

$$< sin(mx), sin(nx) >= \int_{-\pi}^{\pi} sin(mx)sin(nx)dx = 0$$

$$< sin(mx), cos(nx) >= \int_{-\pi}^{\pi} sin(mx)sin(nx)dx = 0$$

Thus, all of the pairs above are orthogonal

$$< 1, 1 >= 2\pi$$

$$< sin(nx), sin(nx) >= \int_{-\pi}^{\pi} sin^2(nx)dx = \pi$$

$$< cos(nx), cos(nx) >= \int_{-\pi}^{\pi} cos^2(nx)dx = \pi$$

Thus we have an orthogonal basis, but not an orthonormal basis

5 Fourier

Let $h(x) \epsilon C^{\infty}(-\pi, \pi)$,

$$h(x)\text{"} = \text{"}\frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k cos(kx) + \sum_{m=1}^{\infty} b_m sin(mx)$$

The constant term has half coefficient because of the 2 in $< 1, 1 >$

$$a_0 = \frac{1}{\pi} < h(x), 1 >= \frac{1}{\pi}\int_{-\pi}^{\pi} h(x)dx$$

$$a_k = \frac{1}{\pi} < h(x), cos(kx) >= \frac{1}{\pi}\int_{-\pi}^{\pi} h(x)cos(kx)dx$$

$$b_m = \frac{1}{\pi} < h(x), sin(mx) >= \frac{1}{\pi}\int_{-\pi}^{\pi} h(x)sim(mx)dx$$

The scalars $a_0, a_1, ...b_1, ...b_m$ are called the fourier coefficients of h(x)
Consider $h(x) = x$ on $(-\pi, \pi)$, and $h(x)$ is an odd function

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} xdx = 0 \text{ since it is an odd function}$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} xcos(kx)dx = 0 \text{ since it is an odd function}$$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} xsin(mx)dx = \frac{2}{\pi}\int_{0}^{\pi} xsin(mx)dx = \frac{2}{\pi}[\frac{xcos(mx)}{-m}|_0^{\pi} + \frac{1}{m}\int_0^{\pi} cos(mx)dx] = \frac{2}{\pi}\frac{\pi cos(m\pi)}{-m}$$

$$= 2(-1)^m / -m = \frac{2(-1)^{m+1}}{m}$$

Thus the fourier series for the function x is

$$2[1sin(1x) - \frac{1}{2}sin(2x) + \frac{1}{3}sin(3x) - \frac{1}{4}sin(4x) + \cdots + \frac{(-1)^{m+1}sin(mx)}{m}]$$

5.1 Partial sums

$$S_1 = 2sin(1x)$$

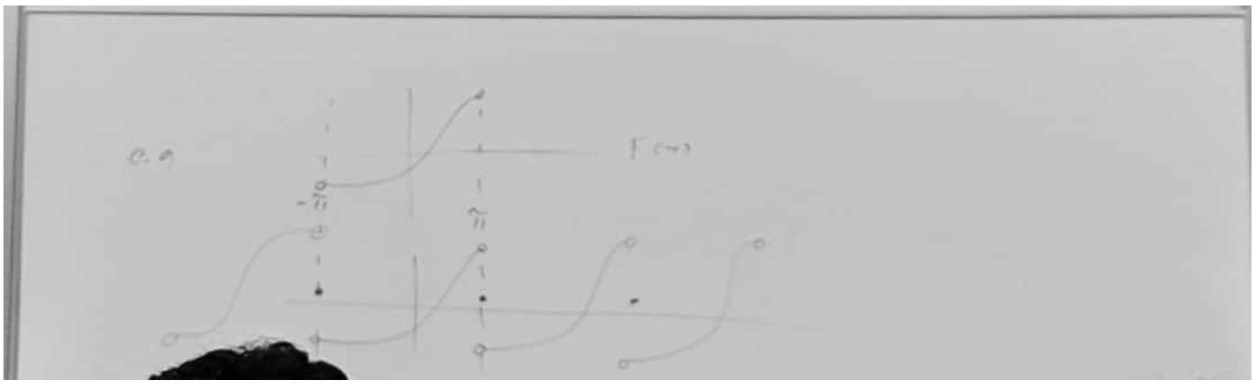
$$S_2 = 2[sinx - \frac{1}{2}sin(2x)]$$

Notice: The original function was defined on $(-\pi, \pi)$ but the Fourier series is defined on the whole of R and has period 2π, so what does it converge to?
Let $F(x)$ be defined on $(-\pi, \pi)$. We define the 2π periodic extension of F, $F_p(x)$ as follows:

$$F_p(x) = F(x) \quad x\epsilon(-\pi, \pi)$$

$$F_p(\pm\pi) = \frac{1}{2}[\lim_{x\to-\pi} F(x) + \lim_{x\to\pi} F(x)]$$

$$F_p(x + 2n\pi) = F_p(x) \; for \; all \; x \; \epsilon \; \mathbb{R}$$



Theorem: Point-wise Convergence of Fourier Series
Suppose $G(x)$ has a period of 2π and is a piecewise in $\mathbb{C}'$ then:

1. The Fourier series of $G(x)$ converges pointwise for all $x_0\epsilon\mathbb{R}$:

$$\frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k cos(kx) + \sum_{m=1}^{\infty} b_m sin(mx) = N_{x_0}$$

2. If $G(x)$ is continuous at x_0 then $N_{x_0} = G(x_0)$
3. If $G(x)$ is not continuous at x_0 then $N_{x_0} = \frac{1}{2}[\lim_{x\to x_0^-} F(x) + \lim_{x\to x_0^+} F(x)]$

In particular the Fourier Series of a function $F(x)$ $[C']$ defined on $(-\pi, \pi)$ will converge to $F_p(x_0)$ at points where $F_p(x)$ is continuous and to the midpoint at points where $F_p(x)$ is not continuous
Note: We have an infinite sum of continuous functions **<u>BUT</u>** the sum is not continuous
Suppose $F(x)$ is defined on $(-L, L)$ instead of $(-\pi, \pi)$
Our Basis become:

$$\{1, cos\frac{n\pi x}{L}, sin\frac{m\pi x}{L}\}$$

$F(x)$ becomes:

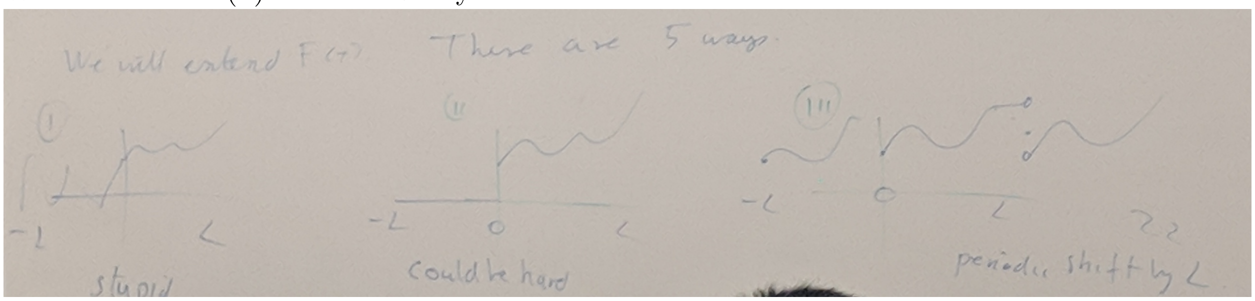
$$F(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k cos(\frac{k\pi x}{L}) + \sum_{m=1}^{\infty} b_m sin(\frac{m\pi x}{L})$$

$$a_k = \frac{1}{L}\int_{-L}^{L} F(x) cos(\frac{k\pi x}{L}) dx$$

$$b_k = \frac{1}{L}\int_{-L}^{L} F(x) sin(\frac{k\pi x}{L})dx$$

In a periodic is often have $F(x)$ on $[0, L)$ and we want a fourier series for this

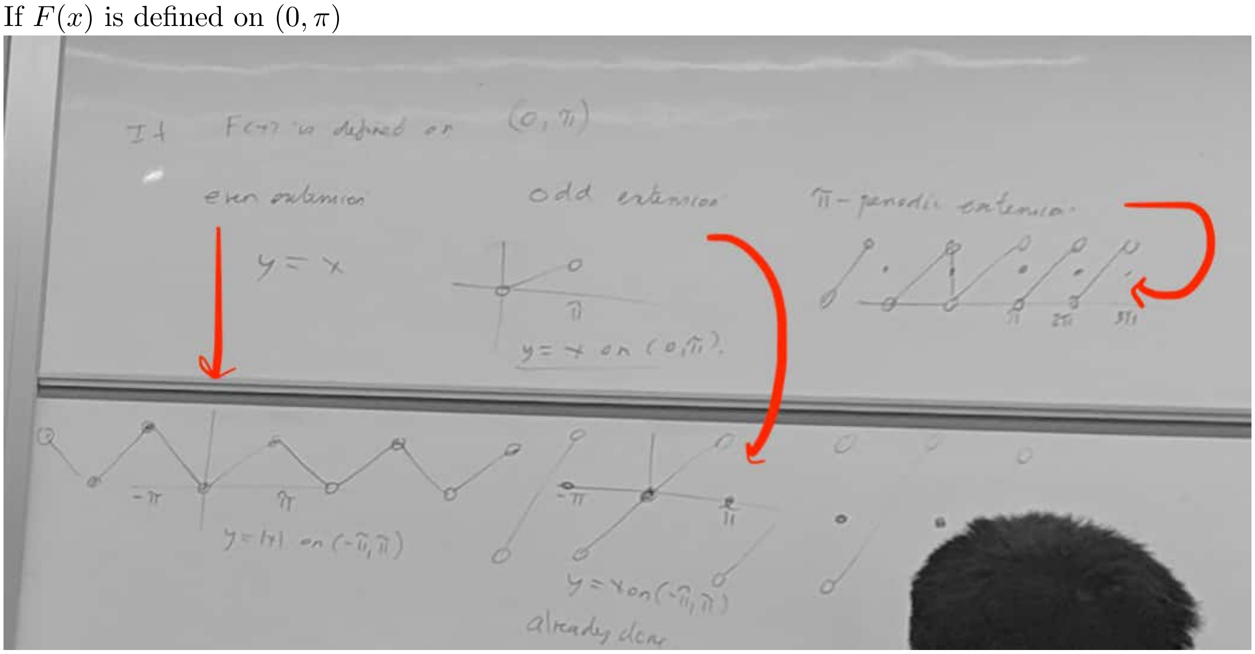
We will extend $F(x)$ there are 5 ways:



$$x = 2[sinx - \frac{sin2x}{2} + \frac{sin3x}{3} + \ldots] \text{ on } [-\pi, \pi]$$

$$\implies \frac{\pi}{2} = 2[1 - \frac{1}{3} + \frac{1}{5}\ldots] \implies \frac{\pi}{4} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$$

If $F(x)$ is defined on $(0, \pi)$



Even Extension: $y = |x|$ on $(-\pi, \pi)$, $b_n = 0 \forall n$
$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| dx = \pi$
$a_n = \frac{1}{\pi}\int_{-pi}^{\pi} |x| cos(n\pi x) dx$, $n = 1, 2, 3$
$a_n = \frac{2}{\pi}\frac{cos(n\pi x)}{n^2}|_0^{\pi} = \frac{2}{\pi n^2}[(-1)^n - 1]$, $n = 1, 2, 3$
$a_n = 0$, when n is even, and is $-\frac{4}{\pi n^2}$
$F(x) = \frac{\pi}{2} - \frac{4}{\pi}[\frac{cosx}{1^2} + \frac{cos3x}{3^2} + \dots]$

$$F(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{cos(2n+1)x}{(2n+1)^2}$$

π Periodic Extension: There are two ways for this

$$F(x) = \begin{cases} x & (0, \pi) \\ x + \pi & (-\pi, 0) \\ 2.5 & x = 0 \end{cases}$$

OR WE CAN DO

$$F(x) = \begin{cases} x & (0, \pi/2) \\ x + \pi & (-\pi/2, 0) \\ 0 & x = 0 \end{cases}$$

$$F(x) = \frac{1}{2}a_0 + 2a_n cos\frac{n\pi x}{\pi/2} + 2b_n sin\frac{n\pi x}{\pi/2}$$

$$F(x) = \frac{1}{2}a_0 + 2a_n cos(2nx) + 2b_n sin(2nx)$$

$$a_0 = \frac{2}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} F(x)dx = \frac{2}{\pi}[\int_{-\pi/2}^{0}(x+\pi)dx + \int_0^{\pi/2}(x)dx] = \pi$$

$$a_n = \frac{2}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} F(x)cos(2nx)dx = \frac{2}{\pi}[\int_{\frac{-\pi}{2}}^{0}(x+\pi)cos(2nx)dx + \int_0^{\frac{\pi}{2}} xcos(2nx)dx]$$

$$\implies \frac{2}{\pi}[\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} xcos(2nx)dx + \int_{\frac{-\pi}{2}}^{0} \pi cos(2nx)dx] = 0$$

$$b_n = \frac{2}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} F(x)sin(2nx)dx = \frac{2}{\pi}[\int_{\frac{-\pi}{2}}^{0}(x+\pi)sin(2nx)dx + \int_0^{\frac{\pi}{2}} xsin(2nx)dx]$$

$$\implies \frac{2}{\pi}[\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} xsin(2nx)dx + \int_{\frac{-\pi}{2}}^{0}\pi sin(2nx)dx] = \frac{2}{\pi}[2(\frac{xcos(2nx)}{-2n})|_0^{\frac{\pi}{2}} + \frac{1}{2n}\int_0^{\frac{\pi}{2}} cos(2nx)dx + \pi\frac{cos(2nx)}{-2n}|_{\frac{-\pi}{2}}^{0}] = \frac{-1}{n}$$

$$\implies F(x) = \frac{\pi}{2} - \sum_{1}^{\infty}\frac{sin(2nx)}{n}$$

Consider a function with period τ defined on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, $2\pi \to \tau$ i.e. $2L = \tau$

$$F(t) = \frac{1}{2}a_0 + \sum_{1}^{\infty} a_n sin(\frac{2n\pi t}{\tau}) + \sum_{1}^{\infty} b_n cos(\frac{2n\pi t}{\tau})$$

$$a_n = \frac{2}{\tau}\int_{\frac{-\tau}{2}}^{\frac{\tau}{2}} F(t)cos(\frac{2n\pi t}{\tau})dt$$

$$\omega_0 = \frac{2\pi}{\tau} \text{ [It is called the fundamental angular frequency]}$$

$$n\omega_0 \text{[These are called the harmonics]}$$

$$a_n = \frac{\omega_0}{\pi}\int_{\frac{-\tau}{2}}^{\frac{\tau}{2}} F(t)cos(n\omega_0 t)dt$$

$$b_n = \frac{\omega_0}{\pi}\int_{\frac{-\tau}{2}}^{\frac{\tau}{2}} F(t)sin(n\omega_0 t)dt$$

We will complexify all this:
Consider the space of complex valued differentiable function defined on $[-\frac{\tau}{2}, \frac{\tau}{2}]$ (this is a vector space).
In this space we define the inner product as:

$$< F, G >= \int_{\frac{-\tau}{2}}^{\frac{\tau}{2}} F(t)\overline{G(t)}dt$$

In this space a basis is $\{1, e^{jn\omega_0 t}\}_{n \,\epsilon\, \mathbb{Z}}$ which is orthogonal i.e.

$$< e^{jn\omega_0 t}, e^{jm\omega_0 t} >= \int_{\frac{-\tau}{2}}^{\frac{\tau}{2}} e^{jn\omega_0 t}e^{-jm\omega_0 t}dt = \int_{\frac{-\tau}{2}}^{\frac{\tau}{2}} e^{j(n-m)\omega_0 t}dt = \frac{e^{j(n-m)\omega_0 t}}{j(n-m)\omega_0}|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} = 0 \text{ given } m \neq n$$

$$< 1, 1 >= \int_{\frac{-\tau}{2}}^{\frac{\tau}{2}} 1dt = \tau$$

$$||e^{jn\omega_0 t}|| =< e^{jn\omega_0 t}, e^{-jn\omega_0 t} >= \int_{\frac{-\tau}{2}}^{\frac{\tau}{2}} e^0 dt = \tau$$

Let function $F(t)$ be a function in the vector space. We define the complex Fourier Series of $F(t)$ as follows

$$\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} = \text{F(t) for most of the time}$$

where $C_n = \frac{1}{\tau}\int_{\frac{-\tau}{2}}^{\frac{\tau}{2}} F(t)e^{-jn\omega_0 t}dt$, $C_n = \frac{1}{\tau} < F(t), e^{jn\omega_0 t} >$
If $F(t)$ is real valued: $\overline{C_n} = \frac{1}{\tau}\int_{\frac{-\tau}{2}}^{\frac{\tau}{2}} F(t)e^{-j(-n)\omega_0 t}dt = -C_n$

eg: $\omega_0 = 1$, t on $(-\pi, \pi)$, $\tau = 2\pi$
$C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} tdt = \frac{1}{2\pi}\frac{1}{2}(\pi^2 - (-\pi)^2) = 0$
$n \neq 0$

$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} te^{-jnt}dt = \frac{1}{2n\pi j}[\pi e^{-jn\pi} - -\pi e^{-jn(-\pi)}] + \frac{1}{2n\pi j}\frac{1}{-jn}[e^{-jn\pi} - e^{-jn(-\pi)}] = \frac{(-1)^n}{-jj} + \frac{1}{2n^2\pi}[0]$
$\implies C_n = \frac{(-1)^n}{-jj} = \frac{(-1)^n j}{n}$

We can also show that $C_{-n} = C_n$
Complex Fourier Series:

$$\sum_{n=1}^{\infty}(\frac{-1^n}{n}j)e^{njt} + \sum_{n=-1}^{-\infty}(\frac{-1^n}{n}j)e^{njt}$$

If the second sum let $m = -n$

$$\sum_{n=1}^{\infty}(\frac{(-1)^n}{n}j)e^{njt} - \sum_{m=1}^{\infty}(\frac{(-1)^m}{m}j)e^{mjt}$$

$$\sum_{n=1}^{\infty}(\frac{(-1)^n}{n}j)\frac{e^{njt} - e^{-njt}}{2j}2j$$

$$2\sum\frac{(-1)^{n+1}}{n}sin(nt)$$

Complex Fourier Series
$f(t)$ is τ periodic and $f(t)$ is defined on $[-\frac{\tau}{2}, \frac{\tau}{2}]$ with $\omega_0 = \frac{2\pi}{\tau}$ Basis: $\{1, e^{jn\omega_0}t\}$

$$\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

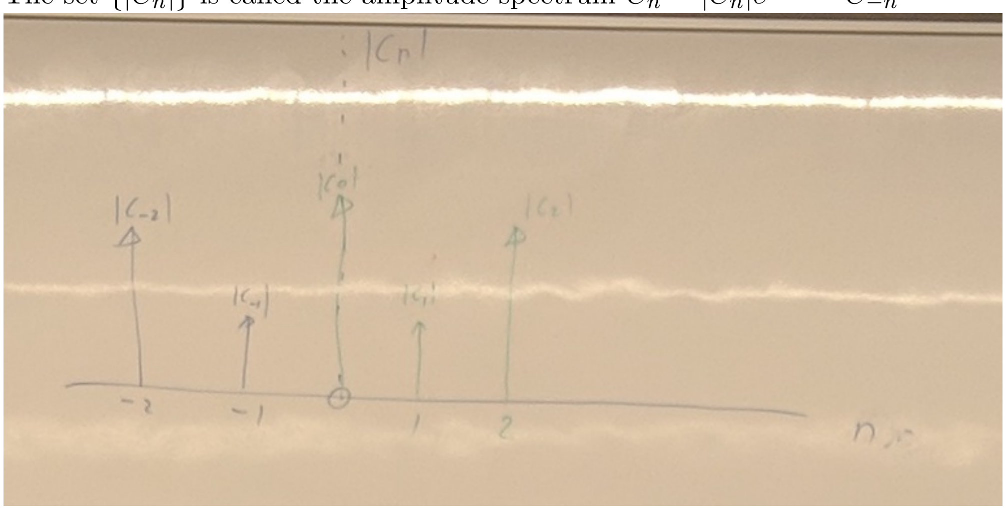
$$C_n = \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t)e^{-jn\omega_0 t}dt$$

If the function $f(t)$ is real $C_{-n} = \overline{C_n}$
Eg: $\omega_0 = 1$, t on $(-\pi, \pi)$, $\tau = 2\pi \implies C_n = \frac{(-1)^n j}{n}$
Relationship between real and complex fourier series

1. $f(t) = \frac{1}{2}a_0 + \sum a_n cos(n\omega_0 t) + \sum b_n sin(n\omega_0 t)$
2. $f(t) = \sum_{n=-\infty}^{\infty} C_n e^{nj\omega_0 t}$

Using these we can get the result: $C_n = \frac{1}{2}(a_n - jb_n)$ and the other way: $a_n = (C_n + \overline{C_n}) = (C_n + C_{-n})$ and $b_n = \frac{1}{j}(\overline{C_n} - C_n) = j(C_n - \overline{C_n})$

In a vector space with a basis B, if $\underline{v} \epsilon V$ and we know the components of $\underline{v}$ in B then we know everything about $\underline{v}$
Suppose we have the complex Fourier coefficients of $f(t)$ then we know everything about $f(t)$
$C_n \,\epsilon\, \mathbb{C}$ and $C_n = |C_n|e^{i\phi}$ where $|C_n|$ is the magnitude and ϕ is the phase.
The set $\{C_n\}$ is sometimes called the Fourier Spectrum of $f(t)$
The set $\{|C_n|\}$ is called the amplitude spectrum $\overline{C_n} = |C_n|e^{-i\phi} = C_{-n}$



Also, the strength of the signal can be defined by $\sum_{n=-\infty}^{\infty} |C_n|^2$
This spectrum illustrates the relative importance of the contribution to $f(t)$ i.e. if $|C_p|$ then both $e^{j\omega_0 pt}$ and $e^{-j\omega_0 pt}$ play an important role in construction of $f(t)$ [equivalently $sin(p\omega_0 t) or cos(p\omega_0 t)$ have large co-efficients]

Parseval's Theorem: Let $f(t)$ be a τ periodic function with $f(t) = \sum_{n=-\infty}^{\infty} C_n e^{nj\omega_0 t}$
Then, $< f(t), f(t) >= \int_{\frac{-\tau}{2}}^{\frac{\tau}{2}} f(t)\overline{f(t)}dt = \tau \sum_{n=-\infty}^{\infty} |C_n|^2$

If you take the inner product of the function then you get the sum of the squares of the components, with τ being added for not having the magnitudes not equal to 1

Eg: t on $(-\pi, \pi)$ with $C_n = \frac{(-1)^n}{j} n$ and $C_0 = 0$

$$< t, t >= \int_{-\pi}^{\pi} t^2 dt = 2\pi \sum_{-\infty}^{\infty} = 2\pi [\sum_{-\infty}^{-1} \frac{1}{n^2} + \sum_{1}^{\infty} \frac{1}{n^2}]$$

$$\implies \int_{-\infty}^{\infty} t^2 dt = \frac{t^3}{3} |_{-\pi}^{\pi} = \frac{2\pi^3}{3} = 2\pi \times 2 \sum_{n=1}^{\infty} \frac{1}{n^2} \implies \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

5.2 Fourier Transformation

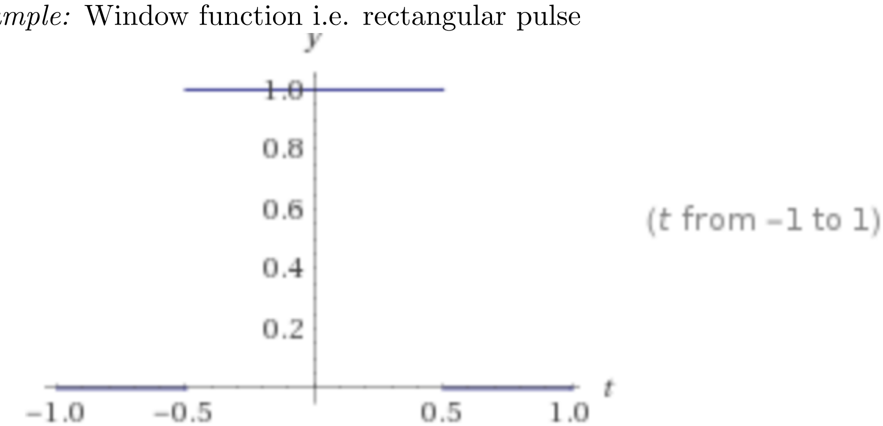
Fourier Transform $f(t)$
need a square integrable function $\int_{-\infty}^{\infty} |f(t)|^2 dt$ is finite

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

Fourier Integral $F(\omega)$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

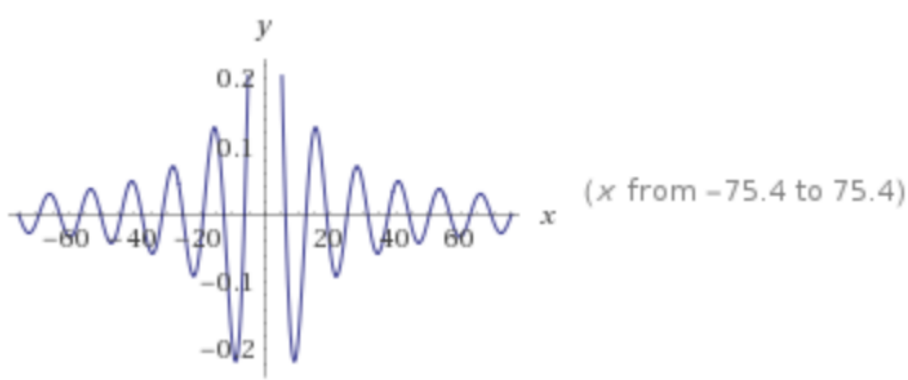
Example: Window function i.e. rectangular pulse



$$f(t) = H(t+\frac{1}{2}) - H(t-\frac{1}{2})$$

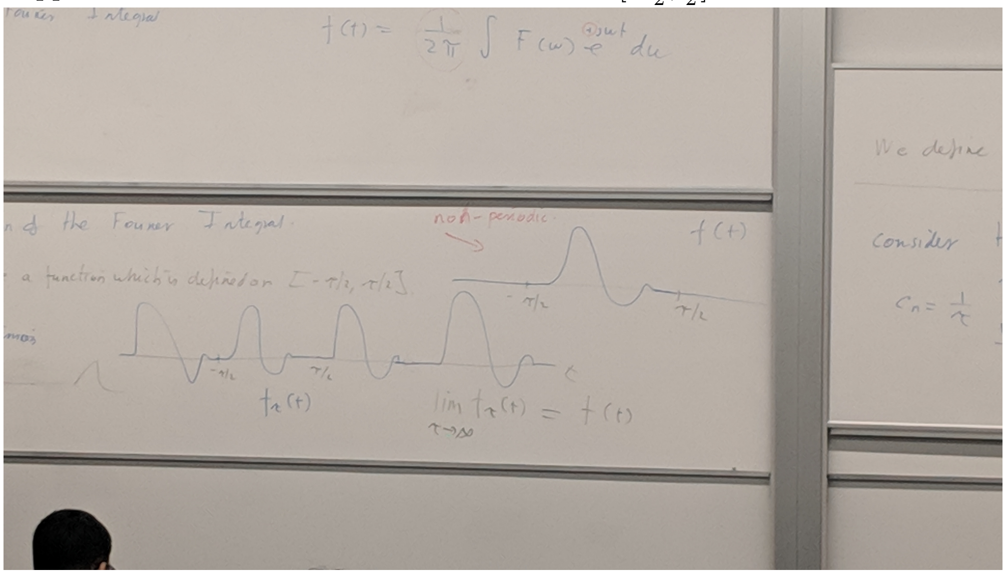
$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} 1e^{-j\omega t}dt = \frac{e^{-j\omega t}}{-j\omega}|_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{2}{\omega}[\frac{e^{\frac{j\omega}{2}} - e^{-\frac{j\omega}{2}}}{2j}] = \frac{sin\frac{\omega}{2}}{\frac{\omega}{2}} = sinc(\frac{\omega}{2})$$

$$[\frac{sinx}{x} = sinc(x), \text{Sinch of x}]$$



5.2.1 Ad Hoc Derivation of the Fourier Integral

Suppose we have a function which is defined on $[-\frac{\tau}{2}, \frac{\tau}{2}]$

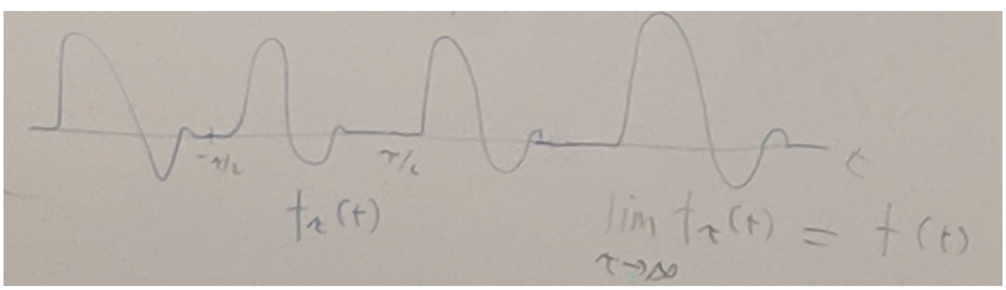


We define

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

Consider the complex Fourier series $f_\tau(t)$

$$\lim_{\tau \to \infty} f_\tau(t) = f(t)$$



We let τ to tend to ∞ because of that $\omega = \frac{2\pi}{\tau} = \Delta\omega$

$$C_n = \frac{1}{\tau} \int_{\frac{-\tau}{2}}^{\frac{\tau}{2}} f_\tau(t) e^{-jn\omega_0 t} dt$$

$$f_\tau(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$C_n = \frac{1}{\tau} f_\tau f(t) e^{-jn\omega_0 t} dr = \frac{1}{\tau} F(n\omega_0) = \frac{1}{\tau} n\Delta\omega$$

$$f_\tau(t) = \sum_{n=-\infty}^{\infty} \frac{1}{\tau} F(n\Delta\omega) e^{jn\omega_0 t} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F(n\Delta\omega) e^{jn\omega_0 t} \Delta\omega$$

$$\text{[Riemann Sum: } \lim_{N\to\infty, \Delta t\to 0} \sum_{n=1}^{N} g(a + n\Delta t)\Delta t = \int_a^b g(t)dt \text{ where } b = a + n\Delta t\text{]}$$

$$\lim_{\tau\to\infty} f_\tau(t) = \frac{1}{2\pi} \lim_{\tau\to\infty, \Delta\omega\to 0} \sum_{n=-\infty}^{\infty} F(n\Delta\omega) e^{jtn\Delta\omega} \Delta\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{jt\omega} d\omega$$

$$= \int_{-\infty}^{\infty} F(\omega) e^{jt\omega} \quad \text{which is known as the Fourier Integral}$$

For τ-periodic function, square integrable function on $\mathbb{R}$ i.e. $\int_{-\infty}^{\infty} |g(t)|^2 dt$ exists then
Fourier Coefficients: $C_n = \frac{1}{\tau}\int_{\frac{-\tau}{2}}^{\frac{\tau}{2}} f(t)e^{-jn\omega_0 t}dt$
Fourier Series: $\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} = f(t)$
Fourier Transform $\mathcal{F}(g(t)) = G(\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt$
Fourier Integral $g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)e^{j\omega t}d\omega$

Continuing example from earlier:

$$H(t+\frac{1}{2}) - H(t-\frac{1}{2}) = g(t)$$

$$\mathcal{F}(g(t)) = G(\omega) = sinc(\frac{1}{2}\omega)$$

Since $\tau = 2\pi, \omega_0 = 1$ and $C_n = \frac{1}{\tau}G(n\omega_0) \implies C_n = \frac{1}{2\pi}sinc(\frac{n}{2})$
Now if $\tau = 4\pi, \omega_0 = \frac{1}{2}$ and $C_n = \frac{1}{\tau}G(n\omega_0) \implies C_n = \frac{1}{4\pi}sinc(\frac{n}{4})$

Another example:

$$T(t) = \begin{cases} 1-|t| & |t| < 1 \\ 0 & |t| \geq 1 \end{cases}$$

$\mathcal{F}(T(t)) = \int_{-\infty}^{\infty} T(t)e^{-j\omega t}dt = \int_{-1}^{1}(1-|t|)e^{-j\omega t}dt = \int_{-1}^{1}(1-|t|)cos(\omega t)dt - j\int_{-1}^{1}(1-|t|)sin(\omega t)dt$
The first term is even and non-zero, while the second term is odd and zero

$$\mathcal{F}(T(t)) = 2[\frac{sin(\omega t)}{\omega}|_0^1 - t\frac{sin(\omega t)}{\omega}|_0^1 + \int_0^1 \frac{(sin\omega t)}{\omega}dt] = -2[\frac{cos\omega - 1}{\omega^2}] = -2[\frac{-2sin^2(\frac{\omega}{2})}{\omega^2}]$$
$$= sinc^2(\frac{\omega}{2})$$

5.3 Properties of the Fourier Transformation

1. **Linearity:** $\mathcal{F}(af(t) + bg(t)) = a\mathcal{F}(f(t)) + b\mathcal{F}(g(t))$
2. **Shift in the time domain:** $\mathcal{F}(g(t-t_0)) = e^{-j\omega t_0}\mathcal{F}(g(t))$
3. **Shift in the frequency domain:** $\mathcal{F}(e^{j\omega_0 t}g(t)) = G(\omega - \omega_0)$ where $\mathcal{F}(g(t)) = G(\omega)\ \forall\ \omega_0 \epsilon \mathbb{R}$
4. **Scaling Property:** $\mathcal{F}(g(at)) = \frac{1}{|a|}G(\frac{\omega}{a})$
5. **Conjugation:** $\mathcal{F}(\overline{g(at)}) = \overline{G(-\omega)}$
6. **Composition:** $\mathcal{F}\circ\mathcal{F}(g(t)) = \mathcal{F}^2[g(t)] = 2\pi g(-\omega)$ where $\mathcal{F}(g(t)) = G(\omega)$
7. **Derivative Property**

$$\mathcal{F}(\frac{dg}{dt}) = j\omega G(\omega)$$

   or extending this:

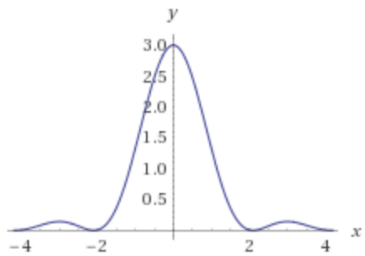
$$\mathcal{F}(\frac{d^n g}{dt^n}) = (j\omega)^n G(\omega)$$

If

$$T(t) = \begin{cases} 1 - |t| & |t| < 1 \\ 0 & |t| \geq 1 \end{cases}$$

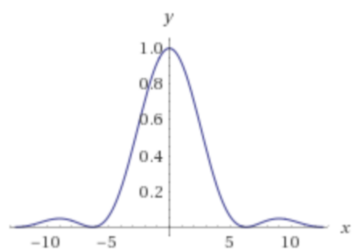
and $\mathcal{F}[T(t)] = (sinc\frac{\omega}{2})^2$ then

1. $\mathcal{F}[T(t-5)] = e^{-5j\omega}(sinc\frac{\omega}{2})^2$
2. $\mathcal{F}[T(t)e^{3jt}] = (sinc\frac{\omega - 3}{2})^2$
3. $\mathcal{F}[T(\frac{t}{3})] = 3(sinc\frac{3\omega}{2})^2$

   (x from −4.2 to 4.2)

   3sinc above, and one sinc down

   (x from −12.6 to 12.6)

   You can see that the function gets compressed

4. $\mathcal{F}^2(T(t)) = 2\pi T(-\omega) = 2\pi T(\omega)$ since T is symmetric
   $\mathcal{F}(F(T(t))) = \mathcal{F}((sinc\frac{\omega}{2})^2) = 2\pi T(\omega)$

$$\mathcal{F}((sinc\frac{\omega}{2})^2) = \begin{cases} 2\pi(1 - |\omega|) & |\omega| < 1 \\ 0 & |\omega| \geq 1 \end{cases}$$

Lemma: Fourier Transformations of even functions are purely real i.e. $g(t)$ is even then $G(\omega)$ is real

$$\mathcal{F}[g(t)] = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} g(t)[cos(\omega t) - j\,sin(\omega t)]dt$$

$$\int_{-\infty}^{\infty} g(t)cos(\omega t) - j\int_{-\infty}^{\infty} g(t)sin(\omega t) = 2\int_{0}^{\infty} g(t)cos(\omega t)dt$$

The second term of the second last step is zero since it is odd

5.4 Fourier Transform of a Gaussian

Lemma: $n(t) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}(\frac{t-\mu}{\sigma})^2}$ [Normal distribution with mean μ and standard deviation σ (spread)] [Side note: the area under this curve is 1]

$$\mathcal{F}(n(t)) = e^{-j\mu\omega}e^{-\frac{1}{2}\sigma^2\omega^2}$$

Proof:

1. It is known that $\int_{-\infty}^{\infty} e^{-u^2}du = \sqrt{\pi}$

2. Let

$$I(\omega) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}cos(\omega x)dx$$

$$\frac{dI}{d\omega} = \frac{d}{d\omega}\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}cos(\omega x) \equiv \int_{-\infty}^{\infty} \frac{\partial}{\partial\omega}(e^{-\frac{1}{2}x^2}cos(\omega x)) = \int_{-\infty}^{\infty} -xe^{-\frac{1}{2}x^2}sin(\omega x)$$

$$e^{-\frac{1}{2}x^2}sin\omega x|_{-\infty}^{\infty} - \omega\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}cos(\omega x)dx = 0 - \omega I(\omega)$$

Thus,

$$\frac{dI}{d\omega} = -\omega I(\omega)$$

and on solving this differential equation we get, either $I = 0$ or $I = I_0 e^{-\frac{1}{2}\omega^2}$

$$I(0) = I_0 = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}dx$$

Let $x^2 = 2u^2 \implies dx = \sqrt{2}du$ and thus

$$(\int_{-\infty}^{\infty} e^{-u^2}du)\sqrt{2} = \sqrt{2\pi}$$

$$\mathcal{F}[e^{-\frac{1}{2}t^2}] = \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}cos(\omega t)dt = \sqrt{2\pi}\ e^{-\frac{1}{2}\omega^2}$$

$$\mathcal{F}[\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}e^{-\frac{1}{2}t^2}] = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}\sqrt{2\pi}\ e^{-\frac{1}{2}\omega^2} = \frac{1}{\sigma}e^{-\frac{1}{2}\omega^2}$$

$$\mathcal{F}[\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}e^{-\frac{1}{2}(\frac{t}{\sigma})^2}] = \frac{1}{\sigma}\sigma e^{-\frac{1}{2}\omega^2\sigma^2} = e^{-\frac{1}{2}\omega^2\sigma^2}$$

$$\mathcal{F}[\frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}e^{-\frac{1}{2}(\frac{t-\mu}{\sigma})^2}] = e^{-j\omega\mu}e^{-\frac{1}{2}\omega^2\sigma^2}$$

5.5 Parseval's Theorem

Let $g(t)$ be a square integral function with Fourier Transformation $G(\omega)$ then

$$\int_{-\infty}^{\infty} g(t)\overline{g(t)} = \int_{-\infty}^{\infty} |g(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\overline{G(\omega)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(\omega)|^2 d\omega$$

If

$$T(t) = \begin{cases} 1-|t| & |t|<1 \\ 0 & |t|\geq 1 \end{cases}$$

and $\mathcal{F}[T(t)] = (sinc\frac{\omega}{2})^2$ then

$$\int_{-\infty}^{\infty} |T(t)|^2 = 2\int_{0}^{\infty} (1-t)^2 dt = \frac{2}{3} \overset{Parseval}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} ((sinc(\frac{1}{2}\omega))^2)^2 d\omega$$

$$\implies \int_{-\infty}^{\infty} \frac{sin(\frac{\omega}{2})^4}{(\frac{\omega}{2})^4} d\omega = \frac{4\pi}{3}$$

After substituting $\frac{\omega}{2} = y$ and simplifying

$$\implies \int_{0}^{\infty} \frac{siny^4}{y^4} dy = \frac{\pi}{3}$$

5.6 Fourier Transformation of Dirac Function

$$\mathcal{F}(\delta(t)) = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = e^0 = 1$$
$$\mathcal{F}(\delta(t-a)) = \int_{-\infty}^{\infty} \delta(t-a)e^{-j\omega t}dt = e^{-j\omega a}$$

5.6.1 Fourier Integral of 1

It gives us the reverse of the transformation above

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} 1e^{j\omega t}dw = \delta(t)$$

5.6.2 Fourier Composition

$$\mathcal{F}(\mathcal{F}(\delta(t))) = 2\pi\delta(-\omega)$$
$$\mathcal{F}(1) = 2\pi\delta(\omega)$$

We can make use of this and our properties of $\mathcal{F}$ to evaluate $F(cos(\alpha t))$ and $F(sin(\alpha t))$

$$cos(\alpha t) = \frac{e^{j\alpha t} + e^{-j\alpha t}}{2} \quad \text{and} \quad sin(\alpha t) = \frac{e^{j\alpha t} - e^{-j\alpha t}}{2j}$$

$$\cos(\alpha t) = \frac{1}{2}[e^{j\alpha t}\cdot 1 + e^{-j\alpha t}\cdot 1]$$

Thus,

$$\mathcal{F}[cos(\alpha t)] = \frac{2\pi}{2}[\delta(\omega-\alpha) + \delta(\omega+\alpha)]$$

Similarly,

$$\mathcal{F}[sin(\alpha t)] = \frac{2\pi}{2j}[\delta(\omega-\alpha) - \delta(\omega+\alpha)]$$

5.7 A Finite Wave Pulse

$$W(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & |t| \geq \frac{1}{2} \end{cases}$$

$$\mathcal{F}(\omega(t)) = sinc(\frac{1}{2}\omega)$$

For $a > 0$,

$$W(at) = \begin{cases} 1 & |at| < \frac{1}{2} \implies |t| < \frac{1}{2a} \\ 0 & |at| \geq \frac{1}{2} \implies |t| \geq \frac{1}{2a} \end{cases}$$

$$\mathcal{F}(\omega(at)) = \frac{1}{a} sinc(\frac{1}{2a}\omega)$$

For,

$$f(t) = cos(\alpha t)W(at) = \frac{1}{2}[e^{j\alpha t} + e^{-j\alpha t}][W(at)]$$

$$\mathcal{F}(f(t)) = \frac{1}{2a}[sinc(\frac{\omega - \alpha}{2a}) + sinc(\frac{\omega + \alpha}{2a})]$$

6 Partial Differential Equation

We not consider the scenario of more than one independent variable for example $z = z(x, y)$. We will now have to consider partial derivatives.
An equation involving the partial derivatives of an unknown function(s) is called a PDE
Eg: Linear PDE's

$$\frac{\partial^2 z}{\partial x^2} + xyz = 5$$

$$\frac{\partial^2 z}{\partial x \partial y} + x^2 \frac{\partial z}{\partial y} + y^2 \frac{\partial z}{\partial x} = 0$$

Non Linear PDE's

$$\frac{\partial^2 z}{\partial x^2 \partial y} + \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y} = xy$$

$$\frac{\partial^2 z}{\partial y^2} = xyz^2$$

6.1 Definitions

The *order* of a PDE is the highest derivative that appears
The PDE is said to be linear when all the terms involving z (and any derivatives) only appear linearly
The PDE is homogeneous when if we collect all the derivative on the left hand side of the PDE then the RHS is zero.

$$\frac{\partial^2 z}{\partial x \partial y} + x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0 \qquad \text{[Homo]}$$

$$\frac{\partial^2 z}{\partial x \partial y} + x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = x^2 + y^2 + 25 \qquad \text{[No-Homo]}$$

In our physical world second order PDEs arise frequently
If we have 2^{nd} independent variables u and v, the most general 2^{nd} order linear PDE has the form:

$$AF_{uu} + BF_{vv} + 2CF_{uv} + DF_u + EF_v + HF = R(u, v)$$

The PDEs are classified according to the sign of the term:

$$\det \begin{pmatrix} A & C \\ C & B \end{pmatrix} = AB - C^2$$

- $AB - C^2 > 0$: The PDE is called **elliptical**.
  examples: Laplace's equation: $\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0$ or Poisson's equation: $\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = e$
- $AB - C^2 = 0$: The PDE is called **parabolic**.
  examples: Heat equation: $\frac{\partial^2 F}{\partial x^2} + 0 \frac{\partial^2 F}{\partial t^2} + \frac{\partial F}{\partial t} = 0$
- $AB - C^2 < 0$: The PDE is called **hyperbolic**.
  examples: Wave equation: $\frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial t^2} = 0$

$$\frac{\partial^2 x}{\partial y^2} = \frac{\partial x}{\partial z}$$

6.2 Vector Differential Operator ∇

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$
$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

Laplacian Del Squared: ∇^2

6.2.1 Elliptical

No time dependence here

$$\text{Laplace's Equation :} \quad \nabla^2\phi = 0$$
$$\text{Poisson's Equation :} \quad \nabla^2\phi = \rho(x, y, x)$$

6.2.2 Parabolic

Time Dependence

$$\text{Heat Equation :} \quad \nabla^2\phi = \frac{\partial\phi}{\partial t}$$

6.2.3 Hyperbolic

Time is involved

$$\text{Wave Equation :} \quad \nabla^2\phi = \frac{\partial^2\phi}{\partial t^2} \implies \nabla^2\phi - \frac{\partial^2\phi}{\partial t^2} = 0$$

Note 1: we often have problems which settle down to some sort of stable state behaviour, we define the equilibrium state to be a solution u_E which satisfies the derivative $\frac{\partial u_E}{\partial t} = 0$
For eg: For the heat eqn: $\frac{\partial\phi_E}{\partial t} = 0$ then $\nabla^2\phi_E = 0$
For eg: For the wave eqn: $\frac{\partial^2\phi_E}{\partial t^2} = 0$ then $\nabla^2\phi_E = 0$

Note 2: Other co-ordinates may be appropriate
eg: 2-D Laplacian - in polar coordinates $(x, y) \to (r, \theta)$
$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$ *and if there were rotational symmetry then:* $\frac{\partial\phi}{\partial\theta} = 0$

6.3 The Heat Equation in one spacial dimension

$$\alpha^2 \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial \theta}{\partial t} \quad [\theta \text{ is the temperature, } \alpha^2 = \frac{k}{\rho c}]$$

where k is the thermal conductivity, ρ is the mass density and c is the specific heat
A general problem is a 1-D rod/bar with perfect insulation on the outside, and length L

1. We need some initial conditions $\theta(x, 0) = g(x)$ on $0 < x < L$
2. We also require boundary conditions, which is typically $\theta(0, t)$ and $\theta(L, t)$ and in the simplest case they might be fixed temperatures T_1 and T_2 respectively

6.3.1 The method of separation of variables

We try to find a solution $\theta(x, t) = T(t) \cdot X(x)$.
Let the endpoints both have zero temperature i.e. $\theta(0, t) = 0$ and $\theta(L, t) = 0$
Obviously $\theta(x, t) \equiv 0$ is a solution and the in the other case:

$$\frac{\partial^2 \theta}{\partial x^2} = T(t) \frac{d^2 X}{dx^2}$$

$$\frac{\partial \theta}{\partial t} = X \frac{dT}{dt}$$

Now our heat equation becomes:

$$\alpha^2 T \frac{\partial^2 X}{\partial x^2} = \frac{\partial T}{\partial t} X$$

$$\frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{\alpha^2} \frac{1}{T} \frac{dT}{dt} = -\lambda$$

The left hand side is a function of only X and the right hand side is a function of only t
`Solving the equation with` t `first:`

$$\frac{dT}{dt} = -\alpha^2 \lambda T \implies T(t) = e^{-\alpha^2 \lambda t}$$

`Solving the equation with` x `now:`
$\theta(0, t) = 0 \implies T(t)X(0) = 0 \implies X(0) = 0$ and $\theta(L, t) = 0 \implies T(t)X(L) = 0 \implies X(L) = 0$

$$\frac{d^2 X}{dx^2} = -\lambda X \implies \frac{d^2 X}{dx^2} + \lambda X = 0$$

This equation is called the `Sturm Liouville equation`

1. Let $\lambda = 0$:
$$\frac{d^2 X}{dx^2} = 0 \implies X = ax + b$$
Solving this using $X(0) = 0$ and $X(L) = 0$ we get $b = 0$ and $a = 0$ respectively.
Thus $X \equiv 0$

2. Let $\lambda < 0$: $\lambda = -\mu^2$
   Thus,
$$\frac{d^2X}{dx^2} - \mu^2 x = 0 \implies \frac{d^2X}{dx^2} = \mu^2 x$$
   $X = Ae^{\mu x} + Be^{-\mu x}$ with $X(0) = 0 \implies A + B = 0$ and $X(L) = 0 \implies Ae^{\mu L} + Be^{-\mu L} = 0$
   Using these we get $X = 0$

3. Let $\lambda > 0 \implies \lambda = \mu^2$ Using these we get X = 0 Thus,
$$\frac{d^2X}{dx^2} + \mu^2 x = 0 \implies \frac{d^2X}{dx^2} = -\mu^2 x$$
   $X = Acos(\mu x) + Bsin(\mu x)$ with $X(0) = 0 \implies A = 0$ and
   $X(L) = 0 \implies Bsin(\mu L) = 0$
   For the second equation, either $B = 0$ or let $\mu = \mu_n$, $\mu_n L = n\pi \implies \mu_n = \frac{n\pi}{L}$ $[n \epsilon \mathbb{Z}]$
   For each $n \epsilon \mathbb{Z}$, $\mu_n = \frac{n\pi}{L}$
$$X_n = sin(\frac{n\pi}{L}x)$$
   From earlier but with $\lambda = (\frac{n\pi}{L})^2$,
$$T_n = e^{-(\frac{\alpha n\pi}{L})^2 t}$$
   And thus,
$$\theta_n(x,t) = sin(\frac{n\pi}{L}x) \cdot e^{-(\frac{\alpha n\pi}{L})^2 t}$$

6.3.2 *Lemma:* Linearity Homogenity and Superposition

Let L be a linear differential operator $[L(t) = 0]$ and suppose that we have homogeneous boundary conditions
For the heat equation, our linear differential operator is:
$$L = \frac{\partial}{\partial t} - \alpha^2 \frac{\partial^2}{\partial x^2} \implies L(\theta) = 0$$
Then the solution to the PDE and BC's has vector space structure
Applications
$$\sum_{n=1}^{\infty} A_n \theta_n(x,t) \quad \text{[Note: } n = 0 \text{ gives nothing and } n < 0 \text{ can be included in } n > 0\text{]}$$
General solution becomes:
$$\theta(x,t) = \sum_{n=1}^{\infty} A_n \theta_n(x,t)$$
This equation above satisfies the PDE's and the Boundary conditions [BC's] What about the initial conditions? $\theta(x,0) = g(x)$
We need $\sum_{n=1}^{\infty} A_n sin(\frac{n\pi x}{L}) = g(x)$ i.e. we need the Fourier Sine series for $g(x)$ on $(0, L)$
$$A_n = \frac{1}{L}\int_{-L}^{L} g(x) sin(\frac{n\pi x}{L})dx = \frac{2}{L}\int_0^L g(x) sin(\frac{n\pi x}{L})dx$$

Example: $g(x) = \theta(x,0) = u_0$
In this case the:

$$A_n = \frac{2u_0}{L}\int_0^L sin(\frac{n\pi x}{L})dx = \frac{2u_0}{L}[\frac{-L}{n\pi} + cos(\frac{n\pi x}{L})]_0^L \implies A_n = \frac{2u_0}{n\pi}[1-(-1)^n]$$

$$A_n = \begin{cases} 0 & n\%2 == 0 \\ \frac{4u_0}{n\pi} & n\%2 == 1 \end{cases}$$

$$\implies A_{2m+1} = \frac{4u_0}{\pi(2m+1)} \quad where \;\; m = 0, 1, 2, \ldots, \infty$$

$$\theta(x,t) = \frac{4u_0}{\pi}[\sum_{m=0}^{\infty}\frac{1}{2m+1}sin(\frac{(2m+1)\pi x}{L})e^{-[\frac{(2m+1)\pi\alpha}{L}]^2 t}]$$

If $L = \pi$

$$\theta(x,t) = \frac{4u_0}{\pi}[\sum_{m=0}^{\infty}\frac{1}{2m+1}sin((2m+1)\pi)e^{-[(2m+1)\alpha]^2 t}]$$

$$\frac{4u_0}{\pi}[sinxe^{-\alpha^2 t} + \frac{1}{3}sin(3x)e^{-9\alpha^2 t} + \frac{1}{5}sin(5x)e^{-25\alpha^2 t}]$$

For $t > 0$,

$$\theta(x,t) \approx \frac{4u_0}{\pi}sinxe^{-\alpha^2 t}$$

We define

$$\theta_E \text{ as one that} \begin{cases} \text{Solves the PDE} \\ \text{Solves the boundary conditions} \\ \frac{\partial\theta_E}{\partial t} = 0 \end{cases}$$

In the problem we have:

$$\frac{\partial^2\theta_E}{\partial x^2} = 0 \implies \theta_E = a + bx$$

Since $\theta_E(0,t) = \theta(L,t) = 0$ then $a = b = 0$
In this case

$$\lim_{t\to\infty}\theta(x,t) = 0 = \theta_E$$

All the solutions are attracted to the equilibrium solution

Continuing from earlier

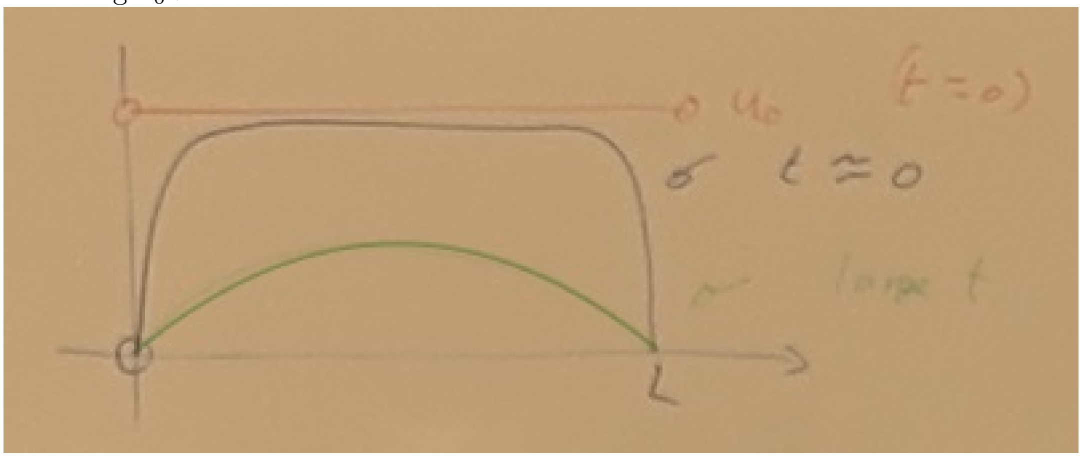
$$\theta = \frac{4u_0}{\pi}[e^{-t}sinx + \frac{1}{3}e^{-9t}sin3x + \frac{1}{5}e^{-25t}sin5x + \ldots]$$

for $t > 0$:

$$\theta = \frac{4u_0}{\pi}[e^{-t}sinx]$$

6.3.3 Temperature Profiles

Assuming $u_0 > 0$



Note that at $t = 0$, $\theta(\frac{\pi}{2}, 0) = u_0$ and $t > 0$, $\theta(\frac{\pi}{2}, t) < u_0$ i.e. instantaneous communication of the signal i.e. temperature at the ends is not physically possible. This shows that this result is idealized equation since the signal propagation is instantaneous, faster than the speed of light

6.3.4 Other Boundary Conditions

Insulation at the Ends

Another type of boundary conditions is when you don't allow the heat to flow out at the ends
Insulation $\iff$ No heat flow $\iff \theta_x = 0$

$$\theta_x(0,t) = 0 \quad \theta_x(L,t) = 0$$

Mixed Conditions

Eg: $\theta(0,t) = 0 \;\; \theta_x(L,t) = 0$

6.3.5 The problem with inhomogeneous boundary conditions

$$\theta_t = \theta_{xx} \;\; , \; \alpha = 1, \quad L = \pi$$

Let the ends be at [boundary conditions]: $\theta(0,t) = 100 \qquad \theta(\pi, 0) = 0$
Initial conditions: $\theta(x, 0) = 50$
Consider, θ_E which satisfies:

1. $(\theta_E)_{xx} = (\theta_E)_t$
2. $\theta_E(0,t) = 100 \quad \theta_E(\pi,t) = 100$
3. $(\theta_E)_t = 0$

If we put 3. into 1., $(\theta_E)_x x = 0 \implies \theta_E = ax + b$
Using this in 2. we get: $a = \frac{-100}{\pi}$ and $b = 100$

$$\theta_E = \frac{100}{\pi}(\pi - x)$$

Let $\theta(x,t) = \theta_E(x,t) + V(x,t)$ where $V(x,t) = \theta(x,t) - \theta_E(x,t)$

1. PDE: $\theta_t = \theta_{xx}$ so $(\theta_E)_t + V_t = (\theta_E)_{xx} + V_{xx} \implies V_t = V_{xx}$
2. Boundary conditions:
$$\theta(0,t) = 100 \;\; and \;\; \theta_E(0,t) = 100 \implies V(0,t) = 0$$
$$\theta(\pi,t) = 0 \;\; and \;\; \theta_E(\pi,t) = 0 \implies V(\pi,t) = 0$$
3. Initial conditions:
$\theta(x,0) = 50$ and $\theta_E(x,0) = \frac{100}{\pi}(\pi - x)$
$V(x,0) = 50 - \frac{100}{\pi}(\pi - x) \implies V(x,0) = \frac{50}{\pi}(2x - \pi)$

Since V satisfies the PDE with the homogeneous BC's

$$V(x,t) = \sum_1^\infty A_n sin(nx) e^{-n^2 t}$$

and we need $V(x,0) = \frac{50}{\pi}(2x - \pi)$ i.e.

$$\sum_1^\infty A_n sin(nx) = \frac{50}{\pi}(2x - \pi)$$

i.e. we need the Fourier sin series of $2x - \pi$

$$\theta = \theta_E + V$$

and

$$\lim_{t\to\infty} U = U_E = 0$$

$$V(x,t) = \frac{-100}{\pi} \sum_{n=1}^\infty \frac{sin(2nx)}{n} e^{-(2n)^2 t}$$

FINAL

(Conrad's rant about things on the final and advise on how not to totally fuck it up)

Course is basically 5 topics

1. 1st Order
   - Seperable
   - Integration Factor
   - Linear
   - Exact
2. 2nd Order
   - $ay'' + by' + cy = f(t)$
     - Mass on a spring
     - LCR circuit
3. Laplace
4. Fourier
5. PDE's

More of the last 3 points, since you've already been tested on points 1 and 2 on the midterm.

He's advised to look at the entire final first, and then starting with the easiest question first, which might not necessarily be the first question. Get into the exam in a good sense of mind, and make sure that you have a good nights of sleep. Forget about the exam as soon as you're done with it.

The final is not out to get you, and according to Conrad is very similar to the midterm. He's repeatedly mentioning the previous final so it might make sense to take a look at it.

Apparently the piece of paper that was being passed around is for Conrad to judge who actually attends lectures, and that attendance is basically proof for who attended and who did not once the final is over and he is getting spammed with emails about further lenience

6.4 Wave Equation (1 spatial dimension)

$$c^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$$

6.4.1 Simple Model

Wire with length $= L$ fixed at the ends with linear density ρ and tension T

$$c^2 = \frac{T}{\rho}$$

1. Boundary Conditions: $y(0,t) = y(L,t) = 0$
2. Initial conditions: $y(x,0) = f(x)$, $\frac{\partial y}{\partial t}(x,0) = g(x)$

Solution: Try $y(x,t) = X(x) \cdot T(t)$
putting this into the DE to get:

$$c^2 T \frac{d^2X}{dx^2} = \frac{d^2T}{dt^2} X \implies \frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2}\frac{1}{T}\frac{d^2T}{dt^2} = \lambda \text{ some constant (separation constant)}$$

1. $\lambda = 0 \implies \frac{d^2X}{dx^2} = 0 \implies X = ax+b$ but since $X(0) = X(L) = 0 \implies a = b = 0 \implies X = 0$
2. $\lambda > 0, \lambda = \mu^2 \implies \frac{d^2X}{dx^2} = \mu^2 X \implies X = Ae^{mx} + Be^{-mx}$ but since $X(0) = X(L) = 0 \implies A = B = 0 \implies X = 0$
3. $\lambda < 0, \lambda = -\mu^2 \implies \frac{d^2X}{dx^2} = -\mu^2 X \implies X = Asin(\mu x) + Bcos(\mu x)$ but since $X(0) = 0 \implies B = 0$ $X(L) = 0 \implies Asin(\mu L) = 0 \implies A = 0$ or $\mu L = n\pi \; \forall \; n \; \epsilon \; \mathbb{Z}$
   Let $\mu_n = \frac{n\pi}{L}$ and $n \; \epsilon \; \mathbb{Z}^+$
   This means $\lambda_n = -(\frac{n\pi}{L})^2 \implies X_n = sin(\mu_n x) = sin(\frac{n\pi x}{L})$
   We now solve for the time behaviour:

$$\frac{1}{c^2}\frac{1}{T_n}\frac{d^2T_n}{dt^2} = \lambda_n = -(\frac{n\pi}{L})^2$$

$$\frac{d^2T_n}{dt^2} = -(\frac{n\pi}{L})^2 T_n \implies T_n = A_n sin(\frac{n\pi ct}{L}) + B_n cos(\frac{n\pi ct}{L})$$

$$Y_n(x,t) = A_n sin(\frac{n\pi ct}{L}) sin(\frac{n\pi x}{L}) + B_n cos(\frac{n\pi ct}{L}) sin(\frac{n\pi x}{L})$$

Thus the general solution of the PDE and the boundary conditions is:

$$\sum_{n=1}^{\infty} Y_n(x,t)$$

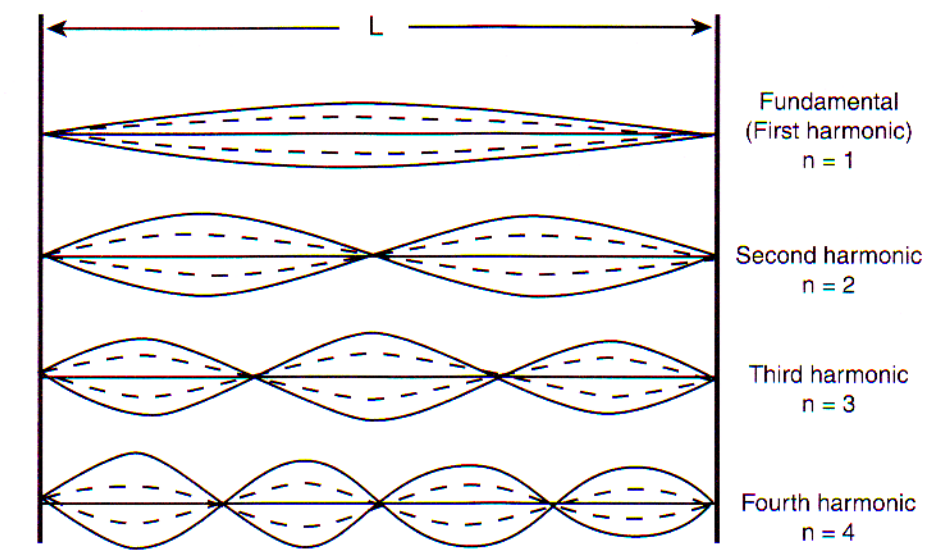
We now consider the initial conditions
$Y(x,0) = \sum_{n=1}^{\infty} B_n sin(\frac{n\pi x}{L}) = f(x)$ i.e. perform the Fourier sine series of $f(x)$
Moreover, $\frac{\partial X(x,0)}{\partial t} = \sum_{n=1}^{\infty} A_n \cdot (\frac{n\pi c}{L}) \cdot sin(\frac{n\pi x}{L})$ i.e. perform the Fourier sine series of $g(x)$

Taking a look at these individual equations

$$Y_n(x,t) = sin(\frac{n\pi x}{L})\Big[A_n sin(\frac{n\pi ct}{L}) + B_n cos(\frac{n\pi ct}{L})\Big] \quad n \,\epsilon\, \mathbb{Z}$$

$$Y_n(x,t) = sin(\frac{n\pi x}{L})\Big[C_n cos(\frac{n\pi c(t-\phi)}{L})\Big] \quad n \,\epsilon\, \mathbb{Z}$$

These functions are called the Normal Modes of Vibration
There are periodic in both space and time and the frequency is $\frac{1}{2\pi}\frac{n\pi c}{L} = \frac{n}{2L}\sqrt{\frac{T}{\rho}} = \nu_n$
$\nu_1 = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$ is called the fundamental frequency and $\nu_n = n\nu_1$
Y_1 is called the first harmonic and Y_n is called the n^{th} harmonic



The n^{th} harmonic has $n-1$ nodes

6.4.2 D'Alemberts Equation

We now investigate the wave equation and its solutions

$$c^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2}$$

If we subsitute $u = x - ct,\ v = x + ct$ it evolves to

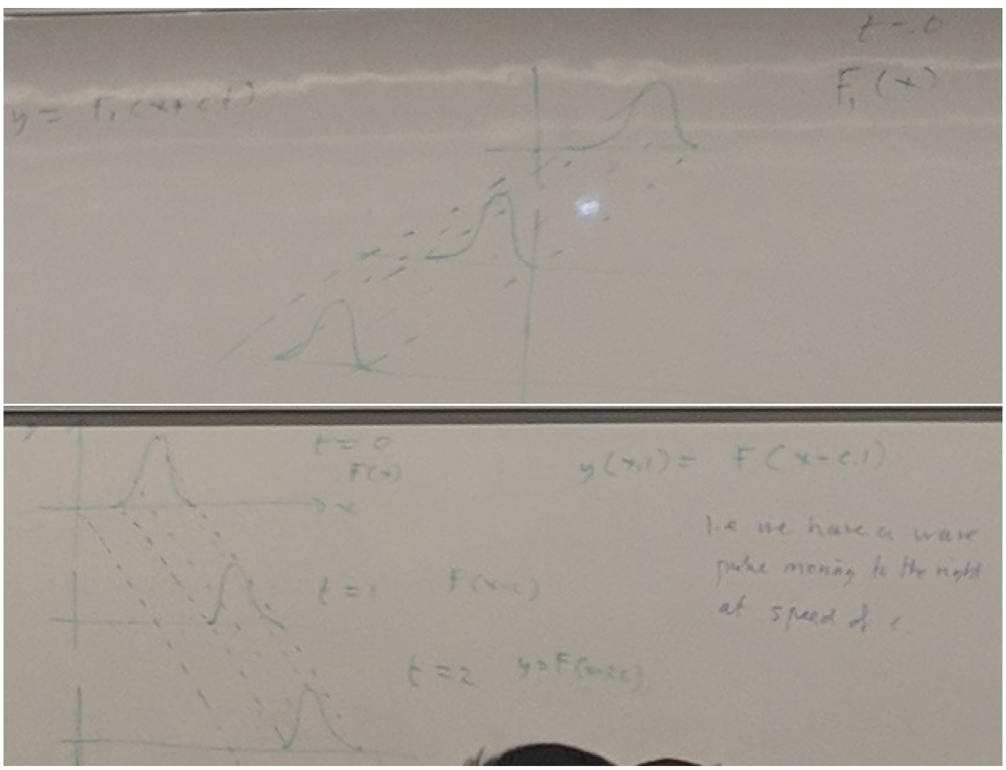
$$\frac{\partial^2 y}{\partial u \partial v} = 0$$

The general solution of the equation above is: $y = F_1(u) + F_1(v) \implies y = F_1(x + ct) + F_2(x - ct)$
Consider $y = F_2(x - ct)$ i.e. $F_1 = 0$ and when $t = 0$, $y = F_2(x) \quad (= f(x))$
If t $= 1$, $y(x, 1) = F_2(x - c.1)$ i.e. we have a wave pulse moving to the right at the speed of c
Similarly for the other term we have
$y = F(x + ct)$ acting like a wave with speed c moving to the left.
In general with $g(x) = 0$

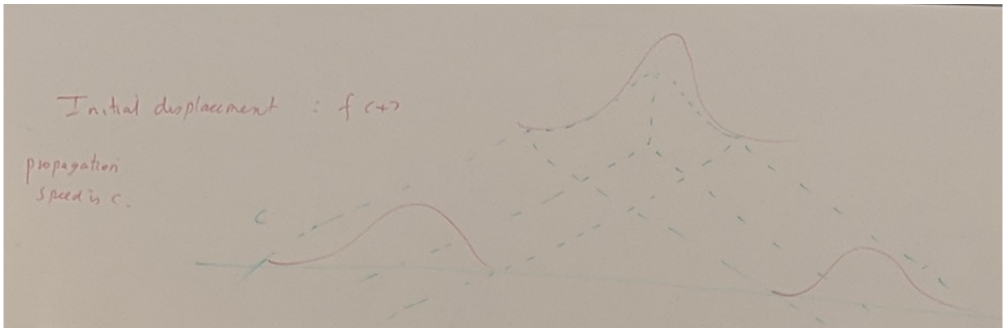
$$y(x, 0) = f(x) \implies F_1(p) = \frac{1}{2} f(p) \ \ and \ \ F_2(p) = \frac{1}{2} f(p)$$

$$y(x, t) = \frac{1}{2}\Big(f(x - ct) + f(x + ct)\Big)$$

In general it seems like the whole wave breaks into two parts, one that goes left with speed c and one that goes right with speed c



Superimposed they look like this:



Capital U (U) and Small u (u) are used interchangably below
For the infinite string (ends as far away so that we don't have to deal with boundary conditions)

$$U_{tt} = c^2 U_{xx} \quad t \geq 0 \, \forall \, -\infty < x < \infty$$

$$U(x,0) = f(x), \;\; U_t(x,0) = g(x)$$

The general solution is :

$$U(x,t) = F_1(x - ct) + F_2(x + ct)$$

1. $u(x,0) = f(x) \implies F_1(x) + F_2(x) = f(x)$
2. $u_t(x,0) = g(x) \implies -cF_1'(x) + cF_2'(x) = g(x)$

We integrate 2. - introduce a dummy variable:

$$-c\int_0^x F_1'(s)ds + c\int_0^x F_2'(s)ds = \int_0^x g(s)ds$$

$$\implies -F_1(x) + F_2(x) = \frac{1}{c}\int_0^x g(s)ds + (F_2(0) - F_1(0)) \quad 3.$$

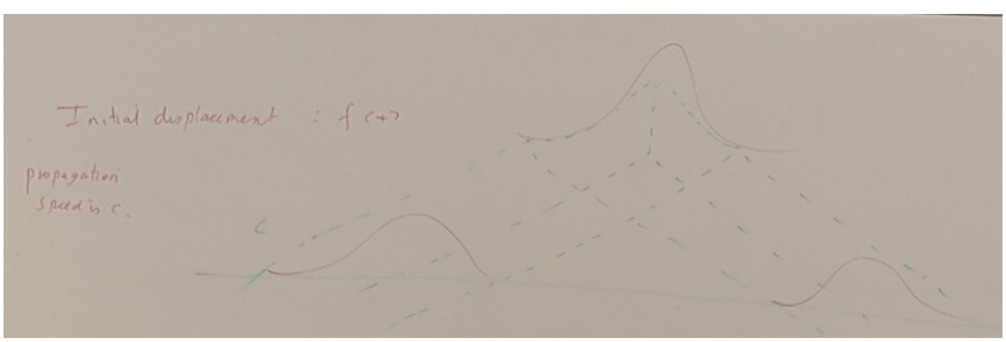
Adding 1. and 3. we get:

$$F_2(x) = \frac{1}{2}[f(x) + \frac{1}{c}\int_0^x g(s)ds + F_2(0) - F_1(0)]$$

Subtracting 3. from 1.

$$F_1(x) = \frac{1}{2}[f(x) - \frac{1}{c}\int_0^x g(s)ds - F_2(0) + F_1(0)]$$

$$U(x,t) = \frac{1}{2}\Big[f(x-ct) + f(x+ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$



The effect of $g(x)$
The integral is only active in a given range, only as time passes do you actually start to feel the signal i.e. if you wait long enough you'll feel the signal $[g(x)]$
This is true in ideal conditions, in reality it often loses energy

6.4.3 D'Alemberts solution on the finite string

$$U_{tt} = c^2 U_{xx} \quad t \geq 0 \; \forall \; -\infty < x < \infty$$
$$U(0,t) = U(L,t) = 0$$
$$U(x,0) = f(x), \;\; U_t(x,0) = 0$$

From earlier recall,

$$U_n(x,t) = A_n cos(\frac{n\pi ct}{L}) sin(\frac{n\pi x}{L}) \implies U_n(x,t) = \frac{1}{2} A_n \Big[sin(\frac{n\pi}{L}(x+ct)) + sin(\frac{n\pi}{L}(x-ct)) \Big]$$
$$U_n(x,t) = F_{n,1}(x+ct) + F_{n,2}(x-ct)$$
$$\implies U(x,t) = \sum_{n=1}^{\infty} U_n(x,t) = \sum_{n=1}^{\infty} F_{n,1}(x+ct) + \sum_{n=1}^{\infty} F_{n,2}(x-ct)$$
$$\implies U(x,t) = F_1(x+ct) + F_2(x-ct)$$

Going backwards you will get:

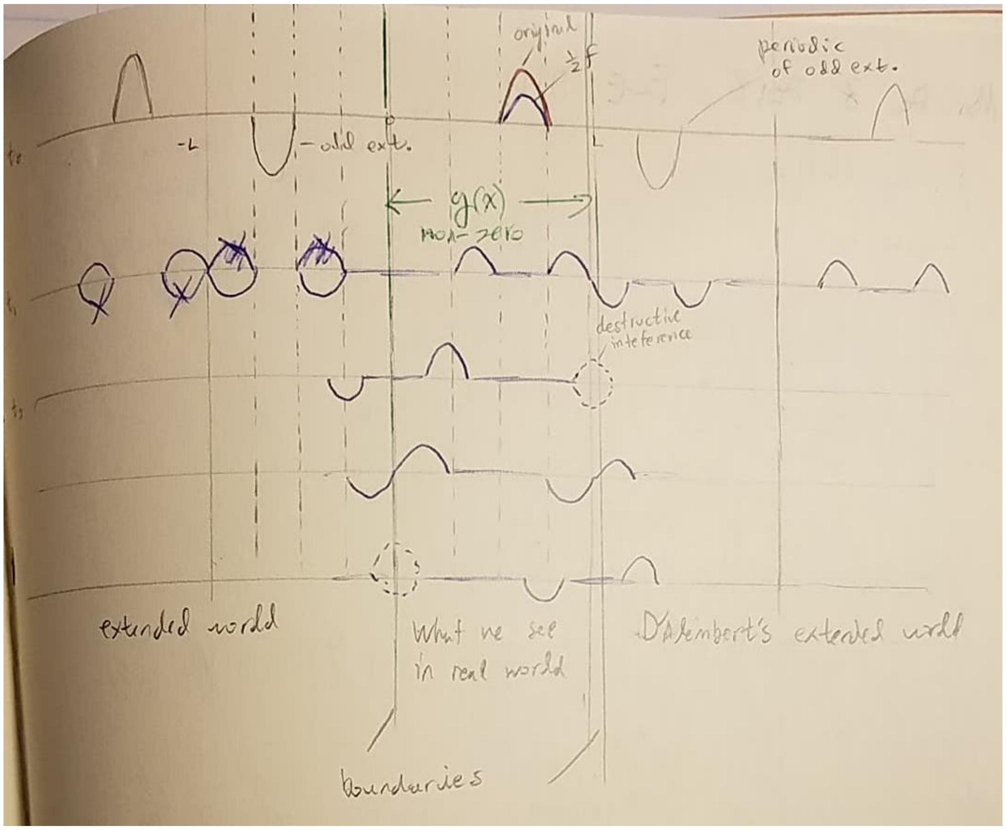
$$F_2(x-ct) = \sum_{n=1}^{\infty} \frac{1}{2} A_n sin(\frac{n\pi}{L}(x+ct))$$
$$F_2(x) = \frac{1}{2} f(x) = \sum_{n=1}^{\infty} \frac{1}{2} A_n sin(\frac{n\pi x}{L})$$

where $f(x)$ was originally defined on the string from $[0, L]$
Let $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ be the $2L$ periodic odd extension of $f(x)$

$$\widetilde{f} = \sum A_n sin(\frac{n\pi s}{L}) \;\; s \,\epsilon\, \mathbb{R}$$

$\widetilde{f}(x-ct) = F_2(x-ct)$ and similarly $\widetilde{f}(x+ct) = F_1(x+ct)$ such that

$$U(x,t) = \frac{1}{2}[\widetilde{f}(x+ct) + \widetilde{f}(x-ct)]$$


original
$\frac{1}{2}f$
periodic
of odd ext.
-L
- odd ext.
L
g(x)
non-zero
destructive
interference
extended world
What we see
in real world
D'Alembert's extended world
boundaries