Inefficiency in Modified Estimator of ES

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Motivation

- Basel III recommends measuring ES at 2.5% tail probability
- Challenge: Measure ES without modeling the underlying distribution of returns
- Is semiparametric estimator of ES effective? How do we measure effectiveness?

1. VaR, ES and Semiparametric Modified Estimator of ES

$VaR_\gamma^{\mu,\sigma} = \mu + \sigma VaR_\gamma^{0,1}$, $ES_\gamma^{\mu,\sigma} = \mu + \sigma ES_\gamma^{0,1}$

Modified VaR (Zangari, 1996)

$$\begin{aligned} mVaR_\gamma &= \mu + \sigma g_\gamma \\ &= \mu + \sigma \left[z_\gamma + \frac{1}{6}\left(z_\gamma^2 - 1\right)\gamma_1 - \frac{1}{36}\left(2z_\gamma^3 - 5z_\gamma\right)\gamma_1^2 + \frac{1}{24}\left(z_\gamma^3 - 3z_\gamma\right)\kappa \right] \end{aligned}$$

Modified ES (Peterson and Boudt, 2008)

$$mES_\gamma = \mu - \frac{\sigma}{\gamma}\phi\left(g_\gamma\right)\left[A_0 + \frac{1}{6}A_1\gamma_1 + \frac{1}{72}A_2\gamma_1^2 + \frac{1}{24}A_3\kappa\right]$$

$A_0 = 1,\ A_1 = g_\gamma^3,\ A_2 = \left(g_\gamma^6 - 9g_\gamma^4 + 9g_\gamma^2 + 3\right),\ A_3 = \left(g_\gamma^4 - 2g_\gamma^2 - 1\right)$

2. Efficiency: Measuring Estimator Effectiveness

- If $\hat{\theta}$ is MLE, then any continuous mapping, $\widehat{g\left(\theta\right)}$ is MLE
- The MLE $\hat{g}$ is consistent, asymptotically unbiased and has the lowest variance
- Asymptotic and Finite Sample Efficiency: $e(\hat{\theta}_{MOD}) = \frac{\text{se}(\hat{\theta}_{MLE})}{\text{se}(\hat{\theta}_{MOD})}$
- Only Finite Sample Efficiency: $e(\hat{\theta}_{MOD}) = \frac{\text{se}(\hat{\theta}_{MLE})}{rmse(\hat{\theta}_{MOD})}$
- If $e(\hat{\theta}_{MOD}) \geq 1$, for all values of $\hat{\theta}_{MOD}$ then $\hat{\theta}_{MOD}$ is less efficienct

3. Multivariate Delta: Calculating Estimator Variance

Consider a sequence, $\mathbf{Y_1} \dots \mathbf{Y}_n$, of k-dimensional random vectors such that

$$\sqrt{n}\left(\mathbf{Y}_n - \boldsymbol{\theta}\right) \xrightarrow{d} \mathcal{N}_k\left(0, \Sigma\right)$$

Then variance of the function $g\left(\mathbf{Y}_n\right)$ is

$$\sqrt{n}\left(g\left(\mathbf{Y}_n\right) - g\left(\boldsymbol{\theta}\right)\right) \xrightarrow{d} \mathcal{N}\left(0, \nabla_g^\top\left(\boldsymbol{\theta}\right)\Sigma\nabla_g\left(\boldsymbol{\theta}\right)\right)$$

4. Modified Risk and its Variance (Martin and Arora, 2015)

The general form of the mVaR and mES risk measures is

$$\begin{aligned} rsk(\boldsymbol{\theta}) &= \mu + C_0\sigma + C_1\sigma\gamma_1 + C_2\sigma\gamma_1^2 + C_3\sigma\kappa \\ &= \mathbf{x}^\top\mathbf{z} = g(\boldsymbol{\theta}) \end{aligned}$$

where

$$\mathbf{x} = \left(1,\ C_0,\ C_1,\ C_2,\ C_3\right)^\top,\ \mathbf{z} = \left(\mu, \sigma, \sigma\gamma_1, \sigma\gamma_1^2, \sigma\kappa\right)^\top,\ \boldsymbol{\theta} = \left(\mu,\ \sigma^2,\ \mu_3,\ \mu_4\right)^\top$$

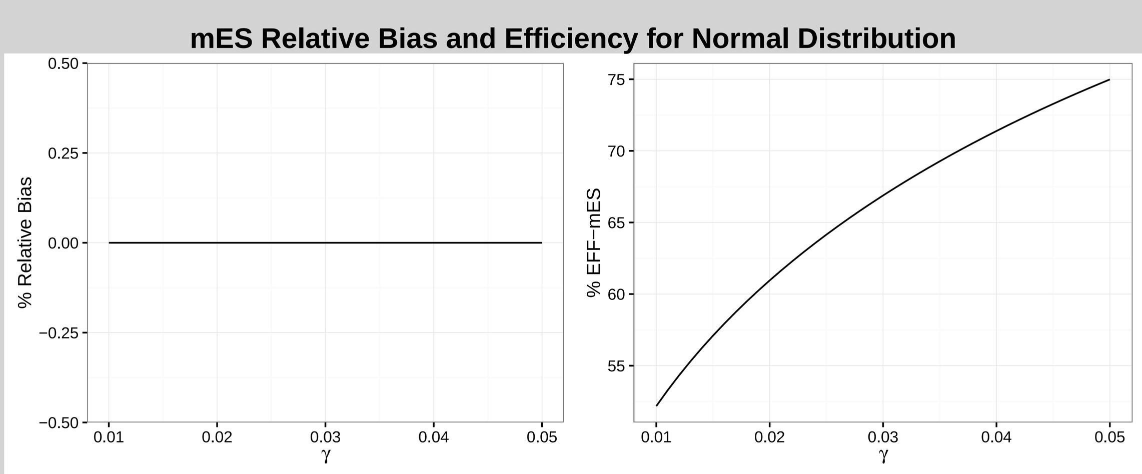
The asymptotic variance of $\widehat{rsk}_n$ is obtained using the delta rule

$$V = V_{\widehat{rsk}_\infty} = \nabla'_{g(\boldsymbol{\theta})} \cdot \Sigma \cdot \nabla_{g(\boldsymbol{\theta})}$$

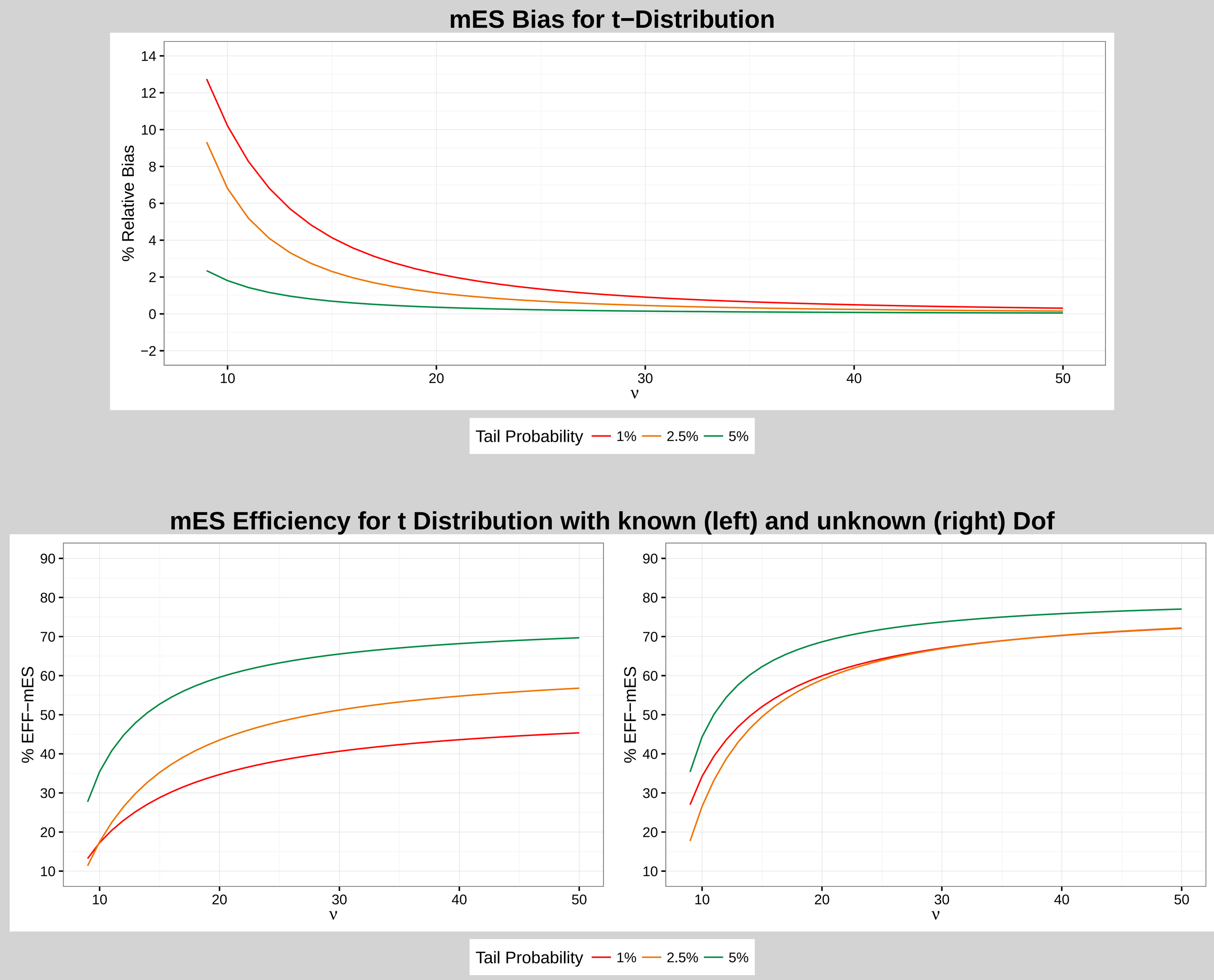
where

$$\begin{aligned} \nabla_{g(\boldsymbol{\theta})} &= \nabla_{\mathbf{x}(\boldsymbol{\theta})^\top\mathbf{z}(\boldsymbol{\theta})} \\ &= \frac{\mathrm{d}\mathbf{z}}{\mathrm{d}\boldsymbol{\theta}}\mathbf{x} + \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\boldsymbol{\theta}}\mathbf{z}\,. \end{aligned}$$

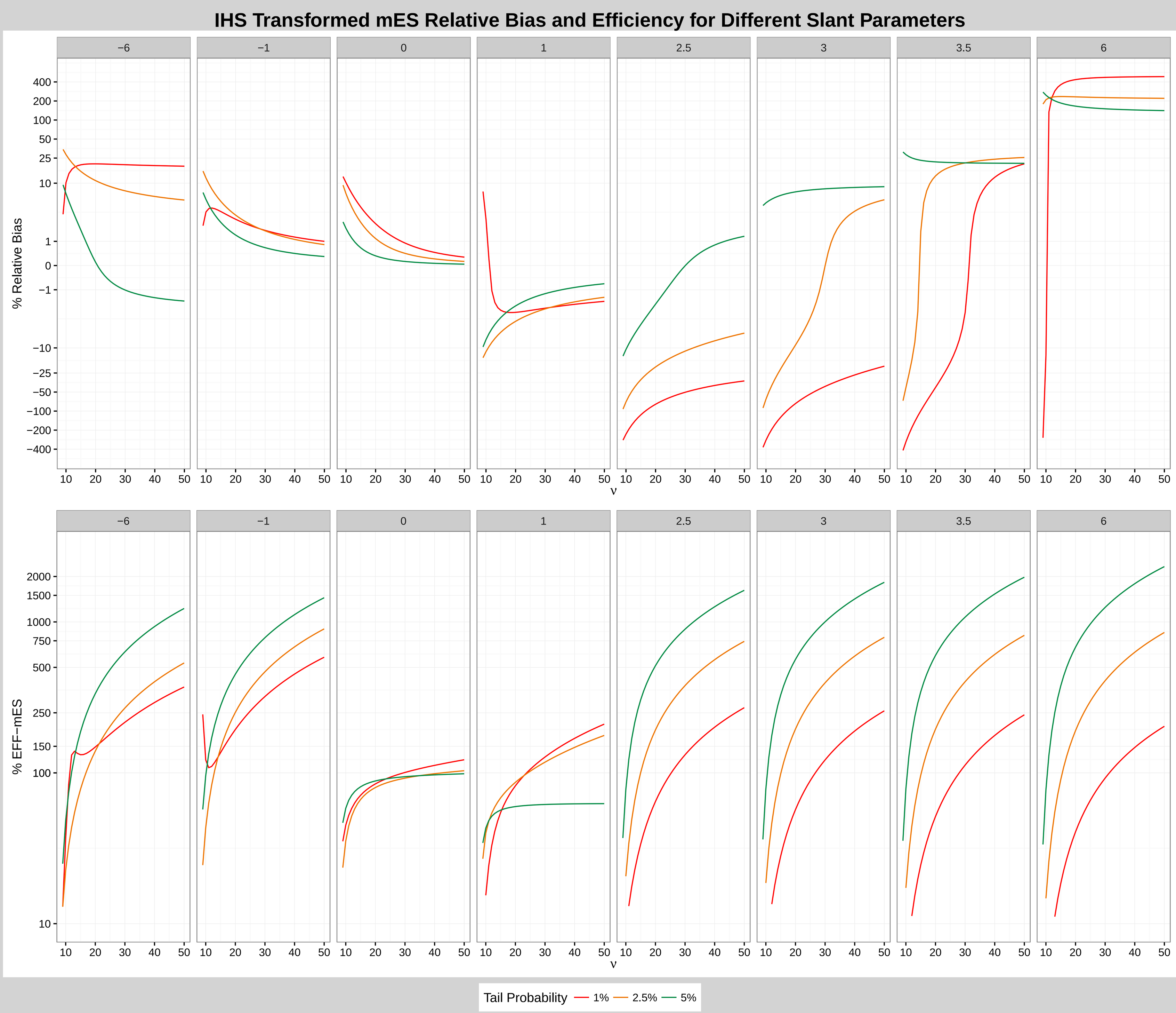
5. Asymptotic Bias and Efficiency: Normal Distribution



6. Asymptotic Bias and Efficiency: t Distribution



7. Asymptotic Bias and Efficiency: Skew-t Distribution



*IHS = Inverse Hyperbolic Sine

Results

- mES efficiencies mostly decrease with decreasing tail probability
- mES efficiencies mostly decrease with increasing tail fatness
- Ordinary estimator variability increases with increasing the number of correlated parameters to be jointly estimated
- Bias or Efficiency alone are insufficient. Relative MSE efficiencies must be evaluated in small sample for comparing estimators
- Asymptotic relative biases vary directly with tail fatness
- In the absence of asymmetry, asymptotic relative biases vary inversely with tail probability

References

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