$$p(\boldsymbol{\beta}, w \mid \mathbf{y}) \propto p(\mathbf{y} \mid \boldsymbol{\beta}, w)\, p(\boldsymbol{\beta} \mid w)\, p(w)$$

$$\mathbf{y} \mid \boldsymbol{\beta}, w \sim \mathcal{N}\left(\mathbf{X}\boldsymbol{\beta}, (w\Lambda)^{-1}\right),\ \boldsymbol{\beta} \mid w \sim \mathcal{N}\left(m, (wK)^{-1}\right),\ w \sim \text{InvGamma}\left(\frac{d}{2}, \frac{\eta}{2}\right)$$

$$\propto \left(w^n |\Lambda|\right)^{1/2} \exp\left(-\frac{1}{2}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})^\top (w\Lambda)(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})\right)$$

$$\left(w^p |K|\right)^{1/2} \exp\left(-\frac{1}{2}(\boldsymbol{\beta}-m)^\top (wK)(\boldsymbol{\beta}-m)\right) w^{d/2-1} \exp\left(-w\frac{\eta}{2}\right)$$

$$\propto w^{(n+d+p)/2-1} \exp\left(-\frac{w}{2}\left((\mathbf{y}-\mathbf{X}\boldsymbol{\beta})^\top \Lambda(\mathbf{y}-\mathbf{X}\boldsymbol{\beta}) + (\boldsymbol{\beta}-m)^\top K(\boldsymbol{\beta}-m)\right)\right) \exp\left(-w\frac{\eta}{2}\right)$$

$$\propto w^{(n+d+p)/2-1} \exp\left(-\frac{w}{2}\left(\begin{array}{l}\left(\mathbf{y}^\top\Lambda\mathbf{y} - 2\mathbf{y}^\top\Lambda\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\beta}^\top\mathbf{X}^\top\Lambda\mathbf{X}\boldsymbol{\beta}\right)\\ +\left(\boldsymbol{\beta}^\top K\boldsymbol{\beta} - 2m^\top K\boldsymbol{\beta} + m^\top Km\right)\end{array}\right)\right) \exp\left(-w\frac{\eta}{2}\right)$$

$$\propto w^{(n+d+p)/2-1} \exp\left(-\frac{w}{2}\left(\boldsymbol{\beta}^\top\left(\mathbf{X}^\top\Lambda\mathbf{X}+K\right)\boldsymbol{\beta} - 2\left(\mathbf{y}^\top\Lambda\mathbf{X} + m^\top K\right)\boldsymbol{\beta}\right)\right)$$

$$\exp\left(-\frac{w}{2}\left(\mathbf{y}^\top\Lambda\mathbf{y} + m^\top Km + \eta\right)\right)$$

$$(C\boldsymbol{\beta}-d)^\top(C\boldsymbol{\beta}-d) = \left(\boldsymbol{\beta}-C^{-1}d\right)^\top\left(C^\top C\right)\left(\boldsymbol{\beta}-C^{-1}d\right) = \boldsymbol{\beta}^\top\left(C^\top C\right)\boldsymbol{\beta} - 2d^\top C\boldsymbol{\beta} + d^\top d$$

$$= \boldsymbol{\beta}^\top\left(C^\top C\right)\boldsymbol{\beta} - 2d^\top C\boldsymbol{\beta} + d^\top d$$

$$C = \left(\mathbf{X}^\top\Lambda\mathbf{X}+K\right)^{1/2},\ d^\top C = \left(\mathbf{y}^\top\Lambda\mathbf{X} + m^\top K\right) \Rightarrow d = \left(\mathbf{X}^\top\Lambda\mathbf{X}+K\right)^{-1/2}\left(\mathbf{y}^\top\Lambda\mathbf{X} + m^\top K\right)^\top$$

$$d^\top d = \left(\mathbf{y}^\top\Lambda\mathbf{X} + m^\top K\right)\left(\mathbf{X}^\top\Lambda\mathbf{X}+K\right)^{-1}\left(\mathbf{y}^\top\Lambda\mathbf{X} + m^\top K\right)^\top$$

$$\boldsymbol{\mu}^* = C^{-1}d = \left(\mathbf{X}^\top\Lambda\mathbf{X}+K\right)^{-1}\left(\mathbf{y}^\top\Lambda\mathbf{X} + m^\top K\right)^\top$$

$$\propto \underbrace{w^{p/2}\exp\left(-\frac{w}{2}\left(\left(\boldsymbol{\beta}-\boldsymbol{\mu}^*\right)^\top C^\top C\left(\boldsymbol{\beta}-\boldsymbol{\mu}^*\right)\right)\right)}_{\mathcal{N}\left(\boldsymbol{\mu}^*, w^{-1}\left(C^\top C\right)^{-1}\right)} \underbrace{w^{(n+d)/2-1}\exp\left(-\frac{w}{2}\left(\mathbf{y}^\top\Lambda\mathbf{y} + m^\top Km + \eta - \boldsymbol{\mu}^{*\top}C^\top C\boldsymbol{\mu}^*\right)\right.}_{\Gamma\left((n+d)/2, \left(\mathbf{y}^\top\Lambda\mathbf{y} + m^\top Km + \eta - \boldsymbol{\mu}^{*\top}C^\top C\boldsymbol{\mu}^*\right)/2\right)}$$

$$\propto w^{d^*/2-1}\exp\left(-\frac{w}{2}\left(\boldsymbol{\beta}-\boldsymbol{\mu}^*\right)^\top\kappa^*\left(\boldsymbol{\beta}-\boldsymbol{\mu}^*\right)\right)\exp\left(-w\frac{\eta^*}{2}\right)$$

$$d^* = n + d + p$$

$$\eta^* = \mathbf{y}^\top\Lambda\mathbf{y} + m^\top Km + \eta - \boldsymbol{\mu}^{*\top}C^\top C\boldsymbol{\mu}^*$$

$$= \mathbf{y}^\top\Lambda\mathbf{y} + m^\top Km + \eta - \left(\mathbf{y}^\top\Lambda\mathbf{X} + m^\top K\right)\left(\mathbf{X}^\top\Lambda\mathbf{X}+K\right)^{-1}\left(\mathbf{y}^\top\Lambda\mathbf{X} + m^\top K\right)^\top$$

$$\kappa^* = C^\top C = \mathbf{X}^\top\Lambda\mathbf{X} + K,\quad \boldsymbol{\mu}^* = \left(\mathbf{X}^\top\Lambda\mathbf{X}+K\right)^{-1}\left(\mathbf{y}^\top\Lambda\mathbf{X} + m^\top K\right)^\top$$

$$\boldsymbol{\beta} \mid w, \mathbf{y} \sim \mathcal{N}\left(\boldsymbol{\mu}^*, w^{-1}\left(\mathbf{X}^\top\Lambda\mathbf{X}+K\right)^{-1}\right)$$