CheeseData

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1. Setup

The data consists of quantities sold and the prices for 88 stores selling cheese. In addition we have a store specific display variable indicating if the cheese was on display. We start of with the following model

$$y_i = X_i\beta + Z_i\gamma_i + \epsilon_i$$

where the coefficients β correspond to the fixed effect across stores and γ_i correspond to the store specific random effects. To setup a fully bayesian model we assume the following set of priors on the parameters in the model

$$\begin{gathered}
y_i|\beta_i, \gamma_i, \sigma^2, \Sigma \sim \mathcal{N}\left(X\beta + Z_i\gamma_i, \sigma^2 I\right) \\
\beta \propto 1 \\
\gamma_i|\Sigma \sim \mathcal{N}\left(0, \Sigma\right) \\
\Sigma \sim \mathcal{IW}\left(d_0, C_0\right) \\
\sigma^2 \propto \frac{1}{\sigma^2}
\end{gathered}$$

β is assumed to have a flat prior. γ_i are considered to have a Jeffrey's prior. Other prior's are standard.

2. Gibbs Updates

Below we specify the posterior updates for the parameters in our model. n_s is the number of stores, n_t is the number of observations per store and n is the total number of stores

- posterior for for fixed effect β

$$\begin{aligned}
p\left(\beta|\mathbf{y}, \gamma, \sigma^2, \Sigma; \Theta\right) &\propto p\left(\beta\right)\prod_{i=1}^{n} p\left(\mathbf{y}_i|\beta, \gamma_i, \sigma^2\right) \\
&\propto 1 \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n_s}\sum_{t=1}^{n_t}\left(y_{it} - x_{it}^\top\beta - z_{it}^\top\gamma_i\right)^2\right) \\
&= \exp\left(-\frac{1}{2}\sum_{i=1}^{n_s}\left(\mathbf{y}_i - X_i\beta - Z_i\gamma_i\right)^\top\left(\sigma^2 I\right)^{-1}\left(\mathbf{y}_i - X_i\beta - Z_i\gamma_i\right)\right) \\
&= \exp\left(-\frac{1}{2}\left(\beta^\top\left(\sigma^{-2}\sum_{i=1}^{n_s}X_i^\top X_i\right)\beta - 2\left(\sum_{i=1}^{n_s}\sigma^{-2}(\mathbf{y}_i - Z_i\gamma_i)^\top X_i\right)\beta\right)\right) \\
&\sim \mathcal{N}\left(\left(\sum_{i=1}^{n_s}X_i^\top X_i\right)^{-1}\left(\sum_{i=1}^{n_s}\left(\mathbf{y}_i - Z_i\gamma_i\right)^\top X_i\right), \left(\sigma^{-2}\sum_{i=1}^{n_s}X_i^\top X_i\right)^{-1}\right)
\end{aligned}$$

- posterior update for random effect γ_i

$$
\begin{aligned}
p\left(\gamma_i|\mathbf{y}_i,\sigma^2,\Sigma;\Theta\right) &\propto p\left(\mathbf{y}_i|\gamma_i,\sigma^2\right)p\left(\gamma_i|\Sigma\right)\\
&\propto \exp\left(-\frac{1}{2\sigma^2}\sum_{t=1}^{n_i}\left(y_{it}-x_{it}^\top\beta-z_{it}^\top\gamma_i\right)^\top\left(y_{it}-x_{it}^\top\beta-z_{it}^\top\gamma_i\right)\right)\exp\left(-\frac{1}{2}\gamma_i^\top\Sigma^{-1}\gamma_i\right)\\
&= \exp\left(-\frac{1}{2\sigma^2}(\mathbf{y}_i-X_i\beta-Z_i\gamma_i)^\top\left(\mathbf{y}_i-X_i\beta-Z_i\gamma_i\right)\right)\exp\left(-\frac{1}{2}\gamma_i^\top\Sigma^{-1}\gamma_i\right)\\
&= \exp\left(-\frac{1}{2}\left((\mathbf{y}_i-X_i\beta-Z_i\gamma_i)^\top\left(\sigma^2 I\right)^{-1}\left(\mathbf{y}_i-X_i\beta-Z_i\gamma_i\right)+\gamma_i^\top\Sigma^{-1}\gamma_i\right)\right)\\
&\propto \exp\left(-\frac{1}{2}\left(-2\sigma^{-2}(\mathbf{y}_i-X_i\beta)^\top Z_i\gamma_i+\sigma^{-2}\gamma_i^\top Z_i^\top Z_i\gamma_i+\gamma_i^\top\Sigma^{-1}\gamma_i\right)\right)\\
&= \exp\left(-\frac{1}{2}\left(\gamma_i^\top\left(\sigma^{-2}Z_i^\top Z_i+\Sigma^{-1}\right)\gamma_i-2\sigma^{-2}(\mathbf{y}_i-X_i\beta)^\top Z_i\gamma_i\right)\right)\\
&\sim \mathcal{N}\left(\left(\sigma^{-2}Z_i^\top Z_i+\Sigma^{-1}\right)^{-1}\sigma^{-2}(\mathbf{y}_i-X_i\beta)^\top Z_i,\left(\sigma^{-2}Z_i^\top Z_i+\Sigma^{-1}\right)^{-1}\right)
\end{aligned}
$$

- posterior update for σ^2

$$
\begin{aligned}
p\left(\sigma^2|\mathbf{y},\gamma,\Sigma;\Theta\right) &\propto p\left(\mathbf{y}|\gamma,\sigma^2\right)p\left(\sigma^2\right) = p\left(\sigma^2\right)\prod_{i=1}^{n_s}p\left(\mathbf{y}_i|\gamma_i,\sigma^2\right)\\
&\propto \sigma^{-2\frac{(n+1)}{2}}\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n_s}\sum_{t=1}^{n_t}\left(y_{it}-z_{it}^\top\gamma_i\right)^\top\left(y_{it}-z_{it}^\top\gamma_i\right)\right)\\
&\sim \mathrm{I}\Gamma\left(\frac{n}{2},\frac{1}{2}\sum_{i=1}^{n_s}\sum_{t=1}^{n_t}\left(y_{it}-x_{it}^\top\beta-z_{it}^\top\gamma_i\right)^\top\left(y_{it}-x_{it}^\top\beta-z_{it}^\top\gamma_i\right)\right)
\end{aligned}
$$

- posterior update for Σ

$$
p\left(\Sigma|\mathbf{y},\gamma,\sigma^2;\Theta\right) \propto \mathcal{IW}\left(d_0+n_s, C_0+\sum_{i=1}^{n_s}\gamma_i\gamma_i^\top\right)
$$