Variance Gamma

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Abstract:

We examine Monte Carlo methods for simulation of Variance Gamma process. We implement algorithms for fitting Variance Gamma density, distribution and quantile functions. We use these to test the empirical evidence of non-normality in financial data. We develop algorithms for pricing European options and geometric Asian options where the underlying evolves as a Variance Gamma process using sequential Monte Carlo methods and benchmark our price against prices from Fourier transform techniques developed in papers based on Variance Gamma process. To reduce variance in our estimates we examine Control Variate, Importance sampling Bridge sampling techniques for Asian options. Finally we use pathwise estimation to develop Greeks for European options.

The paper is organized as follows. In section 1 we motivate the need for Variance Gamma process. In section 2 we briefly describe the notation and results used in the paper. In section 3 we present the algorithms developed and finally we present the results.

1. Introduction

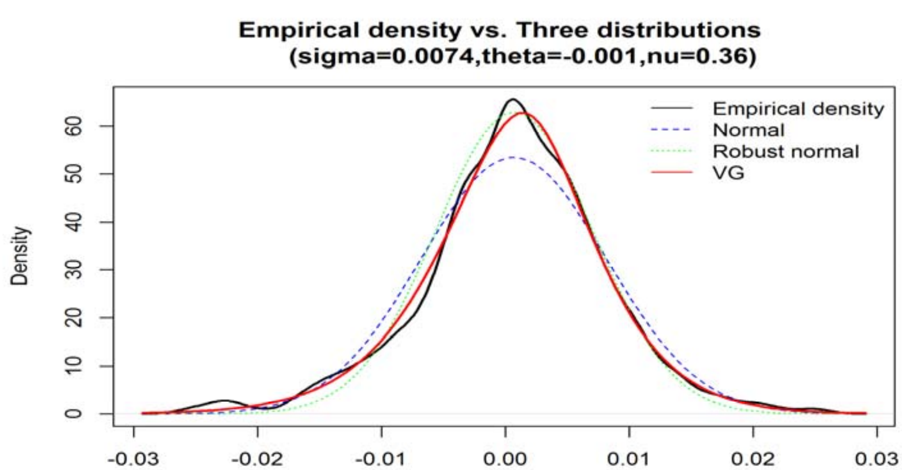
Black-Scholes model for option pricing assumes that the underlying stock price process is log-normally distributed. However, implied stock price density suggests fat tail and skewed behavior. To counter this problem researchers have proposed Variance Gamma distribution as an underlying model for evolution of stock price process.

We developed functionality to fit a $VG(\nu, \theta, \sigma)$ density to data by optimizing log-likelihood. Since we do not have a closed form solution for the density we use the following result

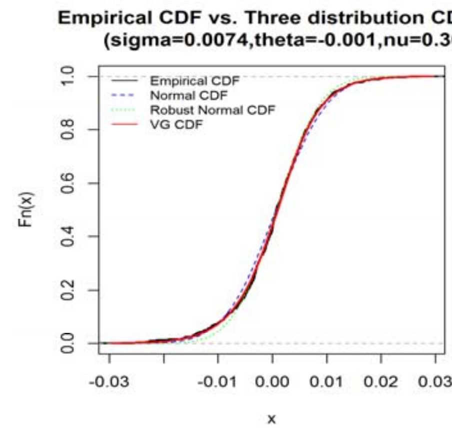
$$F(x) = \frac{1}{2\pi} e^{\alpha x} \int_{-\infty}^{\infty} e^{-iux} \frac{1}{\alpha - iu} \Phi(u + i\alpha) du$$

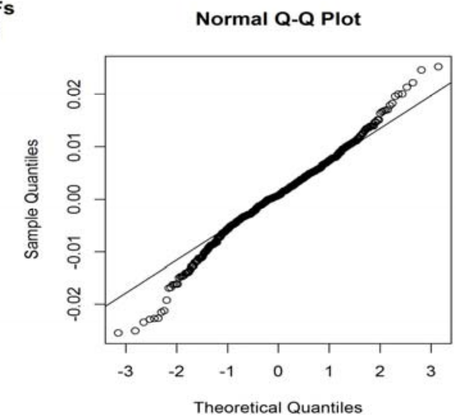
Here Φ is a characteristic function. We then use root finding to develop the quantile function. We use these functions to fit real data and use the fitted density to construct the following graphs.

The adjacent figure shows a comparison of the various parametric densities against empirical density. We see that VG seems to fit the empirical density very well

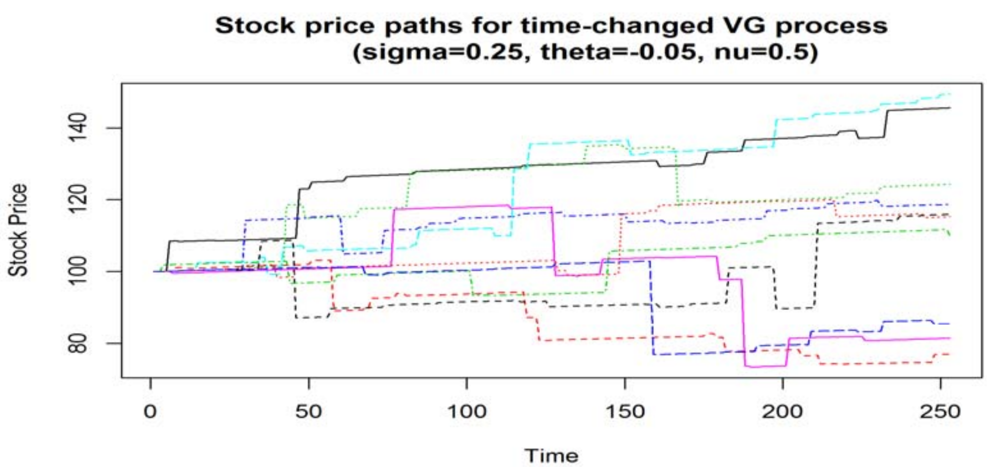


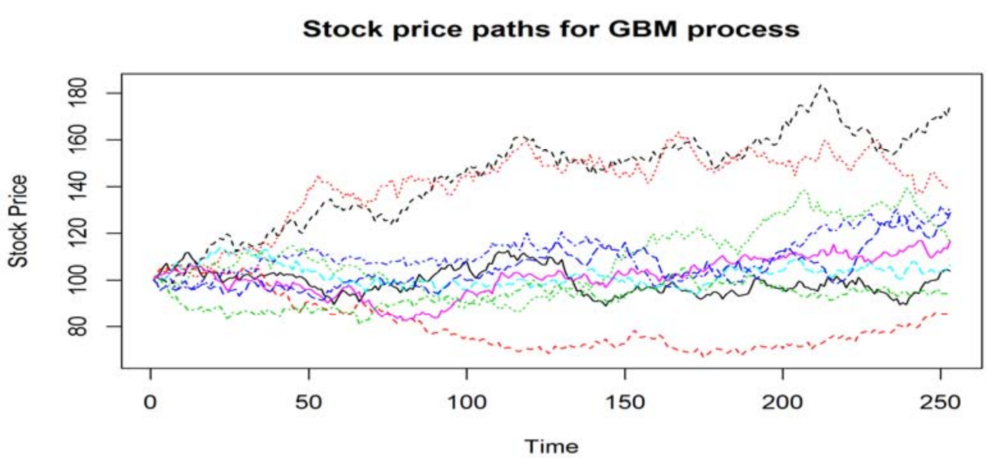
The adjacent figure shows a QQ plot. By examining the tails we can conclude non-normality. We can also see deviation from typical behavior from the overlayed distributions





Brownian motion moves in a uniform way over time and is unable to capture jumps in data. Hence it does not capture smile effects in volatility. On the other hand, Variance Gamma, as seen below is capable of producing jumps. Such a process can produce infinite number of jumps in any interval. It also exhibits a finite variation unlike Geometric Brownian motion.





2. Notation

A random variable which is distributed with respect a Variance Gamma distribution has the following density

$$f(x) = \int_0^\infty \frac{1}{\sigma\sqrt{2\pi g}} \exp\left(-\frac{(x-\theta g)}{2\sigma^2 g}\right) \frac{g^{t/\nu - 1} \exp\left(\frac{-g}{\nu}\right)}{\nu^{t/\nu}\Gamma\left(\frac{t}{\nu}\right)} dg$$

The above form shows that we can also obtain the density by conditioning a Normal density on Gamma density. We exploit this relationship to construct the density of a Variance Gamma random variable. Its characteristic function has the following form

$$\phi(u) = \left(\frac{1}{1-\theta\nu u + \left(\sigma^2 \nu/2\right)u^2}\right)^{t/\nu}$$

The characteristic function severs an important role for pricing contracts using Fourier transform technique as suggested in [4]. We list the main result here

$$C_T(k) = \frac{e^{-\alpha k}}{2\pi} \int_0^\infty e^{-iuk} \Psi_T(u)\, du$$

where Ψ is the characteristic function of the risk-neutral distribution. We use this technique to price European Calls and puts with Variance Gamma as the underlying. We further use results from [5] to price discretely monitored Geometric Asian options. These prices serve as a benchmark for all our pricing results from Monte-Carlo simulation.

3. Methods

a. Simulation using time changed Brownian motion

Variance-Gamma is a class of Levy process and can be described using the following form.

$$S_t = S_0 e^{(r+\omega)t + X_t}, X_t = \theta G_t + \sigma W_{G_t}, \quad G_t = \Gamma\left(\frac{t}{\nu}, \nu\right)$$

$$\omega = \frac{1}{\nu}\ln\left(1 - \theta\nu - \sigma^2 \nu/2\right) \qquad \text{and follows from the martingale condition}$$

$$G_t = \text{subordinator}$$

$$W = \text{weiner process}$$

$$W_{G_t} = o\text{verlying Levy process}$$

We use the above formula to implement the following algorithm

- Generate $\Delta G \sim \Gamma\left(\frac{\Delta t_i}{\nu}, \nu\right)$, $Z_i \sim \mathcal{N}(0,1)$ independently
- Return $\begin{aligned} X_{t_i} &= X_{t_{i-1}} + \theta\Delta G_i + \sigma\sqrt{\Delta G_i} Z_i \\ S_{t_i} &= S_0 \exp\left\{(r+\omega)t_i + X_{t_i}\right\}\end{aligned}$

b. Importance Sampling

Variance reduction is critical when pricing options that are deep out of money to reduce the standard error in estimation. We use the Radon-Nikodym derivative to change the probability measure to a new measure. Such a measure is presented in [2] and has the following form.

$$\frac{dP}{dP'} = \exp\left(-t\int_{-\infty}^{+\infty}\left(k\left(x\right)-k'\left(x\right)\right)dx\right)\cdot\varphi^{+}\left(\tilde{\gamma}_t^{+}\right)\cdot\varphi^{-}\left(-\tilde{\gamma}_t^{-}\right)$$

where $\varphi^{\pm}\left(x\right) = \exp\left(2\cdot\left(\frac{\mu'_{\mp}}{\sigma'^2}-\frac{\mu_{\mp}}{\sigma^2}\right)\cdot\left|x\right|\right)$ and $\tilde{\gamma}_t^{\pm}$ are the independent process generated in the difference-of-gamma representation with new parameters $\left(\sigma',\theta',\nu\right)$.

$$Y = E\left[e^{-rT}payoff\left(S_T\right)\right] = e^{-rT}\int_{-\infty}^{+\infty} payoff\left(S_T\right)\cdot dP = e^{-rT}\int_{-\infty}^{+\infty} payoff\left(S_T\right)\cdot\frac{dP}{dP'}\cdot dP'$$

$$\hat{Y} = \hat{E}\left[e^{-rT}payoff\left(S_T\right)\right] = \frac{1}{n}\sum_{i=1}^{n}\left(e^{-rT}payoff_i\left(S_T\right)\cdot\frac{dP}{dP'}\right)$$

c. Control Variate

Control Variate is an effective technique to reduce variance in an estimator based on variance of another variable whose expected value is known. It has the following general form.

$$X = \bar{X} + \sum_{i=1}^{k}\beta_i\left(Y_i-\bar{Y}\right)+\varepsilon$$

We develop the Control Variate technique to price discreetly monitored geometric Asian option with fixed strike. Since Asian options are path-dependent, we use the price of the same option from the Black-Scholes framework where the analytical price is known as a Control Variate. Further we benchmark the results against the price from the Fourier transform technique.

d. Gamma Bridge Sampling with stratification

Similar to the case of Geometric Brownian Motion we stratify the terminal distribution of the Gamma random variable and use that to generate a Gamma bridge. We use the Time-Changed representation as suggested in [2] for the Gamma Bridge to generate the price path and subsequently the payoff. Then we use the optimal allocation algorithm to generate and estimate for the price of the Asian option.

The central idea is that path for a VG process is increasing. Then for any time $0 \le \tau_1 \le t \le \tau_2$ the conditional distribution of $\gamma\left(t\right)\text{given }\gamma\left(\tau_1\right)\text{ and }\gamma\left(\tau_2\right)\text{ is Beta}\left(\frac{t-\tau_1}{\nu},\frac{\tau_2-t}{\nu}\right)$. Details of the algorithm can be found in [2]

e. Pathwise estimation of Greeks

Suppose the price is given by

$$Y = E\left[e^{-rT} payoff\left(S_T\right)\right],$$

$\therefore$ the sensitivity is

$$\frac{dY\left(\chi\right)}{d\chi} = \frac{dE\left[e^{-rT} payoff\left(S_T\right)\right]}{d\chi} = E\left[\frac{d\left(e^{-rT} payoff\left(S_T\right)\right)}{d\chi}\right].$$

Under certain conditions the expectation operator and the derivative operator can be interchanged. These conditions are satisfied for European options. Differentiating the payoff and using stock price derivatives suggested in [2] we get the following results

$$\Delta = \frac{de^{-rT}\left(S_T - K\right)^+}{dS_0} = e^{-rT}\mathbf{1}_{\{S_T > K\}}\frac{S_T}{S_0}$$

$$\rho = \frac{de^{-rT}\left(S_T - K\right)^+}{dr} = e^{-rT}\mathbf{1}_{\{S_T > K\}}\frac{dS_T}{dr} - Te^{-rT}\mathbf{1}_{\{S_T > K\}}\left(S_T - K\right) = e^{-rT}\mathbf{1}_{\{S_T > K\}}TK$$

$$\theta = \frac{de^{-rT}\left(S_T - K\right)^+}{d\theta} = e^{-rT}\mathbf{1}_{\{S_T > K\}}\frac{dS_T}{d\theta} = e^{-rT}\mathbf{1}_{\{S_T > K\}}S_T\left(T\frac{d\omega}{d\theta} + \gamma_T^{(\nu)}\right)$$

$$\kappa = \frac{de^{-rT}\left(S_T - K\right)^+}{d\sigma} = e^{-rT}\mathbf{1}_{\{S_T > K\}}\frac{dS_T}{d\sigma} = e^{-rT}\mathbf{1}_{\{S_T > K\}}S_T\left(T\frac{d\omega}{d\sigma} + W_{\gamma_T^{(\nu)}}\right)$$

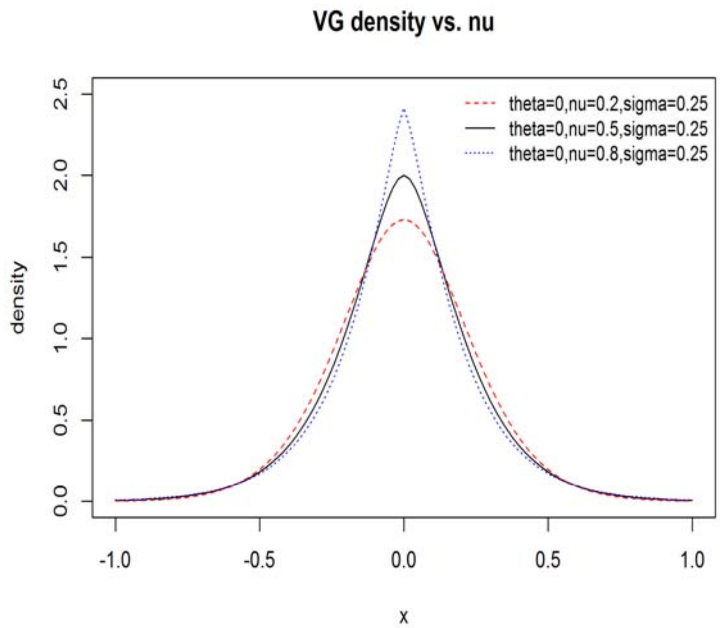
$$\frac{d\omega}{d\theta} = -\frac{1}{1 - \theta\nu - \sigma^2\nu/2}$$

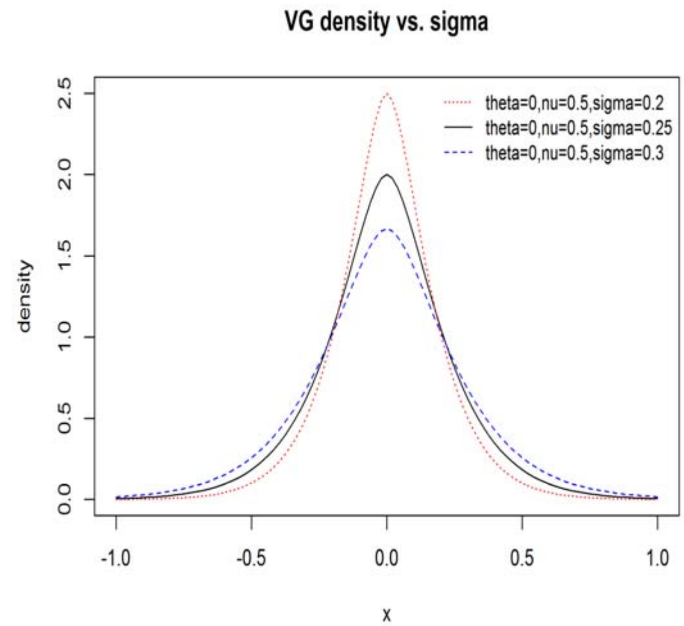
$$\frac{d\omega}{d\sigma} = -\frac{\sigma}{1 - \theta\nu - \sigma^2\nu/2}$$

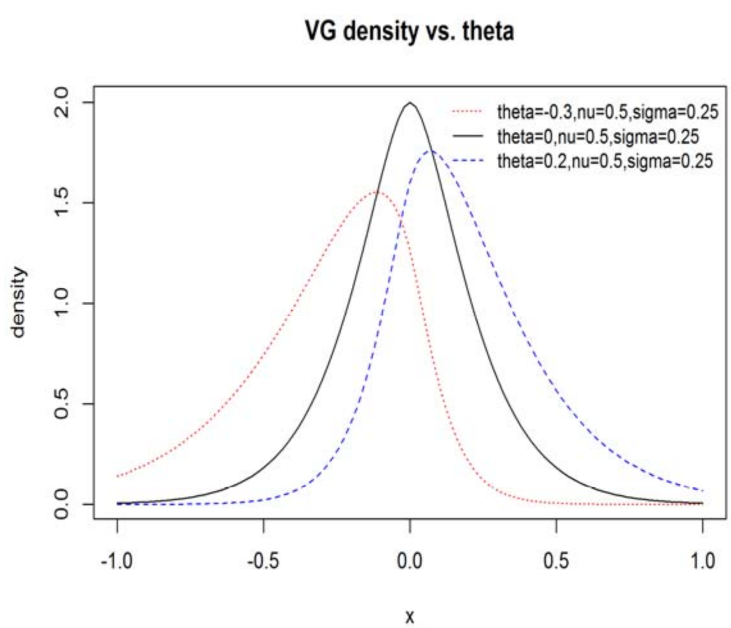
We benchmark the results from the pathwise method with results from crude Monte Carlo using finite difference method. We also use Fourier based estimates for Delta, Gamma and Rho for benchmarking results.

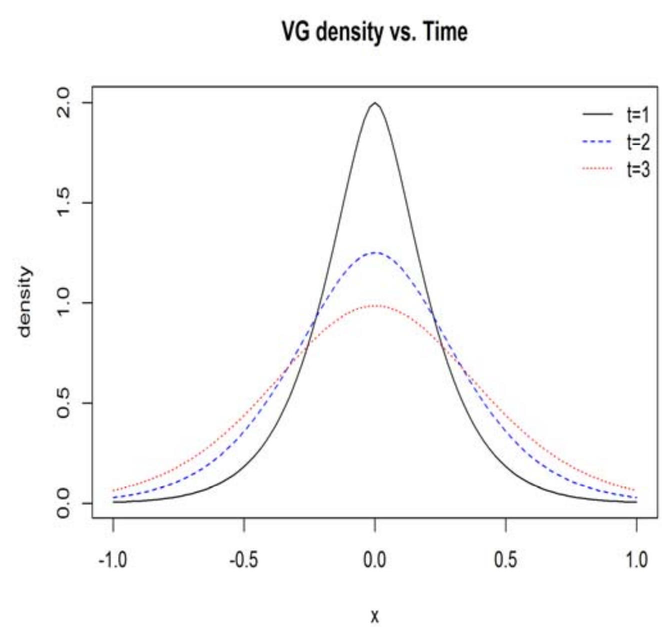
4. Results

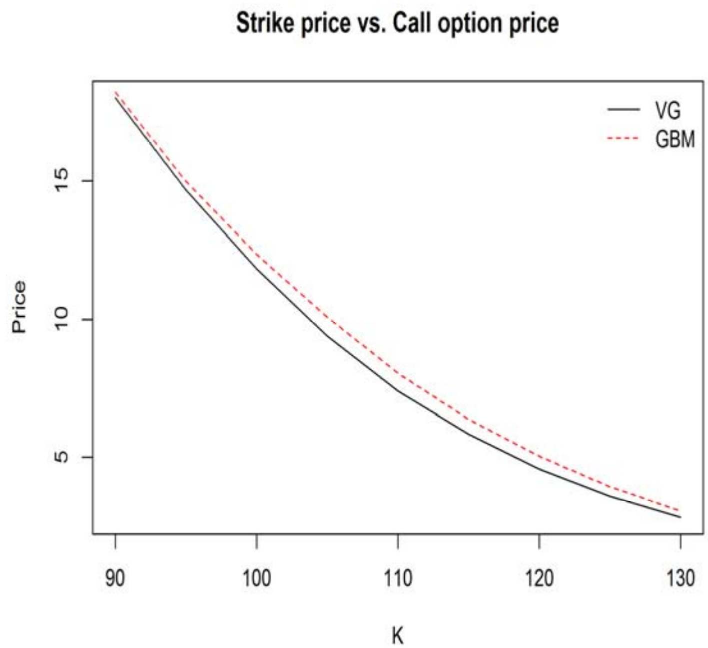
As ν increases the VG density becomes more peaked. As σ increases the density becomes wider. As θ increases the density becomes more skewed. As time increases the density becomes less peaked and we can see the density behaves more like Gaussian density. We can see this behavior in the following graphs.



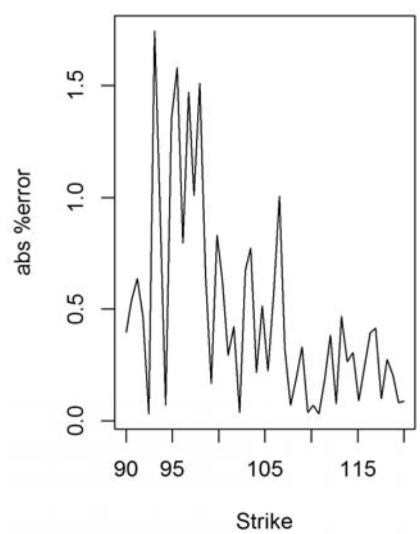


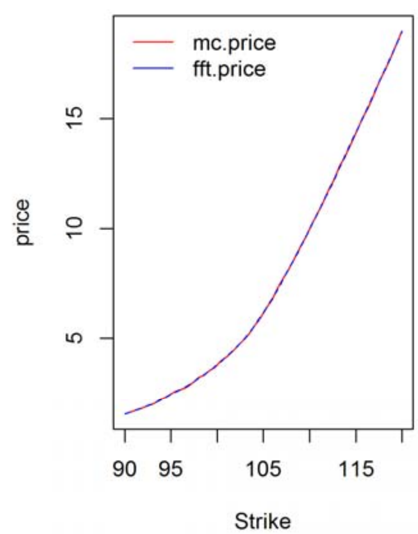






In the above graph we compare the Black-Scholes options prices against VG option price for European calls and puts. We notice that Black-Scholes over estimates the price.





To the left we compare the price of our crude Monte Carlo estimate against Carr-Madan formula. We see that the % absolute error in our price estimate is small.

Below we show results of prices for European Calls, Puts and discretely monitored Geometric Asian option using crude Monte Carlo estimation. We benchmark the prices against Carr-Madan estimates for prices. We see that our price estimates are close.

	European Call		**European Put**	
	Estimated price	**Standard error**	**Estimated price**	**Standard error**
VG Time Changed	7.3624611	0.0502044	12.00348248	0.04563806
VG Diff Gamma	7.43042828	0.05110586	11.95911826	0.04570866
VG FFT	7.402404	NA	12.03764	NA
GBM	8.05231697	0.04896859	12.73824066	0.04516489
GBM FFT	8.026385	NA	12.66162	NA

	Asian Call		Asian Put	
	Estimated price	**Standard error**	**Estimated price**	**Standard error**
VG	2.39295990	0.02190057	10.0600321	0.0318425
VG FFT	2.393005	NA	10.00268	NA
GBM	2.73848509	0.02005997	10.37265283	0.03154842
GBM FFT	2.76867	NA	10.38007	NA

Variance Reduction

Importance Sampling

Here we try to price the deep-out-of money European put with S0=100, and K=45. As new parameters $\left(\sigma', \theta', \nu\right)$ for change of measure, we decrease original θ, increase original σ and keep the ν parameter constant. The parameters are as follows

σ	θ	ν	σ'	θ'	ν
0.25	-0.05	0.5	0.35	-0.1	0.5

Pricing results are as follows

	Estimated price	Standard error
VG Diff Gamma	0.025807280	0.001583751
VG Importance Sampling	0.0237484936	0.0005620372

As we can see in the table, there is variance reduction for the price, but not very significant. This is probably due to choice of parameters for the new probability measure. We are still working on how to make an efficient selection of new measure parameters $\left(\sigma', \theta', \nu\right)$.

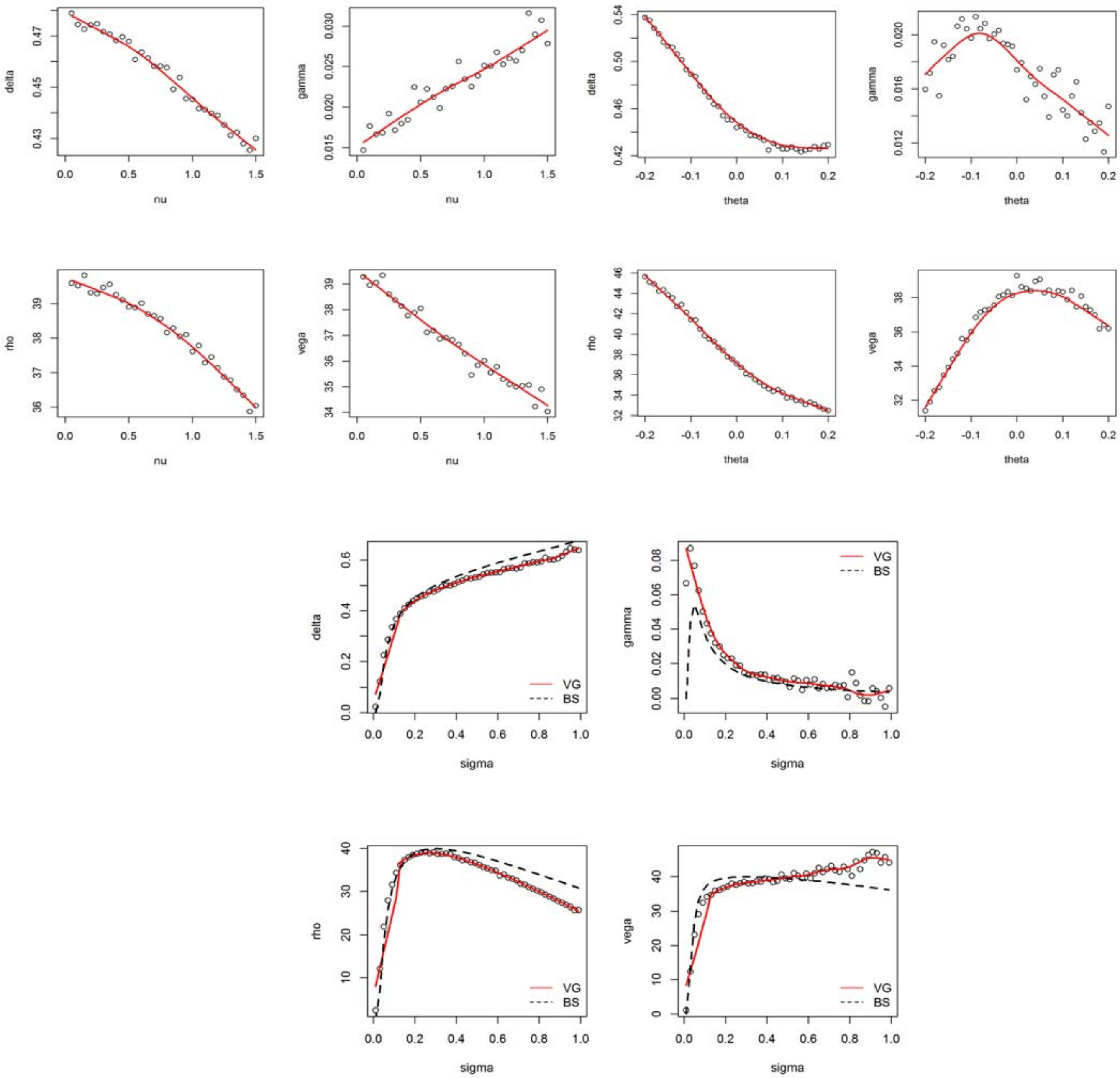
Control Variate

Below are results for price of discretely monitored fixed strike Geometric option. We see significant variance reduction in our prices compared to crude Monte Carlo estimates.

	Asian Call		Asian Put	
	Estimated price	**Standard error**	**Estimated price**	**Standard error**
VG	2.395449	8.785859e-05	10.03663	0.0001258223

Greeks

In the graphs below we examine the behavior of changing VG parameters on Greeks. For Vega we examine a comparison between Black Scholes Vega and VG Vega. Except Gamma, increasing nu deceases the sensitivity. In case of varying Theta, Delta and Rho as moronically decreasing but Gamma and Vega peak at the origin and decrease on either sides. For sigma, we notice the dissimilarity between sensitivities of Black-Scholes and VG as sigma becomes larger. In case of Vega the effect is more pronounced in all ranges of sigma.



The following table shows the results for calculating Greeks using finite difference Monte Carlo, IPA and Fourier transform methods. We see that our IPA estimates are close to the Fourier estimates and finite difference estimates except for Gamma. In case of Gamma we see a high standard error and that is expected as we did not calculate an optimal step size for Gamma. Standard errors of finite difference method are higher than pathwise method. Therefore, pathwise derivative performs better than finite difference.

	European Call Greek Value	Standard Error
Delta diff	0.47934777	0.03566071
Delta pathwise	0.465494395	0.001945311
Delta Fourier	0.4653853	NA
Gamma diff	0.09000361	0.12400942
Gamma pathwise	0.018325135	0.001374918
Gamma Fourier	0.01978011	NA
Rho diff	39.6033748	0.3656007
Rho pathwise	38.9640694	0.1599638
Rho Fourier	39.13612	NA

References

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