Proof of Proposition 3.1

a) W is a vector space over F

~~b)~~ For any function f ~~any c~~ and g from V to W, and any scalar c, we define the function $(f+g)$ and (cf) by:

- $(f+g)(\bar{u}) = f(\bar{u}) + g(\bar{u}) \quad \forall\ \bar{u} \in V$
- $(cf)(\bar{u}) = c\, f(\bar{u}) \quad \forall\ \bar{u} \in V$

1) $\cancel{f}(f+g)(\bar{u}) = f(\bar{u}) + g(\bar{u}) = \cancel{h(\bar{u})} \quad \forall\ \bar{u} \in V$

$\because f : V \to W$ & $g : V \to W$

$\Rightarrow f(\bar{u}), g(\bar{u})$ in W

~~$\Rightarrow f$~~

$\because W$ is a vector space

$\Rightarrow f(\bar{u}) + g(\bar{u})$ ~~&~~ is in W

$\Rightarrow$ Closure is proved

2) ~~a~~ Associativity

Let f, g & h be ~~a~~ any functions from V to W

~~$(f+g+h)(\bar{u}) = f(\bar{u}) + g(\bar{u}) + h(\bar{u})$~~

~~$= (f(\bar{u}) + g(\bar{u})) + h(\bar{u})$~~

~~$f(\bar{u}) + g(\bar{u})$~~

$$((f+g)+h)(\bar{u}) = (f+g)(\bar{u}) + h(\bar{u})$$
$$= f(\bar{u}) + g(\bar{u}) + h(\bar{u})$$
$$= f(\bar{u}) + \bar{u}(g+h)\bar{u}$$
$$= (f + (g+h))\bar{u}$$

$\Rightarrow W^V$ is associative

Similarly, for other properties.

ii) To show that $L(V,W)$ is a subspace :

1) The zero function is a linear transformation, hence belongs to $L(V,W)$

2) (Closure under addition) Suppose that T & U are two linear transformations. Then:

$$\begin{aligned}(T+U)(\bar{u}+\bar{v}) &= T(\bar{u}+\bar{v}) + U(\bar{u}+\bar{v}) \quad \text{(by definition of add}^n\text{)}\\ &= T(\bar{u})+T(\bar{v}) + U(\bar{u}) + U(\bar{v}) \quad (\because T \,\&\, U \text{ are linear})\\ &= T(\bar{u}) + U(\bar{u}) + T(\bar{v}) + U(\bar{v})\\ &= (T+U)(\bar{u}) + (T+U)(\bar{v})\end{aligned}$$

Similarly, $(T+U)(c\bar{u}) = T(c\bar{u}) + U(c\bar{u}) = cT(\bar{u}) + cU(\bar{u})$
$(\because T \,\&\, U \text{ are linear})$

$$\begin{aligned}&= c(T(\bar{u}) + U(\bar{u})) \quad (\because W \text{ is a vector space})\\ &= c(T+U)(\bar{u})\end{aligned}$$

3) Closure under scaler multiplication)

$$\begin{aligned}(cT)(\bar{u}+\bar{v}) &= cT(\bar{u}+\bar{v})\\ &= c(T(\bar{u}) + T(\bar{v}))\\ &= cT(\bar{u}) + cT(\bar{v}) = (cT)(\bar{u}) + (cT)(\bar{v})\end{aligned}$$

Similarly, $(cT)(d\bar{u}) = cT(d\bar{u}) = cdT(\bar{u}) = d(c(T(\bar{u})))$
$= d(cT)(\bar{u})$

Neelgagan