LU Algorithm

- Input an $m \times n$ matrix A

- Step 1: Row reduce A, if possible, to an echelon form matrix U, using only row replacement ops that add a multiple of a row to a row below

- Step 2: Place entries in L such that the same seq of row ops reduces L to I

- Remark: a) The computational efficiency of the LU approach depends on the fact that L is obtained without doing any significant work.

  b) Step 1 is not always possible. But if step 1 is possible, we will give the theoretical justification for why an LU factorization is easily obtained.

Eg:- $A = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 3 & 2 \end{bmatrix}$ $\quad A \xrightarrow{e_1, e_2, e_3} U$ $\quad e_1 : R_1 \to R_2 - R_1$

$R_3 \to R_3 - 2R_1$

$R_3 \to R_3 - 5R_2$

$\to U$ (echelon form of A) $= \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

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We record the elementary row ops e_i & then inverse f_i:

$e_1: R_2 \to R_2 - R_1$ $f_1: R_1 \to R_2 + R_1$

$R_3 \to R_3 - 2R_1$ $f_2: R_3 \to R_3 + 2R_1$

$R_3 \to R_3 - 5R_2$ $f_3: R_3 \to R_3 + 5R_2$

The theory: the same steps when take A to U take L to I :-

$$I = (E_p \cdots E_1) L \Rightarrow L = (E_p \cdots E_1)^{-1} I$$

$$L = (E_1^{-1} \cdots E_p^{-1}) I$$

$\therefore$ To get L, apply f_3, then f_2, then f_1 to I

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\;3)f_1]{\substack{1)f_3 \\ 2)f_2}} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 5 & 1 \end{bmatrix} = L$$

lower tar with 1's on diagonal.

Check : $LU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 3 & 2 \end{bmatrix} = A$

Note: If we have to solve for only one given RHS, say $\vec{b}_1$, then, it is better to do directly by ~~RREF~~ augmented matrix. But if we have to solve for $\vec{b}_1, \vec{b}_2, \ldots \vec{b}_n$, you can use the same LU for all of them.

Solve $A\bar{x} = \bar{b}$ where $\bar{b} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Now $L\bar{y} = \bar{b}$ is:

$$y_1 = 2$$
$$y_1 + y_2 = -1$$
$$2y_1 + 5y_2 + y_3 = 1$$

Rather than "row-red", we just do forward substitution

$$y_1 = 2$$
$$y_2 = -3$$
$$y_3 = 12$$

Now we solve $U\bar{x} = \bar{y} = \begin{bmatrix} 2 \\ -3 \\ 12 \end{bmatrix}$

$$U = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$x_1 - x_2 - 2x_3 = 2$$
$$x_2 + x_3 = -3$$
$$x_3 = 12$$

to solve this, we use back-substitution

$\Rightarrow x_3 = 12$

$x_2 = -15$

$x_1 = 11$