MTH100 **MATHS 1** **Monsoon Semester – 2021-22**

Tutorial exercise for the Week Commencing Monday 20220328

1. Let A be an m×n matrix and let B be a 1×m matrix. Show that BA $\in$ Row A.

2. Let T: V $\rightarrow$ W be a bijective linear transformation. Since T is bijective, the inverse function of T, T^{-1} : W $\rightarrow$ V is well-defined. Prove that is also a linear transformation. (***Remark: This result holds for finite-dimensional as well as infinite-dimensional spaces. Your proof cannot make use of bases.***)

3. **Definition:** A linear transformation T from V into W is said to be **non-singular** if Ker T = $\{\mathbf{0}\}$. Prove:

   a. T is non-singular if and only if T is injective.
   b. T is non-singular if and only if T carries every lin. indep. subset of V into a lin. indep. subset of W.
   c. If V and W are finite-dimensional with dim V = dim W, then T is non-singular if and only if T is invertible.

4. Show that if A^2 is the zero matrix, then 0 is the only eigenvalue of A.

5. Show that λ is an eigenvalue of A if and only if λ is an eigenvalue of A^T.

6. Suppose A is an n×n square matrix such that all the row sums equal the same number s. Show that s is an eigenvalue of A.

7. Suppose A is an n×n square matrix and Rank (A) = k. Show that A can have at most (k + 1) distinct eigenvalues.

8. a) Find the characteristic polynomial $q(\lambda)$ of the matrix A given below, and verify that A satisfies its characteristic polynomial.

   b) Show that both the polynomials $p(\lambda) = \lambda^2 - 3\lambda + 2$ and $r(\lambda) = \lambda^2 - 4\lambda + 4$ are divisors of $q(\lambda)$. Does A satisfy either $p(\lambda)$ or $r(\lambda)$?

   c) What conclusion can you derive from b) ? Explain briefly.

   d) Verify that $\lambda = 1$ and $\lambda = 2$ are both eigenvalues of A, and determine at least three linearly independent eigenvectors of A

$$A = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

Q1

$$A = \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \vdots \\ \bar{a}_m \end{bmatrix}$$, where ~~$\bar{a}_i$ is a n-tuple vector~~ $\bar{a}_i \in \mathbb{R}^n$

$$B = \begin{bmatrix} b_1 & b_2 & b_3 \dots & b_m \end{bmatrix}$$, where $b_i \in \mathbb{R}$

$$BA = B \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \vdots \\ \bar{a}_m \end{bmatrix} = \cancel{\begin{bmatrix} B\bar{a}_1 \\ B\bar{a}_2 \\ \vdots \\ B\bar{a}_m \end{bmatrix}}$$

$$= \begin{bmatrix} b_1 & b_2 & \dots & b_m \end{bmatrix} \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \vdots \\ \bar{a}_m \end{bmatrix}$$

$$BA = \begin{bmatrix} b_1\bar{a}_1 & b_2\bar{a}_2 & \dots & b_m\bar{a}_m \end{bmatrix}$$

$\Rightarrow$ BA is a ~~matrix~~ m-tuple, ~~made up of~~ which is a linear combination of Rows of A

$\Rightarrow BA \in \text{Row } A$

Q2 Let $T: V \to W$ be a linear transformation

$\because T$ is bijective

$\Rightarrow T$ is invertible.

Let $T^{-1}: W \to V$ be its inverse function

i) Let $\bar{w}_1, \bar{w}_2 \in W$

Consider the vector $\bar{v} = T^{-1}\bar{w}_1 + T^{-1}\bar{w}_2$ (1)

& $\bar{u} = T^{-1}(\bar{w}_1 + \bar{w}_2)$

Clearly $\bar{u}, \bar{v} \in V$

$$\begin{aligned} \text{Let } T\bar{v} &= T(T^{-1}\bar{w}_1 + T^{-1}\bar{w}_2) \\ &= T(T^{-1}\bar{w}_1) + T(T^{-1}\bar{w}_2) \quad (\because T \text{ is a linear transformat}) \\ &= (TT^{-1})\bar{w}_1 + (TT^{-1})\bar{w}_2 \\ &= \bar{w}_1 + \bar{w}_2 \quad (3) \end{aligned}$$

$$\begin{aligned} \text{Also } T\bar{u} &= T(T^{-1}(\bar{w}_1 + \bar{w}_2)) \\ &= (TT^{-1})(\bar{w}_1 + \bar{w}_2) = \bar{w}_1 + \bar{w}_2 \quad (4) \end{aligned}$$

$\because T$ is injective $\bar{v} = \bar{u}$ from (3)

$\Rightarrow T^{-1}(\bar{w}_1) + T^{-1}(\bar{w}_2) = T^{-1}(\bar{w}_1 + \bar{w}_2)$ as required

Similarly for scalar multiplication

Q3 a) $\because$ T is said to be non~~s~~ singular if $\text{Ker } T = \{\bar{0}\}$

Also, by Remarks of linear transformation T is also injective when $\text{Ker } T = \{\bar{0}\}$

$\Rightarrow$ T is non-singular if and only if T is injective

b) [$\Rightarrow$] Let T be non-singular let $\bar{v}_1, \ldots, \bar{v}_n$ be linearly ind. in V

Consider $T\bar{v}_1, \ldots, T\bar{v}_n \in W$

$\Rightarrow c_1 T\bar{v}_1 + \cdots + c_n T\bar{v}_n = \bar{0}$
$T(c_1\bar{v}_1 + \cdots + c_n\bar{v}_n) = 0$

$\Rightarrow c_1\bar{v}_1 + \cdots + c_n\bar{v}_n \in \text{Ker } T$

$\Rightarrow c_1\bar{v}_1 + \cdots + c_n\bar{v}_n = \bar{0}$
$\Rightarrow c_1 = \cdots = c_n = 0 \quad \because \bar{v}_i$ is are lin. ind.

[$\Leftarrow$] Suppose T carries every lin. indep. subset of V to a lin indep. subset of W

Let $\bar{v} \in V$ ~~&~~, $\bar{v} \neq 0$

$\therefore$ $\{\bar{v}\}$ is linearly independent in V

$\Rightarrow$ By hypothesis $\{T\bar{v}\}$ is lin indep. in W

$\therefore$ $T\bar{v} \neq \bar{0}$ — (1)

$\Rightarrow \ker T = \{\bar{0}\}$ ~~follows~~ (By (1))

Q4 Given A^2 is the zero matrix

T.P: 0 is the only eigen value of A

Proof:- Let A be a $n \times n$ matrix

Characteristic polynomial of $A = \det(A - \lambda I)$

$$= a_0 + a_1\lambda + \cdots + a_n \lambda^n$$

$\because$ A satisfies its characteristic polynomial

$$\Rightarrow 0 = a_0 + a_1 A + a_2 A^2 + \cdots + a_n A^n$$

$$= a_0 + a_1 A + \cancel{a_2 0}\ 0 + \cdots + 0$$

Q4 Let λ be a non zero eigen value of A. then, there exists a non zero vector $\bar{x}$ st.

$$A\bar{x} = \lambda\bar{x}$$

$$\Rightarrow A^2\bar{x} = A(\lambda x) = \lambda(A\bar{x}) = \lambda \cdot \lambda x = \lambda^2 x$$

$$\Rightarrow A^2 x = \lambda^2 x$$

$\Rightarrow \lambda^2$ is an eigen value of $A^2 = O_{n \times n}$

$$\Rightarrow \lambda^2 = 0$$

$$\Rightarrow \lambda = 0$$

Hence, we get a contradiction,

$\Rightarrow \lambda = 0$ is the only eigen value of A.

~~Q5~~

Q5 $\because (A-\lambda I)^T = (A^T - \lambda I)$

$\Rightarrow |A-\lambda I| = |A^T - \lambda I|$

Let λ be an eigenvalue of A

$|A-\lambda I| = 0$
$|(A-\lambda I)^T| = 0$
$|A^T - \lambda I| = 0$

$\Rightarrow \lambda^T$ is an eigenvalue of A^T

Q(4) Let λ be a non zero eigen-value of A. Then, there exists a non zero vector x s.t.

$$Ax = \lambda x$$

$$\Rightarrow A^2x = A(\lambda x) = \lambda(Ax) = \lambda\cdot\lambda x = \lambda^2 x$$

$$\Rightarrow A^2x = \lambda^2 x$$

$$\Rightarrow \lambda^2 \text{ is an eigen value of } A^2 = O_{n\times n}$$

$$\Rightarrow \lambda^2 = 0 \quad (\text{as } \lambda=0 \text{ is the only eigen-value of a zero matrix})$$

$$\Rightarrow \lambda = 0$$

Hence, we get a contradiction, that is, $\lambda = 0$ is the only eigen value of A.

Q(5) Clearly, $(A-\lambda I)^T = A^T - \lambda I_{n\times n}$ as $I_{n\times n}$ is a symmetric matrix.

$$\Rightarrow |(A-\lambda I)^T| = |A^T - \lambda I_{n\times n}| \quad \text{———} (1)$$

Let λ be an eigen-value of A. Then

$$|A - \lambda I_{n\times n}| = 0$$

$$\Leftrightarrow |(A-\lambda I_{n\times n})^T| = 0 \qquad (\text{as } |A| = |A^T|)$$

$$\Leftrightarrow |A^T - \lambda I_{n\times n}| = 0 \qquad (\text{By } (1))$$

$$\Leftrightarrow \lambda \text{ is an eigen- value of } A^T.$$

(6) Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}$. Given that

$$\sum_{j=1}^{n} a_{ij} = s, \quad \forall\ 1\le i\le n$$

that is,

$$a_{11} + a_{12} + \cdots + a_{1n} = s$$
$$a_{21} + a_{22} + \cdots + a_{2n} = s$$
$$\vdots$$
$$a_{n1} + a_{n2} + \cdots + a_{nn} = s$$

Let λ be an eigen-value of A. Then,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} \underbrace{a_{11}+a_{12}+\cdots+a_{1n}-\lambda} & a_{12} & \cdots & a_{1n} \\ s-\lambda & a_{22}-\lambda & & a_{2n} \\ s-\lambda & & \ddots & \\ \vdots & & & \\ s-\lambda & & & a_{nn}-\lambda \end{vmatrix} = 0 \qquad C_1 \to C_1 + C_2 + \cdots + C_n$$

$s - \lambda$

$$\Rightarrow (s-\lambda) \begin{vmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 1 & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \\ 1 & \cdots & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

$\Rightarrow \underline{\underline{\lambda = s}}$ and hence, s is an eigen-value of A.

Q(7) Let A has $k+2$ distinct eigen values, say $\{\lambda_1, \lambda_2, \dots \lambda_{k+2}\}$. Then, it has at least $k+1$ nonzero eigen values. Without loss of generality, we may assume that $\{\lambda_1, \lambda_2, \dots, \lambda_{k+1}\}$ are all non-zero eigen values. This further implies that for each $1 \le i \le k+1$, $\exists$ a non-zero vector v_i such that

$$A v_i = \lambda_i v_i$$

$$\Rightarrow A\left(\frac{v_i}{\lambda_i}\right) = v_i$$

$$\Rightarrow v_i \in \mathrm{Col}\, A$$

$$\Rightarrow \dim \mathrm{Col}\, A \ge k+1$$

{as $\{v_1, \dots v_{k+1}\}$ are linearly independent}
{By Proposition 36}

Q(8) (a) Characteristic polynomial of $A : q(\lambda) = |A - \lambda I|$

$$= \begin{vmatrix} 3-\lambda & -1 & -1 \\ 1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = (3-\lambda)(1+\lambda^2-2\lambda-1) + 1(1-\lambda+1) - 1(-1-1+\lambda)$$

$$= (3-\lambda)(\lambda^2-2\lambda) + 2-\lambda + 2-\lambda$$

$$= 3\lambda^2 - 6\lambda - \lambda^3 + 2\lambda^2 + 4 - 2\lambda = -\lambda^3 + 5\lambda^2 - 8\lambda + 4$$

Verification: $A^2 = \begin{bmatrix} 7 & -3 & -3 \\ 3 & 1 & -3 \\ 3 & -3 & 1 \end{bmatrix}$ $A^3 = \begin{bmatrix} 15 & -7 & -7 \\ 7 & 1 & -7 \\ 7 & -7 & 1 \end{bmatrix}$

$$\underbrace{-A^3 + 5A^2}\underbrace{-8A + 4I} = \begin{bmatrix} -20 & 8 & 8 \\ -8 & -4 & 8 \\ -8 & 8 & -4 \end{bmatrix} + \begin{bmatrix} 20 & -8 & -8 \\ 8 & 4 & -8 \\ 8 & -8 & 4 \end{bmatrix}$$

$$= 0_{3\times3}$$

b) Clearly $q(\lambda) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4$

$$= (-\lambda+2)(\lambda^2 - 3\lambda + 2) = -(\lambda-2)^2(\lambda-1)$$

Also, $r(\lambda) = (\lambda-2)^2$

Hence, $p(\lambda)$ and $r(\lambda)$ both are divisors of $q(\lambda)$.

Verification: $p(A) = A^2 - 3A + 2I$

$$= \begin{bmatrix} 7 & -3 & -3 \\ 3 & 1 & -3 \\ 3 & -3 & 1 \end{bmatrix} + \begin{bmatrix} 7 & 3 & 3 \\ -3 & -1 & -3 \\ -3 & 3 & 1 \end{bmatrix} = 0_{3\times3}$$

are divisors.

$$r(A) = (A-2I)(A-2I)$$

$$= \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \neq 0_{3\times3}$$

A satisfies $p(\lambda)$ but does not satisfy $r(\lambda)$.

(c) We observe that a matrix satisfies ~~the~~ its characteristic polynomial but not necessarily satisfies the divisors of $q(\lambda)$. However, we can observe that A satisfies a divisor of $q(\lambda)$ consisting all of the eigen values of A.

d) Eigen Values of A: $|A - \lambda I| = 0$

$$\Rightarrow (\lambda-2)^2(\lambda-1) = 0$$

$\Rightarrow \lambda = 1, 2, 2$ are the eigenvalues.

$(A-\lambda I)X = 0$

For $\lambda = 1$:

$$\Rightarrow \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ (Solve this system)

After solving it we get:

$$\begin{bmatrix} 2 & -1 & -1 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2x_1 - x_2 - x_3 = 0, \quad \frac{x_2}{2} - \frac{x_3}{2} = 0$$

$$\Rightarrow 2x_1 = 2x_2 \Rightarrow \boxed{x_1 = x_2} \qquad \boxed{x_2 = x_3}$$

So, $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigen vector of A wrt $\lambda = 1$.

<u>For $\lambda = 2$</u>

$$(A - 2I)x = 0$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{(Solve it)}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 - x_2 - x_3 = 0,$$

$$\Rightarrow x_1 = x_2 + x_3$$

Hence $v_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are eigen vectors of A wrt. $\lambda = 2$.

<u>Check:</u> $\{v_2, v_3\}$ is Linearly independent.

Let $\alpha \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\alpha, \beta \in \mathbb{R}$

$\Rightarrow 2\alpha + \beta = 0$, $\alpha = 0$, $\alpha + \beta = 0$
$\Rightarrow \beta = 0$

Hence, $\{v_2, v_3\}$ is linearly independent.

Since v_1 is an eigen vector corresponding to a distinct eigen value so $\{v_1, v_2, v_3\}$ are linearly independent.