Let $K = U \cap W$

Put $\dim U = m$, $\dim W = n$, $\dim K = k$

We exclude the obvious cases when $U \subseteq W$ or $W \subseteq U$ or $U = \{0\}$ or $W = \{0\}$

So let $B_1 = \{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_k\}$ be a basis for K.

In case $k=0$, we let $B_1 = \emptyset$

So we extend B_1 to a basis of U by adjoining the vectors $\bar{u}_1, \dots, \bar{u}_{m-k}$, ~~and~~

And, we extend B_1 to a basis of W by adjoining the vectors $\bar{w}_1, \dots, \bar{w}_{n-k}$

Here we have used Proposition 15.

Consider the set $B = \{\bar{v}_1, \dots, \bar{v}_k, \bar{u}_1, \dots, \bar{u}_{m-k}, \bar{w}_1, \dots, \bar{w}_{n-k}\}$

We claim B is a basis for $U + W$.

Clearly, $\operatorname{Span} B = U + W$

So we only need to show B is linearly independent.

Let $\bar{u} \in U$

$\bar{u} = c_1\bar{v}_1 + c_2\bar{v}_2 + \dots + c_k\bar{v}_k + d_1\bar{u}_1 + \dots + d_{m-k}\bar{u}_{m-k}$

& $\bar{w} \in W$

$\bar{w} = g_1\bar{v}_1 + \dots + g_k\bar{v}_k + h_1\bar{w}_1 + \dots + h_{n-k}\bar{w}_{n-k}$

So $\bar{u}+\bar{w} = \underbrace{(\dots\dots)}_{\bar{v}'s \cdot \bar{u}'s} + \underbrace{(\dots\dots)}_{\bar{v}s \quad \bar{w}s}$

is a linear comb of

~~$\bar{v}_1$ upd~~ $\bar{v}_1, \dots \bar{v}_k, \bar{u}_1, \dots \bar{u}_{m-k}, w_1, \dots, \bar{w}_{n-k}$

Now

$$a_1\bar{v}_1 + \dots + a_k\bar{v}_k + b_1\bar{u}_1 + \dots + b_{m-k}\bar{u}_{m-k} + c_1\bar{w}_1 + \dots + c_{n-k}\bar{w}_{n-k} = 0 \quad (a)$$

$$\Rightarrow c_1\bar{w}_1 + \dots + c_{n-k}\bar{w}_{n-k} = -a_1\bar{v}_1 - \dots - a_k\bar{v}_k - b_1\bar{u}_1 - \dots - b_{m-k}u_{m-k}$$

$c_1\bar{w}_1 + \dots + c_{n-k}\bar{w}_{n-k} \in K$

$\Rightarrow c_1\bar{w}_1 + \dots + c_{n-k}\bar{w}_{n-k} = d_1\bar{v}_1 + \dots + d_k\bar{v}_k$ for som ~~dis~~. d_i (b)

Substitute (b) in (a)

$$(a_1+d_1)\bar{v}_1 + \dots + (a_k+d_k)\bar{v}_k + b_1\bar{u} + \dots + b_{m-k}\bar{u} = 0 \quad -(c)$$

$\Rightarrow$ It becomes ~~for~~ a basis for U

$\Rightarrow b_1 = b_2 = \dots = b_{m-k} = 0 \quad -(4)$

$\Rightarrow$ (a) becomes

$$a_1\bar{v}_1 + \cdots + a_k\bar{v}_k + c_1\bar{w}_1 + \cdots + c_{n-k}\bar{w}_{n-k} = \bar{0}$$

$$\Rightarrow a_1 = \cdots = a_k = c_1 = \cdots = c_{n-k} = 0 \quad \text{--- (5)}$$

$\Rightarrow$ B is l.i., using (4) & (5).

$$\begin{aligned}\therefore \dim(U+W) &= k + m-k + n-k \\ &= m+n-k \\ &= \dim U + \dim W - \dim(U\cap W)\end{aligned}$$ as req.