MTH100B Math 1 (Linear Algebra) Monsoon Semester – 2021-22 Tutorial

Exercise for the Week Commencing Monday 20220124

1. Determine the inverse of the given matrix A ***using row reduction***.

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$$

2. TRUE or FALSE ?

a) The sum of two invertible matrices (square matrices of the same order) is always invertible.

b) If matrices A and B commute, then invertibility of A implies invertibility of B.

Justify your answer – proof if TRUE or counter-example if FALSE.

3. Suppose AB = AC, where B and C are $n \times p$ matrices and A is an invertible $n \times n$ matrix. Show that B = C. Is this true, in general, when A is not invertible ? Justify your answer (proof if true, counter-example if false).

4. **Observation 1 in Invertible Matrices - Quick Review** (L07 on Monday 20220117) states that if the inverse of A exists, it is unique. Can you prove this ?

5. Consider a general 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

a) Using Theorem 1 (VIT) and Corollary 1.1, show that A is invertible if and only if $ad - bc \neq 0$.

b) Hence determine an expression (formula) for A^{-1}.

6. Construct a 2×2 matrix A with all non-zero entries such that the solution set of the system $A\mathbf{x} = \mathbf{0}$ is the line in $\mathbb{R}^2$ through $(5,-3)$ and the origin. Now find a non-zero vector $\mathbf{b}$ such that the solution set of $A\mathbf{x} = \mathbf{b}$ **is not a line in** $\mathbb{R}^2$ **parallel to the solution** set of $A\mathbf{x} = \mathbf{0}$. Explain why this does not contradict Observation 6 (see lecture slides for L06

on Thursday 20220113).

7. Prove Proposition 5 in the general case, i.e. for any row operation e and any matrix A. (*NB: The three cases of scaling, replacement and interchange require separate proofs.*)

   Hint: Recall that if **u** = $(u_1,u_2,\ldots..,u_n)$ and **v** = $(v_1,v_2,\ldots..,v_n)$ are vectors in $\mathbb{R}^n$, then the dot product **u•v** is the scalar (real number) given by **u•v** = $u_1v_1u_2v_2\ldots..u_nv_n$. If A is an m×n matrix and B is an n×p matrix, then the rows of A and columns of B are vectors in $\mathbb{R}^n$. The general i,j-th term of the product C = AB is $c_{i,j}$ = dot product of i-th row of A with j-th column of B. This tip is useful while doing the algebraic calculations required in proofs of results involving multiplication of matrices.

8. Given an m×n matrix A and an n×p matrix B, the product AB is given by the rule AB = [A$\mathbf{v}_1$ A$\mathbf{v}_2$ ….. A$\mathbf{v}_p$] in column form where B = [$\mathbf{v}_1$ $\mathbf{v}_2$ ….. $\mathbf{v}_p$] in column form. Construct an example to illustrate this rule. The matrix A in your example should be at least 3×3 and B should be at least 3×2. Then prove the rule in the general case.

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Math 1

Tut - 3

Q1

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$$

$$A:I = \left[\begin{array}{ccc:ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 5 & 2 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 - \frac{5}{2}R_1} \left[\begin{array}{ccc:ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & -\frac{5}{2} & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 + \frac{R_2}{4}} \left[\begin{array}{ccc:ccc} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{4} & -\frac{5}{2} & +\frac{1}{4} & 1 \end{array}\right]$$

$$\xrightarrow[\substack{R_2 = R_2/2 \\ R_1 = R_1/2}]{R_3 = -4R_3} \left[\begin{array}{ccc:ccc} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 10 & -1 & -4 \end{array}\right]$$

$$\xrightarrow[R_1 = R_1 + \frac{1}{2}R_3]{R_2 = R_2 - \frac{1}{2}R_3} \left[\begin{array}{ccc:ccc} 1 & \frac{1}{2} & 0 & \frac{11}{2} & -\frac{1}{2} & -2 \\ 0 & 1 & 0 & -5 & 1 & 2 \\ 0 & 0 & 1 & 10 & -1 & -4 \end{array}\right]$$

$$\xrightarrow{R_1 = R_1 - \frac{1}{2}R_2} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 8 & -1 & -3 \\ 0 & 1 & 0 & -5 & 1 & 2 \\ 0 & 0 & 1 & 10 & -1 & -4 \end{array}\right]$$

$$= [I : A^{-1}] \Rightarrow A^{-1} = \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & 4 \end{bmatrix}$$

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Q2

a) False, the sum of two invertible matrix need not be invertible

eg:- let A be $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ & ~~let~~ B be $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Since both A & B both are invertable but ~~A+B~~

$A+B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertable.

b) ~~False~~, If matrices A & B ~~are~~ commute, then if A is invertible B need not also be invertible

For eg:- Let A be $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ & B be $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow AB = BA$ ($\because$ A is identity matrix)

~~But~~
And A is invertible but
B is not invertible

Hence, Proved

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Q3 $AB = AC$

~~Let A be an~~

~~$B=C$ is only true when it is~~

Let A be an invertable matrix

multiplying on the left by A^{-1}

$$A^{-1}AB = A^{-1}AC$$
$$\Rightarrow B = C$$

~~But~~ $\Rightarrow B=C$ is true when A is invertible

But false when A is not invertible.

eg:- $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\Rightarrow$ A is not invertible

$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow AB = AC$

but $B \neq C$.

Q4 Let A have 2 inverse B & C

$\Rightarrow \quad AB = I$

$AC = I$

$\Rightarrow \quad AB = AC$

& $\because$ A is invertable

& By the proof of prev question

$B = C$

$\Rightarrow$ the inverse of A, if exists is unique

Q5 $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Let A be invertible

Extended matrix $= [A:I] = \left[\begin{array}{cc:cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$

$\xrightarrow{R_2 = R_2 - \frac{c}{a}R_1} \left[\begin{array}{cc:cc} a & b & 1 & 0 \\ 0 & d - \frac{c}{a} & -\frac{c}{a} & 1 \end{array}\right]$

$\xrightarrow[R_2 = R_2/\ldots]{R_1 = R_1/a} \left[\begin{array}{cc:cc} 1 & b/a & 1/a & 0 \\ \ldots \end{array}\right]$

$\Rightarrow a$

d

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Q5 a) $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

given A is invertible

$$A \xrightarrow{R_2 = R_2 - \frac{c}{a}R_1} \begin{bmatrix} a & b \\ 0 & d - \frac{cb}{a} \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & \frac{ad-cb}{a} \end{bmatrix} = B$$

$\Rightarrow$ A is row equivalent to B

$\Rightarrow$ For A to be invertible

A must be row equivalent to I

$\Rightarrow$ B must be row equivalent to I

& For B to be row equivalent to I

$a \neq 0$

& $ad - cb \neq 0$

Hence, Proved

b) $[A : I] = \begin{bmatrix} a & b & : & 1 & 0 \\ c & d & : & 0 & 1 \end{bmatrix}$

$$\xrightarrow{R_2 \Rightarrow R_2 - \frac{c}{a}R_1} \begin{bmatrix} a & b & : & 1 & 0 \\ 0 & \frac{ad-cb}{a} & : & -\frac{c}{a} & 1 \end{bmatrix}$$

$\because ad - cb \neq 0$ & $a \neq 0$

$$\xrightarrow[R_1 \to R_1/a]{R_2 \to R_2 \frac{a}{ad-cb}} \begin{bmatrix} 1 & b/a & : & 1/a & 0 \\ 0 & 1 & : & \frac{-c}{ad-cb} & \frac{a}{ad-cb} \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - \frac{b}{a}R_2} \begin{bmatrix} 1 & 0 & : & \frac{ad-cb+cb}{a(ad-cb)} & -\frac{b}{ad-cb} \\ 0 & 1 & : & \frac{-c}{ad-cb} & \frac{a}{ad-cb} \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{d}{ad-cb} & \frac{-b}{ad-cb} \\ \frac{-c}{ad-cb} & \frac{a}{ad-cb} \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-cb} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

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Q6 Let ~~A~~ the ~~be~~ a 2×2 matrix A be

$$A = \begin{bmatrix} 3 & 5 \\ 6 & 10 \end{bmatrix}$$

sol^n set of
$\Rightarrow A\bar{x} = \bar{0}$ is a line in $\mathbb{R}^2$ passing through $(5,-3)$ & the origin.

Now, let $b = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \neq \bar{0}$. Then $A\bar{x} = \bar{b}$ will be reduced to

$$\begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$3x_1 + 5x_2 = 0$$
$$0 = 2 \Rightarrow \text{not possible}$$
$\Rightarrow$ system has no solution

Hence, solution set of $Ax = b$ is not a line in $\mathbb{R}^2$ parallel to the sol^n set of $A\bar{x} = \bar{0}$. However, if $A\bar{x} = \bar{b}$ is consistent then sol^n set of $A\bar{x} = \bar{b}$ will be a line in $\mathbb{R}^2$ ∥ to the sol^n set of $A\bar{x} = \bar{0}$

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Q7 Proposition 5

If A is a $m \times n$ matrix & e is an e-row, then $e(A) = EA$ where E_n is the elementary matrix ~~of~~ obtained from I_m by applying e.

Proof

For replacement

eg:- $e = R_x \to R_x + kR_y$

$A = [a_{ij}]_{m \times n}$

$e(A) = B = [b_{ij}]_{m \times n}$

$EA = C = [c_{ij}]_{m \times n}$

for B

For any i, $i \neq x$

$a_{ij} = b_{ij}$ — 1

For $i = x$

$b_{xj} = a_{xj} + ka_{yj}$ — 2

for C

For any i, $i \neq x$

$\Rightarrow$ i^{th} row of E

$$\begin{array}{c} 0\ 0\ 0 \ldots \ 1\ 0 \ldots 0 \\ \qquad\quad\ \uparrow \\ \qquad\quad i^{th} \text{ element} \end{array}$$

~~$\Rightarrow$~~ for

$\Rightarrow$ j^{th} column of A

$$\begin{bmatrix} a_{ij} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

$\Rightarrow$ $c_{ij} = 0 \cdot a_{ij} + 0 \cdot a_{2j} + \cdots + 1 \cdot a_{ij} + \cdots 0 \cdot a_{mj}$

$c_{ij} = a_{ij} = b_{ij}$ — ③ (By ①)

for $i = x$

$\Rightarrow$ x^{th} row

y^{th} element

$0 \; 0 \; 0 \cdots 1 \cdots k \cdots 0$

x^{th} element

$\Rightarrow$ j^{th} column of A

$\Rightarrow$ $c_{xj} = 0 \cdot a_{ij} + 0 \cdot a_{2j} + \cdots + 1 \cdot a_{xj} + \cdots + k \cdot a_{yj} + \cdots 0 \cdot a_{mj}$

$c_{xj} = a_{xj} + k a_{yj} = b_{xj}$ — ④ (By ②)

$\Rightarrow$ By ③ & ④

$c_{ij} = b_{ij}$ (for any i, j)

$\Rightarrow$ $C = B$

$\Rightarrow$ $e(A) = EA$ for replacement — Ⓐ

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For scaling

$A = [a_{ij}]_{m\times n}$

$e :- \quad R_x \rightarrow kR_x$

$e(I) = E$

$\Rightarrow e(A) = B \quad = [b_{ij}]_{m\times n}$

$EA = C \quad [c_{ij}]_{m\times n}$

<u>For B</u>

for any $i \neq x$

$a_{ij} = b_{ij}$ — ①

for $i = x$

$b_{ij} = k\,a_{ij}$ — ②

<u>For C</u>

for any $i \neq x$

$\Rightarrow i^{th}$ row of E

$0 \;\; 0 \;\cdots\; 1 \;\cdots\; 0$

i^{th} element

$\Rightarrow j^{th}$ column of A

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

$c_{ij} = 0\cdot a_{1j} + 0\cdot a_{2j} + \cdots + 1\cdot a_{ij} + \cdots + 0\cdot a_{mj}$

$c_{ij} = a_{ij} = b_{ij}$ — ③ (From ①)

for $i = x$

$\Rightarrow x^{th}$ row of E

$0 \; 0 \; \cdots \; k \cdots 0$

$\uparrow$

x^{th} element

$\Rightarrow j^{th}$ column of A

$c_{xj} = 0 \cdot a_{1j} + 0 \cdot a_{2j} + \cdots + k \cdot a_{xj} + \cdots + 0 \cdot a_{mj}$

$c_{xj} = k a_{xj} = b_{xj}$ — ④ (From ②)

From ③ & ④

$b_{ij} = c_{ij}$ (for any i, j)

$\Rightarrow$ ~~C = B~~ $B = C$

$\Rightarrow e(A) = EA$ for scaling — B

For interchange

$A = [a_{ij}]_{m \times n}$

$e :-$ $R_x \leftrightarrow R_y$

$e(I) = E$

$e(A) = B = [b_{ij}]_{m \times n}$

$EA = C = [c_{ij}]_{m \times n}$

For B

for any $i \neq x, y$

$c_{ij} = b_{ij}$ — ①

for $i = x$

$a_{xj} = b_{yj}$ — ②

for $i = y$

$a_{yj} = b_{xj}$ — ③

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For C

for any $i \neq x, y$

$\Rightarrow i^{th}$ row of E

$$0\ 0\ 0 \cdots \underset{\uparrow\, i^{th} \text{ element}}{1} \cdots 0$$

$\Rightarrow j^{th}$ column of A

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

$$c_{ij} = 0 \cdot a_{1j} + \cdots + 1 \cdot a_{ij} + \cdots + 0 \cdot a_{mj}$$

$$c_{ij} = a_{ij} = b_{ij} \quad -④ \quad (\text{From } ①)$$

for $i = x$

$\Rightarrow x^{th}$ row of E

$$0\ 0\ 0 \cdots \underset{\uparrow\, y^{th} \text{ element}}{1} \cdots 0$$

$\Rightarrow j^{th}$ column of A

$$c_{xj} = 0 \cdot a_{1j} + \cdots + 1 \cdot a_{yj} + \cdots + 0 \cdot a_{mj}$$

$$\Rightarrow c_{xj} = a_{yj} = b_{xj} \quad -⑤ \quad (\text{from } ②)$$

Similarly for $i = y$

$$c_{yj} = a_{xj} = b_{yj} \quad -⑥ \quad (\text{from } ③)$$

$\Rightarrow$ From ⑤ & ⑥

$$b_{ij} = c_{ij} \quad (\text{For any } i, j)$$

$$\Rightarrow B = C$$

$$\Rightarrow e(A) = EA \quad \text{for Interchange} \quad -Ⓒ$$

$\Rightarrow$ From Ⓐ, Ⓑ, Ⓒ

$$e(A) = EA \quad \text{for any row e-ops.}$$

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Q8 Let A is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

& B is $\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$

$\Rightarrow B = [b_1 \ \ b_2]$ where $b_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ & $b_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

$\Rightarrow$ ~~AB = [~~

$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 6 \\ 3 & 4 \end{bmatrix}$ —(1)

~~& $A[b_1 \ \ b_2] = Ab_1 \ A$~~

& $Ab_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$

$Ab_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$

$\Rightarrow A[b_1 \ \ b_2] = [Ab_1 \ \ Ab_2]$

$= \begin{bmatrix} 1 & 2 \\ 4 & 6 \\ 3 & 4 \end{bmatrix} = AB$ (By (1))

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T.P: $AB = [Av_1, Av_2 \ldots Av_p]$

Proof Let A be a $m \times n$ matrix $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

Let B be a $n \times p$ matrix $\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix}$

$\Rightarrow v_1 = \begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{bmatrix} \Rightarrow v_i = \begin{bmatrix} b_{i1} \\ b_{2i} \\ \vdots \\ b_{ni} \end{bmatrix}$ & $B = [v_1 \; v_2 \ldots v_p]$

LHS

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots a_{1n}b_{n1} & \cdots\cdots & a_{11}b_{1p} + \cdots + a_{1n}b_{np} \\ \vdots \\ a_{m1}b_{11} + a_{m2}b_{21} + \cdots a_{mn}b_{n1} & \cdots\cdots & a_{m1}b_{1p} + \cdots + a_{mn}b_{np} \end{bmatrix} \quad -①$$

RHS

$$Av_1 = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1n}b_{n1} \\ \vdots \\ a_{m1}b_{11} + a_{m2}b_{21} + \cdots + a_{mn}b_{n1} \end{bmatrix}$$

$$Av_i = \begin{bmatrix} a_{11}b_{1i} + a_{12}b_{2i} + \cdots + a_{1n}b_{np} \\ \vdots \\ a_{m1}b_{1i} + a_{m2}b_{2i} + \cdots + a_{mn}b_{ni} \end{bmatrix}$$

$\Rightarrow AB = [Av_1 \; Av_2 \ldots Av_p]$ (By ①)

$\Rightarrow$ LHS = RHS

Hence, Proved