Singular Value Decomposition – 3a

- **Proof of Proposition 51:**
- Clearly the vectors $A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_n$ belong to Col A.
- Also, for $j > r$, we have $\|A\mathbf{v}_j\| = \sqrt{\lambda_j} = \sigma_j = 0$, so $A\mathbf{v}_j = 0$.
- For $i,j \le r$, we have:

$A\mathbf{v}_i \cdot A\mathbf{v}_j = (A\mathbf{v}_i)^T(A\mathbf{v}_j) = \mathbf{v}_i^T(A^TA)\mathbf{v}_j$

$= \mathbf{v}_i^T\lambda_j\mathbf{v}_j$ (since $\mathbf{v}_j$ is an eigenvector of A^TA for λ_j)

$= \lambda_j(\mathbf{v}_i \cdot \mathbf{v}_j) = 0$, since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthonormal basis for R^n.

Thus, the vectors $A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_r$ form an orthogonal set of non-zero vectors and are therefore linearly independent.

Singular Value Decomposition – 3b

- **Proof of Proposition 51 (continued):**
- Finally suppose that $\mathbf{y}$ is in Col A. Then $\mathbf{y} = A\mathbf{x}$ for some vector $\mathbf{x}$.

Then $\mathbf{x}$ can be expressed in terms of the basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$,

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n.$$

Then $\mathbf{y} = A\mathbf{x} = A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n)$

$= c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + \ldots + c_rA\mathbf{v}_r$, since remaining terms are $\mathbf{0}$, as noted at the start of the proof.

Thus, the vectors $A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_r$ also span Col A.