MTH100 MATHS I Monsoon Semester – 2021-22

Tutorial Exercise for Week Commencing Monday 20220228.

1. a) Let $\{\mathbf{v_1}, \mathbf{v_2}, \ldots\ldots, \mathbf{v_k}\}$ be an <u>ordered</u> list of $k \geq 2$ vectors in a vector space V. Prove that the list is linearly dependent if and only if either $\mathbf{v_1} = \mathbf{0}$ or there is some j, $2 \leq j \leq k$, such that $\mathbf{v_j}$ is a linear combination of the preceding vectors in the list. ***Note: This is an advanced version of Remark 4 about linear dependence/independence.***

   b) Is it true that if an ordered list is linearly dependent, then every vector, other than the first in the list, can be expressed as a linear combination of the preceding vectors (YES/NO) ? Justify your answer briefly.

2. Let U and W be subspaces of the vector space V. Then the **sum** of U and W is defined as $U + W = \{\mathbf{u} + \mathbf{w}: \mathbf{u} \in U, \mathbf{w} \in W\}$. Show that U + W is a subspace of V. Show further that U + W is the smallest subspace of V containing both U and W.

3. Let U and W be subspaces of the vector space V. Show by means of a suitable counter-example that $U \cup W$ (set-theoretic union) need not be a subspace of V. Then, prove that $U \cup W$ is a subspace **if and only if** either $U \subset W$ or $W \subset U$. (***NB: This result holds whether V is finite-dimensional or infinite-dimensional. Hence, you cannot use any propositions related to basis or dimension in the proof.***)

4. Given any m×m (square) matrix A and any polynomial $p(t) = a_0 + a_1t + a_2t^2 + \ldots.. + a_nt^n$ $(a_n \neq 0)$ of degree n, we say that A **satisfies the polynomial p(t)** if $p(A) = a_0I_m + a_1A + a_2A^2 + \ldots.. + a_nA^n = \mathbf{0}$, i.e. the zero matrix. Show that any non-zero m×m matrix A must satisfy at least one (non-zero) polynomial of degree $\leq m^2$.

5. Consider the space $\mathbb{C}$ of complex numbers as a vector space over the field $\mathbb{R}$ of real numbers.

   a) Is $\mathbb{C}$ finite-dimensional (YES/NO) ? If YES, determine the dimension of $\mathbb{C}$.

   b) Prove or disprove: There exists a field F lying strictly between $\mathbb{R}$ and $\mathbb{C}$, i.e. there is a field F such that $\mathbb{R} \subseteq F$ but $\mathbb{R} \neq F$, and $F \subseteq \mathbb{C}$ but $F \neq \mathbb{C}$.

6. Let $V = \mathbb{R}^\infty$, $W = \{< a_n >$: only finitely many of the terms in $< a_n >$ are non-zero$\}$. (NB: *In future, we will use the standard notation* $\mathbf{p}_\infty$ *for the subspace W defined here.)*

   a) Show that W is a subspace of V.
   b) Is W finite-dimensional ? Justify your answer.
   c) Is V finite-dimensional ? Justify your answer.
   d) Consider *c*, the vector space of all convergent sequences in $\mathbb{R}^\infty$, Is c finite-dimensional? Justify your answer.

Q 1(a) Show that an <u>ordered list</u> $\{\bar v_1, \bar v_2, \ldots, \bar v_k\}$, $k \geq 2$, is lin. dep. if and only if either $\bar v_1 = \bar 0$ or for some j, $2 \leq j \leq k$, $\bar v_j$ is a lin. comb. of the preceding vectors in the list. [$\Leftarrow$] is trivial and omitted.

Answer: [$\Rightarrow$] Given $\{\bar v_1, \ldots, \bar v_k\}$ is lin. dep.

RTP: Either $\bar v_1 = \bar 0$ or for some j, $2 \leq j \leq k$, $\bar v_j$ is a lin. comb. of preceding vectors.

Now, if $\bar v_1 = 0$, we are done. So suppose $\bar v_1 \neq \bar 0$.

Put j = <u>least positive integer</u> such that $\bar v_1, \ldots, \bar v_{j-1}$ are lin. indep., but $\bar v_1, \ldots, \bar v_{j-1}, \bar v_j$ are lin. dependent.

Clearly, $j \geq 2$, since $\bar v_1$ is lin. indep. Also, $j \leq k$, since $\bar v_1, \ldots, \bar v_k$ are given to be lin. dependent.

So, then the expression (1)

$$c_1 \bar v_1 + \ldots + c_j \bar v_j = \bar 0$$

has a non-trivial solution. Suppose $c_j = 0$. Then (1) becomes

$$c_1 \bar v_1 + \ldots + c_{j-1} \bar v_{j-1} = \bar 0 \quad (2).$$

But since these vectors are lin. indep., $c_1 = c_2 = \ldots = c_{j-1} = 0$

But this contradicts the fact that (1) has a non-trivial solution.

Hence $c_j \neq 0$.

$\therefore$ (1) can be re-written as

$$c_j \bar v_j = -c_1 \bar v_1 - \ldots - c_{j-1} \bar v_{j-1}$$

$$\Rightarrow \bar v_j = -c_j^{-1} c_1 \bar v_1 - \ldots - c_j^{-1} c_{j-1} \bar v_{j-1}$$

and so $\bar v_j$ can be expressed as a lin. comb. of the preceding vectors.

Remark: In $\mathbb{R}^3$, consider the ordered list $\{\bar 0, \bar e_1, \bar e_2, \bar e_3\}$. Clearly l.d., but no vector can be expressed as a lin. comb. of the preceding. Hence, the condition: either $\bar v_1 = \bar 0$ or .... This only applies to $\bar v_1$. If we had taken the list $\{\bar e_1, \bar e_2, \bar e_3, \bar 0\}$, then clearly $\bar 0 = 0.\bar e_1 + 0.\bar e_2 + 0.\bar e_3$ i.e. a lin. comb. of the preceding.

(b) NO. In the list $\{\bar e_1, \bar e_2, \bar 0, \bar e_3\}$, it is <u>not true</u> that every vector other than first is a lin. comb. of the preceding vectors.

Maths

Tut 07

Q1 a) [$\Leftarrow$]

Let $\bar{v}_1 = 0$

taking $c_1\bar{v}_1 + c_2\bar{v}_2 + \dots + c_k\bar{v}_k = 0$, where all c_i not equal to zero

let c_j for $2 \le j \le k = 0$

& $c_1 = 1$

$\Rightarrow 1\bar{v}_1 + 0 + \dots + 0$

$= 1 \cdot \bar{0} + 0 + \dots + 0 = 0$

$\Rightarrow$ The set is l.d.

let $\bar{v}_j$ be a linear combination of the preceding vectors

$\Rightarrow \vec{v_j} = a_1\bar{v}_1 + a_2\bar{v}_2 + \dots + a_{j-1}\bar{v}_{j-1}$ — (1)

$\Rightarrow$ taking $c_1\bar{v}_1 + c_2\bar{v}_2 + \dots + c_j\bar{v}_j + \dots + c_k\bar{v}_k = 0$

Let c_i for $1 \le i \le j-1 = a_i$

& $c_j = -1$ & c_i for $k \ge i > j = 0$

$\Rightarrow a_1\bar{v}_1 + a_2\bar{v}_2 + \dots + a_{j-1}\bar{v}_{j-1} - \bar{v}_j + 0 + \dots + 0 = 0$ (By (1))

$\Rightarrow$ the set is l.d.

[$\Rightarrow$] $\because$ The set is l.d

$\Rightarrow c_1\bar{v}_1 + c_2\bar{v}_2 + \dots + c_k\bar{v}_k = 0$, where all c_i not equal to k

$\Rightarrow$ Either any element is $\bar{0}$, ~~or other element~~

$\Rightarrow$ only $\bar{v}_1$ can be $\bar{0}$ as the ~~list~~ is ordered

or v_i is a vector combination of ~~any~~ other vectors.

$\therefore$ the list is ordered the vectors preceding $\bar{v}_j$ form a ~~vector~~ linear combination $= \bar{v}_j$

H.P.

b) NO, not every element can be expressed as a linear combination of the preceding vectors as for the list to be ~~to~~ l.d only one element has to be ~~expressed~~ expressed as linear combination of others elements.

Q2 $U+W = \{\bar{u}+\bar{w} : \bar{u} \in U, \bar{w} \in W\}$

$\because$ Both U & W contains $\bar{0}$

$\Rightarrow \bar{0}+\bar{0}=\bar{0}$

$\Rightarrow$ ~~$U+W$~~ $\bar{0} \in U+W$ — ①

Let $\bar{u}, \in U$, $\bar{w} \in W$ & $c \in R$

$\Rightarrow c\bar{u} \in U$

$c\bar{w} \in W$

$\Rightarrow c\bar{u} + c\bar{w} \in U+W$

$c(\bar{u}+\bar{w}) \in U+W$ & Also $\bar{u}+\bar{w} \in U+W$

~~$\Rightarrow$ Scalar multip~~ $\Rightarrow$ Scalar multiplication holds true ②

Let $\bar{u}_1, \bar{u}_2 \in U$ & $\bar{w}_1, \bar{w}_2 \in W$

$\Rightarrow \bar{u}_1+\bar{u}_2 \in U$

$\bar{w}_1+\bar{w}_2 \in W$

$\Rightarrow \bar{u}_1+\bar{u}_2+\bar{w}_1+\bar{w}_2 \in U+W$

$(\bar{u}_1+\bar{w}_1)+(\bar{u}_2+\bar{w}_2) \in U+W$ & $(\bar{u}_1+\bar{w}_1), (\bar{u}_2+\bar{w}_2) \in U+W$

$\Rightarrow$ Vector add^n holds true — ③

$\Rightarrow$ By (1), (2) & (3)

$U+W$ is also a subspace of V.

Let X be the smallest subspace of V ~~where both U & W are~~ containing both U & W,

$\Rightarrow$ For $\bar{u} \in U$ & $\bar{w} \in W$

$\bar{u} + \bar{w} \in X$

$\Rightarrow \because X$ is the smallest subspace not other element other than $\bar{u} + \bar{w}$ should $\in X$

$\Rightarrow X = U + W$

H.P.

Q3 Let U be x-axis

& W be y-axis

$\Rightarrow U \cup W$ is all the points on x, y-axis

Let $\bar{x} \in U$, $\bar{y} \in W$, where $\bar{x}, \bar{y} \neq \bar{0}$

$\Rightarrow \bar{x} + \bar{y}$ is a pt. in x-y plane but

$\bar{x} + \bar{y} \notin U \cup W$.

[$\Leftarrow$]

$\because$ $U \subset W$ or $W \subset U$

$\Rightarrow$ $U \cup W = W$ or $U \cup W = U$

which is a subspace.

H.P.

[$\Rightarrow$]

Let $U \cup W$ be subspace of V

$\Rightarrow$ For $\bar{x}, \bar{y} \in U \cup W$

$\bar{x} + \bar{y} \in U \cup W$

only when both $\bar{x}, \bar{y} \in U$ or $\in W$

$\Rightarrow$ Either $U \subset W$ or $W \subset U$.

H.P.

Q4 We use the fact that dimension of $\mathbb{R}^{m \times m}$ is m^2

So consider the matrices $= I_m, A, A^2, \dots, A^{m^2}$

Since there are m^2+1 matrices, they cannot be linearly independent. (Prop 12 & 17)

Hence, they are l.d.

Hence, there exist scalars $\alpha_0, \alpha_1, \dots, \alpha_{m^2}$ not all zero such that

$$\alpha_0 I_n + \alpha_1 A + \dots + \alpha_{m^2} A^{m^2} = \bar{0} \quad -(1)$$

Hence, A satisfies the non-zero polynomial.

Q5 a) Yes, $\mathbb{C}$ is ~~a~~ finite-dimensional

~~dim C = 2~~

Basis of $\mathbb{C} = \{1, i\}$.

$\Rightarrow \dim \mathbb{C} = 2$.

b) No, since, $\dim \mathbb{C} = 2$ & $\dim \mathbb{R} = 1$

Since F is also a vector space over $\mathbb{R}$, it follows from prop 18 that $1 < \dim F < 2$

But F is a +ve int.

$\Rightarrow$ F cannot exist.

Q6 a) $V = \mathbb{R}^{\infty}$

$W = \{ \langle a_n \rangle$: only finitely many of the terms in $\langle a_n \rangle$ are non-zero $\}$

(i) $\langle \vec{0} \rangle = \langle 0, 0, 0, \dots, 0 \rangle$

$\Rightarrow \langle 0 \rangle \in W$ — ①

ii) Let $\langle a_n \rangle \in W$ & $\langle b_n \rangle \in W$

$\langle a_n \rangle = \langle a_1, a_2, \dots, a_n \rangle$

$\langle b_n \rangle = \langle b_1, b_2, \dots, b_n \rangle$

$\langle a_n \rangle + \langle b_n \rangle = \langle a_1 + b_1, a_2 + b_2, \dots, a_n + b_n \rangle$

$= \langle a_n + b_n \rangle$

$\therefore \langle a_n + b_n \rangle \in W$

$\Rightarrow \langle a_n \rangle + \langle b_n \rangle \in W$ — ②

iii) Let $\langle a_n \rangle \in W$ & $c \in \mathbb{R}$

$c \langle a_n \rangle = c \langle a_1, a_2, \dots, a_n \rangle$

$= \langle c a_1, c a_2, \dots, c a_n \rangle$

$= \langle c a_n \rangle$

$\therefore \langle c a_n \rangle \in W$

$\Rightarrow c \langle a_n \rangle \in W$ ③

$\therefore$ By ①, ② & ③ & Prop 8

W is a subspace of V.

b) Let us assume that W is finite dimensional having $B = \{s_1, s_2, \dots, s_n\}$ as its basis

~~$k =$ Max $\{$ no. of~~

$k =$ last position of of non-zero dim in $\{s_1, s_2, \dots, s_n\}$

Let s be a seq. in which all the terms are 0 except the $k+1^{th}$ term.

$\therefore s \in W$

But $s \notin$ span(B)

$\Rightarrow$ W is ~~not~~ infinite dimensional.

c) V is not finite dimensional

$\because$ W is a sub space of V

& W is infinite dimensional

$\therefore$ V is also infinite dimensional.

d) No, clearly all sequences in W are converging to 0.