Proposition 28

Let $B = \{\bar{u}_1, \ldots, \bar{u}_n\}$ & $C = \{\bar{v}_1, \ldots, \bar{v}_n\}$ be two ordered basis of a vector space V. Then, there is an invertible $n \times n$ matrix P such that $[\bar{x}]_C = P[\bar{x}]_B$ for all $\bar{x} \in V$.

Proof: Let $\bar{x} \in V$; since B is a basis for V, we can write
$\bar{x} = b_1\bar{u}_1 + \cdots + b_n\bar{u}_n$ — (1)

$$\Rightarrow [\bar{x}]_B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \quad \text{— (2)}$$

Since, C is also a basis of V

$$\left.\begin{aligned} \bar{u}_1 &= A_{11}\bar{v}_1 + A_{21}\bar{v}_2 + \cdots + A_{n1}\bar{v}_n \\ \bar{u}_2 &= A_{12}\bar{v} + \cdots + A_{n2}\bar{v}_n \\ &\vdots \\ \bar{u}_n &= A_{1n}\bar{v}_1 + \cdots + A_{nn}\bar{v}_n \end{aligned}\right\} \text{— (3)}$$

where A_{ij} are scalars belonging to the underlying field F.

From (3), we get that $[\bar{u}_i]_C = \begin{bmatrix} A_{1i} \\ \vdots \\ A_{ni} \end{bmatrix}$ for $i = 1, \ldots, n$

Substituting from ③ in ①, we get :

$$\bar{x} = d_1(A_{11}\bar{v}_1 + \dots + A_{n1}\bar{v}_n) + d_2(A_{12}\bar{v}_1 + \dots + A_{n2}\bar{v}_n) + \dots + d_n(A_{1n}\bar{v}_1 + A_{2n}\bar{v}_2 + \dots + A_{nn}\bar{v}_n) \quad \text{⑤}$$

By rearranging terms & collecting coefficients of $\bar{v}_1, \dots, \bar{v}_n$ we get

$$\bar{x} = (A_{11}d_1 + A_{12}d_2 + \dots + A_{1n}d_n)\bar{v}_1 + \dots + (A_{n1}d_1 + A_{n2}d_2 + \dots + A_{nn}d_n)\bar{v}_n \quad \text{⑥}$$

Hence $[x]_C = \begin{bmatrix} A_{11}d_1 + A_{12}d_2 + \dots + A_{1n}d_n \\ \vdots \\ A_{n1}d_1 + A_{n2}d_2 + \dots + A_{nn}d_n \end{bmatrix}$

$$= [A_{ij}] \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = P[\bar{x}]_B \quad \text{⑦}$$

By ④, the ~~coordin~~ columns $\bar{p}_1, \dots, \bar{p}_n$ of the matrix P are coordinate vectors of the old basis vectors in terms of the new basis.

$\Rightarrow$ P must be invertible, as $\bar{p}_1, \dots, \bar{p}_n$ are the images of vectors $\bar{u}_1, \dots, \bar{u}_n$ under the coordinate mapping relative to the basis B.

By obs 2 of coordinate mapping $V \to F^n$ is an isomorphism

$\Rightarrow$ $\bar{p}_1, \dots, \bar{p}_n$ form a basis of F^n. Thus by VIT, P is invertible. ($\because \bar{p}_1, \dots, \bar{p}_n$ are linearly independent)