MTH100 Section B   MATHS I   Monsoon Semester – 2021-22

Tutorial Exercise for Week Commencing Monday 7th February 2022.

1. Let V be a vector space. Prove the Cancellation Law, i.e. show that if $\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w}$, for $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in V$, then $\mathbf{v} = \mathbf{w}$.

2. Prove Proposition 7: Let V be a vector space. Then:
   a) The zero vector is unique.
   b) The additive inverse vector of any vector $\mathbf{u}$ is unique.
   c) $0\mathbf{u} = \mathbf{0}$ for every vector $\mathbf{u}$
   d) $c\mathbf{0} = \mathbf{0}$ for every scalar c   e) $-\mathbf{u} = (-1)\mathbf{u}$ for every vector $\mathbf{u}$

3. Give an example of a set X and an operation involving elements of X, which does not satisfy the cancellation law. Briefly justify your answer.

4. Verify the axioms (properties) of a vector space for the space $\mathbb{R}_n[t]$ of all real polynomials of degree $\leq$ n, with base field $\mathbb{R}$. Here n is any fixed but arbitrary non-negative integer. Briefly explain the case n = 0. ***Remark:*** *You must start by defining the addition and scalar multiplication operations for* $\mathbb{R}_n[t]$.

5. Is the set X_n of all real polynomials with degree = n a vector space over $\mathbb{R}$ (YES/NO) ? Justify your answer. Here n is any fixed but arbitrary **positive** integer. What happens in the case n = 0 ? ***Note:*** *You may assume that the zero polynomial is a member of* X_n *for all n.*

6. a) Is $\mathbb{R}$ a vector space over $\mathbb{Q}$? Justify your answer in brief.
   b) Is $\mathbb{C}$ a vector space over $\mathbb{R}$? Justify your answer in brief.
   c) Can you generalize the answers to a) and b) above to a statement about fields and vector spaces ? Explain briefly.

Tut-5
Maths

Q1 $\bar{u}+\bar{v} = \bar{u}+\bar{w}$

Let $\bar{a}$ ~~be~~ such that $\bar{u}+\bar{a} = \bar{0}$

~~$\Rightarrow$ Adding a~~

$\Rightarrow \cancel{\bar{u}+\bar{a}}+\bar{v} = \bar{v}+(\bar{u}+\bar{a}) = (\bar{v}+\bar{u})+\bar{a} = (\bar{u}+\bar{w})+\bar{a}$ (given)

$= \bar{w}+(\bar{u}+\bar{a}) = \bar{w}$

$\Rightarrow \bar{v} = \bar{w}$

$\Rightarrow$ H.P.

Q2 a) The zero vector is unique

$\Rightarrow$ Suppose $\bar{u}$ & $\bar{v}$ be ~~zero~~ 2 distinct zero vectors

$\Rightarrow \bar{u}+\bar{v} = \bar{u}$ —① ($\because \bar{v}$ is a zero vector)

& $\bar{u}+\bar{v} = \bar{v}$ —② ($\because \bar{u}$ is a zero vector)

$\Rightarrow$ From ① & ②

$\bar{u} = \bar{v}$

$\Rightarrow$ By contradiction (a) is proved

b) ~~Let~~ Let $\bar{a}$ & $\bar{b} \in V$ be addition inverse of $\bar{u} \in V$

$\Rightarrow \bar{u}+\bar{a} = \bar{0}$ —①

& $\bar{u}+\bar{b} = \bar{0}$ —②

$\Rightarrow \cancel{\bar{u}}+\bar{a} = \cancel{\bar{u}}+\bar{b}$ (From ①&②)

$\bar{a} = \bar{b}$ (From cancellation law)

$\Rightarrow$ addition inverse of $\bar{u}$ is unique &

c) $0\bar{u} = \bar{0}$

LHS

$0\bar{u} = (0+0)\bar{u} = 0\bar{u} + 0\bar{u}$

Adding $-0\bar{u}$ to ~~bo~~

$0\bar{u} - 0\bar{u} = 0\bar{u} + 0\bar{u} - 0\bar{u}$

$\bar{0} = 0\bar{u}$

Hence, proved

d) $c\bar{0} = \bar{0}$

~~LHS~~

$\cancel{c\bar{0} = \bar{0}}$

~~$c\bar{0}$ adding ac~~

$\therefore \cancel{c(\bar{0} - \bar{0}) = \bar{0}}$ $\quad \because -\bar{0}$ is ~~additive inverse of $\bar{0}$~~

$\Rightarrow \cancel{c\bar{0} - c\bar{0} = \bar{0}}$

$\therefore \bar{0} + \bar{0} = \bar{0}$

$c\bar{0} + c\bar{0} = c\bar{0}$ (Multiplying by c)

adding $-c\bar{0}$ to both sides

$c\bar{0} + c(\bar{0} - \bar{0}) = c(\bar{0} - \bar{0})$

$c\bar{0} + \bar{0} = \bar{0}$ $\quad (\because \bar{a} - \bar{a} = \bar{0})$

$c\bar{0} = \bar{0}$ $\quad (\because \bar{a} + \bar{0} = \bar{a})$

Hence, proved.

e) Frame N.B

Q3 Let X be a set
with ~~two~~ three matrices A, B, C which aren't invertible

~~AP~~ Let $AB = AC$
$\because$ A is not invertible
$\Rightarrow$ A cannot be cancelled

Q4

Q5 NO, X_n is the set of all real poly. with degree $\leq n$ over $\mathbb{R}$.

$\because$ ~~a~~ Zero polynomial a part of X_n

$\Rightarrow$ X_0 ~~can~~ only contains zero polynomial.

Q6a) Yes, $\mathbb{R}$ is a vector space over $\mathbb{Q}$

$\because$ Let $\bar{a} \in \mathbb{R}$ & $b \in \mathbb{Q}$

& $b\bar{a} \in \mathbb{R}$ vector space.

b) Similar to (a)

c) Yes,

Let $\bar{u}, \bar{v} \in$ Vector space

& $a \in$ Field

such that, $a\bar{u} = \bar{v}$