Proof of proposition 12

So, let $\bar{v}_1, \ldots, \bar{v}_n$ be linearly independent and let $\{\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_m\}$ form a spanning set for V.

Then, $\bar{v}_1 = c_1\bar{w}_1 + \cdots + c_m\bar{w}_m$ ① for some scalars $c_1, \ldots, c_m$

Then $c_i \neq 0$ for all i — ②

~~For~~

If $c_1 = c_2 = \cdots = c_m = 0$, then $\bar{v}_1 = \bar{0}$, (not possible) (Remark 1)

Renumbering the w_j so that $c_1 \neq 0$

From ① we get

$$c_1\bar{w}_1 = \bar{v}_1 - c_2\bar{w}_2 - \cdots - c_m\bar{w}_m$$
$$\bar{w}_1 = c_1^{-1}\bar{v}_1 - c_1^{-1}c_2\bar{w}_2 - \cdots - c_1^{-1}c_m\bar{w}_m$$
$$= d_1\bar{v}_1 + d_2\bar{w}_2 + \cdots + d_m\bar{w}_m \quad ③$$

for some scalars d_i

It follows from ③, that $\{\bar{v}_1, \bar{w}_2, \ldots, \bar{w}_m\}$ is a spanning set for V. — ④

For $\bar{v} \in V \Rightarrow \bar{v} = f_1\bar{w}_1 + \cdots + f_m\bar{w}_m$ & substituting for w_1 from ③, we get $\bar{v}$ as a linear combination of $\bar{v}_1$ along with $\bar{w}_2, \ldots, \bar{w}_m$.

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We proceed by repeating the above process

$\bar{v}_2 = g_1\bar{v}_1 + g_2\bar{w}_2 + \dots + g_m\bar{w}_m$ ① for some scalar g_i. ~~the~~ ~~the~~ $\forall i \neq 1$

Here, not all g_i are zero, because if so, $\bar{v}_2 = g_1\bar{v}_1$ +
$\Rightarrow$ either $\bar{v}_2$ is linearly dependent on $\bar{v}_1$ or $g_1 = 0$
$\Rightarrow \bar{v}_2 = \bar{0}$

So, renumbering the $\bar{w}_j$ if necessary, we may assume $g_2 \neq 0$
By rearranging & multiplying by g_2^{-1}, we get

$\bar{w}_2 = h_1\bar{v}_1 + h_2\bar{v}_2 + \dots + h_m\bar{w}_m$ ⑥ for suitable scalar h_i

Arguing as before, we get that
$\{\bar{v}_1, \bar{v}_2, \bar{w}_3, \dots, \bar{w}_m\}$ is a spanning set of V ⑦

Proceeding in this way, inserting the $\bar{v}_i$'s in order one-by-one in the spanning set, & ejecting a different $\bar{w}_j$ at each step, we must eventually come to a halt, since there are only finitely many $\bar{v}$'s & $\bar{w}$'s.

There are only 2 possibilities at the halting stage:

Case 1: $n \leq m$, & so we are able to insert all the $\bar{v}$'s into the spanning set.

So in this case, post proposition 12 is proved.

Case 2 : $n > m$

Then, we have the following situation:-

$\bar{v}_1, \bar{v}_2, \ldots, \bar{w}_m, \ldots, \bar{v}_n$ are linearly independent &

$\{\bar{v}_1, \ldots, \bar{v}_m\}$ form a spanning set.

Since, we have been able to replace all of the $\bar{w}_j$'s by $\bar{v}_1$ to $\bar{v}_m$.

But then, $\bar{v}_{m+1}$ is a linear combination of $\bar{v}_1, \ldots, \bar{v}_m$

which contradicts our assumption.

$\Rightarrow$ Case 2 can't happen.