Proof: Suppose that $\dim V = n$ & nullity $(T) = k$. Then let $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k$ be a basis for $\ker T$ and expand this to a basis B of V by inserting the additional vectors $\bar{v}_{k+1}, \ldots, \bar{v}_n$. We claim that the vectors $T(\bar{v}_{k+1}), \ldots, T(\bar{v}_n)$ form a basis for Range (T)

Firstly, all the vectors $T(\bar{v}_1), \ldots, T(\bar{v}_n)$ surely span Range (T) & since $T(\bar{v}_1) = T(\bar{v}_2) = \cdots = T(\bar{v}_k) = \bar{0}$,

$\Rightarrow T(\bar{v}_{k+1}), \ldots, T(\bar{v}_n)$ span Range T

Secondly, suppose that $c_{k+1} T(\bar{v}_{k+1}) + \cdots + c_n T(\bar{v}_n) = \bar{0}$

Then $T(c_{k+1}\bar{v}_{k+1} + c_{k+2}\bar{v}_{k+2} + \cdots + c_n\bar{v}_n) = \bar{0}$

$\Rightarrow c_{k+1}\bar{v}_{k+1} + \cdots + c_n\bar{v}_n \in \ker T$.

$\Rightarrow c_{k+1}\bar{v}_{k+1} + \cdots + c_n\bar{v}_n = b_1\bar{v}_1 + \cdots + b_k\bar{v}_k$

$\Rightarrow 0 = b_1\bar{v}_1 + \cdots + b_k\bar{v}_k - c_{k+1}\bar{v}_{k+1} - \cdots - c_n\bar{v}_n$

But since $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_k, \ldots, \bar{v}_n$ form a basis for V, all the coefficients must be 0.

This proves the claim.

$\text{rank}(T) = \dim \text{Range}(T) = n - k$ & finally.

$\text{rank}(T) + \text{nullity}(T) = \dim V$.