THEOREM 6

Proof: We will prove the following by induction on k :- For $k = 1, 2, \ldots, p$, the following hold: (3)

(i) $\bar{v}_1, \ldots, \bar{v}_k$ are non-zero vectors

(ii) $\{\bar{v}_1, \ldots, \bar{v}_k\}$ is an orthogonal set

(iii) $\text{Span}\{\bar{v}_1, \ldots, \bar{v}_k\} = \text{Span}\{\bar{x}_1, \ldots, \bar{x}_k\}$

<u>Base Case:</u> $k=1$

(i) holds since $\bar{v}_1 = \bar{x}_1 \neq \bar{0}$ since $\bar{x}_1$ is a basis vector.

(ii) and (iii) hold trivially.

<u>Inductive Step</u>: Suppose (i), (ii), (iii) hold for some k, $k = 1, 2, \ldots, p-1$ and consider the case for $k+1$.

(IH = Induction Hypothesis)

Theorem 6: Continued.

(i) Suppose $\bar{v}_{k+1} = \bar{x}_{k+1} - \left(\frac{\langle \bar{x}_{k+1}, \bar{v}_1 \rangle}{\langle \bar{v}_1, \bar{v}_1 \rangle}\right) \bar{v}_1 - \cdots\cdots - \left(\frac{\langle \bar{x}_{k+1}, \bar{v}_k \rangle}{\langle \bar{v}_k, \bar{v}_k \rangle}\right) \bar{v}_k$

$= \bar{0}$ (1)

Switching terms, we get:

$\bar{x}_{k+1} = c_1 \bar{v}_1 + \cdots + c_k \bar{v}_k$ (2)

where $c_i = \frac{\langle \bar{x}_{k+1}, \bar{v}_i \rangle}{\langle \bar{v}_i, \bar{v}_i \rangle}$ for $i = 1, 2, \ldots, k$ (3)

(2) $\Rightarrow \bar{x}_{k+1} \in \text{Span}\{\bar{v}_1, \ldots, v_k\}$
$= \text{Span}\{\bar{x}_1, \ldots, \bar{x}_k\}$ since

(iii) holds for k

$\Rightarrow \{\bar{x}_1, \ldots, \bar{x}_{k+1}\}$ is lin. dep. (Remark 4)

$\Rightarrow \Leftarrow$ since the vectors $\bar{x}_i$ form a basis.

(ii) We need to show $\langle \bar{v}_{k+1}, \bar{v}_j \rangle = 0$ for all $j = 1, 2, \ldots, k$

Consider:-

$\langle \bar{v}_{k+1}, \bar{v}_j \rangle = \langle \bar{x}_{k+1} - c_1 \bar{v}_1 - c_2 \bar{v}_2 - \cdots - c_k \bar{v}_k, \bar{v}_j \rangle$ using (3)

Theorem 6 - cont'd :- (5)

$$= \langle \bar{x}_{k+1}, \bar{v}_j \rangle - c_1 \langle \bar{v}_1, \bar{v}_j \rangle - \cdots$$

$$- c_k \langle \bar{v}_k, \bar{v}_j \rangle \quad (4)$$

Since $\{\bar{v}_1, \ldots, \bar{v}_k\}$ is an orthogonal set, all terms in (4) are zero, except :-

$$\langle \bar{x}_{k+1}, \bar{v}_j \rangle - c_j \langle \bar{v}_j, \bar{v}_j \rangle$$

$$= \langle \bar{x}_{k+1}, \bar{v}_j \rangle - \frac{\langle \bar{x}_{k+1}, \bar{v}_j \rangle}{\langle \bar{v}_j, \bar{v}_j \rangle} \langle \bar{v}_j, \bar{v}_j \rangle$$

substituting from (3)

$= 0$, as reqd.

(iii) For convenience, put $W_{k+1} = \text{Span}\{\bar{v}_1, \ldots, \bar{v}_k, \bar{v}_{k+1}\}$

Now, from the expression for $\bar{v}_{k+1}$, we get:

$$\bar{v}_{k+1} \in \text{Span}\{\bar{x}_{k+1}, \bar{v}_1, \ldots, \bar{v}_k\}$$

$$\Rightarrow W_{k+1} \subseteq \text{Span}\{\bar{x}_{k+1}, \bar{v}_1, \ldots \bar{v}_k\} \quad (5)$$

But, since $\{\bar{v}_1, \ldots, \bar{v}_{k+1}\}$ is a an orthogonal set of non-zero

Theorem ~~5~~ - cont'd (6)

vectors, it is linearly independent (by Prop. 46).

$\therefore \bar{v}_1, \dots, \bar{v}_{k+1}$ form a basis for

$W_{k+1} \Rightarrow \dim W_{k+1} = k+1$ (5)

But now, since $\text{Span}\{\bar{x}_{k+1}, \bar{v}_1, \dots, \bar{v}_k\}$ has a spanning set of $k+1$ vector, $\dim \text{Span}\{\bar{x}_{k+1}, \bar{v}_1, \dots, \bar{v}_k\} \leq k+1$ (by Prop. 16)

Using (5) and (6),

$k+1 \leq \dim W_{k+1} \leq \dim$ ~~$\{\bar{x}_k$~~ $\text{Span}\{\bar{x}_{k+1}, \bar{v}_1, \dots, \bar{v}_k\} \leq k+1$ (7)

From (7), it follows that

$W_{k+1} = \text{Span}\{\bar{x}_{k+1}, \bar{v}_1, \dots, \bar{v}_k\}$

$= \text{Span}\{\bar{x}_{k+1}, \bar{x}_1, \dots, \bar{x}_k\}$

since $\text{Span}\{\bar{v}_1, \dots, \bar{v}_k\} = \text{Span}\{\bar{x}_1, \dots, \bar{x}_k\}$ by the IH. We have shown that if IH holds for k, it holds for $k+1$.

$\therefore$ by Principle of Mathematical Induction (PMI), it holds for all $k = 1, 2, \dots, p$.