MTH100 Math 1 (Linear Algebra) Section B Monsoon Semester – 2021-2

Tutorial Exercise for Week Commencing Monday 20220131

1. a) Show that an elementary matrix E obtained by replacement of a row R_i of I by $R_i + kR_j$, where $j < i$, is a unit lower triangular matrix.

   b) Show that the product of two unit lower triangular matrices is again a unit lower triangular matrix.

   c) Show that if A is a unit lower triangular matrix, then A is invertible and A^{-1} is also a unit lower triangular matrix.

2. a) Obtain an LU decomposition of the matrix A given below.

   b) Solve the non-homogeneous system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b}$ is given below, using the LU decomposition obtained in part a).

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

3. For each of the following, clearly state TRUE or FALSE. Then, justify your answer (proof if TRUE, counter-example if FALSE).

   a) For any square matrix A, if A^k is invertible for some positive integer $k > 1$, then A itself is invertible.

   b) If a 3×3 square matrix A satisfies $A^3 = \mathbf{0}$, then $A = \mathbf{0}$. Here $\mathbf{0}$ indicates the zero matrix.

4. Consider the system $\mathbb{R}^{3\times3}$ of 3×3 (square) matrices with real entries. A non-zero matrix A is said to be a **zero-divisor** if there exists some non-zero matrix B such that $AB = \mathbf{0}$, the zero matrix.

   a) If A is invertible, then it cannot be a zero-divisor. TRUE or FALSE ? Justify your answer.

   b) If A is not invertible, then it must be a zero-divisor. TRUE or FALSE ? Justify your answer.

5. a) Obtain an LU decomposition of the matrix A given below.

b) Solve the non-homogeneous system $A\mathbf{x} = \mathbf{b}$, for $\mathbf{b}_1$ and $\mathbf{b}_2$ given below, using the LU decomposition obtained in part a). Take $\mathbf{b}_1$ and $\mathbf{b}_2$ as column vectors. Explain the difference in the answers for these two vectors $\mathbf{b}_1$ and $\mathbf{b}_2$.

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 6 & 16 \\ 3 & 8 & 21 \end{bmatrix} \qquad \mathbf{b}_1 = (1, 4, 5) \quad \mathbf{b}_2 = (3, 7, 15)$$

6. a) Obtain an LU decomposition of the matrix A given below.

b) Solve the non-homogeneous system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b}$ is given below, using the LU decomposition obtained in part a). Take $\mathbf{b}$ as a column vector.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 1 & 1 \\ 1 & 7 & 2 & 1 \end{bmatrix} \qquad \mathbf{b} = (4, 9, 14)$$

7. a) Obtain an LU decomposition of the matrix A given below.

b) Solve the non-homogeneous system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b}$ is given below, using the LU decomposition obtained in part a). Take $\mathbf{b}$ as a column vector.

$$A = \begin{bmatrix} 2 & -4 & 2\pi^2 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{bmatrix} \qquad \mathbf{b} = (-4\pi^2, -12\pi^2 + 7, -4\pi^2 + 8)$$

Maths

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Q1 a) $E \quad I \xrightarrow{R_i \to R_i + kR_j} E$

$j < i$

~~since after replace~~
~~I is a unit diagon~~
since after replacement
the i^{th} row of E

$$\begin{array}{cccccccccc} 0 & 0 & 0 & \cdots & k & 0 & 0 & \cdots & 1 & \cdots & 0 \\ & & & & \uparrow & & & & i^{th} \text{ column} & & \\ & & & & j^{th} \text{ column} & & & & & & \end{array}$$

$\Rightarrow$ k is in the lower half of the diagonal
& $\because$ other rows are similar to I
$\Rightarrow$ There is no element in the upper half of the diagonal
$\Rightarrow$ E is a lower Δar matrix

b) Let A & B be

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ \vdots & & & & \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mm} \end{bmatrix} \quad \& \quad \begin{bmatrix} b_{11} & 0 & 0 & \cdots & 0 \\ b_{21} & b_{22} & 0 & \cdots & 0 \\ b_{m1} & b_{m2} & \cdots & & b_{mm} \end{bmatrix}$$

~~AB =~~ ~~Consider~~
Let $C = AB$
Consider c_{xy} element where $x < y$

$$c_{xy} = a_{x1} b_{1y} + a_{x2} b_{2y} + \cdots + a_{xm} b_{my}$$

$$= a_{x1} \cdot 0 + a_{x2} \cdot 0 + \cdots + a_{xy} b_{yy} + \cdots + a_{xx} b_{xy} + \cdots + 0_{xx} b_{my} \quad \left(\because \text{for } j > i \; a_{ij} = 0 \; \& \; b_{ij} = 0 \right)$$

$$c_{xy} = a_{xy} b_{yy} + \cdots + a_{xx} b_{xy}$$

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Consider c_{xy} where $y > x$

$$c_{xy} = a_{x1} b_{1y} + a_{x2} b_{2y} + \cdots + a_{xm} b_{my}$$

$(\because$ for $j > i$; $a_{ij}, b_{ij} = 0)$

$$= a_{x1} b_{1y} \cdot 0 + \cdots + \underset{x^{th}\text{ element}}{0 \cdot 0_{,y}} + \cdots + \underset{y^{th}\text{ element}}{0 \cdot b_{yy}} + \cdots + 0 \cdot b_{ym}$$

$$= 0$$

$\Rightarrow$ Product of 2 lower Δar matrix is also Δar

e)

c) Let $A = \begin{bmatrix} 1 & & & O \\ a_{21} & 1 & & \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & 1 \end{bmatrix}$ be a unit lower triangular matrix.

Let us find soln of

$A\bar{x} = \bar{O}$

$\Rightarrow x_1 = 0$

$a_{21}x_1 + x_2 = 0 \quad \Rightarrow x_2 = 0$

$\vdots$

$a_{n1}x_1 + a_{n2}x_2 + \cdots + x_n = 0 \Rightarrow x_n = 0$

$\Rightarrow$ ∵ A ~~is a~~ has a trivial solution

$\Rightarrow$ A is invertible.

$\Rightarrow AA^{-1} = I = [c_{ij}]$

& Let $A^{-1} = [b_{ij}]$

For $i>j$

$\Rightarrow c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$

$= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ij}\cdot b_{jj} + \cdots + 0\cdot b_{ij} + 0\cdot b_{nj}$

For $i<j$

$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + 0\cdot b_{ij} + \cdots + 0\cdot b_{jj} + \cdots + 0\cdot b_{nj}$

$\because c_{ij} = 0$ $\quad$ ($\because [c_{ij}]$ is an identity matrix)

$\Rightarrow a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ii}b_{ij} = 0$

$\because a_{xy} \neq 0$ when $x<y$

$\Rightarrow b_{xj} = 0$

$\Rightarrow b_{ij} = 0$ when $j>i$ $\quad$ —(1)

~~for $x=n$~~

for $x=y$

$c_{xx} = a_{x1} b_{1x} + a_{x2} b_{2x} + \cdots + a_{xx} b_{xx} + \cdots + a_{xn} b_{nx}$

$= a_{x1} \cdot 0 + a_{x2} \cdot 0 + \cdots + a_{xx} b_{xx} + \cancel{a_{xn}}\, 0 \cdot b_{x+1\,x} + \cdots + 0 \cdot b_{nx}$ (By ① $i > j$)

$= a_{xx} b_{xx}$

$\therefore c_{xx} = 1$ ($\because [c_{ij}]$ is an identity matrix)

$\Rightarrow a_{xx} b_{xx} = 1$

$1 \cdot b_{xx} = 1$ ($\because [a_{ij}]$ is a unit lower matrix)

$\Rightarrow b_{xx} = 1$ — ②

$\Rightarrow$ By ① & ②

A^{-1} is a unit lower matrix

Q2 a) $A = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

$$\xrightarrow[\substack{R_3 \to R_3 - \frac{1}{2}R_1 \\ R_4 \to R_4 - \frac{1}{2}R_1}]{R_2 \to R_2 - \frac{1}{2}R_1} \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 3/2 & 1/2 & 1/2 \\ 0 & 1/2 & 3/2 & 1/2 \\ 0 & 1/2 & 1/2 & 3/2 \end{bmatrix} \xrightarrow[R_4 \to R_4 - \frac{1}{3}R_3]{R_3 \to R_3 - \frac{1}{3}R_2} \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 3/2 & 1/2 & 1/2 \\ 0 & 0 & 4/3 & 1/3 \\ 0 & 0 & 1/3 & 4/3 \end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4 - \frac{1}{4}R_3} \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 3/2 & 1/2 & 1/2 \\ 0 & 0 & 4/3 & 1/3 \\ 0 & 0 & 0 & 5/4 \end{bmatrix} = U$$

e:- $e_1 = R_2 \to R_2 - \frac{1}{2}R_1$
$e_2 : R_3 \to R_3 - \frac{1}{2}R_1$
$R_4 \to R_4 - \frac{1}{2}R_1$
$e_2 = R_3 \to R_3 - \frac{1}{3}R_2$
$R_4 \to R_4 - \frac{1}{3}R_2$
$e_3 = R_4 \to R_4 - 1/4\, R_3$

f:- $f_1 = R_2 \to R_2 + \frac{1}{2}R_1$
$R_3 \to R_3 + \frac{1}{2}R_1$
$R_4 \to R_4 + \frac{1}{2}R_1$
$f_2 = R_3 \to R_3 + \frac{1}{3}R_2$
$R_4 \to R_4 + \frac{1}{3}R_2$
$f_3 = R_4 \to R_4 + \frac{1}{4}R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{f_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/4 & 1 \end{bmatrix} \xrightarrow{f_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1/3 & 1 & 0 \\ 0 & 1/3 & 1/4 & 1 \end{bmatrix}$$

$$\xrightarrow{f_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/2 & 1/3 & 1 & 0 \\ 1/2 & 1/3 & 1/4 & 1 \end{bmatrix} = L$$

$$\Rightarrow LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/2 & 1/3 & 1 & 0 \\ 1/2 & 1/3 & 1/4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & 3/2 & 1/2 & 1/2 \\ 0 & 0 & 4/3 & 1/3 \\ 0 & 0 & 0 & 5/4 \end{bmatrix} = A$$

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b) $LU\bar{x} = \bar{b}$ $\quad$ ~~$L\bar{y} = B$~~ $\quad$ Let $\bar{y} = L^{-1}B$

$U\bar{x} = L^{-1}\bar{b}$ $\quad$ ~~$\bar{y} = L^{-1}B$~~ $\quad$ $\Rightarrow L\bar{y} = B$

$\Rightarrow y_1 = 1$

$y_1/2 + y_2 = -1$ $\quad \Rightarrow y_2 = -3/2$

$y_1/2 + y_2/3 + y_3 = -1$ $\quad \Rightarrow y_3 = -1$

$y_1/2 + y_2/3 + y_3/4 + y_4 = 1$ $\quad \Rightarrow y_4 = 5/4$

$\Rightarrow y = \begin{bmatrix} 1 \\ -3/2 \\ -1 \\ 5/4 \end{bmatrix}$ $\quad$ & $U\bar{x} = \bar{y}$

$\Rightarrow$ ~~$U\bar{x} = \bar{y}$~~

~~$U\bar{x} =$~~

$\Rightarrow 2x_1 + x_2 + x_3 + x_4 = y_1 = 1$ $\quad \Rightarrow x_1 = 1$

$\frac{3}{2}x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 = y_2 = -3/2$ $\quad \Rightarrow x_2 = -1$

$\frac{4x_3}{3} + \frac{1}{3}x_4 = y_3 = -1$ $\quad \Rightarrow x_3 = -1$

$\frac{5}{4}x_4 = y_4 = 5/4$ $\quad \Rightarrow x_4 = 1$

$\Rightarrow x = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

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Q3) a) True,

~~$\because$ For A^k to be invertible~~

$\because$ From Corollary 1.3

for A^k to be invertible A should be invertible.

b) False,

Let A be $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\Rightarrow A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Q4 a) True,

$\because$ A is invertible

$AB = 0$

$B = A^{-1}0$

$B = 0$

b) ~~False~~ ~~True~~, False

$\because AB = 0$

& $A \neq 0$

but nothing can be said about B, it can be 0 or ~~any~~ another matrix.

$\Rightarrow$ A may or may not be zero-divisor

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Q5 a) $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 6 & 16 \\ 3 & 8 & 21 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 5 \\ 0 & 2 & 6 \\ 0 & 2 & 6 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 2 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix} = U$

e :- $e_1 = R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$
$e_2 = R_3 \to R_3 - R_2$

f :- $f_1 = R_2 \to R_2 + 2R_1$
$R_3 \to R_3 + 3R_1$
$f_2 = R_3 \to R_3 + R_2$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{f_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{f_1} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} = L$

$\Rightarrow LU = A$

b) $\cancel{L\bar{y} = b_1}$ For $A\bar{x} = \bar{b}$

$U\bar{x} = \bar{y}$ & $L\bar{y} = \bar{b}$

$\Rightarrow$ For b_1

$y_1 = 1 \quad \Rightarrow y_1 = 1$
$2y_1 + y_2 = 4 \quad \Rightarrow y_2 = 2$
$3y_1 + y_2 + y_3 = 5 \quad \Rightarrow y_3 = 0$

$\Rightarrow x_1 + 2x_2 + 5x_3 = 1 \quad \Rightarrow x_1 = x_3 - 1$
$2x_2 + 6x_3 = 2 \quad \Rightarrow x_2 = 1 - 3x_3$
$x_3 = x_3 \quad \Rightarrow x_3 = x_3$

$\Rightarrow \bar{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$ for b_1

$\Rightarrow$ For b_2

$y_1 = 3 \Rightarrow y_1 = 3$

$2y_1 + y_2 = 7 \Rightarrow y_2 = 1$

$3y_1 + y_2 + y_3 = 15 \Rightarrow y_3 = 5$

$\Rightarrow x_1 + 2x_3 + 5x_3 = 3$

$2x_3 + 6x_3 = 2$

$0 = 5$

$\because 0 \neq 5$

$\Rightarrow$ The system $A\bar{x} = \bar{b}_2$ is inconsistant.

Q6 d) $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 1 & 1 \\ 1 & 7 & 2 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 6 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = U$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 + 2R_1} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -3 & 1 \end{bmatrix}$

$LU = A$

ii) $L\bar{y} = \bar{b}$

$y_1 = 4 \Rightarrow y_1 = 4$

$2y_1 + y_2 = 9 \Rightarrow y_2 = 1$

$y_1 - 3y_2 + y_3 = 14 \Rightarrow y_3 = 13$

$\Rightarrow \bar{y} = \begin{bmatrix} 4 \\ 1 \\ 13 \end{bmatrix}$

$\Rightarrow U\bar{x} = \bar{y}$

$x_1 + x_2 + x_3 + x_4 = 4$

$2x_2 = 1$

$x_3 = 13$

$x_4 = x_4$

$\Rightarrow x = \begin{bmatrix} -19/2 \\ 1/2 \\ 13 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$