MTH100 **MATHS 1** **Monsoon Semester – 2021-22**

Tutorial exercise for the Week Commencing Monday 20220404

1. Find the eigenvalues and corresponding eigenvectors for the matrix A given below. Is A diagonalizable ? Justify your answer in at most one sentence.

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{bmatrix}$$

2. For each matrix find all eigenvalues and a basis of each eigenspace. Which matrix can be diagonalized and why? If yes, indicate the diagonal matrix D and the invertible matrix P such that $A = PDP^{-1}$. [*Hint: $\lambda = 4$ is an eigenvalue.*]

a) $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$

b) $A = \begin{bmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix}$

3. A 7×7 matrix A has three eigenvalues. One eigenspace is 2-dimensional and one of the others is 3-dimensional. Is it possible for A to be not diagonalizable? Justify your answer.

4. a) If A is row-equivalent to the identity matrix, then A must be diagonalizable. Is this statement TRUE or FALSE?

b) Justify your answer to a). Give a proof if TRUE or a concrete counter-example if FALSE. In the second case, you should verify that your counter-example is row-equivalent to identity matrix but not diagonalizable.

5. Let V = C[a,b]. Verify the inner product properties for the inner product given by:

$$\langle f,g\rangle = \int_a^b f(t)g(t)\,dt$$

6. Use the Gram-Schmidt process to find an orthonormal basis for $\mathbb{R}^3$ given the basis $\{\mathbf{x_1} = (2, 1, 2), \mathbf{x_2} = (4, 1, 0), \mathbf{x_3} = (3, 1, -1)\}$.

7. Let V be the vector space $\mathbb{R}_2[t]$ of polynomials of degree ≤ 2 with real coefficients with the inner product $\langle p,q\rangle = p(-2)q(-2) + p(0)q(0) + p(2)q(2)$, i.e. the interpolation inner product.

   a) Find an orthogonal basis for V starting from the standard basis $\{1,t,t^2\}$ using the Gram-Schmidt process.

   b) Find the coordinates of $p(t) = 1 + 2t + 3t^2$ with respect to the orthogonal basis found in part a).

8. Let W be the subspace of R^3 spanned by the vector $\mathbf{v} = (1,2,3)$. Find orthogonal bases for W and $W^{\perp}$ respectively. Is the union of these two bases a basis for $\mathbb{R}^3$?

9. Let S be a (finite) subset of an inner product vector space V, and define $S^{\perp} = \{\mathbf{v} \in V: \langle \mathbf{v},\mathbf{u}\rangle = 0 \text{ for every } \mathbf{u} \in S\}$, i.e. $S^{\perp}$ is the set of vectors orthogonal to S. Show that in fact $S^{\perp}$ is a subspace of V. If W = Span S, what is the relationship between $S^{\perp}$ and $W^{\perp}$? Justify your answer.

10. Let $A \in \mathbb{R}^{m\times n}$, i.e. A is an $m\times n$ matrix with real entries. Show that Nul A is the orthogonal complement of Row A.

11. Let $V = C^{\infty}[\mathbb{R}]$, the vector space of real functions having continuous derivatives of all orders. Let D be the differentiation operator on V. Determine the eigenvalues and corresponding eigenvectors of D.

12. Let $\{(x_i, y_i): i = 1, 2,\ldots,n\}$ be n points in R^2. Show that:

$$|x_1y_1 + x_2y_2 + \ldots + x_ny_n| \leq (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2} (y_1^2 + y_2^2 + \ldots + y_n^2)^{1/2}$$

Tutorial 12 Solutions

Q (1) $|A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 & -1 \\ -12 & 0-\lambda & 5 \\ 4 & -2 & -1-\lambda \end{vmatrix} = 0$

$\Rightarrow (3-\lambda)[(+\lambda)(\lambda+1)+10] + (12(1+\lambda)-20) - 1(24+4\lambda) = 0$

$\Rightarrow (3-\lambda)[\lambda^2+\lambda+10] + 12 + 12\lambda - 20 - 24 - 4\lambda = 0$

$\Rightarrow -\lambda^3 + 3\lambda^2 + 3\lambda - \lambda^2 + 30 - 10\lambda - 8 + 8\lambda - 24 = 0$

$\Rightarrow -\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$

$\Rightarrow \lambda^2(-\lambda+2) - 1(-\lambda+2) = 0$

$\Rightarrow (\lambda^2-1)(-\lambda+2) = 0 \Rightarrow \lambda = \pm 1, 2$

Algebraic multiplicity of all $\lambda_i = 1$, $i = 1, 2, 3$

i) For $\lambda_1 = 1$. $A - \lambda_1 I = \begin{vmatrix} 2 & -1 & -1 \\ -12 & -1 & 5 \\ 4 & -2 & -2 \end{vmatrix}$

$R_2 \to R_2 + 6R_1$
$R_3 \to R_3 - 2R_1$ $\begin{vmatrix} 2 & -1 & -1 \\ 0 & 5 & -1 \\ 0 & 0 & 0 \end{vmatrix}$ (RREF)

So, $(A - \lambda_1 I)x = 0 \Rightarrow 2x_1 - x_2 - x_3 = 0$

$5x_2 - x_3 = 0$

$\Rightarrow x_2 = \frac{x_3}{5}$

$2x_1 = +x_2 + x_3$

$v_1 = [3 \quad 1 \quad 5]$ is an eigen vector corresponding to $\lambda_1 = 1$

ii) $\lambda_2 = -1$ $[A - \lambda_2 I] = \begin{vmatrix} 4 & -1 & -1 \\ -12 & 1 & 5 \\ 4 & -2 & 0 \end{vmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 + 3R_1} \begin{vmatrix} 4 & -1 & -1 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{vmatrix}$

②

$$\xrightarrow{R_3 \to R_3 - \frac{1}{2}R_1} \begin{vmatrix} 4 & -1 & -1 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{vmatrix}$$

So, $(A - \lambda_2 I)X = 0 \Rightarrow 4x_1 - x_2 - x_3 = 0, \quad -2x_2 - 2x_3 = 0$

$x_1 = \dfrac{x_2 + x_3}{4} \qquad \Rightarrow x_2 = -x_3$

$v_2 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$ is an eigen vector corresponding to $\lambda_2 = -1$

iii) $\lambda_3 = 2$, $|A - \lambda_3 I| = \begin{vmatrix} 1 & -1 & -1 \\ -12 & -2 & 5 \\ 4 & -2 & -3 \end{vmatrix} \xrightarrow[R_3 \to R_3 - 4R_1]{R_2 \to R_2 + 12R_1} \begin{vmatrix} 1 & -1 & -1 \\ 0 & -14 & -7 \\ 0 & 2 & 1 \end{vmatrix}$

$$\xrightarrow{R_3 \to R_3 + \frac{R_2}{7}} \begin{vmatrix} 1 & -1 & -1 \\ 0 & -14 & -7 \\ 0 & 0 & 0 \end{vmatrix}$$

So, $(A - \lambda_3 I)X = 0 \Rightarrow x_1 - x_2 - x_3 = 0, \quad -14x_2 - 7x_3 = 0$

$x_1 = x_1 + x_2 \qquad \Rightarrow x_2 = \dfrac{-x_3}{2}$

$v_3 = \begin{bmatrix} -1 & 1 & -2 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_3 = 2$.

Since geometric multiplicity of all $\lambda_i = 1$

$= $ algebraic multiplicity of λ_i

$\Rightarrow A$ is diagonalizable.

Note: A has 3 distinct eigen values so A will be diagonalizable.

Q(2)

a) $|A-\lambda I| = \begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} = 0$

$\Rightarrow -\lambda^3 + 12\lambda + 16 = 0$

$\Rightarrow (\lambda-4)(-\lambda^2-4\lambda-4) = 0$

$\Rightarrow (\lambda-4)(\lambda+2)^2 = 0$

$\Rightarrow \lambda = 4, -2, -2$

For $\lambda_1 = 4$. $A-\lambda_1 I = \begin{vmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{vmatrix} \xrightarrow[R_3 \to R_3+2R_1]{R_2 \to R_2+R_1} \begin{vmatrix} -3 & -3 & 3 \\ 0 & -12 & 6 \\ 0 & -12 & 6 \end{vmatrix}$

$\xrightarrow[R_3 \to R_3 - R_2]{} \begin{vmatrix} -3 & -3 & 3 \\ 0 & -12 & 6 \\ 0 & 0 & 0 \end{vmatrix}$

So, $(A-\lambda_1 I)x = 0 \Rightarrow -3x_1 - 3x_2 + 3x_3 = 0$, $-12x_2 + 6x_3 = 0$

$\Rightarrow x_1 = -x_2 + x_3$ $\quad \Rightarrow x_2 = \frac{x_3}{2}$

$v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is an eigen vector corr. to $\lambda_1 = 4$

For $\lambda_2 = -2$. $|A-\lambda_2 I| = \begin{vmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{vmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \begin{vmatrix} 3 & -3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}$

So, $(A-\lambda_2 I)x = 0 \Rightarrow 3x_1 - 3x_2 + 3x_3 = 0$

$\Rightarrow x_1 = x_2 - x_3$

$\bar{X} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are eigen vector corr. to $\lambda_2 = -2$.

As geometric multiplicity (G.M) of λ_1 = Algebraic mult of λ_1
$= 1$

and G.M of λ_2 = A.M of $\lambda_2 = 2$

$\Rightarrow A$ is diagonalizable.

$\rightarrow P = \begin{bmatrix} 1 & 1 & -1 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ consisting eigen vectors as columns.

$\rightarrow D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

<u>Verification:</u> $PDP^{-1} = A$, $P^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ -1 & 1 & 0 \end{bmatrix}$

b) $A = \begin{bmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix}$

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -3-\lambda & 1 & -1 \\ -7 & (5-\lambda) & -1 \\ -6 & 6 & (-2-\lambda) \end{vmatrix} = 0$

$\Rightarrow -(3+\lambda)\left[(2+\lambda)(5-\lambda)-6\right] - \left(7(2+\lambda)-6\right) - 1\left(42+6(5-\lambda)\right) = 0$

$\Rightarrow (3+\lambda)\left[-\lambda^2+3\lambda+10-6\right] - [7\lambda+8) - (12-6\lambda) = 0$

$\Rightarrow -\lambda^3 - \cancel{3\lambda^2} + 9\lambda + \cancel{3\lambda^2} + 30 + 10\lambda - 18 - \cancel{6\lambda} - 7\lambda - 8 + 12 + \cancel{6\lambda} = 0$

$\Rightarrow -\lambda^3 + 12\lambda + 16 = 0$

$\Rightarrow (\lambda-4)(-\lambda^2-4\lambda-4) = 0 \Rightarrow \lambda = 4, -2, -2$

i) For $\lambda_1 = -2$, $A - \lambda I = \begin{vmatrix} -1 & 1 & -1 \\ -7 & 3 & -1 \\ -6 & 6 & 0 \end{vmatrix} \xrightarrow[R_3 \to R_3 - 6R_1]{R_2 \to R_2 - 7R_1} \begin{bmatrix} -1 & 1 & -1 \\ 0 & -4 & 6 \\ 0 & 0 & 6 \end{bmatrix}$

So, $(A-\lambda_1 I)X = 0 \Rightarrow$ $-x_1 + x_2 - x_3 = 0$ $x_1 = 0$

$-4x_2 + 6x_3 = 0 \Rightarrow x_2 = 0$

$6x_3 = 0 \Rightarrow x_3 = 0$

So, dim (Eigen space corr. to λ_1) $= 0$

$\Rightarrow$ G.M of $\lambda_1 = 0 \neq$ Algebraic multiplicity of λ_1

$\Rightarrow A$ is not diagonalizable.

Q (3) Let W_1, W_2 and W_3 be the eigen spaces corresponding to the 3 distinct eigen values. Let $\dim W_1 = 2$ and $\dim W_2 = 3$. Yes, it is possible for A to be diagonalizable if

$\dim W_3 = 7 - (2+3) = 2$ (Use Proposition 40 (b) to justify)

Q(4) a) False. b) Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$\underset{R_1 \to R_1 - 2R_2}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$ [A is Row equivalent to I_2]

$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 = 0$

$\Rightarrow \lambda = 1, 1$

Algebric multiplicity of $\lambda (=1) = 2$

$(A - \lambda I)X = 0 \Rightarrow \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$\Rightarrow -x_1 + 2x_2 = 0$ and $x_2 = 0$

$\Rightarrow x_1 = 0$

$\therefore$ dim of eigen space corresponding to $\lambda = 1$ $= 0$

ie geometric multiplicity of $\lambda = 0 \neq$ Algebraic multiplicity

Hence, A is not diagonalizable.

Q(5) $\langle f, g\rangle = \int_a^b f(t)\, g(t)\, dt \quad \in \mathbb{R}$

i) $\langle g, f\rangle = \int_a^b g(t) f(t)\, dt = \int_a^b f(t)\, g(t)\, dt$

$= \langle f, g\rangle$

ii) Let $f, g, h \in C[a,b]$.

$\langle f+g, h\rangle = \int_a^b \left[(f+g)(t)\right] h(t)\,dt = \int_a^b \left[f(t)+g(t)\right] h(t)\, dt$

$= \int_a^b \left[f(t)\, h(t) + g(t)\, h(t)\right] dt$

$= \int_a^b f(t)\, h(t)\, dt + \int_a^b g(t)\, h(t)\, dt$

$= \langle f, h\rangle + \langle g, h\rangle$

{as $f(t), g(t), h(t)$ are real numbers and R is a field so distributive laws hold

(iii) $\langle cf, g\rangle = \int_a^b \left(cf(t)\right) g(t)\, dt$

$= \int_a^b c\, f(t) \cdot g(t)\, dt$

$= c \int_a^b f(t)\, g(t)\, dt$

$= c\langle f, g\rangle$

(iv) $\langle f, f\rangle = \int_a^b \left(f(t)\right)^2 dt$

≥ 0

{as $(f(t))^2 \geq 0$
$\Rightarrow \int_a^b (f(t))^2 dt$ represents area under the graph so will be non-ve)

$\langle f, f\rangle = 0 \Leftrightarrow \int_a^b (f(t))^2 = 0$

$\Leftrightarrow (f(t))^2 = 0 \quad \forall\, t \in [a,b]$ [as area $= 0$

$\Leftrightarrow f = 0 \qquad \Leftrightarrow f(t) = 0 \quad \forall\, t \in [a,b]$ [$\Rightarrow$ function = zero function

Q(6) $\bar{x}_1 = [2 \ 1 \ 2]$ $\quad \bar{x}_2 = [4 \ 1 \ 0]$ $\quad \bar{x}_3 = [3 \ 1 \ -1]$

$\bar{v}_1 = \bar{x}_1 = [2 \ 1 \ 2]$ (1)

$$\bar{v}_2 = \bar{x}_2 - \frac{\langle x_2, \bar{v}_1 \rangle}{\langle \bar{v}_1, \bar{v}_1 \rangle} \bar{v}_1 = [4 \ 1 \ 0] - \frac{9}{9}[2 \ 1 \ 2]$$

$$= [2 \ 0 \ -2]$$

$$\bar{v}_3 = \bar{x}_3 - \frac{\langle \bar{x}_3, \bar{v}_1 \rangle}{\langle \bar{v}_1, \bar{v}_1 \rangle} \bar{v}_1 - \frac{\langle \bar{x}_3, \bar{v}_2 \rangle}{\langle \bar{v}_2, \bar{v}_2 \rangle} \bar{v}_2$$

$$= [3 \ 1 \ -1] - \frac{5}{9}[2 \ 1 \ 2] - \frac{8}{8}[2 \ 0 \ -2]$$

$$= [1 \ 1 \ 1] - \frac{5}{9}[2 \ 1 \ 2]$$

$$= \left[\frac{-1}{9} \ \frac{4}{9} \ \frac{-1}{9}\right]$$

$\langle v_1, v_2 \rangle = 4 - 4 = 0$ ✓

$\langle v_1, v_3 \rangle = -\frac{2}{9} + \frac{4}{9} - \frac{2}{9} = 0$ ✓

$\langle v_2, v_3 \rangle = -\frac{2}{9} + 0 + \frac{2}{9} = 0$ ✓

Orthonormal basis: $\left\{ \frac{\bar{v}_1}{\|\bar{v}_1\|}, \frac{\bar{v}_2}{\|\bar{v}_2\|}, \frac{\bar{v}_3}{\|\bar{v}_3\|} \right\}$

$$= \left\{ \left[\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right], \left[\frac{2}{2\sqrt{2}}, 0, \frac{-2}{2\sqrt{2}}\right], \left[\frac{-1}{9} \times \frac{3}{\sqrt{2}}, \frac{4}{9} \times \frac{3}{\sqrt{2}}, \frac{-1}{9} \times \frac{3}{\sqrt{2}}\right.\right.$$

$$= \left\{ \left[\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right], \left[\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right], \left[\frac{-1}{3\sqrt{2}}, \frac{4}{3\sqrt{2}}, \frac{-1}{3\sqrt{2}}\right] \right\}$$

Q (7) a) $p_1(t) = 1 \qquad p_2(t) = t \qquad p_3(t) = t^2$

$q_1(t) = p_1(t) = 1$

(*)

$$q_2(t) = p_2(t) - \frac{\langle p_2(t), q_1(t) \rangle}{\langle q_1(t), q_1(t) \rangle} q_1(t)$$

$$= t - 1 \cdot \frac{[p_2(-2)q_1(-2) + p_2(0)q_1(0) + p_2(2)q_1(2)]}{q_1(-2)q_1(-2) + q_1(0)q_1(0) + q_1(2)q_1(2)}$$

$$= t - \frac{\cancel{(-2)\times 1} + 0\times 1 + \cancel{2\times 1}}{1+1+1}$$

$$= t - 0 = t$$

$$q_3(t) = p_3(t) - \frac{\langle p_3(t), q_2(t) \rangle}{\langle q_2(t), q_2(t) \rangle} q_2(t) - \frac{\langle p_3(t), q_1(t) \rangle}{\langle q_1(t), q_1(t) \rangle} q_1(t)$$

$$= t^2 - \frac{[4\times -2 + 0\times 0 + 4\times 2]t}{4+0+4} - \frac{[4+0+4]}{1+1+1} \times 1$$

$$= t^2 - 0 - \frac{8}{3}$$

$$= t^2 - \frac{8}{3}$$

Check: $\langle q_1, q_2 \rangle = \langle q_1, p_2 \rangle = 0$ (from (*))

$\langle q_1, q_3 \rangle = \left(4 - \frac{8}{3}\right) - \frac{8}{3} + \left(4 - \frac{8}{3}\right) = 8 - \frac{24}{3} = 8 - 8 = 0$

$\langle q_2, q_3 \rangle = (-2\times 4) + 0 + (2\times 4) = 0$

Orthogonal basis: $\left\{1,\ t,\ t^2 - \frac{8}{3}\right\}$

coordinate of $p(t)$ $= [9, 2, 3]$

b) $p(t) = 1 + 2t + 3t^2 = \alpha \times 1 + \beta t + \gamma\left(t^2 - \frac{8}{3}\right)$

$\Rightarrow \alpha - \frac{8}{3}\gamma = 1,\ \beta = 2 \qquad \gamma = 3 \quad \Rightarrow \alpha = 1 + 8 = 9$

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Q(8) Let $(u_1, u_2, u_3) \in W^{\perp}$

$$\Rightarrow \langle (u_1, u_2, u_3), (1,2,3) \rangle = 0 \Rightarrow u_1 + 2u_2 + 3u_3 = 0$$

$$\Rightarrow (-1, -1, 1) \text{ is orthogonal to } (1,2,3).$$

Let (y_1, y_2, y_3) be orthogonal to $(-1,-1,1)$ and $(1,2,3)$. Then

$$-y_1 - y_2 + y_3 = 0 \text{ and } y_1 + 2y_2 + 3y_3 = 0$$

$$\Rightarrow y_2 + 4y_3 = 0 \Rightarrow y_2 = -4y_3$$

$$\text{and } y_1 = -y_2 + y_3 = 4y_3 + y_3 = 5y_3$$

So, $(5, -4, 1)$ is orthogonal to both $(-1,-1,1)$ and $(1,2,3)$.

Hence, $\{(5,-4,1), (-1,-1,1), (1,2,3)\}$ is a set of orthogonal vectors in $\mathbb{R}^3$. By Proposition 41 it will be linearly independent

$\Rightarrow$ It forms a basis of $\mathbb{R}^3$, as $\dim \mathbb{R}^3 = 3$.

Also, $W^{\perp} = \text{Span}\{(5,-4,1), (-1,-1,1)\}$

Basis for $W = \{(1,2,3)\}$ and $W^{\perp} = \{(5,-4,1), (-1,-1,1)\}$

Q(9) $S^{\perp} = \{ v \in V \mid \langle v, u \rangle = 0 \ \forall u \in S \}$

i) $\langle 0, u \rangle = 0 \ \forall u \in S \Rightarrow 0 \in S^{\perp}, \ S^{\perp} \neq \phi$

ii) Let $v_1, v_2 \in S^{\perp}$. Then,

$\langle v_1, u \rangle = 0$ and $\langle v_2, u \rangle = 0 \quad \forall u \in S$

$\Rightarrow \langle v_1 + v_2, u \rangle = \langle v_1 + u \rangle + \langle v_2 + u \rangle \quad \forall u \in S$

$= 0 + 0 = 0$

$\Rightarrow v_1 + v_2 \in S^{\perp}$

iii) Let $c \in F$ and $v_1 \in S^{\perp}$.

$\langle c v_1, u \rangle = c \langle v_1, u \rangle = c.0 = 0$

$\Rightarrow c v_1 \in S^{\perp}$

and $S^{\perp}$ is a subspace of V.

Let $W =$ span S. Then, $S^{\perp} = W^{\perp}$.

Let $v \in S^{\perp} \Rightarrow \langle v, u \rangle = 0 \quad \forall u \in S$

Let $w \in W$. Then, $w = \sum_{i=1}^{m} \alpha_i u_i$, $\alpha_i \in \mathbb{F}$ $\quad S = \{u_1, u_2 - u_m\}$

$\Rightarrow \langle v, w \rangle = \langle v, \sum_{i=1}^{m} \alpha_i u_i \rangle = \sum_{i=1}^{m} \alpha_i \langle v, u_i \rangle$

$= 0$ as $\langle v, u_i \rangle = 0 \ \forall i$

$\Rightarrow v \in W^{\perp}$ (as $w \in W$ is arbitrarly)

$\Rightarrow \boxed{S^{\perp} \subseteq W^{\perp}}$

Let $v \in W^{\perp}$. Then, $\langle v, w \rangle = 0 \ \forall w \in W =$ span$S$

$\Rightarrow \langle v, u_i \rangle = 0 \ \forall 1 \le i \le m$ $\quad \left\{ \begin{array}{l} u_i \in S \subseteq W \\ \Rightarrow u_i \in W \ \forall i \end{array} \right.$

$\Rightarrow v \in S^{\perp}$

$\Rightarrow \boxed{W^{\perp} \subseteq S^{\perp}}$ Hence, $\boxed{S^{\perp} = W^{\perp}}$

Q 1a Let A be an $m \times n$ matrix with real entries.

Claim: $\text{Nul } A = (\text{Row } A)^{\perp}$

Let $z \in \text{Nul } A \Rightarrow Az = 0$

$\Rightarrow a_i^T z = 0 \quad , \quad \forall \; 1 \leq i \leq m$

For any $y \in \text{Row } A$,

$$A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix}$$

$$y = \sum_{i=1}^{m} a_i x_i \quad , \quad x_i \in \mathbb{R}$$

$$\Rightarrow y^T z = \left(\sum_{i=1}^{m} a_i x_i \right)^T z = \left(\sum_{i=1}^{m} x_i a_i^T \right) z$$

$$= \sum_{i=1}^{m} x_i a_i^T z = 0$$

$\Rightarrow z \in (\text{Row } A)^{\perp}$

$\Rightarrow \boxed{\text{Nul } A \subseteq (\text{Row } A)^{\perp}}$

Let $z \in (\text{Row } A)^{\perp} \Rightarrow \langle z, y \rangle = 0 \quad \forall y \in \text{Row } A$

$\Rightarrow \langle z, a_i \rangle = 0$ (for particular choices of $y = a_i$) $\forall \; 1 \leq i \leq m$

$\Rightarrow z \in \text{Nul } A$

$\Rightarrow \boxed{(\text{Row } A)^{\perp} \subseteq \text{Nul } A}$

$\Rightarrow \boxed{(\text{Row } A)^{\perp} = \text{Nul } A}$

Q 12 Let $u = (x_1, x_2, --, x_n)$ and $v = (y_1, y_2 -- y_n)$ $\in \mathbb{R}^n$.

$\left.\begin{array}{l}\text{As } \mathbb{R}^n(\mathbb{R}) \text{ is an inner product space with } \langle u,v\rangle = \sum_{i=1}^{n} x_i y_i \\ \text{By Cauchy-Schwarz Inequality we have the result directly}\end{array}\right\}$

If $u = 0$ then

$$\langle u, v\rangle = \sum_{i=1}^{n} x_i y_i = 0 \text{, i.e } |x_1y_1 + x_2y_2 + --- + x_ny_n| = 0$$

and $(x_1^2 + x_2^2 + --- + x_n^2)^{\frac{1}{2}} = 0$

$$\Rightarrow |x_1y_1 + -- + x_ny_n| = (x_1^2 + x_2^2 + -- + x_n^2)^{\frac{1}{2}} (y_1^2 + y_2^2 + --- + y_n^2)^{\frac{1}{2}}$$

Let $u \neq 0$. Then, $(x_1^2 + x_2^2 + -- + x_n^2) > 0$, i.e $\langle u, u\rangle > 0$.

Let us choose,

$$w = v - \lambda u \text{ , } \lambda = \frac{\langle v, u\rangle}{\langle u, u\rangle} \quad — (1)$$

Now, $\langle w, w\rangle \geq 0$ (by def^n of inner product space $\mathbb{R}^n(\mathbb{R})$)

$\Rightarrow \langle v - \lambda u, v - \lambda u\rangle \geq 0$

$\Rightarrow \langle v, v - \lambda u\rangle - \lambda\langle u, v - \lambda u\rangle \geq 0$

$\Rightarrow \langle v, v\rangle - \lambda\langle v, u\rangle - \lambda\langle u, v\rangle + \lambda^2\langle u, u\rangle \geq 0$

$\Rightarrow \langle v, v\rangle - \cancel{\lambda^2\langle u, u\rangle} - \lambda\langle u, v\rangle + \cancel{\lambda^2\langle u, u\rangle} \geq 0$ (using (1))

$\Rightarrow \langle v, v\rangle \geq \lambda\langle u, v\rangle = \frac{\langle u, v\rangle\langle u, v\rangle}{\langle u, u\rangle}$

$\Rightarrow |\langle u, v\rangle|^2 \leq \langle v, v\rangle\langle u, u\rangle$ $\quad \{ |x|^2 = x^2$

$\Rightarrow |\langle u, v\rangle| \leq \sqrt{\langle v, v\rangle\langle u, u\rangle}$

$\Rightarrow |x_1y_1 + x_2y_2 + -- + x_ny_n| \leq \sqrt{(x_1^2 + x_2^2 + -- + x_n^2)^{\frac{1}{2}} (y_1^2 + y_2^2 + -- + y_n^2)^{1/2}}$

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Q11.

Suppose $\lambda \in \mathbb{R}$ is an eigen value for D.

Let $y = f(x)$ be corresponding eigenvector in the situation the vectors are actually function $f: R \longrightarrow R$

So, we must have $Dy = \lambda y$.

NB: Note the similarity with Q 2(b) of Grade Improvement Exam.

Q11 (cont'd)

ie $\frac{dy}{dx} = \lambda y$

or $\frac{dy}{y} = \lambda dx$

Solving for y in terms of x, we get

$\ln y = \lambda x + c$, where c is arbitrary constant.

let $c = \ln c_1$

$\ln y - \ln c_1 = \lambda x$

$y/c_1 = e^{\lambda x}$

$\Rightarrow y = c_1 e^{\lambda x}$

The corresponding eigen function is $y = e^{\lambda x}$

$\therefore$ Every $\lambda \in \mathbb{R}$ is an eigen value.

The corresponding eigen function is $y = e^{\lambda x}$

[NB: If $\lambda = 0$, we get the constant function $y = e^0 = 1$ which is an eigen function with eigen value 0.]