VIT

~~(1) ⇒ (3)~~

(1) ⇒ (3) Given A is invertible

To prove: $A\bar{x}=\bar{0}$ has only the trivial solⁿ

Suppose $\bar{y}$ is a solⁿ of $A\bar{x}=\bar{0}$

$\therefore A\bar{y}=\bar{0}$

Multiplying A^{-1} to the ~~of~~ left

$A^{-1}A\bar{y}=A^{-1}\bar{0}$

$\therefore \bar{y}=\bar{0}$

Hence, Proved

(3) ⇒ (2) Given: the homogeneous system $A\bar{x}=\bar{0}$ has only a trivial solⁿ

T.P: A is row equivalent to I

Now, if R is the RREF matrix of A
then $R\bar{x}=\bar{0}$ has only the trivial solⁿ.

⇒ R has no free variables

⇒ R has only basic variables

⇒ R has a leading 1 in each row

⇒ R has exactly one 1 in each column

⇒ R is I_n

(2) $\Rightarrow$ (1) Given: A is row equivalent to I

T.P: A is invertible

Now, A is row equivalent to I

$\Rightarrow$ There are e-ops

$e_p, e_{p-1}, e_{p-2}, \ldots, e_2, e_1$ such that

$e_p(e_{p-1}(\ldots(e_1(A))\ldots)) = I$ (1)

If E_i is the elementary matrix of e-ops e_i

From (1)

$E_p E_{p-1} \cdots E_1 A = I$

Putting $B = E_p \ldots E_1$, we get from prop 6 & ols 4 for invertible matrices

that B is invertible (2)

From (1), $BA = I$

Multiplying by B^{-1} on left

$B^{-1}(BA) = B^{-1}$

$\Rightarrow A = B^{-1}$

Hence, A being the inverse of an invertible matrix, A is also invertible.

H.P

(1)$\Rightarrow$(4) Given: A is invertible
T.P: The non homogeneous system $A\bar{x} = \bar{b}$ has atleast one sol^n for every $\bar{b}$

Put $\bar{y} = A^{-1}\bar{b}$ (since A is invertible)

$A\bar{y} = A(A^{-1}\bar{b}) = AA^{-1}\bar{b} = I_m\bar{b} = \bar{b}$

$\Rightarrow$ $\bar{y}$ is a sol^n of $A\bar{x} = \bar{b}$ as required

(4)$\Rightarrow$(1) Given: $A\bar{x} = \bar{b}$ has atleast one sol^n for every $\bar{b} \in \mathbb{R}^m$
T.P: A is invertible

Consider the m vector $\bar{e}_1, \dots, \bar{e}_m$ in $\mathbb{R}^m$, has 1 in the i^{th} posi
0 else where, for eg:- $\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ & $\bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Now, let $\bar{v}_i$ be a sol^n of the eq^n $A\bar{x} = \bar{e}_i$ (1)
$\Rightarrow$ $A\bar{v}_i = \bar{e}_i$ (2)

Now let B be the $m \times m$ matrix with the $\bar{v}_i$ as its columns, i.e.,
$B = [\bar{v}_1 \, \bar{v}_2 \, \dots \, \bar{v}_m]$ in column form
$\therefore AB = A[\bar{v}_1 \, \bar{v}_2 \dots \bar{v}_m] = [A\bar{v}_1 \, A\bar{v}_2 \dots A\bar{v}_m]$
$= [\bar{e}_1 \, \bar{e}_2 \dots \bar{e}_m]$ (Using (2))
$AB = I_m$ (3)
$\Rightarrow$ A is invertible
H.P