THEOREM 7

Remark: We will prove the original statement, from which the alternative statement easily follows. Theorem 7 is more informative than Theorem 7A, since it provides an expression for $\hat{y}$. But Theorem 7A is simpler and easier to remember, and is sufficient in most circumstances.

Proof: Since W is finite-dimensional, it has a finite basis, and hence, by Theorem 6, we may

GRAM-SCMIDT PROCESS

Theorem ~~3~~ 7 (8)

assume W has an orthogonal basis $\{\bar{u}_1, \dots, \bar{u}_p\}$. (1)

We first show that any vector $\bar{y} \in V$ cannot be expressed in two different ways as a sum of vectors in W and $W^\perp$, i.e. we prove uniqueness first.

So suppose $\bar{y} = \bar{y}_1 + \bar{z}_1$

and $\bar{y} = \bar{y}_2 + \bar{z}_2$ (2)

where the $\bar{y}_i \in W$, and the $\bar{z}_i \in W^\perp$,

Subtracting, we get

$$\bar{0} = (\bar{y}_1 - \bar{y}_2) + (\bar{z}_1 - \bar{z}_2)$$

or $\bar{y}_1 - \bar{y}_2 = -(\bar{z}_1 - \bar{z}_2)$ (3)

Now, the LHS vector in (3) is in W, and the RHS vector is in $W^\perp$. Since the two are equal, both belong to $W \cap W^\perp$.

However, by Prop. 42(4), $W \cap W^\perp = \{\bar{0}\}$.

$\therefore$ from (3),

$\bar{y}_1 - \bar{y}_2 = \bar{0} \Rightarrow \bar{y}_1 = \bar{y}_2$

and $\bar{z}_1 - \bar{z}_2 = \bar{0} \Rightarrow \bar{z}_1 = \bar{z}_2$

This proves uniqueness. ~~Provided~~

Theorem ~~8~~ - ~~concluded~~ continued:-

However, we need to prove that every $\bar{y} \in V$ can be expressed in this way.

We use the statement of the Theorem. Putting

$$\hat{y} = c_1\bar{u}_1 + \cdots + c_p\bar{u}_p \quad (4)$$

where $c_i = \langle \bar{y}, \bar{u}_i \rangle / \langle \bar{u}_i, \bar{u}_i \rangle$ for $i = 1, \ldots, p$, we see that $\hat{y} \in W$. If we put $\bar{z} = \bar{y} - \hat{y}$,

$$\text{clearly } \bar{y} = \hat{y} + \bar{z} \quad (5)$$

It only remains to prove that $\bar{z} \in W^{\perp}$.

For this, we use Prop. 42 (a).

Now, for any $i = 1, \ldots, p$

$$\langle \bar{z}, \bar{u}_i \rangle = \langle \bar{y} - \hat{y}, \bar{u}_i \rangle$$

$$= \langle \bar{y}, \bar{u}_i \rangle - \langle \hat{y}, \bar{u}_i \rangle$$

$$= \langle \bar{y}, \bar{u}_i \rangle - \langle c_1\bar{u}_1 + \cdots + c_p\bar{u}_p, \bar{u}_i \rangle$$

using (4)

Theorem 7 (conclusion)

$$= \langle \bar{y}, \bar{u}_i \rangle - c_1 \langle \bar{u}_1, \bar{u}_i \rangle - c_2 \langle \bar{u}_2, \bar{u}_i \rangle - \dots - c_p \langle \bar{u}_p, \bar{u}_i \rangle$$

$$= \langle \bar{y}, \bar{u}_i \rangle - c_i \langle \bar{u}_i, \bar{u}_i \rangle \qquad (6)$$

since all other terms are zero (as ~~the~~ $\bar{u}_i \perp \bar{u}_j$ for $j \neq i$).

Now, putting in the value of c_i from (4), we get:

$$\langle \bar{z}, \bar{u}_i \rangle = \langle \bar{y}, \bar{u}_i \rangle - \frac{\langle \bar{y}, \bar{u}_i \rangle}{\langle \bar{u}_i, \bar{u}_i \rangle} \langle \bar{u}_i, \bar{u}_i \rangle$$

$$= 0, \text{ as reqd.}$$

This completes the proof.

Remark: You would have noticed the similarity in the calculations in the proofs of Theorem 6 and Theorem 7.