Proposition 20

[⇒] Suppose $V = U \oplus W$. Then by definition, every vector $\bar{v} \in V$ is uniquely expressible as $\bar{v} = \bar{u} + \bar{w}$, with $\bar{u} \in U$ & $\bar{w} \in W$.

It only remains to show that $U \cap W = \{\bar{0}\}$.
Let us suppose $\bar{x} \in U \cap W$.

$\Rightarrow \bar{x} = \bar{x} + \bar{0}$, where $\bar{x} \in U$ & $\bar{0} \in W$
and $\bar{x} = \bar{0} + \bar{x}$, where $\bar{0} \in U$ & $\bar{x} \in W$

$\Rightarrow$ By uniqueness of expression, $\bar{x} = \bar{0}$ as required.

[⇐] Suppose $V = U + W$ with $U \cap W = \{\bar{0}\}$
we need to show that every $\bar{v} \in V$ is uniquely expressible as $\bar{v} = \bar{u} + \bar{w}$ with $\bar{u} \in U$ & $\bar{w} \in W$.

Let $\bar{v} \in V$. Since, $V = U + W$, $\bar{v}$ is expressible as $\bar{v} = \bar{u} + \bar{w}$

Suppose the expression is not unique, i.e., we also have $\bar{v} = \bar{u}_1 + \bar{w}_1$. ~~with $\bar{u}_1 \in$~~

Subtracting, $\bar{0} = \bar{v} - \bar{v} = (\bar{u} + \bar{w}) - (\bar{u}_1 + \bar{w}_1)$
$\Rightarrow \bar{w} - \bar{w}_1 = \bar{u}_1 - \bar{u}$ (1)

The vectors on LHS of ① is in U & the vectors on RHS of ① is in W, i.e.,
each one of them is in $U \cap W = \{\bar{0}\}$.