MTH100 **MATHS 1 Section B** **Monsoon Semester – 2021-22**

Tutorial exercise for the Week Commencing Monday 20220314

1. Given any m×n matrix A, show that rank (A) ≤ min {m, n}. Give an example in which equality is achieved (i.e. rank (A) = min{m,n}), and an example in which strict inequality holds (i.e. rank (A) < min{m,n}). (***NB: Your examples A and B for Q6, 7 and 8, should be non-zero and at least 2×2 in size.)***

2. Given any two m×n matrices A and B, prove that rank (A + B) ≤ rank (A) + rank (B). Give an example in which equality is achieved, and an example in which strict inequality holds.

3. Given any m×n matrix A and any n×k matrix B, show that rank (AB) ≤ min {rank (A), rank (B)}. Give an example in which equality is achieved, and an example in which strict inequality holds.

4. Determine whether the following are linear transformations (yes or no). Justify your answers.
   a. $T: \mathbb{R}^3 \to \mathbb{R}^2$ given by $T(x,y,z) = (x + y, x - z)$
   b. $T: \mathbb{R}^3 \to \mathbb{R}^2$ given by $T(x,y,z) = (x + y, z^2)$
   c. $U: \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ given by $U(A) = A^T$. Here A^T indicates the transpose of the matrix A.
   d. $M: \mathbb{R}[t] \to \mathbb{R}[t]$ given by $M(p(t)) = tp(t)$ for all polynomials $p(t) \in \mathbb{R}[t]$.

5. Determine all linear transformations $T: \mathbb{R}^1 \to \mathbb{R}^1$. (NB: $\mathbb{R}^1$ is the vector space consisting of all 1-tuples with real entries; it is essentially the same as $\mathbb{R}$, however regarded as only a vector space rather than a field.)

6. Consider the space $V = C[\mathbb{R}]$ and consider the mapping $D_\varepsilon: V \to V$ given by $D_\varepsilon(f) = f_\varepsilon$, where $f_\varepsilon(x) = f(x + \varepsilon)$ for all x. Here ε is an arbitrary but fixed real number. Is D_ε a linear transformation? Justify your answer.

7. Prove that there does not exist a linear transformation $T: R^5 \to R^2$ such that
   $\mathrm{Ker}\, T = \{(x_1,x_2,x_3,x_4,x_5) \in R^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}$.

8. Consider the field $\mathbb{C}$ of complex numbers as a vector space over the field $\mathbb{R}$. Show that the function $\phi: \mathbb{C} \to \mathbb{C}$ given by $\phi(z) = \overline{z}$ is a linear transformation. Here $\overline{z}$ indicates the complex conjugate of z, i.e. if $z = a + bi$, then $\overline{z} = a - bi$. Show that complex conjugation is actually a multiplicative function, i.e. if $w, z \in \mathbb{C}$, then $\phi(wz) = \phi(w)\phi(z)$. Finally, show that ϕ is the only multiplicative linear transformation on $\mathbb{C}$ to $\mathbb{C}$, other than the zero and identity linear transformations.

Tut 09

Q1 $\because$ Rank of a matrix is the dim(Col A), where ~~set~~ Col A is the is the set of all linear combinations of the columns of A.

$\Rightarrow$ By proposition 23 dim(col(A)) is the no of pivot column

$\because$ pivot column can not be greater than the min of m or n

$\Rightarrow$ $\dim(\text{col} A) \leq \min\{m, n\}$

H.P.

g: For equality

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\dim(\text{Col} A) = 2$

For inequality

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \dim(\text{Col}(A)) = 1$$

Q2 Let $v \in Col(A+B)$. Then $\exists x$ s.t.

$\bar{v} = (A+B)\bar{x} = A\bar{x} + B\bar{x}$

$\bar{v} = \bar{z} + \bar{w}$, $\bar{z} \in A\bar{x}$, $\bar{w} \in B\bar{x}$

i.e. $\bar{z} \in Col A$ & $\bar{w} \in Col B$

$v = Col A + Col B$

$Col(A+B) \subseteq Col A + Col B$

$\Rightarrow \dim(Col(A+B)) \leq \dim(Col A + Col B)$

$\leq \dim Col A + \dim Col B$

$\Rightarrow$ $Rank(A+B) \leq Rank(A) + Rank(B)$

Q3 Let $\bar{v} \in Col(AB)$ then $\exists \bar{x}$ s.t

$\bar{v} = \cancel{(AB)\bar{x}}\ (AB)\bar{x}$

Let $A\bar{y} = \bar{v}$ for $\bar{y} = B\bar{x}$

$\Rightarrow \bar{v} \in Col A$

$\Rightarrow Col AB \subset Col A$

$\dim(Col(AB)) \leq \dim(Col A)$

$\Rightarrow Rank(AB) \leq \dim A$ ①

$\because Row AB = Col(AB)^T = Col B^T A^T \subseteq Col B^T$ (~~from~~ similarly as above)

$= Row B$

$\Rightarrow Row AB \subseteq Row B$

$\Rightarrow Rank AB \leq Rank B$ ②

$\Rightarrow$ From ① & ②

$\text{Rank}(AB) \leq \text{Min}\{\text{Rank}A, \text{Rank}B\}$

Q4 a) $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$

$T(x, y, z) = (x+y, x-z)$

Let $\bar{u} = (x_1, y_1, z_1) \in \mathbb{R}^3$

& $\bar{v} = (x_2, y_2, z_2) \in \mathbb{R}^3$

Let $c \in \mathbb{R}$

$T(u+v) = T(x_1+x_2, y_1+y_2, z_1+z_2) = x_1+x_2+y_1+y_2, x_1+x_2-z_1-z_2$ ①

$T(u)+T(v) = T(x_1, y_1, z_1) + T(x_2, y_2, z_2)$

$= (x_1+y_1, x_1-z_1) + (x_2+y_2, x_2-z_2)$

$T(u)+T(v) = (x_1+x_2+y_1+y_2, x_1+x_2-z_1-z_2) = T(u+v)$ ①

$T(cu) = \cancel{T}(cx_1, cy_1 \cancel{z_1}) = \cancel{c(x_1} (cx_1+cy_1, \cancel{c}x_1 - cz_1)$

$cT(u) = c(x_1+y_1, x_1-z_1)$

$cT(u) = (cx_1+cy_1, cx_1-cz_1) = T(cu)$ ②

$\Rightarrow$ By ① & ②

T is a linear transformation

b) Not a linear transformation

c) Is a linear transformation

d) Is a linear transformation.

Q5 Suppose $T: \mathbb{R}^1 \to \mathbb{R}^1$ is any linear transformation

Now $T(1) = c$; $c \in \mathbb{R}^1$

But for $x \in \mathbb{R}^1$

$$T(x) = T(x \cdot 1) = xT(1)$$
$$= x \cdot c$$
$$= cx$$

$\Rightarrow T(x) = cx$ is the only linear transformation from $\mathbb{R}^1 \to \mathbb{R}^1$

Q6 Let $f, g \in C[\mathbb{R}]$

$\Rightarrow D_E(f+g) = h$

$$h(x) = (f+g)(x+E)$$
$$= f(x+E) + g(x+E)$$
$$= f_E + g_E$$

$\Rightarrow D_E(f+g) = D_E(f) + D_E(g)$

Similarly for

$$D_E(cf) = (cf)(x+E) = c f(x+E) = c D_E(f)$$

Q7 $T: \mathbb{R}^5 \to \mathbb{R}^2$

$\text{Ker}\, T = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2 \ \& \ x_3 = x_4 = x_5\}$

$\text{Ker}\, T = \{\bar{v} \in \mathbb{R}^5 : T\bar{v} = \bar{0} \in \mathbb{R}^2\}$

Suppose T exist for a homogenous system

$$\Rightarrow \begin{aligned} x_1 - 3x_2 &= 0 \\ x_3 - x_4 &= 0 \\ x_3 - x_5 &= 0 \\ x_4 - x_5 &= 0 \end{aligned}$$

$\Rightarrow$ Coeff matrix of this is

$$A = \begin{bmatrix} 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Its RREF $= R =$

$$\begin{bmatrix} 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ rank $A = 3$
& nullity $A = 2$

$\Rightarrow$ rank $T = 3 = \dim(\text{Range } T)$ $\quad$ ($\because$ rank A = rank T)
But $\dim(\text{Range } \mathbb{R}^2) \leq 2$
$\Rightarrow$ Not possible

Q8 Let $B = \{1, i\}$ be the basis for C

If $z = a+ib$ & $w = c+id \in C$ & ~~a, b, c, d~~, $r \in R$ then

$$\phi(z+w) = \phi((a+c)+i(b+d))$$
$$= \cancel{\phi\ (b+d) + i(a+c)}\ (a+c) - i(b+d)$$
$$\phi(z+w) = (a-ib) + (c-id) = \phi(z) + \phi(w) \quad (1)$$

& $\phi(rz) = \phi(ra + irb) = ra - irb = r(a-ib) = r\phi(z) \quad (2)$

$\Rightarrow$ By (1) & (2)
ϕ is a linear transformation

~~Let~~

$$\Rightarrow \phi(zw) = \overline{zw} = \overline{(a+ib)(c+id)} = \overline{ac + iad + ibc - bd}$$
$$= \overline{(ac-bd) + i(ad+bc)}$$
$$\phi(zw) = (ac-bd) - i(ad+bc) \quad (3)$$

$\phi(z) = a - ib$
$\phi(w) = c - id$

$$\phi(z)\phi(w) = (a-ib)(c-id) = ac - iad - ibc + -bd$$
$$= (ac-bd) - i(ad+bc) \quad (4)$$

$\Rightarrow$ By (3) & (4)
$\phi(zw) = \phi(z)\phi(w)$
$\Rightarrow \phi$ is a multiplicative function