MTH100 MATHS I Monsoon Semester – 2021-22

Tutorial Exercise for the Week Commencing Monday 7th March 2022.

1. Given the standard basis B = {$\mathbf{e}_1$,$\mathbf{e}_2$,$\mathbf{e}_3$} of $\mathbb{R}^3$ and the linearly independent vectors $\mathbf{v}_1$ = (0,1,1) and $\mathbf{v}_2$ = (1,1,1), apply the method of the Steinitz Exchange Lemma (Proposition 12) to exchange two of the vectors in B and obtain a basis C which includes $\mathbf{v}_1$ and $\mathbf{v}_2$. Show your calculations in detail.

2. Expand the linearly independent set S = {**u**, **w**} to a basis of $\mathbb{R}^3$, using the approach of Propositions 14 and 15. Here **u** = (3,3,7) and **w** = (10, 9, 21). Justify your answer.

3. V is a vector space with dim (V) = n. W_1 and W_2 are subspaces of V such that dim (W_1) = dim (W_2) = n – 1 and $W_1 \cap W_2 = \{\mathbf{0}\}$. Find n.

4. Prove **Proposition 20**: If U and W are subspaces of the vector space V, then $V = U \oplus W$ if and only if V = U + W, and $U \cap W = \{\mathbf{0}\}$.

5. Given the vector space $\mathbb{R}^3$, let W_1 be the set of vectors of the form (x,y,0) and let W_2 be the set of vectors of the form (0,a,b).

   a) Show that W_1 and W_2 are subspaces of $\mathbb{R}^3$.

   b) Find the dimensions of W_1, W_2, $W_1 + W_2$ and $W_1 \cap W_2$.

   c) Find two distinct subspaces U_1 and U_2 of $\mathbb{R}^3$ such that $\mathbb{R}^3 = W_1 \oplus U_1 = W_1 \oplus U_2$, i.e. find two distinct complements of V. Justify your answer.

6. Given the matrix A below:
   a) Find a basis for each of the spaces Nul A, Col A and Row A.
   b) Find a basis for Row A consisting of rows of the given matrix A, different from the one in part a).
   c) Is A invertible? Justify your answer with reference to VIT.

$$A = \begin{bmatrix} 2 & 6 & 3 \\ 4 & 12 & 5 \\ 13 & 39 & 17 \end{bmatrix}$$

7. Given the matrices A and B below.

   a) Find a basis for the row space of A and a basis for the row space of B, showing your calculations.
   b) Let U = Span {(1,2, –1, 3), (2,4, –1, 2), (3,6,3, –7)} and let W = Span {1,2, –4,11), (2,4, –5,14)}. Is U = W ? Justify your answer.

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & -4 & 11 \\ 2 & 4 & -5 & 14 \end{bmatrix}$$

Tut 08

Maths

Q1 $B = \{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$

$\bar{v}_1 = (0,1,1)$

$\bar{v}_2 = (1,1,1)$

$\bar{v}_1 = c_1\bar{e}_1 + c_2\bar{e}_2 + c_3\bar{e}_3$

$= 0\bar{e}_1 + 1\bar{e}_2 + 1\bar{e}_3$

$\Rightarrow \bar{e}_2 = \bar{v}_1 - \bar{e}_3$ — ①

$\bar{v}_2 = c_1\bar{e}_1 + c_2\bar{e}_2 + c_3\bar{e}_3$

$= c_1\bar{e}_1 + c_2(\bar{v}_1 - \bar{e}_3) + c_3\bar{e}_3$

~~$= c_1 e_1 + c_2(v_1 - e_3) + c_3 e_3$~~

$\bar{v}_2 = \bar{e}_1 + (\bar{v}_1 - \bar{e}_3) + \bar{e}_3$

$\bar{v}_2 = e_1 + \bar{v}_1$

$\Rightarrow e_1 = \bar{v}_2 - \bar{v}_1$ — ②

$\Rightarrow$ ~~$0 \; c_1\bar{e}_1$~~

$\because \bar{0} = c_1\bar{e}_1 + c_2\bar{e}_2 + c_3\bar{e}_3$ (for all $c_i = 0$)

$= c_1(\bar{v}_1 - \bar{v}_2) + c_2(\bar{v}_2 - e_3) + c_3\bar{e}_3$

$\bar{0} = c_1\bar{v}_2 + (c_2 - c_1)\bar{v}_1 + (c_3 - c_2)\bar{e}_3$ (Is also only true for all coff $= 0$)

$\Rightarrow C = \{\bar{v}_1, \bar{v}_2, \bar{e}_3\}$ is a basis for $\mathbb{R}^3$

Q2 Let $\bar{v}_s = (0,1,0)$

~~Let $c_3\bar{v}$~~,

Let $c\bar{v}_s = c_1\bar{u} + c_2\bar{w}$

which is true ~~for~~ only for $c = c_1 = c_2 = 0$

$\Rightarrow \bar{v}$ is not in span S but in $\mathbb{R}^3$

$\Rightarrow$ By Prop 14 & 15

Let ~~Set~~ $R = \{\bar{u}, \bar{v}, \bar{w}\}$

$\Rightarrow$ Span $R = \mathbb{R}^3$

& $\{\bar{u}, \bar{v}, \bar{w}\}$ are linearly independent

$\Rightarrow$ ~~the~~ set R is a basis for $\mathbb{R}^3$.

Q3 $\because$ W_1 & W_2 are subspaces of V

$\Rightarrow W_1 + W_2$ is also a subspace of V — ①

& $\because$ $W_1 \cap W_2 = \{0\}$ — ②

$\Rightarrow$ By ① & ②

$\dim(W_1 + W_2) \leq \dim(V)$ (Prop 12)

$\Rightarrow \dim(W_1) + \dim(W_2) \leq \dim(V)$ (By ② & Prop 19)

$\Rightarrow 2n - 2 \leq n$

$n \leq 2$ — ③

$\Rightarrow$ Case 1: $n=0$

$\Rightarrow \dim(W_1) = \dim(W_2) = n-1 = -1$

which is not possible

Case 2: $n=1$

$\Rightarrow \dim(W_1) = \dim(W_2) = 1-1 = 0$

$\Rightarrow W_1 = W_2 = \{0\}$

but W_1 & W_2 are distinct

$\Rightarrow$ not possible

Case 3: $n=2$

$\Rightarrow \dim(W_1) + \dim(W_2) = 2-1 = 1$

$\Rightarrow n=2$

Q4 [$\Rightarrow$] Suppose $V = U \oplus W$. Then by definition, every vector $\bar{v} \in V$ is uniquely expressible as $\bar{v} = \bar{u} + \bar{w}$, with $\bar{u} \in U$ & $\bar{w} \in W$.

It only remains to show that $U \cap W = \{\bar{0}\}$.
Let us suppose $\bar{x} \in U \cap W$.

$\Rightarrow \bar{x} = \bar{x} + \bar{0}$, where $\bar{x} \in U$ & $\bar{0} \in W$
and $\bar{x} = \bar{0} + \bar{x}$, where $\bar{0} \in U$ & $\bar{x} \in W$

$\Rightarrow$ By uniqueness of expression, $\bar{x} = \bar{0}$ as required.

[$\Leftarrow$] Suppose $V = U + W$ with $U \cap W = \{\bar{0}\}$
we need to show that every $\bar{v} \in V$ is uniquely expressible as $\bar{v} = \bar{u} + \bar{w}$ with $\bar{u} \in U$ & $\bar{w} \in W$.

Let $\bar{v} \in V$. Since, $V = U + W$, $\bar{v}$ is expressible as $\bar{v} = \bar{u} + \bar{w}$

Suppose the expression is not unique, i.e., we also have $\bar{v} = \bar{u}_1 + \bar{w}_1$. ~~with $\bar{u}_1 \in$~~

Subtracting, $\bar{0} = \bar{v} - \bar{v} = (\bar{u} + \bar{w}) - (\bar{u}_1 + \bar{w}_1)$
$\Rightarrow \bar{w} - \bar{w}_1 = \bar{u}_1 - \bar{u}$ — (1)

The vectors on LHS of ① is in U & the vectors on RHS of ① is in W, i.e.,
each one of them is in $U \cap W = \{\vec{0}\}$.

Q5 a) $\because$ W_1 is the set of vectors of the form $(x, y, 0)$

$\Rightarrow W_1 = \text{Span}\{\bar{e}_1, \bar{e}_2\}$

$\because$ W_1 is a span of 2 linearly independent vectors W_1 is a subspace of $\mathbb{R}^3$

where
$\bar{e}_1 = (1, 0, 0)$
$\bar{e}_2 = (0, 1, 0)$
$\bar{e}_3 = (0, 0, 1)$

Similarly for W_2
$W_2 = \text{Span}\{\bar{e}_2, \bar{e}_3\}$
$\Rightarrow W_1$ is a subspace of $\mathbb{R}^3$

b) $\because W_1 = \text{Span}\{\bar{e}_1, \bar{e}_2\}$
$\Rightarrow \dim W_1 = 2$
& $W_2 = \text{Span}\{\bar{e}_2, \bar{e}_3\}$
$\Rightarrow \dim W_2 = 2$

$W_1 \cap W_2 = \text{Span}\{\bar{e}_2\}$
$\Rightarrow \dim(W_1 \cap W_2) = 1$

$\Rightarrow \dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$ (Prop 19)
$= 2 + 2 - 1 = 3$

c) By ~~Prop 20~~ direct sum definition
$\mathbb{R}^3 = W_1 \oplus U_1$ ~~=~~ $= W_1 \oplus U_2$ is ~~possible for~~ only ~~possible~~ ~~for~~ uniquely expressible

$\Rightarrow U_1 = U_2 = U$ can't be distinct.

$\Rightarrow \mathbb{R}^3 = W_1 \oplus U$

Where $U = \text{span}\{\vec{e}_3\}$

Q6 a) $A = \begin{bmatrix} 2 & 6 & 3 \\ 4 & 12 & 5 \\ 13 & 39 & 17 \end{bmatrix} \xrightarrow[R_3 \to R_3 - \frac{13}{2}R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 2 & 6 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & -5/2 \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 - \frac{5}{2}R_2} \begin{bmatrix} 2 & 6 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_2 \to -1R_2]{R_1 \to R_1/2} \begin{bmatrix} 1 & 3 & 3/2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{R_1 \to R_1 - \frac{3}{2}R_2} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = R$

$\Rightarrow$ ~~Basis~~ Nul $A = \text{Span}\{(3,0,0)\}$

$\Rightarrow$ Basis of Nul $A = \{(3,0,0)\}$

$\Rightarrow$ Col $A = \text{Span}\{(1,0,0), (0,1,0)\}$

$\Rightarrow$ Basis of Col $A = \{(1,0,0), (0,1,0)\}$

$\Rightarrow$ Row $A = \text{Span}\{(1,3,0), (0,0,1)\}$

$\Rightarrow$ Basis of Row $A = \{(1,3,0), (0,0,1)\}$

b) $A^T = \begin{bmatrix} 2 & 4 & 13 \\ 6 & 12 & 39 \\ 3 & 5 & 17 \end{bmatrix}$

$\because$ Row A = Col A^T

$\Rightarrow A^T = \begin{bmatrix} 2 & 4 & 13 \\ 6 & 12 & 39 \\ 3 & 5 & 17 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3/2 R_1]{R_2 \to R_2 - 3R_1} \begin{bmatrix} 2 & 4 & 13 \\ 0 & 0 & 0 \\ 0 & -1 & -5/2 \end{bmatrix}$

$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 2 & 4 & 13 \\ 0 & -1 & -5/2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_2 \to (-1)R_2]{R_1 \to R_1/2} \begin{bmatrix} 1 & 2 & 13/2 \\ 0 & 1 & 5/2 \\ 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 1 & 5/2 \\ 0 & 0 & 0 \end{bmatrix} = R'$

$\Rightarrow$ Row A^* = Col A^T = Span$\{(1,0,0), (0,1,0)\}$

$\Rightarrow$ Basis of Row A = $\{(1,0,0), (0,1,0)\}$

c) $\because$ RREF of $A \neq I$

$\Rightarrow$ ~~By VIT~~ A is not row equivalent

$\Rightarrow$ ~~A is~~ By VIT A is not invertible.

Q7 a) $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & 7 \end{bmatrix}$

$\xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 6 & -2 \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 14 \end{bmatrix}$

$\xrightarrow[R_3 \to R_{3/14}]{R_2 \to R_{2/3}} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & -8/3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_1 \to R_1 - 3R_3]{R_2 \to R_2 + 8/3 R_3} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\xrightarrow{R_1 \to R_2 + R_1} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\rightarrow$ Basis of Row A = $\{(1,2,0,0), (0,0,1,0), (0,0,0,1)\}$

$$B = \begin{bmatrix} 1 & 2 & -4 & 11 \\ 2 & 4 & -5 & 14 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -4 & 11 \\ 0 & 0 & 3 & -8 \end{bmatrix} \xrightarrow{R_2 \to R_2/3} \begin{bmatrix} 1 & 2 & -4 & 11 \\ 0 & 0 & 1 & -8/3 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 + 4R_2} \begin{bmatrix} 1 & 2 & 0 & 1/3 \\ 0 & 0 & 1 & -8/3 \end{bmatrix}$$

Basis of Row $B = \{(1, 2, 0, 1/3), (0, 0, 1, -8/3)\}$

b)