MTH100 SECTION B MATHS I Monsoon Semester – 2021-22

Tutorial Exercise for Week Commencing Monday 20220214.

1. Given the following vectors in $\mathbb{R}^3$: $\mathbf{u} = (1,3,5)$, $\mathbf{v} = (1,4,6)$, $\mathbf{w} = (2, -1, 3)$ and $\mathbf{b} = (6,5,17)$.
   a) Does $\mathbf{b} \in W = \text{span}\ \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$
   b) If the answer to b) is yes, express $\mathbf{b}$ as a linear combination of $\mathbf{u},\mathbf{v},\mathbf{w}$.

2. Let U and W be two subspaces of the vector space V. Show that $U \cap W$ is also a subspace of V.

3. In the following is W a subspace of V? Base field is $\mathbb{R}$ in all. Justify your answer.
   a. $V = \mathbb{R}_n[t]$ = vector space of polynomials of degree $\leq n$, $W = \{p(t) \in V: \deg p(t) = n\} \cup \{\mathbf{0}(t)\}$. Here $\mathbf{0}(t)$ indicates the zero polynomial.
   b. $V = \mathbb{R}^3$, $W = \{(x,y,z): x, y, z \in \mathbb{Q}\}$.
   c. $V = \mathbb{R}^3$, $W = \{(x,y,z): xy = 0\}$.
   d. $V = \mathbb{R}^3$, $W = \{(x,y,z): x^2 + y^4 + z^6 = 0\}$

4. Consider the space V of all 2×2 matrices over R. Which of the following sets of matrices A in V are subspaces of V? Justify (prove) your answers.
   a. All upper triangular matrices
   b. All A such that AB = BA where B is some fixed matrix in V
   c. All A such that $BA = \mathbf{0}$ where B is some fixed matrix in V
   d. Would the above results hold for all $n\times n$ matrices where n is a general positive integer.

5. Let $V = \{x \in \mathbb{R} : x > 0\}$. Define addition for V by $x \oplus y = xy$, and scalar multiplication by any $\alpha \in \mathbb{R}$ by $\alpha * x = x^{\alpha}$.

   (a) (7 marks) Verify the closure axioms, the commutative, zero and inverse properties for addition, and the property $1 * x = x$ for all $x \in V$.
   (***Remark***: *V is in fact a vector space over the field* $\mathbb{R}$. *However, you need not verify the other properties of a vector space.)*
   (b) (3 marks) Is V a subspace of $\mathbb{R}$ regarded as a vector space over itself (YES/NO) ? Justify your answer clearly.

   (*This question was given as an exam problem for a previous batch*.)

Remark: Remaining problems relate to linear dependence/independence. You may try them after Monday's lecture.

6. Prove Remark 6 related to linear dependence/independence : Any list which contains a linearly dependent list is linearly dependent.

7. Prove Remark 7 related to linear dependence/independence : Any subset of a linearly independent set is linearly independent .

8. Determine whether the given matrices in the vector space $R^{2\times 2}$ are linearly dependent or linearly independent.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

9. In the vector space $V = C[0, 2\pi]$, determine whether the given vectors (i.e. functions) are linearly dependent or linearly independent :

   $f_1(x) = 1$, $f_2(x) = \sin(x)$, $f_3(x) = \sin(2x)$.

   (You must justify your answer.)

Tut 6

Q1 a) Consider the non-homo system

$[\bar{u}\ \bar{v}\ \bar{w}]\bar{x} = \bar{b}$

$\Rightarrow$ Augmented matrix $A = \left[\begin{array}{ccc:c} 1 & 1 & 2 & 6 \\ 3 & 4 & -1 & 5 \\ 5 & 6 & 3 & 17 \end{array}\right]$

$\xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 - 3R_1} \left[\begin{array}{ccc:c} 1 & 1 & 2 & 6 \\ 0 & 1 & -7 & -13 \\ 0 & 1 & -7 & -13 \end{array}\right] \xrightarrow{R_3 \to R_3 - R_2} \left[\begin{array}{ccc:c} 1 & 1 & 2 & 6 \\ 0 & 1 & -7 & -13 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc:c} 1 & 0 & 9 & 19 \\ 0 & 1 & -7 & -13 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\Rightarrow x_1 + 9x_3 = 19 \quad \Rightarrow x_1 = 19 - 9x_3$

$x_2 - 7x_3 = -13 \quad \Rightarrow x_2 = 7x_3 - 13$

$\Rightarrow (19 - 9x_3)\bar{u} + (7x_3 - 13)\bar{v} + x_3\bar{w} = \bar{b}$ ①

$\therefore \bar{b}$ can be represented as a linear function of $\bar{u}, \bar{v}$ & $\bar{w}$

$\Rightarrow \bar{b} \in \text{Span}\{\bar{u}, \bar{v}, \bar{w}\}$

b) One combination can be by putting $x_3 = 2$

$\Rightarrow \bar{u} + \bar{v} + 2\bar{w} = \bar{b}$

In general

$\bar{b} = (19 - 9a)\bar{u} + (7a - 13)\bar{v} + a\bar{w} = \bar{b}$, where $a \in \mathbb{R}$.

Q2 Let $\bar{a}, \bar{b} \in U \cap W$

$\Rightarrow \bar{a}, \bar{b} \in U$ & $\bar{a}, \bar{b} \in W$

$\Rightarrow \bar{a} + c\bar{b} \in U$ & $\bar{a} + c\bar{b} \in W$ ($\because$ U & W are subspaces) ~~(Inverse of Proposition 9)~~

$\Rightarrow \bar{a} + c\bar{b} \in U \cap W$

$\Rightarrow$ By Proposition 9

~~$\bar{a}, \bar{b}$~~ $U \cap W$ is a subspace of V

Q3 a) $W = \{ p(t) \in V : \deg(p(t)) = n \}$

Let $p_1(t) = a_1 + a_2 t + \cdots + a_n t^n$

& $p_2(t) = b_1 + b_2 t + \cdots + (-a_n) t^n$

$\Rightarrow p_1(t) + p_2(t) \notin W$ ($\because \deg(p_1(t) + p_2(t)) \neq n$)

$\Rightarrow$ W is not a subspace

b) $W = \{ (x, y, z) : x, y, z \in \mathbb{Q} \}$ & $V = \mathbb{R}^3$

$\because W \subseteq \mathbb{R}^3 \Rightarrow W \leq V$

Let $x = y = z = 0$ & $\because 0 \in \mathbb{Q}$

$\Rightarrow (0,0,0)$ ~~$\in \mathbb{Q}$~~ $\in W$ — ①

Let $x, y, z \in \mathbb{Q}$

~~$\Rightarrow 2x, 2y, 2z \in \mathbb{Q}$~~ $\Rightarrow cx, cy, cz \notin \mathbb{Q}$

$\Rightarrow c(x, y, z) \notin W$

$\Rightarrow$ W is not a subspace of V

c) $V = \mathbb{R}^3$ & $W = \{(x, y, z) : xy = 0\}$

$\Rightarrow$ either x or $y = 0$

~~$\because$ $(0,0,0)$ satisfy $xy = 0$~~

$\because$ $0 \cdot 0 = 0$ &

$\Rightarrow (0,0,0) \in W$

Let (x_1, y_1, z_1) & $(x_2, y_2, z_2) \in W$

$\Rightarrow \therefore (x_1, y_1, z_1) + (x_2, y_2, z_2)$

$= (x_1 + x_2, y_1 + y_2, z_1 + z_2) \notin W$ (As $x_1 + x_2$ & $y_1 + y_2$ might not be zero)

$\Rightarrow$ W is not a subspace of V

d) $W = \{(x, y, z) : x^2 + y^4 + z^6 = 0\}$

$\because$ All the powers in the condition are even

$\Rightarrow$ The eq^n is satisfied only for $x = y = z = 0$

$\Rightarrow W = \{(0,0,0)\}$

$\Rightarrow$ W is a subspace of V but not a proper subspace

As $\{\bar{0}\}$ & $\{V\}$ are always subspaces of V.

Q4 $V = R^{2\times 2}$

a) Let $A, B \in V$ & are all upper Δar matrix
Let W be subset of V such that all elements are upper Δar matrices

$\because$ Sum of upper Δar matrices is also upper Δar matrices
$A+B \in W$ — ①

$\because$ 0 matrix is also upper Δar matrix ②

& Let c be a scalar
$\Rightarrow cA \in W$
$\Rightarrow$ By proposition 8
W is a vector subspace.

b) ~~AB~~ Let W be subset of V such that
$AB = BA$ where B is fixed.
$\because BO = OB = 0$
$\Rightarrow 0 \in W$ — ①
Let $A, C \in W$
~~A+C~~ $(A+C)B = AB + CB = BA + BC$ (given)
$= B(A+C)$
$\Rightarrow A+C \in W$ ②

Let c be a scalar
$\because AB = BA$
$cAB = cBA = B(cA)$ — ③
$\Rightarrow$ By ①, ②, & ③ & Prop 8.
W is a subspace of V.

c) Let $W = \{A : BA = 0\}$ be a subset of V.

$\because B0 = 0$

$\Rightarrow$ ~~A=~~ W contains 0 elem matrix ①

Let $A, C \in W$

$\Rightarrow$ ~~BC~~ $BA = 0$ & $BC = 0$

$\Rightarrow BA + BC = 0$

$B(A+C) = 0$ —②

~~$\Rightarrow A+$~~

Let c be a scalar

~~$\Rightarrow cA$~~

$\because BA = 0$

$cBA = 0 \cdot c$

$\Rightarrow B(cA) = 0$ —③

$\Rightarrow$

$\Rightarrow$ By ①, ② & ③ & Prop. 8

W is a ~~super~~ subspace of V.

d) Yes, since in the above parts we didn't used the fact that the elements were 2×2.

$\Rightarrow$ These will be true for general $n \times n$.

Q5 $V = \{x \in \mathbb{R} : x > 0\}$

$x + y = xy$

$\alpha^* x = x^\alpha$

a) ~~Let this~~

Closure axioms

Let $\vec{x}, \vec{y} \in V$

$\Rightarrow$ ~~$x+y$~~ $\vec{x}, \vec{y} > 0$

~~$\Rightarrow x+y>0$~~ $\Rightarrow \vec{x}\vec{y} > 0 \Rightarrow \vec{x} \oplus \vec{y} > 0$

$\Rightarrow (\vec{x}+\vec{y}) \in V$, — (A)

Let c be a scale & $\vec{x} \in V$

~~Let $x \in V$~~

$\Rightarrow \vec{x} > 0$

$\Rightarrow c\vec{x} = x^c > 0$ (B) ($\because$ any power of +ve number is

~~$\Rightarrow$ By (A) & (B)~~

Closure is proved

Addition

Let $\vec{x}, \vec{y} \in V$

$\because \vec{x} \oplus \vec{y} = \vec{y} \oplus \vec{x} = \vec{x}\vec{y} = \vec{y}\vec{x}$ ($\because x, y$ also $\in \mathbb{R}$)

$\Rightarrow$ commutation prop is hold

~~Let y be a zero element~~

~~$\Rightarrow x + y = x = x + 0$~~

~~$\Rightarrow$ By cancellation law~~

~~$y = 0$~~

~~But $y > 0$~~

~~$\Rightarrow$ there is no zero element in V~~ — (D)

~~Let $x, y \in V$ & y be inverse of x~~

~~$\Rightarrow x+y=0$~~

~~$\Rightarrow y=-x$~~

~~But $x, y>0$~~

~~$\Rightarrow$ Elements of V aren't commutative~~

Multiplication

Let $\tilde{x}, \tilde{y} \in V$ & y is the zero element

$\Rightarrow \tilde{x} \oplus \tilde{y} = \tilde{x}\tilde{y} = \tilde{x}$

~~for y~~

which is true for $\tilde{y}=1>0$

~~$\Rightarrow$ Commutative law holds true for V~~

$\Rightarrow$ There is a zero element for V — C

~~Let $x, y \in V$ & y be inverse of x~~

~~$\Rightarrow x \oplus y = xy$~~ -

Let $\tilde{x}, \tilde{y} \in V$ & $\tilde{y}$ be inverse of $\tilde{x}$

$\Rightarrow \tilde{x} \oplus \tilde{y} = \tilde{x}\tilde{y}$ = zero element = 1

$\Rightarrow xy = 1$

$\Rightarrow x = \frac{1}{y}$

~~x>0~~ & $y>0 \Rightarrow 1/y > 0$

$\Rightarrow$ Inverse prop. holds true for V.

Multiplication

T.P $1 * \tilde{x} = \tilde{x}$

Let $\tilde{x} \in V$

$\Rightarrow 1 * x = x'$ $\quad$ & $\because x$ also $\in R$

$x' = x$

$\Rightarrow 1 * x = x$

$\Rightarrow$ Identity property holds true.

(b) By (A), (B), (C) in part (a) & prop (D)

V is a subspace of $\mathbb{R}$.

Q6 let L_1 be linearly ~~in~~dependent list

& L be a list containing L_1

let $L_1 = [\bar{v}_1, \bar{v}_2, \bar{v}_3, \ldots \bar{v}_p]$

~~$\because L_1$ is l.d $\Rightarrow$~~

& $L = [\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_p, \bar{w}_{p+1}, \bar{w}_{p+2}, \ldots \bar{w}_n]$

$\because L_1$ is l.d

$\Rightarrow c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_p\bar{v}_p = 0$, where not all $c_1, c_2, c_3, \ldots, c_p = 0$.

$\Rightarrow$ for L to be l.d

$$d_1\bar{v}_1 + d_2\bar{v}_2 + \cdots + d_p\bar{v}_p + d_{p+1}\bar{w}_{p+1} + \cdots + d_n\bar{w}_n = 0$$

$\Rightarrow$ Let $d_{p+1} = d_{p+2} = \cdots = d_n = 0$

& $d_1 = c_1$; $d_2 = c_2$; $\ldots$; $d_p = c_p$

$\Rightarrow c_1\bar{v}_1 + c_2\bar{v}_2 + \cdots + c_p\bar{v}_p + 0\bar{w}_{p+1} + \cdots + 0\bar{w}_n \neq 0$

$\because c_1, c_2, \ldots, c_p$ are not all zeros.

$\Rightarrow L$ is l.d.

H.P

Q7 Let L be a l.i list

$\Rightarrow$ & L_1 be subset of ~~L~~ L

~~∵ for~~ $\Rightarrow$ Let $L = [\bar{v}_1, \bar{v}_2, \dots, \bar{v}_p]$

& $L_1 = [\bar{v}_k, \dots, \bar{v}_l]$

∵ L is l.i

$c_1\bar{v}_1 + c_2\bar{v}_2 + \dots + c_p\bar{v}_p = 0$ only when all $c_1, c_2, \dots, c_p = 0$

$\Rightarrow$ ~~for~~ $c_k\bar{v}_k + \dots + c_l\bar{v}_l = 0$ will ~~all~~ also hold true only when $c_k, \dots, c_l$ are all zero.

$\Rightarrow$ ~~L~~ L_1 is also l.i

H.P

Q8 Let $xA + yB + zC = 0$

$$\Rightarrow \begin{bmatrix} x+y+z & x+z \\ x & x+y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} x+y+z &= 0 \\ x+z &= 0 \\ x &= 0 \\ x+y &= 0 \end{aligned}$$

$\Rightarrow x = y = z = 0$ is the only solution

$\Rightarrow$ The ~~system~~ matrices are l.i

Q9 $V = C[0, 2\pi]$

$f_1(x) = 1$, $f_2(x) = \sin(x)$, $f_3(x) = \sin(2x)$

$\Rightarrow \alpha + \beta \sin(x) + \gamma \sin(2x) = 0$ ①

Now there are 3 unknown in ~~the~~ ① , so we need atleast 3 eq^{n}s to solve.

Since, ① must hold ~~true for all~~ identically for all values of x.

$\Rightarrow$ for $x = \cancel{\pi/2}\ \pi/2, \pi, \cancel{3\pi/2}\ \pi/4$

$\Rightarrow \alpha + \beta + \cancel{\gamma} = 0$

$\alpha = 0$

$\cancel{\alpha - \beta = 0}$ $\cancel{\alpha + \beta +}$ $\alpha + \frac{1}{\sqrt{2}}\beta + \gamma = 0$

$\Rightarrow \alpha = \beta = \gamma = 0$

$\Rightarrow$ ① only have trivial sol^n.

so the functions are l.i