MTH100 – Section B **Math 1 (Linear Algebra)** **Monsoon Semester – 2021-22**

Tutorial exercises for Tuesday 18th January 2022

1. Find the solution set in vector form for the homogeneous system A**x** = **0** given A below. NB: A must be row-reduced to an RREF matrix in order to give the solution in standard form.

$$A = \begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{bmatrix}$$

2. a) Row reduce the augmented matrix of the system given below to an RREF matrix:

$$\begin{aligned} 3x + 2y + 7z + 9w &= 7 \\ 6x + 14y + 22z + 15w &= 13 \\ x + 4y + 5z + 2w &= 2 \end{aligned}$$

b) Is the system consistent or inconsistent ? If consistent, express the solution in the form of a vector **u** which is a solution of the non-homogeneous system plus scalar multiples of vector(s) which are solutions of the associated homogeneous system.

3. Repeat Q2, both parts a) and b), for the non-homogeneous system A**x** = **b**, where A and **b** are given below.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \\ 0 & 2 & -2 \end{bmatrix}$$ **b** = (3, –3, 1) taken as a column vector

4. Row reduce the augmented matrix of the system given below to an RREF matrix:

$$\begin{aligned} x + 5y - 3z &= -4 \\ -x - 4y + z &= 3 \\ -2x - 7y &= a \end{aligned}$$

b) For what values of a is the above system consistent and for what values of a is it inconsistent ? Justify your answer.

5. Is it possible for a non-homogeneous system A**x** = **b**, **b** ≠ **0**, to be inconsistent when the associated homogeneous system A**x** = **0** has a unique solution (i.e. only the trivial solution) ? Answer YES or NO, and justify your answer. If YES, construct an example and verify. If NO, explain with reference to suitable propositions and theorems.

6. a) Find the values of x for which the following matrix is an augmented matrix corresponding to a consistent system.

$$A = \begin{bmatrix} 1 & -2 & 1 & x \\ 0 & 5 & -2 & x^2 \\ 4 & -23 & 10 & x^3 \end{bmatrix}$$

b) Find the RREF of the matrix formed by replacing x in A by π.

7. Let A and B be $m \times n$ matrices that are both in reduced row echelon form (RREF), such that $A \neq B$. Suppose that the first $n - 1$ columns of A and B are identical. Assume further that neither A nor B have pivot positions in the last column.
Prove or disprove: There exists a vector $\mathbf{x}$ in $\mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{0}$ but $B\mathbf{x} \neq \mathbf{0}$.

(*Remark: The above problem was set in an MTH100 exam a few years ago. It is rather difficult, and is given as an extra challenge. The solution will not be discussed in the tutorial, but will be considered later.*)

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Tut - 2

Q 1

$$A = \begin{bmatrix} 1 & -2 & 3 & -1 \\ 2 & -1 & 2 & 2 \\ 3 & 1 & 2 & 3 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$A \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -4 & 4 \\ 0 & 7 & -7 & 6 \end{bmatrix}$$

$$\xrightarrow{R_3 = R_3 - 7/3 R_2} \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -4 & 4 \\ 0 & 0 & 7/3 & -10/3 \end{bmatrix}$$

$$\xrightarrow[R_2 = \frac{R_2}{3}]{R_3 = 3/7 R_3} \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 1 & -4/3 & 4/3 \\ 0 & 0 & 1 & -10/7 \end{bmatrix}$$

$$\xrightarrow[R_1 = R_1 - 3R_3]{R_2 = R_2 + 4/3 R_3} \begin{bmatrix} 1 & -2 & 0 & 23/7 \\ 0 & 1 & 0 & -4/7 \\ 0 & 0 & 1 & -10/7 \end{bmatrix}$$

$$\xrightarrow{R_1 = R_1 + 2R_2} \begin{bmatrix} 1 & 0 & 0 & 15/7 \\ 0 & 1 & 0 & -4/7 \\ 0 & 0 & 1 & -10/7 \end{bmatrix} = R$$

$x_1 = -15/7 \, x_4$

$x_2 = 4/7 \, x_4$ $\Rightarrow$ Solⁿ in vector form

$x_3 = 10/7 \, x_4$

$x_4 = x_4$

$$x = x_4 \begin{bmatrix} -15/7 \\ 4/7 \\ 10/7 \\ 1 \end{bmatrix}$$

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Q2 a)

$$[A:b] = \left[\begin{array}{cccc:c} 3 & 2 & 7 & 9 & 7 \\ 6 & 14 & 22 & 15 & 13 \\ 1 & 4 & 5 & 2 & 2 \end{array}\right] = \text{augmented matrix}$$

$$\xrightarrow[R_3 = R_3 - R_1/3]{R_2 = R_2 - 2R_1} \left[\begin{array}{cccc:c} 3 & 2 & 7 & 9 & 7 \\ 0 & 10 & 8 & -3 & -1 \\ 0 & 10/3 & 8/3 & -1 & -1/3 \end{array}\right]$$

$$\xrightarrow{R_3 = R_3 - R_2/3} \left[\begin{array}{cccc:c} 3 & 2 & 7 & 9 & 7 \\ 0 & 10 & 8 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_1 = R_1/3]{R_2 = R_2/10} \left[\begin{array}{cccc:c} 1 & 2/3 & 7/3 & 3 & 7/3 \\ 0 & 1 & 8/10 & -3/10 & -1/10 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 = R_1 - 2/3 R_2} \left[\begin{array}{cccc:c} 1 & 0 & 9/5 & 16/5 & 12/5 \\ 0 & 1 & 8/10 & -3/10 & -1/10 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] = R$$

The system is consistent $\because$ The right most column is not pivot column

$$\Rightarrow \begin{aligned} x_1 &= 12/5 - 9/5 x_3 - 16/5 x_4 \\ x_2 &= -1/10 - 8/10 x_3 + 3/10 x_4 \\ x_3 &= 1x_3 + 0x_4 \\ x_4 &= 0x_3 + 1x_4 \end{aligned}$$

$$x = \begin{bmatrix} 12/5 \\ -1/10 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -9/5 \\ -8/10 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} \cancel{-10}\ -16/5 \\ 3/10 \\ 0 \\ 1 \end{bmatrix}$$

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Q3 a) $[A:b] = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 1 & 2 & -1 & -3 \\ 0 & 2 & -2 & 1 \end{array}\right]$ = Augmented matrix $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$\xrightarrow{R_2 = R_2 - R_1} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 3 & -3 & -6 \\ 0 & 2 & -2 & 1 \end{array}\right]$

$\xrightarrow{R_3 = R_3 - 2/3 R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 3 & -3 & -6 \\ 0 & 0 & 0 & 5 \end{array}\right]$

$\xrightarrow[R_3 = R_3/5]{R_2 = R_2/3} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{array}\right]$

$\xrightarrow[R_1 = R_1 - 3R_3]{R_2 = R_2 + 2R_3} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

$\xrightarrow{R_1 = R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = R$

∵ The rightmost column is a pivot column

∴ The ~~e~~ system is in consistant

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Q4 a) $[A:b] = \left[\begin{array}{ccc:c} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & a \end{array}\right]$

$\xrightarrow[R_3 = R_3 + 2R_1]{R_2 = R_2 + R_1} \left[\begin{array}{ccc:c} 1 & 5 & 3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & a-8 \end{array}\right]$

$\xrightarrow{R_3 = R_3 - 3R_2} \left[\begin{array}{ccc:c} 1 & 5 & 3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & a-5 \end{array}\right]$

$\xrightarrow{R_1 = R_1 - 5R_2} \left[\begin{array}{ccc:c} 1 & 0 & 13 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & a-5 \end{array}\right] = R$

i) The ~~that~~ above system is consistant for

$a - 5 = 0$
$\Rightarrow a = 0$

$\because$ The right most column shouldn't be a pivot column for the system to be consistant.

$\Rightarrow$ For $a \in R - \{5\}$ the system is inconsistent

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Q5 Yes, the non-homo-system $A\bar{x} = \bar{b}$ can be inconsistant if the associated homo system $A\bar{x} = \bar{0}$ has a unique solⁿ.

As the non-homo system is inconsistant if the right most column of RREF of Augmented matrix is a pivot column. ~~which is~~

Eg:- let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ & $b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$A\bar{x} = \bar{0}$ has a unique solⁿ

$\Rightarrow [A : b] = \begin{bmatrix} 1 & 0 & 0 & : & \phi \\ 0 & 1 & 0 & : & \phi \\ 0 & 0 & 1 & : & 1 \\ 0 & 0 & 0 & : & 1 \end{bmatrix}$

but $A\bar{x} = \bar{b}$ is inconsistant.

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Q6 a) $A = \begin{bmatrix} 1 & -2 & 1 & x \\ 0 & 5 & -2 & x^2 \\ 4 & -23 & 10 & x^3 \end{bmatrix}$

$\xrightarrow{R_3 = R_3 - 4R_1} \begin{bmatrix} 1 & -2 & 1 & x \\ 0 & 5 & -2 & x^2 \\ 0 & -15 & 6 & x^3 - 4x \end{bmatrix}$

$\xrightarrow{R_3 = R_3 + 3R_2} \begin{bmatrix} 1 & -2 & 1 & x \\ 0 & 5 & -2 & x^2 \\ 0 & 0 & 0 & x^3 + 3x^2 - 4x \end{bmatrix} = B$

For the system to be consistent

$x^3 + 3x^2 - 4x = 0$

$\Rightarrow x = 0$

& $x^2 + 3x - 4 = 0$

$x^2 + 4x - x - 4 = 0$

$(x+4)(x-1) = 0$

$\Rightarrow$ for $x = 0, -4, 1$ the system is consistent.

b) puttin $x = \pi$ in B

$\begin{bmatrix} 1 & -2 & 1 & \pi \\ 0 & 5 & -2 & \pi^2 \\ 0 & 0 & 0 & \pi^3 + 3\pi^2 - 4\pi \end{bmatrix}$

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Q6 (cont.)

$$\xrightarrow[R_2 = R_2/5]{R_3 = R_3/\pi^3+3\pi^2-4\pi} \begin{bmatrix} 1 & -2 & 1 & \pi \\ 0 & 1\cancel{5} & -2/5 & \pi^2/5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

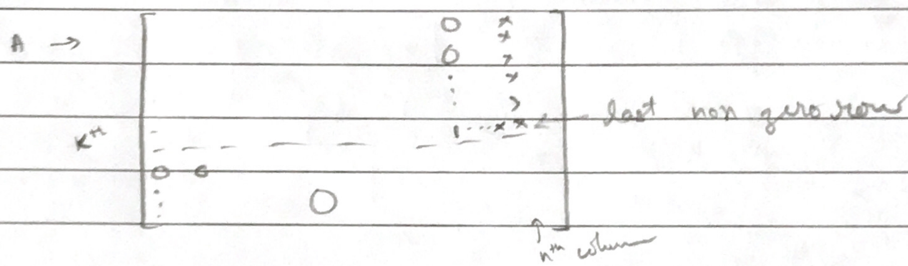
$$\xrightarrow[R_1 = R_1 - \pi R_3]{R_2 = R_2 - \frac{\pi^2}{5} R_3} \begin{bmatrix} 1 & -2 & \phi & 0 \\ 0 & 1 & -2/5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

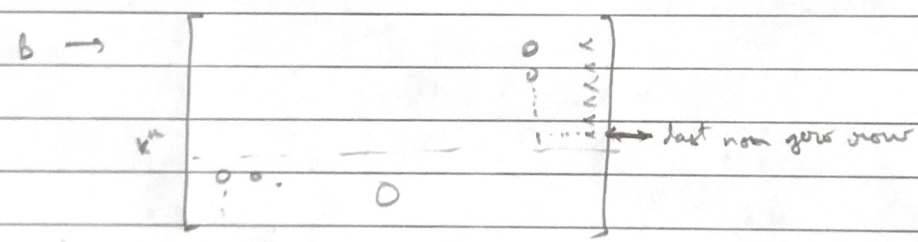
$$\xrightarrow{R_1 = R_1 + 2R_2} \begin{bmatrix} 1 & 0 & 1/5 & 0 \\ 0 & 1 & -2/5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Q7

Q7 Let $A = [a_{ij}]$ & $B = [b_{ij}]$ & Ir as given. Since both are RREF matrices each non-zero row has a leading 1 corresponding to a basic variable. Since the n^{th} column does not have a pivot position, it corresponds to a free variable.

∴ Both A and B have $\leq n-1$ non zero rows. Furthermore, they both have the same no° of non zero rows. We will ignore non-zero rows.





Key pt. → in last non zero row (in both A & B) there is a leading ~~1in~~ 1 in some column before n^{th} column

Since $A \neq B$, let $a_{ij} = b_{ij}$ for any $j < n$, $a_{kn} \neq b_{kn}$ for some row k. ①

Let us now consider the homogeneous system $A\bar{x} = \bar{0}$

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Recall that when we proceeded to solve the system, we did the following:

i) In each eq^n, we kept the basic variables on the LHS, & expressed it in terms of the free variables, which were shifted to RHS

ii) We inserted ~~new~~ dummy eq^n of the form $x_i = x_i$ for each free variable x_i, in the correct order, so as to get exactly n eq^n

iii) We finally take each free variable as a parameter and expressed its coefficients in the n eq^n as a column vector.

Now, we know that x_n for sure is a free variable (there may be some others, but we are not interested in them)

So, on the RHS we get a term like $x_n \bar{v}$, where $\bar{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ is a sol^n of $A\bar{x} = \bar{0}$

Now that

$$v_i = \begin{cases} -a_{in} & \text{if } x_i \text{ is a basic variable} \\ 0 & \text{if } x_i \text{ is a free variable} \\ 1 & \text{if } i = n \end{cases}$$

We claim that

$A\bar{v} = 0$ but $B\bar{v} \neq \bar{0}$ — (2) in a different form(?)

This will prove the desired result

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Proving the claim

Consider the k-th entry in $A\bar{v}$:-

$$a_{k1}v_1 + a_{k2}v_2 + \cdots + a_{kn}v_n$$
$$= a_{k1}v_1 + \cdots + a_{k(n-1)}v_{n-1} + a_{kn} \quad \text{—③}$$
since $v_n = 1$

$= 0$, since $A\bar{v} = \bar{0}$

the k^{th} entry in $B\bar{v}$ is:-

$$b_{k1}v_1 + \cdots + b_{kn}v_n =$$
$$b_{k1}v_1 + \cdots + b_{kn-1}v_{n-1} + b_{kn} \quad \text{—④}$$

Suppose by the way of contradiction (BWOC) that $B\bar{v} = \bar{0}$ also. Then, its k^{th} entry must be 0 and hence eq^n ③ & ④ and recalling that $a_{kj} = b_{kj}$ for $j < n$ we get :- $a_{kn} = b_{kn}$

which contradicts our assumption since, $a_{kn} \neq b_{kn}$ by hypothesis!

Hence, Proved