Q 12 Let $u = (x_1, x_2, \ldots, x_n)$ and $v = (y_1, y_2, \ldots, y_n)$ $\in \mathbb{R}^n$.

{ As $\mathbb{R}^n(\mathbb{R})$ is an inner product space with $\langle u,v\rangle = \sum_{i=1}^{n} x_i y_i$. By Cauchy-Schwarz Inequality we have the result directly }

If $u = 0$ then

$$\langle u,v\rangle = \sum_{i=1}^{n} x_i y_i = 0 \text{, i.e } |x_1 y_1 + x_2 y_2 + \cdots + x_n y_n| = 0$$

and $(x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}} = 0$

$$\Rightarrow |x_1 y_1 + \cdots + x_n y_n| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}} (y_1^2 + y_2^2 + \cdots + y_n^2)^{\frac{1}{2}}$$

Let $u \neq 0$. Then, $(x_1^2 + x_2^2 + \cdots + x_n^2) > 0$, i.e $\langle u,u\rangle > 0$.

Let us choose,

$$w = v - \lambda u, \quad \lambda = \frac{\langle v,u\rangle}{\langle u,u\rangle} \quad \text{--- (1)}$$

Now, $\langle w,w\rangle \geq 0$ (by def^n of inner product space $\mathbb{R}^n(\mathbb{R})$)

$\Rightarrow \langle v - \lambda u, v - \lambda u\rangle \geq 0$

$\Rightarrow \langle v, v - \lambda u\rangle - \lambda\langle u, v - \lambda u\rangle \geq 0$

$\Rightarrow \langle v,v\rangle - \lambda\langle v,u\rangle - \lambda\langle u,v\rangle + \lambda^2\langle u,u\rangle \geq 0$

$\Rightarrow \langle v,v\rangle - \cancel{\lambda^2\langle u,u\rangle} - \lambda\langle u,v\rangle + \cancel{\lambda^2\langle u,u\rangle} \geq 0$ (using (1))

$\Rightarrow \langle v,v\rangle \geq \lambda\langle u,v\rangle = \dfrac{\langle u,v\rangle\langle u,v\rangle}{\langle u,u\rangle}$

$\Rightarrow |\langle u,v\rangle|^2 \leq \langle v,v\rangle\langle u,u\rangle$ $\quad$ { $|x|^2 = x^2$

$\Rightarrow |\langle u,v\rangle| \leq \sqrt{\langle v,v\rangle\langle u,u\rangle}$

$$\Rightarrow |x_1 y_1 + x_2 y_2 + \cdots + x_n y_n| \leq \sqrt{(x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}} (y_1^2 + y_2^2 + \cdots + y_n^2)^{1/2}}$$