Tutorial exercise for the Week Commencing Monday 20220321

1. Let V and W be vector spaces, and let $T: V \to W$ be a linear transformation. Prove:
   a. If T is an isomorphism, then T takes every basis of V to a basis of W.
   b. Conversely if T takes any one basis of V to a basis of W, then T is an isomorphism. (***Hint: Do not try to use Proposition 27; you may use Proposition 26 and the Rank Theorem.)***

2 Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by $T(x_1, x_2, x_3) = (x_1 + x_2, 2x_3 - x_1)$.

a) Find the matrix of T with respect to the standard bases for $\mathbb{R}^3$ and $\mathbb{R}^2$.

b) Verify that $\beta = \{(1,0,-1), (1,1,1), (1,0,0)\}$ is a basis for $\mathbb{R}^3$.

c) Now, determine the matrix of T with respect to the ordered bases β and $\beta' = \{(0,1), (1,0)\}$ for $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively.

3 Find the matrix relative to the standard basis of the linear operator T on R^3 given by: $T(x_1,x_2,x_3) = (x_1 + x_3, x_1 + 2x_2 + x_3, -x_1 + x_2)$.

b) Find the matrix of the same linear operator T relative to the ordered basis $\beta = \{(1,1,1), (1,2,3), (1,3,6)\}$.

4 a) Let $V = F^{n\times n}$ for a fixed $n \geq 2$, and let $P \in V$ be a fixed but arbitrary invertible matrix. Then the mapping $S_P: V \to V$ given by $S_P(A) = PAP^{-1}$ is known as the similarity transformation induced by P. Show that S_P that is an isomorphism. Further, show that S_P is a multiplicative transformation, i.e. $S_P(AB) = S_P(A)\,S_P(B)$ for all $A, B \in V$.

b) Prove or disprove: There exist square matrices (at least 2×2) A and B such that B is row-equivalent to A, but B is not similar to A.

c) Prove or disprove: There exist square matrices (at least 2×2) A and B such that B is similar to A, but B is not row-equivalent to A.

5 In the 2-dimensional plane, i.e. the vector space $\mathbb{R}^2$, let R_θ indicate rotating a vector (regarded as a directed line segment with its tail at the origin) by an angle of θ (radians) in the positive (anti-clock-wise) direction. It is geometrically intuitive that this rotation is a linear transformation. Prove this rigorously by constructing the matrix of the operator $R_\theta: \mathbb{R}^2 \to \mathbb{R}^2$ relative to the standard basis $S = \{\mathbf{e_1}, \mathbf{e_2}\}$. If $\mathbf{v_i} = R_\theta(\mathbf{e_i})$, find the change of basis matrix $P_{S\to\beta}$, where $\beta = \{\mathbf{v_1}, \mathbf{v_2}\}$. Finally, find $[\mathbf{v}]_\beta$ for any arbitrary $\mathbf{v} \in \mathbb{R}^2$.

6 Let $V = \mathbb{R}^{2\times 2}$ = vector space of 2×2 matrices with real entries, and consider the function $U: V \to V$ given by $U(A) = A + A^T$, for all $A \in V$, where A^T indicates the transpose of A.

a) Show that U is a linear operator.
b) Determine the matrix of U with regard to any suitable ordered basis β of V. (***Remark: the choice of ordered basis is left to you, but should be clearly specified***.)
c) Find bases for Ker U and Range U, and hence determine the nullity and rank of U.
d) Show that $Sym_n(\mathbb{R})$, the set of symmetric n×n matrices with real entries is a subspace of $\mathbb{R}^{n\times n}$, and determine its dimension. Briefly explain your answer.

7 Show that a linear transformation $T: V \to W$, where V and W are finite-dimensional with dim V = dim W, is injective if and only if it is surjective.

8 Give an example of a vector space V, and two linear operators $T, U: V \to V$, such that T is surjective but not injective, and U is injective but not surjective.

Tutorial 10 Solutions

Q(1) Let $B = \{v_1, v_2 \dots v_n\}$ be a basis of V.

Claim: $B' = \{T(v_1), T(v_2), \dots, T(v_n)\}$ is a basis of W.

First we show B' is linearly independent.

Let $\alpha_1 T(v_1) + \alpha_2 T(v_2) + \dots + \alpha_n T(v_n) = 0_W$

$\Rightarrow T(\alpha_1 v_1 + \alpha_2 v_2 \dots + \alpha_n v_n) = \cancel{T(0)}\ 0_W$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \in \operatorname{Ker} T = \{0_V\}$ (as T is an isomorphism)

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0_V$

$\Rightarrow \alpha_1 = \alpha_2 = \dots = \alpha_n = 0_F$ (as $\{v_1, - v_n\}$ is linearly independent)

$\Rightarrow \{T(v_1), T(v_2), \dots T(v_n)\}$ is Linearly independent.

Now, next we show that $\operatorname{span}(B') = W$.

$$\begin{aligned} W &= \text{Range } T \quad (\text{as } T \text{ is onto}) \\ &= \{T(v) \mid v \in V\} \\ &= \{T(\alpha_1 v_1 + \dots + \alpha_n v_n) \mid \alpha_i \in F, 1 \le i \le n\} \\ &= \{\alpha_1 T(v_1) + \dots + \alpha_n T(v_n) \mid \alpha_i \in F, 1 \le i \le n\} \\ &\quad (\text{as } T \text{ is a linear trans}) \\ &= \operatorname{Span}\{B'\} \end{aligned}$$

Hence B' is a basis of W.

b) Let T takes $B = \{v_1, v_2 - - v_n\}$ basis of V to a basis $B' = \{w_1, w_2 - w_n\}$ of W. Then, by Porposition 26, there exists a linear transformation T such that $T(v_1) = w_1$, $T(v_2) = w_2$ - - - and $T(v_n) = w_n$.

Now, we will show that T is one-one and onto.

<u>One-one:</u> Let $v \in \mathrm{Ker}(T)$

$$\Rightarrow T(v) = 0$$

$$\Rightarrow T(\alpha_1 v_1 + - - - + \alpha_n v_n) = 0, \text{ where } v = \alpha_1 v_1 + - - + \alpha_n v_n, \ \alpha_i \in F, \ 1 \le i \le n$$

$$\Rightarrow \alpha_1 T(v_1) + - - + \alpha_n T(v_n) = 0$$

$$\Rightarrow \alpha_1 w_1 + \alpha_2 w_2 + - - + \alpha_n w_n = 0 \text{ (as } T \text{ is Linear transformation)}$$

$$\Rightarrow \alpha_i = 0 \ \forall \ 1 \le i \le n \text{ as } \{\cancel{T(v_1), T(v_2)} \ w_1, w_2 - - w_n\} \text{ is linearly independent}\}$$

$$\Rightarrow v = \alpha_1 v_1 + - - + \alpha_n v_n = 0$$

$$\Rightarrow v = 0 \quad \Rightarrow \mathrm{Ker}(T) = \{0\}$$

$$\Rightarrow \mathrm{Nullity}(T) = 0$$

Using Rank Theorem we have

$$\mathrm{rank}(T) + \mathrm{nullity}(T) = \dim V$$

$$\Rightarrow \mathrm{rank}(T) = \dim V$$

$\Rightarrow$ T is onto and hence, T is an isomorphism.

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Q1 a) $T: V \to W$
$V \cong W$

By Prop 26 (25)

If $B = \{\bar{v}_1, \dots, \bar{v}_n\}$ is a basis of V
& $C = [\bar{w}_1, \dots, \bar{w}_n]$ is a list of n vectors in W.
Then there is a unique linear transformation T
such that $T(\bar{v}_1) = \bar{w}_1, \dots, T(\bar{v}_n) = \bar{w}_n$

$\because V \cong W$
Then ~~the~~ ~~fn~~ T is injective and surjective

$\Rightarrow$ For every $\bar{v}_i$ vector in B $\bar{w}_i$ is unique
$\Rightarrow$ C is a set — (1)

$\because$ rank(T) = range of T = dim W (2) ($\because$ ~~Ker~~ T is surjective)

& nullity (T) = 0 (3) ($\because$ Ker T $= \{\bar{0}\}$ by definition of injective)

By (2), (3) & Rank Theorem

$\text{rank}(T) + \text{nullity}(T) = \dim V$
$\dim W + 0 = \dim V$
$\Rightarrow \dim W = \dim V$
$\Rightarrow \dim W = n$
$\Rightarrow$ set C is a basis for W (By proposition 17)

$\Rightarrow$ T takes every basis of V to a basis of W.

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iii) If T takes one basis of ~~V~~ V to a basis of W

$\Rightarrow$ rank(T) = dim W — ①

~~& Also since T takes every vector~~

~~& Also T is injective~~

~~& Also : $T(\bar{v}_1) = \bar{w}_1, \ldots, T(\bar{v}_n) = \bar{w}_n$~~

& Also since T maps basis of V to basis of W

$\Rightarrow T(\bar{v}_1) = \bar{w}_1, \ldots, T(\bar{v}_n) = \bar{w}_n$

Where $B = \{\bar{v}_1, \ldots, \bar{v}_n\}$ is a basis of V
& $C = \{\bar{w}_1, \ldots, \bar{w}_n\}$ is a basis of W

$\Rightarrow$ dim V = dim W — ②

$\therefore$ By ① & ② & Rank theorem

$$\text{Rank}(T) + \text{nullity}(T) = \dim V$$
$$\dim W + \text{nullity}(T) = \dim V$$
$$\Rightarrow \text{nullity}(T) = 0$$
$$\Rightarrow \text{Ker}(T) = \{0\}$$

~~$\therefore$ By~~

$\Rightarrow$ By definition of Ker(T) (remark)

T is injective — ③

$\Rightarrow$ By ① & ③
T is both injective & surjective
Making T an isomorph

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Q2 $T(x_1, x_2, x_3) = (x_1 + x_2,\ 2x_3 - x_1)$

~~$\Rightarrow T(e_1$~~

a) $\Rightarrow T(1,0,0) = (1, -1) = 1e_1 + -1e_2$

$T(0,1,0) = (1, 0) = 1e_1 + 0e_2$

$T(0,0,1) = (0, 2) = 0e_1 + 2e_2$

$$\Rightarrow A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix} = \text{Matrix of } T$$

b) $\beta = \{(1,0,-1), (1,1,1), (1,0,0)\} = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$

$$\text{Let } B = \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \bar{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_1 \to R_1 + R_3]{R_2 \to R_2 - 2R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R$$

$\because$ RREF $= I_3$

$\Rightarrow \beta$ is a ~~not~~ basis of $\mathbb{R}^3$

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i) $\beta' = \{(0,1),(1,0)\} = \{\bar{v}_1, \bar{v}_2\}$

$\Rightarrow T(1,0,-1) = (1,-3) = -3\bar{v}_1 + \bar{v}_2$

$T(1,1,1) = (2,1) = 1\bar{v}_1 + 2\bar{v}_2$

$T(1,0,0) = (1,-1) = -\bar{v}_1 + \bar{v}_2$

$\Rightarrow C = \text{matrix of } T = \begin{bmatrix} -3 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix}_{\beta}$

Q3 $T(x_1,x_2,x_3) = (x_1+x_3,\ x_1+2x_2+x_3,\ -x_1+x_2)$

a) $T(1,0,0) = (1,1,-1) = 1e_1 + 1e_2 - 1e_3$

$T(0,1,0) = (0,2,1) = 0e_1 + 2e_2 + e_3$

$T(0,0,1) = (1,1,0) = 1e_1 + 1e_2 + 0e_3$

$\Rightarrow \text{matrix of } T,\ A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

b) Let Q be the change of base matrix from new to old basis

$\Rightarrow Q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$

$\Rightarrow$ The change of basis matrix

$$P = P_{old \to new} = Q^{-1}$$

$$\Rightarrow [Q : I] = \left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 3 & 6 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 2 & 5 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 - R_3]{R_2 \to R_2 - 2R_3} \left[\begin{array}{ccc:ccc} 1 & 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - R_2} \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -3 & 5 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right] = [I : Q^{-1}]$$

$$\Rightarrow P = Q^{-1} = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

$\Rightarrow$ B, matrix of T wrt new β basis = PAP^{-1}

$$B = \begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -5 & 0 \\ 4 & 8 & 2 \\ -2 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} -6 & -11 & -16 \\ 14 & 26 & 38 \\ -6 & -11 & -17 \end{bmatrix} = \text{matrix of T wrt } \beta$$

Q4 a) $V = F^{n \times n}$

$S_P(A) = PAP^{-1} = B$

$\because$ for a fixed P
& a var A there exist only ~~one B~~ one B.

$\Rightarrow S_P$ is injective

$\because B = PAP^{-1}$
$\Rightarrow$ A & B are similar

b) Let A be any invertible $n \times n$ sq. matrix
$\Rightarrow$ A is row equivalent to I_n — (1)
& let ~~be~~ B be $n \times n$ identity matrix $= I_n$

$\Rightarrow$

$\because$ I_n is ~~sim~~ similar to only I_n

$\because (PI)P^{-1}$ (for any arbitrary, invertible P)
$= PP^{-1} = I$

$\Rightarrow$ A ~~is~~ is not similar to B
but A & B are row equivalent.

Hence, Proved

Q4) By Proposition 30

Let $\alpha = \{e_1, e_2\}$

& $\beta = \{e_2, e_1\}$

$\Rightarrow$ Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ (for any linear operator T)

$\Rightarrow B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

~~$\Rightarrow$ change of matrix~~

$\therefore$ A & B are not row equivalent. ①

But By Prop 30

there exist a change of base matrix $P_{\alpha \to \beta}$

$\Rightarrow$ By solving $P_{\alpha \to \beta} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\Rightarrow B = PAP^{-1}$ ②

By ① & ②

$\Rightarrow$ A, B are not row-equivalent but ~~not~~ are similar.

Q5 $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$

$R_\theta(1,0) = (\cos\theta, \sin\theta)$

$R_\theta(0,1) = \cancel{(\cos\theta, -\sin\theta)}\ (-\sin\theta, \cos\theta)$

$\Rightarrow$ A, matrix of R_θ = $\begin{bmatrix} \cos\theta & \cancel{\cos\theta}\ -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$Q_{B\to S} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = P^{-1}_{S\to B}$

$\Rightarrow [Q_{B\to S} : I]$

$$\left[\begin{array}{cc:cc} \cos\theta & -\sin\theta & 1 & 0 \\ \sin\theta & \cos\theta & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - \tan\theta R_1} \left[\begin{array}{cc:cc} \cos\theta & -\sin\theta & 1 & 0 \\ 0 & \frac{\cos^2\theta + \sin^2\theta}{\cos\theta} & -\tan\theta & 1 \end{array}\right]$$

$$\xrightarrow[R_2 \to R_2 \cos\theta]{R_1 \to R_1/\cos\theta} \left[\begin{array}{cc:cc} 1 & -\tan\theta & \sec\theta & 0 \\ 0 & 1 & -\sin\theta & \cos\theta \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 + \tan\theta R_2} \left[\begin{array}{cc:cc} 1 & 0 & \frac{1-\sin^2\theta}{\cos\theta} & \sin\theta \\ 0 & 1 & -\sin\theta & \cos\theta \end{array}\right] = [I : Q^{-1}] = [I : P]$$

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$$P_{S\to B} = \begin{bmatrix} \frac{1-\sin^2\theta}{\cos\theta} & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Any arbitrary $\bar{v} \in \mathbb{R}^2 = [\bar{v}]_S$

Let $\bar{v} = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\Rightarrow [v]_B = P_{S\to B} [\bar{v}]_S$$

$$= \begin{bmatrix} \frac{1-\sin^2\theta}{\cos\theta} & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$[v]_B = \left(\left(\frac{1-\sin^2\theta}{\cos\theta}\right)x + \sin\theta y, \left(-\sin\theta x + \cos\theta y\right) \right)$$

Q6 b) $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$

$U(E_{11}) = 2E_{11}$

$U(E_{12}) = \cancel{2}E_{12} + E_{21}$

$U(E_{21}) = E_{12} + E_{21}$

$U(E_{22}) = 2E_{22}$

$$[U]_\beta = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

b) ~~Ker~~ Let B be the basis of Ker U

$$\Rightarrow B = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

$\Rightarrow \dim(\text{Ker } U) = 3$

Let C be the basis of Range U

$$C = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$\Rightarrow \dim(\text{Range } U) = 4$

d) $Sym_n(\mathbb{R}) = \{A : A = A^T \ \forall A \in \mathbb{R}^{n \times n}\}$

$$\Rightarrow \text{Basis of } Sym_n(\mathbb{R}) = \left\{ \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & & 0 \\ \vdots & & & \vdots \\ 0 & \cdots & & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & & 0 \\ 1 & 0 & \cdots & & 0 \\ \vdots & & & & \\ 0 & \cdots & & & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & 1 \\ 0 & \cdots & 1 & 0 \end{bmatrix} \right\}$$

$$\Rightarrow \dim(Sym_n(\mathbb{R})) = \frac{n(n-1)}{2}$$

Q7 let $\dim V = \dim W = n$ &
let $B = \{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n\}$ be a basis of V

[$\Rightarrow$]

Suppose T is injective

We claim that $\{T\bar{v}_1, T\bar{v}_2, \ldots, T\bar{v}_n\}$ are linearly independent in W.

For $c_1 T\bar{v}_1 + \ldots + c_n T\bar{v}_n = \bar{0}$
$\Rightarrow T(c_1\bar{v}_1 + \cdots + c_n\bar{v}_n) = \bar{0}$
$\Rightarrow c_1 = \cdots = c_n = 0$ ($\because$ B is linearly independent)

$\therefore C = \{T\bar{v}_1, \ldots, T\bar{v}_n\}$ is a basis of W

if $\bar{w} \in W$, then
$\bar{w} = c_1 T\bar{v}_1 + \cdots + c_n T\bar{v}_n$ for some scalar c_i
$= T(c_1\bar{v}_1 + \cdots + c_n\bar{v}_n)$
$=$ Range (T)
$\Rightarrow$ T is surjective

[$\Leftarrow$] Conversely, Suppose T is surjective
let $C = \{\bar{w}_1, \ldots, \bar{w}_n\}$ be a basis of V. Since T is surjective $\exists$ vectors $\bar{u}_1, \ldots, \bar{u}_n \in V$
s.t. $T(u_i) = w_i$

We claim that $D = \{\bar{u}_1, \ldots, \bar{u}_n\}$ is a basis of V.

For $c_1\bar{u}_1 + c_2\bar{u}_2 + \cdots + c_n\bar{u}_n = \bar{0}$

$T(c_1\bar{u}_1 + c_2\bar{u}_2 + \cdots + c_n\bar{u}_n) = 0$

$c_1\bar{\omega}_1 + \cdots + c_n\bar{\omega}_n = 0$

$\Rightarrow c_1 = \cdots = c_n = 0$ ($\because$ C_is linearly independent)

so if $\bar{v} \in V$, ~~but~~

$\bar{v} = c_1\bar{u}_1 + \cdots + c_n\bar{u}_n$ for $c_i \in F$

$\therefore T(\bar{v}) = 0$

$T(c_1\bar{u}_1 + \cdots + c_n\bar{u}_n) = \bar{0}$

$c_1\bar{\omega}_1 + \cdots + c_n\bar{\omega}_n = \bar{0}$

$c_1 = c_2 = \cdots = c_n = 0$

$\Rightarrow \bar{v} = 0$ as desired

$\Rightarrow \ker T = \{\bar{0}\}$

$\Rightarrow T$ is injective

H.P

Q8 ~~Ex~~

By ques 7, we must search a V, which is not finite dimensional.

Let $V = \mathbb{R}[t]$ = span of polynomials

Let T = differentiation operation

Clearly, T is surjective

$\because$ $p(t) = a_0 + a_1 t + \dots + a_n t^n$

$q(t) = a_0 t + \frac{a_1}{2} t^2 + \dots + \frac{a_n}{n+1} t^{n+1}$

then $T_q(t) = p(t)$

But T is not injective

as $T(1+t) = T(2+t) = 1$