A Not So Simple Full-history Recurrence Relation

$$T(n) = n - 1 + \frac{2}{n} \sum_{i=1}^{n-1} T(i), \text{ (for } n \geq 2); \; T(1) = 0.$$

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- **Idea:** Use the shifting and canceling terms technique, s.t., most of the $T(i)$ terms gets canceled out.

A Not So Simple Full-history Recurrence Relation (Cont.)

Multiplying both sides by n, we get:

$$\begin{aligned} nT(n) &= n(n-1) + 2\sum_{i=1}^{n-1} T(i), \\ (n+1)T(n+1) &= n(n+1) + 2\sum_{i=1}^{n} T(i). \end{aligned}$$

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Therefore,

$$(n+1)T(n+1) - nT(n) = n(n+1) - n(n-1) + 2T(n)$$

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Therefore,

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\Rightarrow \quad T(n+1) &= \frac{n+2}{n+1}T(n) + \frac{2n}{n+1} \\
&\leq \frac{n+2}{n+1}T(n) + 2 \quad \text{(A close approx.)}
\end{aligned}
$$

A Not So Simple Full-history Recurrence Relation (Cont.)

Expanding, we get

$$T(n) \quad \leq \quad 2 + \frac{n+1}{n}\left[2 + \frac{n}{n-1}\left[2 + \frac{n-1}{n-2}\left[\cdots\frac{4}{3}\right]\right]\right]$$

A Not So Simple Full-history Recurrence Relation (Cont.)

Expanding, we get

$$\begin{aligned} T(n) &\leq 2 + \frac{n+1}{n}\left[2 + \frac{n}{n-1}\left[2 + \frac{n-1}{n-2}\left[\cdots\frac{4}{3}\right]\right]\right] \\ &= 2\left[1 + \frac{n+1}{n} + \frac{n+1}{n}\cdot\frac{n}{n-1} + \frac{n+1}{n}\cdot\frac{n}{n-1}\cdot\frac{n-1}{n-2} + \right. \\ &\qquad \left. \cdots + \frac{n+1}{n}\cdot\frac{n}{n-1}\cdot\frac{n-1}{n-2}\cdots\frac{4}{3}\right] \end{aligned}$$

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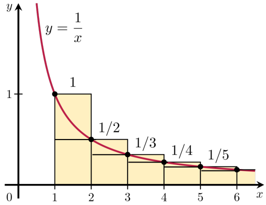
&= 2(n+1)\left[\frac{1}{n+1} + \frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \cdots + \frac{1}{3}\right] \\
&= 2(n+1)(H(n+1) - 1.5);
\end{aligned}
$$

where $H(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$ is the Harmonic series.

Harmonic Series Approximation



$$H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \quad > \quad \int_1^{n+1} \frac{dx}{x} = \ln(n+1) \quad \text{and}$$

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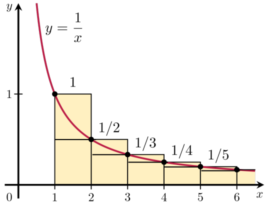

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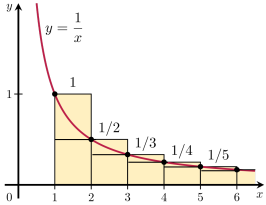

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Combining: $\ln(n+1) < H(n) < 1 + \ln(n)$.

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Let $\delta_n = H(n) - \ln n$.

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i.e., δ_n is monotone decreasing.

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$$\gamma = \lim_{n\to\infty} \delta_n = \lim_{n\to\infty} (H(n) - \ln n).$$

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$\therefore\ H(n) \approx \ln n + \gamma$.

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Using the Harmonic series approximation, we get

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$\therefore$ **Quicksort is indeed quick on the average!!**