Matrix multiplication Analysis

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Algorithms Compared

- Naive
- Strassens' algorithm

Naive Matrix

For two matrices *a* and *b* that can be multiplied

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj} \,.$$

Naive Matrix multiplication

For two matrices of size n x n

Time complexity:

$$O(n^3)$$

Strassens's Method of Matrix Multiplication

Key insight

- Multiplication is computationally more expensive than addition
- Utilise Divide and Conquer

Preparation for Divide and Conquer

Let us break down the problem into subparts as:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \mathbf{B}_{1,1} & \mathbf{B}_{1,2} \\ \mathbf{B}_{2,1} & \mathbf{B}_{2,2} \end{bmatrix}, \mathbf{C} = \begin{bmatrix} \mathbf{C}_{1,1} & \mathbf{C}_{1,2} \\ \mathbf{C}_{2,1} & \mathbf{C}_{2,2} \end{bmatrix}$$

Naive Algorithm

Now, the standard algorithm does the follows:

$$\mathbf{C}_{1,1} = \mathbf{A}_{1,1}\mathbf{B}_{1,1} + \mathbf{A}_{1,2}\mathbf{B}_{2,1}$$
$$\mathbf{C}_{1,2} = \mathbf{A}_{1,1}\mathbf{B}_{1,2} + \mathbf{A}_{1,2}\mathbf{B}_{2,2}$$
$$\mathbf{C}_{2,1} = \mathbf{A}_{2,1}\mathbf{B}_{1,1} + \mathbf{A}_{2,2}\mathbf{B}_{2,1}$$
$$\mathbf{C}_{2,2} = \mathbf{A}_{2,1}\mathbf{B}_{1,2} + \mathbf{A}_{2,2}\mathbf{B}_{2,2}$$

i.e. 8 multiplications, 4 Additions

Strassens's Algorithm

Strassen's algorithm does a neat trick:

$$
\begin{aligned}
\mathbf{M}_1 &:= (\mathbf{A}_{1,1} + \mathbf{A}_{2,2})(\mathbf{B}_{1,1} + \mathbf{B}_{2,2}) \\
\mathbf{M}_2 &:= (\mathbf{A}_{2,1} + \mathbf{A}_{2,2})\mathbf{B}_{1,1} \\
\mathbf{M}_3 &:= \mathbf{A}_{1,1}(\mathbf{B}_{1,2} - \mathbf{B}_{2,2}) \\
\mathbf{M}_4 &:= \mathbf{A}_{2,2}(\mathbf{B}_{2,1} - \mathbf{B}_{1,1}) \\
\mathbf{M}_5 &:= (\mathbf{A}_{1,1} + \mathbf{A}_{1,2})\mathbf{B}_{2,2} \\
\mathbf{M}_6 &:= (\mathbf{A}_{2,1} - \mathbf{A}_{1,1})(\mathbf{B}_{1,1} + \mathbf{B}_{1,2}) \\
\mathbf{M}_7 &:= (\mathbf{A}_{1,2} - \mathbf{A}_{2,2})(\mathbf{B}_{2,1} + \mathbf{B}_{2,2})
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{C}_{1,1} &= \mathbf{M}_1 + \mathbf{M}_4 - \mathbf{M}_5 + \mathbf{M}_7 \\
\mathbf{C}_{1,2} &= \mathbf{M}_3 + \mathbf{M}_5 \\
\mathbf{C}_{2,1} &= \mathbf{M}_2 + \mathbf{M}_4 \\
\mathbf{C}_{2,2} &= \mathbf{M}_1 - \mathbf{M}_2 + \mathbf{M}_3 + \mathbf{M}_6
\end{aligned}
$$

i.e. **7 multiplications**, 18 Additions

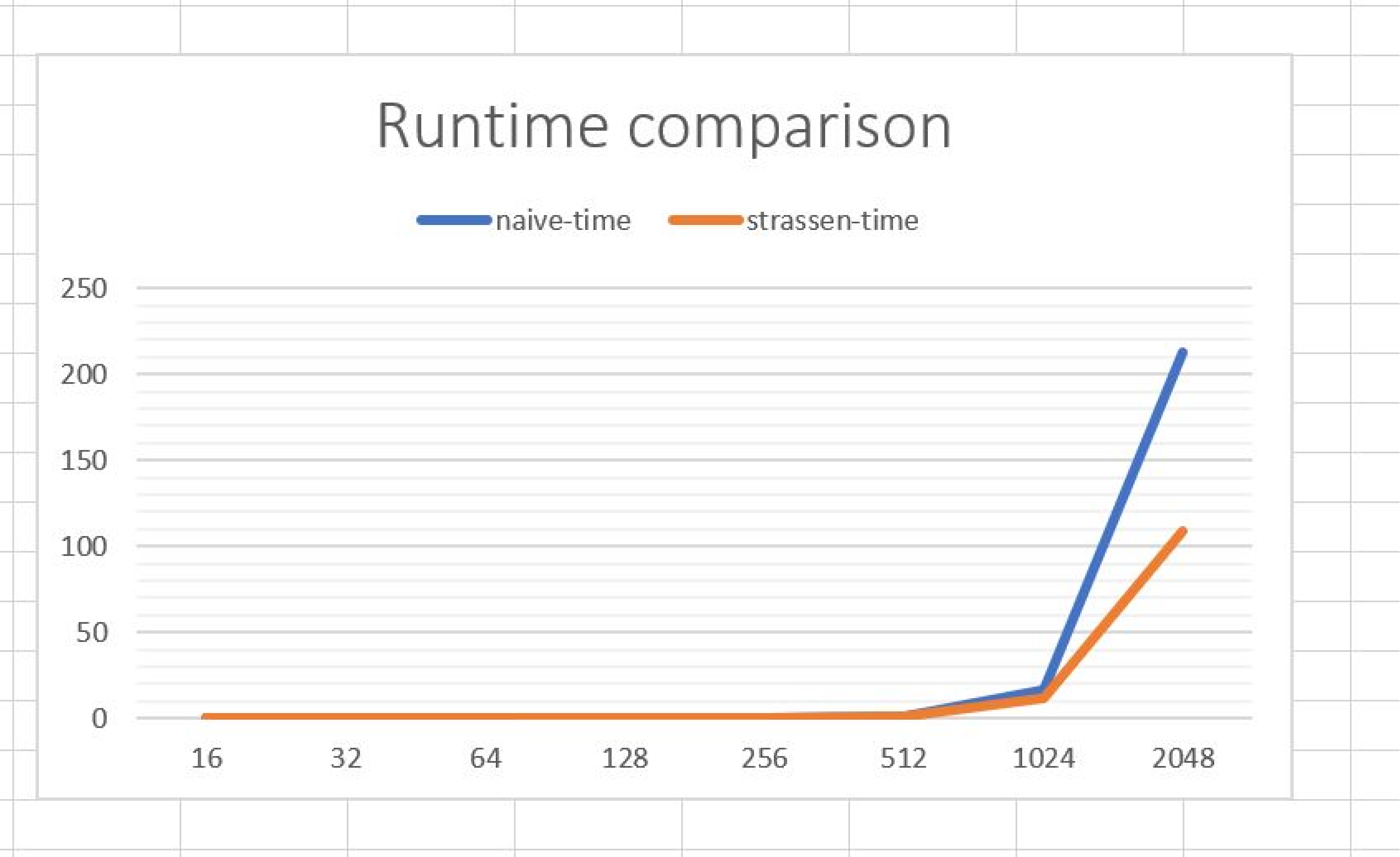
Strassen Algorithm

For two matrices of size n x n

Time complexity:

$$O(n^{log(7)}) \approx O(n^{2.81})$$

Results



Runtime comparison
naive-time
strassen-time
250
200
150
100
50
0
16
32
64
128
256
512
1024
2048

Results

size	naive-time	strassen-time	isEqual
16	0	0	TRUE
32	0	0	TRUE
64	0	0	TRUE
128	0.03125	0.03125	TRUE
256	0.1875	0.25	TRUE
512	1.65625	1.671875	TRUE
1024	16.53125	11.625	TRUE
2048	213.098587	109.159874	TRUE

Some notes

- Current fastest algorithms has performance : "Modified" Coppersmith–Winograd algorithm

  $$O(n^{2.373})$$

- However, algorithms better than strassens have very large overhead, and not used in practice
- The lower bound on Matrix Multiplication is proved to be:

  $$O(n^2)$$

Future work

- Utilise parallelization
- Utilise distributed systems
- Use GPU's