NCERT Math 11.9.2 Q8

EE23BTECH11009 - AROSHISH PRADHAN*

Question: An input voltage in the form of a square wave of frequency $1\ kHz$ is given to a circuit, which results in the output shown schematically below. Which one of the following options is the CORRECT representation of the circuit?

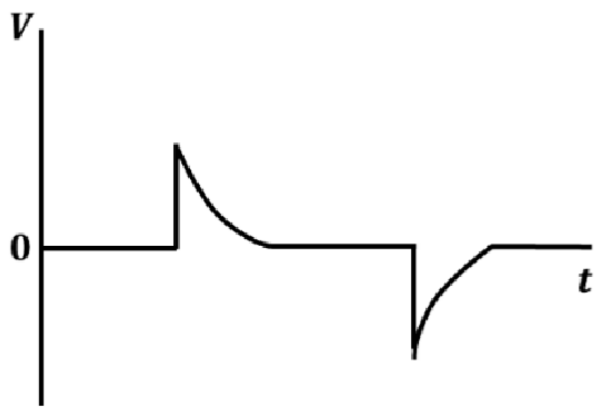
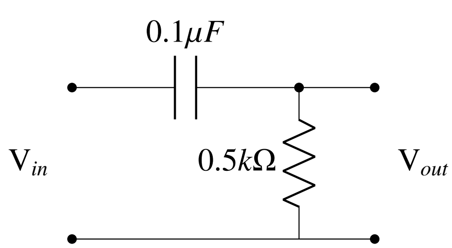
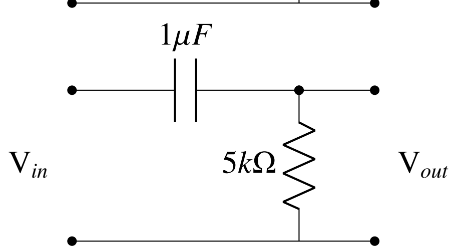


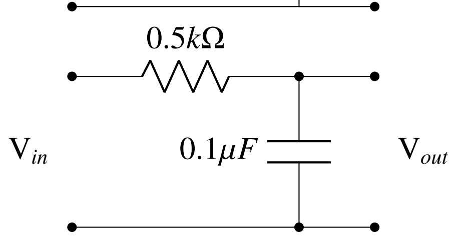
Fig. 1



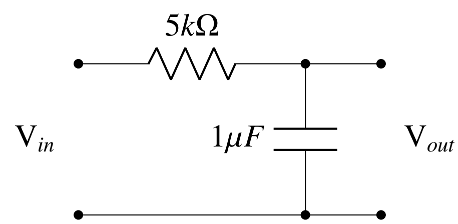
(a)



(b)



(c)



(d)

Solution:

Symbol	Value	Description
$V_{in}(t)$		Input Voltage
$\mathcal{V}_{in}(j\omega)$		Fourier Transform of $V_{in}(t)$
$V_{out}(t)$		Output Voltage
$\mathcal{V}_{out}(j\omega)$		Fourier Transform of $V_{out}(t)$
$I(t)$		Current
$\mathcal{I}(j\omega)$		Fourier Transform of $I(t)$
f	$\frac{\omega}{2\pi} = 1000Hz$	Input Wave Frequency
T	$\frac{2\pi}{\omega} = 10^{-3}s$	Input Wave Time Period
R	(a) $0.5k\Omega$	Resistance
	(b) $5k\Omega$	
C	(a) $0.1\mu F$	Capacitance
	(b) $1\mu F$	
τ	RC	Time Constant
Z	$R + \frac{1}{j\omega C}$	Impedance
$H(j\omega)$	$\frac{V_{out}}{V_{in}}$	General Transfer Function
$H_R(j\omega)$	$\frac{V_{R,out}}{V_{in}}$	Transfer Function for Resistor
$H_C(j\omega)$	$\frac{V_{C,out}}{V_{in}}$	Transfer Function for Capacitor

TABLE I: Given Parameters

Input waveform is a square wave (Fig. 2), so we take its Fourier Transform as shown in Fig. 4

$$V_{in}(t) = 2\left(2\left\lfloor\frac{\left(t - \frac{T}{4}\right)}{T}\right\rfloor - \left\lfloor\frac{2\left(t - \frac{T}{4}\right)}{T}\right\rfloor\right) + 1 \tag{1}$$

Fourier Series Coefficient:

$$c_k = \frac{1}{T}\int_T V_{in}(t)e^{-jk2\pi ft}dt \tag{2}$$

As square wave is even, $\sin(k2\pi ft)$ terms become zero. Cosine coefficients are:

$$a_n = \frac{2}{T}\int_T V_{in}(t)\cos\left(\frac{2\pi nt}{T}\right) \tag{3}$$

$$= \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos(n\pi) \tag{4}$$

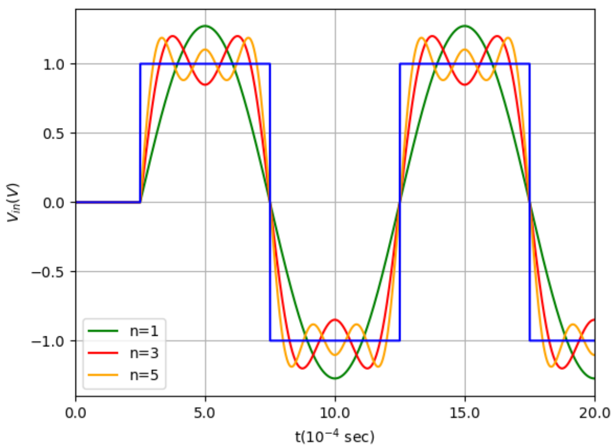


Fig. 2: Input Square Waveform ($V_{in}(t)$)

Fourier Series of $V_{in}(t)$, visualized in Fig. 2:

$$V_{in}(t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T}\right) \tag{5}$$

Taking Fourier Transform of $V_{in}(t)$, plotted in Fig. 4:

$$V_{in}(t) \overset{\mathcal{F}}{\longleftrightarrow} \mathcal{V}_{in}(j\omega) \tag{6}$$

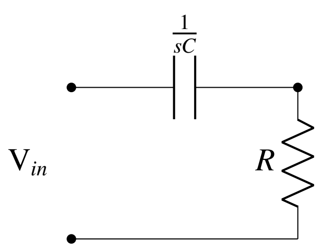


Fig. 3: Series RC Circuit in s-domain

$$s = j\omega \tag{7}$$
$$\implies Z = R + \frac{1}{sC} \tag{8}$$
$$= R + \frac{1}{j\omega C} \tag{9}$$

Applying KVL:

$$\mathcal{V}_{in}(j\omega) = \mathcal{I}(j\omega)\left(R + \frac{1}{j\omega C}\right) \tag{10}$$
$$\implies \mathcal{I}(j\omega) = \frac{\mathcal{V}_{in}(j\omega)}{\left(R + \frac{1}{j\omega C}\right)} \tag{11}$$

$\therefore$ output across R:

$$= R\mathcal{I}(j\omega) \tag{12}$$
$$= \frac{R}{\left(R + \frac{1}{j\omega C}\right)}\mathcal{V}_{in}(j\omega) \tag{13}$$

and output across C:

$$= \frac{1}{j\omega C}\mathcal{I}(j\omega) \tag{14}$$
$$= \frac{\frac{1}{j\omega C}}{\left(R + \frac{1}{j\omega C}\right)}\mathcal{V}_{in}(j\omega) \tag{15}$$

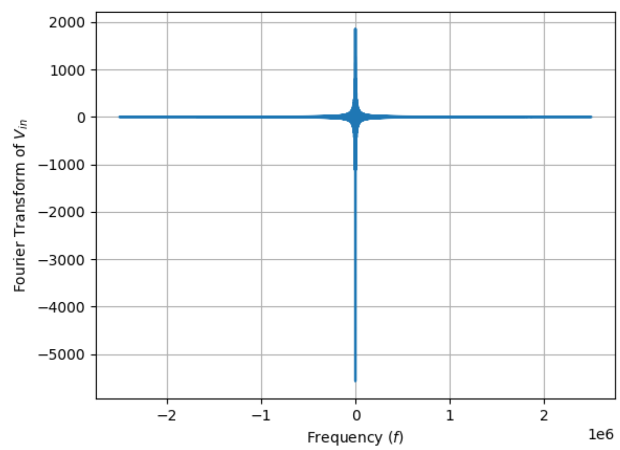


Fig. 4: $\mathcal{V}_{in}(j\omega)$ (Fourier Transform of $V_{in}(t)$)

$\mathcal{V}_{in}(j\omega)$ was input into all four circuits and Inverse Fourier Transform was taken of the response. All responses are plotted below:

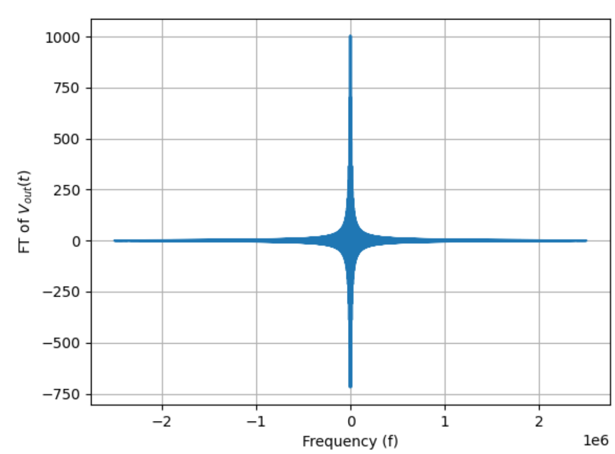


Fig. 5: Opt A: Fourier Transform of $V_{out}(t)$

As Fig. 6 resembles question Fig. 1, option (a) is the correct answer.

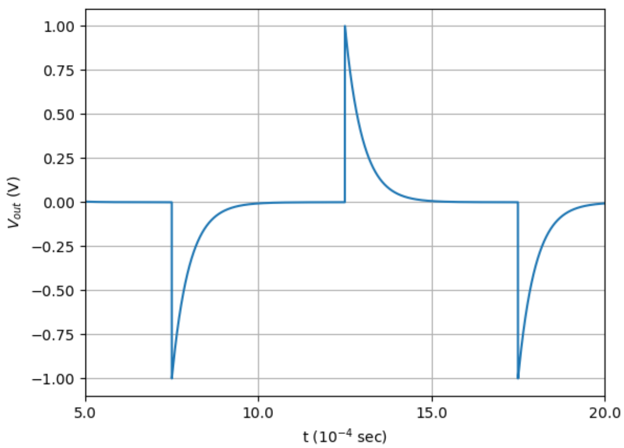


Fig. 6: Opt A: Response

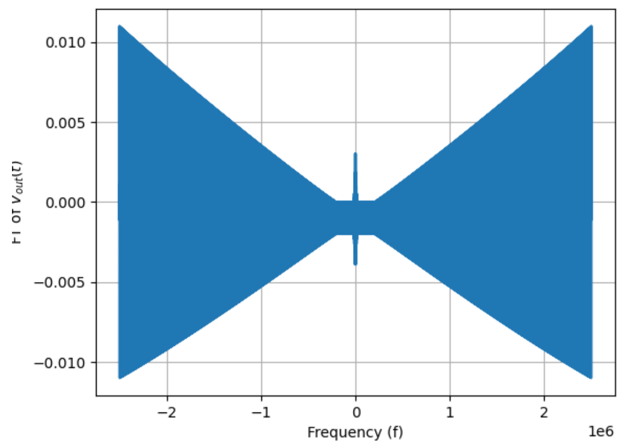


Fig. 9: Opt C: Fourier Transform of $V_{out}(t)$

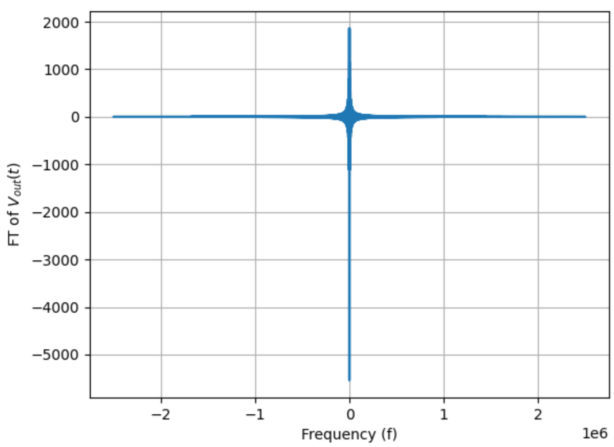


Fig. 7: Opt B: Fourier Transform of $V_{out}(t)$

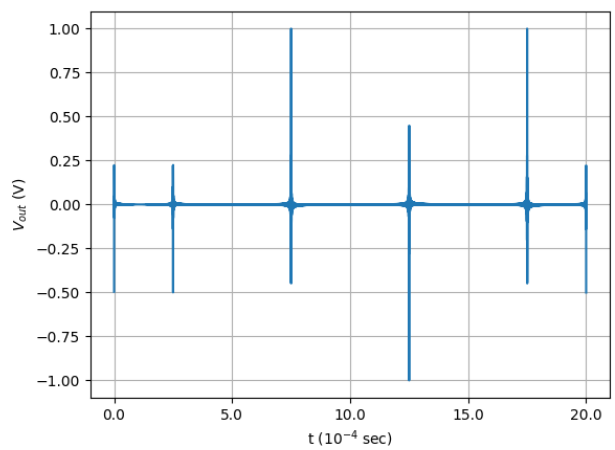


Fig. 10: Opt C: Response

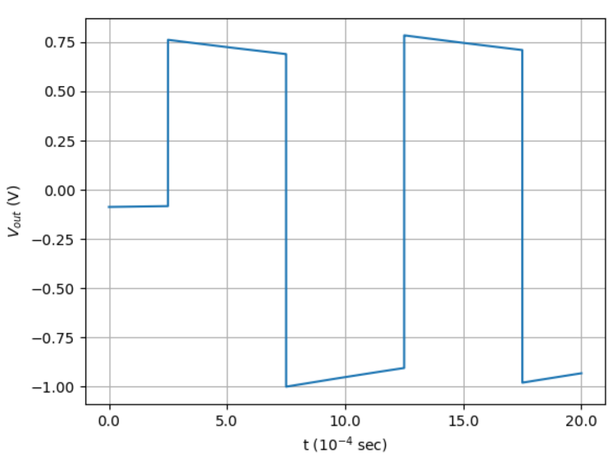


Fig. 8: Opt B: Response

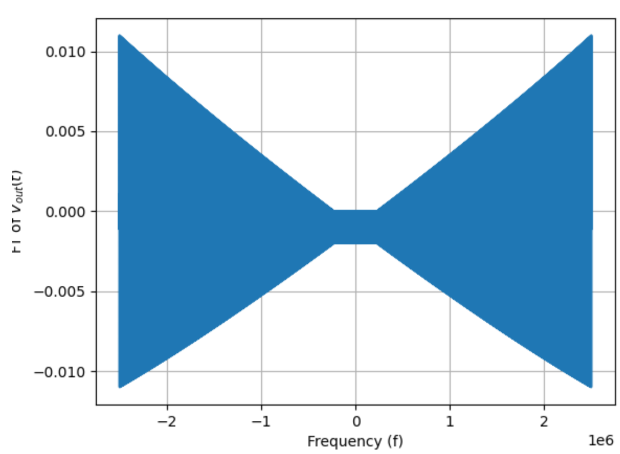


Fig. 11: Opt D: Fourier Transform of $V_{out}(t)$

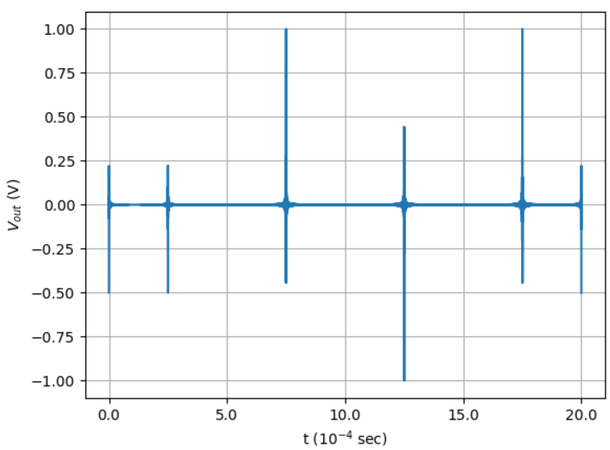


Fig. 12: Opt D: Response