NCERT Physics 12.7 Q21

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Question: Obtain the resonant frequency and Q-factor of a series LCR circuit with L = 3.0 H, C = 27 μF, and R = 7.4 Ω. It is desired to improve the sharpness of the resonance of the circuit by reducing its 'full width at half maximum' by a factor of 2. Suggest a suitable way.

Solution: Given parameters are:

Parameter	Value
L	3.0 H
C	27 μF
R	7.4 Ω

TABLE I: Given Parameters

Resonance Frequency (ω_0) is given by:

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Substituting values of L and C gives:

$$\omega_0 = \frac{1}{\sqrt{3 \cdot 27 \times 10^{-6}}} \tag{1}$$
$$= \frac{10^3}{9}\, s^{-1} \tag{2}$$
$$= 111.12\, s^{-1} \tag{3}$$

Quality Factor (Q) is given by:

$$Q = \frac{1}{R} \cdot \sqrt{\frac{L}{C}}$$

Substituting values of R, L and C gives:

$$Q = \frac{1}{7.4} \cdot \sqrt{\frac{3}{27 \times 10^{-6}}} \tag{4}$$
$$= \frac{10^3}{22.2} \tag{5}$$
$$\approx 45 \tag{6}$$

To reduce the full width at half maximum by a factor of 2, the quality factor needs to be doubled. One way of doing this is to reduce the resistance by a factor of 2.

$$R' = \frac{R}{2} \tag{7}$$
$$= \frac{7.4}{2}\Omega \tag{8}$$
$$= 3.7\Omega \tag{9}$$

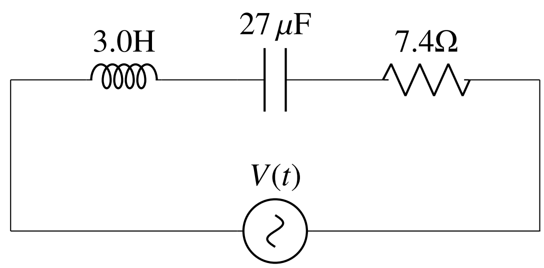

Fig. 1: LCR Circuit

Frequency Response of the Circuit

This is a series LCR circuit, with the elements in series with the voltage source. Applying Kirchhoff's Voltage Law (KVL), we get:

$$V_R + V_L + V_C = V(t) \tag{10}$$

where V_R, V_L and V_C are the voltages across R, L and C respectively and $V(t)$ is the time-varying voltage source.

Substituting,

$$V_R = R \cdot I(t) \tag{11}$$
$$V_L = L\frac{dI(t)}{dt} \tag{12}$$
$$V_C = V(0) + \frac{1}{C}\int_0^t I(\tau)\, d\tau \tag{13}$$

into equation (10), we get:

$$R \cdot I(t) + L\frac{dI(t)}{dt} + V(0) + \frac{1}{C}\int_0^t I(\tau)\, d\tau = V(t) \tag{14}$$

The response of the circuit can be analysed at the transient and steady state by using the Laplace

Transform.

The Laplace Transform $F(s)$ of a function $f(t)$, defined for all real $t > 0$, is defined by

$$\mathcal{L}\{f\} = F(s) = \int_0^\infty f(t)\mathrm{e}^{-st} dt \tag{15}$$

where s is a complex frequency domain parameter, i.e. $s = \alpha + \iota\omega$ $(\alpha, \omega \in \mathbf{R})$

Applying the Laplace Transform to equation (14), we get

$$V(s) = I(s)\left(R + Ls + \frac{1}{sC}\right) \tag{16}$$

$$\Rightarrow I(s) = \frac{V(s)}{\left(R + Ls + \frac{1}{sC}\right)} \tag{17}$$

The term $\dfrac{I(s)}{V(s)}$ is called the Laplace Admittance Y(s).

$$\Rightarrow Y(s) = \frac{I(s)}{V(s)} = \frac{s}{L\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)} \tag{18}$$

We now define two terms: Neper Frequency (α) and Angular Resonance Frequency (ω_0).

Neper Frequency or Attenuation is a measure of how fast the transient response of a circuit will die out after the source has been removed.

For a series LCR circuit,

$$\alpha = \frac{R}{2L} \tag{19}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{20}$$

Equation (18) can then be written as

$$Y(s) = \frac{s}{L\left(s^2 + 2\alpha s + \omega_0^2\right)} \tag{21}$$

The poles of Y(s) are the values of s for which $Y(s) \to \infty$, i.e.

$$s^2 + 2\alpha s + \omega_0^2 = 0 \tag{22}$$

$$\Rightarrow s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \tag{23}$$

which are identical to the roots of the characteristic equation of equation (14).

For the given values of R, L and C we get

$$\alpha = \frac{R}{2L} = 1.234\, s^{-1} \tag{24}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = 111.12\, s^{-1} \tag{25}$$

As $\omega_0 > \alpha$, values of s are imaginary. The frequency response of this circuit is therefore underdamped. The solution for $I(t)$ is given by inverse Laplace transform of $I(s)$:

$$I(t) = \frac{1}{L}\int_0^t V(t-\tau)\mathrm{e}^{-\alpha\tau}\left[\cosh(\omega_d\tau) - \frac{\alpha}{\omega_d}\sinh(\omega_d\tau)\right] d\tau \tag{26}$$

where $\omega_d = \sqrt{\omega_0^2 - \alpha^2} = 111.11\, s^{-1}$ is the damped frequency of oscillation.

The integral yields the transient response:

$$I(t) = \mathrm{e}^{-\alpha t}\left[B_1\cos(\omega_d t) + B_2\sin(\omega_d t)\right] \tag{27}$$

where B_1 and B_2 are constants. Substituting $\alpha = 1.234\, s^{-1}$ and $\omega_d = 111.11\, s^{-1}$ we get

$$I(t) = \mathrm{e}^{-1.234t}\left[B_1\cos(111.11t) + B_2\sin(111.11t)\right] \tag{28}$$

where t is in seconds. Constants B_1 and B_2 can be determined by the initial conditions of the given circuit.

The steady-state response of the circuit will depend on the nature of input $V(t)$. The transient and steady-state responses of the circuit can be added to give the net equation of $I(t)$.

The current $I(t)$ will fluctuate in a sinusoidal manner and decay over time. The frequency response of the circuit is underdamped response.