NCERT Math 11.9.2 Q8

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Question: If the sum of n terms of an AP is $(pn + qn^2)$, where p and q are constants, find the common difference.

Solution:

Symbol	Value	Description
$s(n)$	$(pn + qn^2)$	Sum of n terms
$x(n)$		n^{th} term of AP
d	$x(n+1) - x(n)$	Common Difference

TABLE I: Given Parameters

Sum of n terms, as a discrete signal:

$$s(n) = (pn + qn^2)u(n) \tag{1}$$

Taking the Z-Transform,

1) $\mathcal{Z}\{nu(n)\}$
   Using GP summation,

$$\sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}} \tag{2}$$

$$nu(n) \xleftrightarrow{\mathcal{Z}} -zU'(z) \tag{3}$$

$$\implies \sum_{n=0}^{\infty} nz^{-n} = \frac{z^{-1}}{(1 - z^{-1})^2} \{|z| > 1\} \tag{4}$$

2) $\mathcal{Z}\{n^2u(n)\}$
   From (3),

$$n^2u(n) \xleftrightarrow{\mathcal{Z}} -z\left(\mathcal{Z}\{nu(n)\}\right)' \tag{5}$$

$$\implies \sum_{n=0}^{\infty} n^2z^{-n} = \frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3} \{|z| > 1\} \tag{6}$$

Taking the Z-Transform of (1) using (4) and (6)

$$S(z) = p\left(\frac{z^{-1}}{(1 - z^{-1})^2}\right) + q\left(\frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3}\right) \tag{7}$$

Now,

$$s(n) = x(n) * u(n) \tag{8}$$

$$\implies S(z) = X(z)U(z) \tag{9}$$

$$\implies X(z) = \frac{S(z)}{U(z)} \tag{10}$$

where,

$$U(z) = \frac{1}{1 - z^{-1}} \tag{11}$$

Using (11) in (10),

$$X(z) = p\left(\frac{z^{-1}}{(1 - z^{-1})}\right) + q\left(\frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^2}\right) \tag{12}$$

Using contour integration for inverse Z-Transform:

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz \tag{13}$$

$$= \frac{1}{2\pi j}\oint_C \left(p\left(\frac{z^{-1}}{(1 - z^{-1})}\right) + q\left(\frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^2}\right)\right)z^{n-1}dz \tag{14}$$

$$R_m = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}((z - a)^m f(z)) \tag{15}$$

For first term, pole $(z = 1)$ is repeated once so $m = 1$

$$\implies R_1 = \frac{1}{0!}\lim_{z\to 1}(z - 1)\left(p\left(\frac{z^{-1}}{1 - z^{-1}}\right)\right)z^{n-1} \tag{16}$$

$$= p \tag{17}$$

For second term, pole $(z = 1)$ is repeated twice so $m = 2$

$$\implies R_2 = \frac{1}{1!}\lim_{z\to 1}\frac{d}{dz}\left((z - 1)^2 q\left(\frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^2}\right)\right)z^{n-1} \tag{18}$$

$$= q\lim_{z\to 1}\frac{d}{dz}\left(z^n + z^{n-1}\right) \tag{19}$$

$$= q(2n - 1) \tag{20}$$

$$\implies x(n) = R_1 + R_2 \tag{21}$$

$$= p + q(2n - 1) \tag{22}$$

Writing x(n) as a discrete signal we get:

$$x(n) = (p - q)u(n) + 2qnu(n) \tag{23}$$

To simplify, use $n = 0$:

$$s(0) = x(0) \tag{24}$$

$$\implies 0 = (p - q)u(0) + 2q(0)u(0) \tag{25}$$

$$\implies p = q \tag{26}$$

$\therefore$ (23) can be written as:

$$x(n) = 2qnu(n) \tag{27}$$

Common difference is given by:

$$d = x(n+1) - x(n) \tag{28}$$
$$= 2q \tag{29}$$

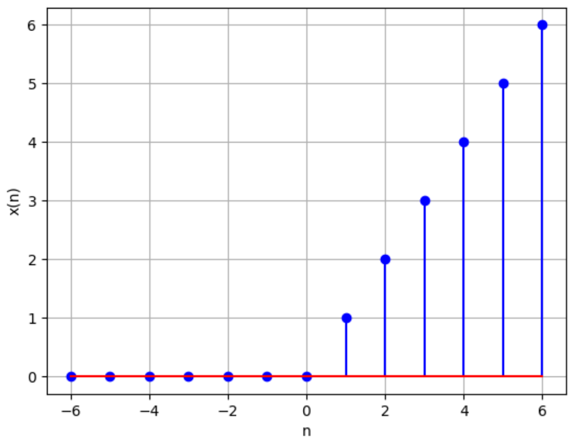

Fig. 1: Plot of x(n) vs n for $p = q = 0.5$