NCERT Physics 12.7 Q21

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Question: Obtain the resonant frequency and Q-factor of a series LCR circuit with L = 3.0 H, C = 27 μF, and R = 7.4 Ω. It is desired to improve the sharpness of the resonance of the circuit by reducing its 'full width at half maximum' by a factor of 2. Suggest a suitable way.

Solution: Given parameters are:

Symbol	Value	Description
L	3.0 H	Inductance
C	27 μF	Capacitance
R	7.4 Ω	Resistance
Q	$\frac{1}{R}\sqrt{\frac{L}{C}}$	Quality Factor
ω_0	$\frac{1}{\sqrt{LC}}$	Angular Resonant Frequency

TABLE I: Given Parameters

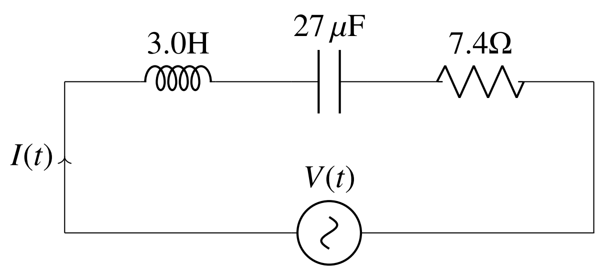

Fig. 1: LCR Circuit

Frequency Response of the Circuit

Applying Kirchhoff's Voltage Law (KVL), we get:

$$V_R + V_L + V_C = V(t) \tag{1}$$

where V_R, V_L and V_C are the voltages across R, L and C respectively and $V(t)$ is the time-varying voltage source.

Elements and their corresponding reactances are given in the table below where s is a complex variable.

Symbol	Reactance	Description
R	R	Resistance
L	sL	Inductance
C	$\frac{1}{sC}$	Capacitance

TABLE II: Reactances

The circuit can now be redrawn as:

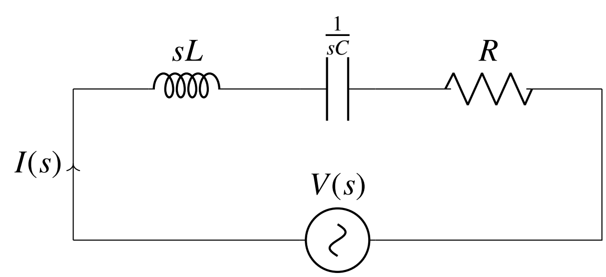

Fig. 2: LCR Circuit

Using the reactances of R, L and C from TABLE II in equation (1), we get

$$V(s) = RI(s) + sLI(s) + \frac{1}{sC}I(s) \tag{2}$$

$$\Rightarrow V(s) = I(s)\left(R + Ls + \frac{1}{sC}\right) \tag{3}$$

$$\Rightarrow I(s) = \frac{V(s)}{\left(R + Ls + \frac{1}{sC}\right)} \tag{4}$$

At resonance, the circuit becomes purely resistive. The reactances of capacitor and inductor cancel out as follows:

$$Ls + \frac{1}{sC} = 0 \tag{5}$$

$$\Rightarrow s = j\frac{1}{\sqrt{LC}} \tag{6}$$

s can be expressed in terms of angular resonance frequency as

$$s = j\omega_0 \tag{7}$$

Comparing equations (6) and (7), we get

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{8}$$

Quality Factor

Quality Factor (Q) of an LCR circuit is defined as the ratio of voltage across inductor or capacitor to that across the resistor at resonance.

$$Q = \left(\frac{V_L}{V_R}\right)_{\omega_0} = \frac{|sLI(s)|}{|RI(s)|} \tag{9}$$

$$\Rightarrow Q = \frac{1}{\sqrt{LC}}\frac{L}{R} \tag{10}$$

$$\Rightarrow Q = \frac{1}{R}\sqrt{\frac{L}{C}} \tag{11}$$

Plot of Impedance vs Angular Frequency

Impedance is defined as

$$H(s) = \frac{V(s)}{I(s)} \tag{12}$$

Using equation (4),

$$H(s) = R + sL + \frac{1}{sC} \tag{13}$$

$$\Rightarrow H(j\omega) = R + j\omega L + \frac{1}{j\omega C} \tag{14}$$

$$\Rightarrow |H(j\omega)| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \tag{15}$$

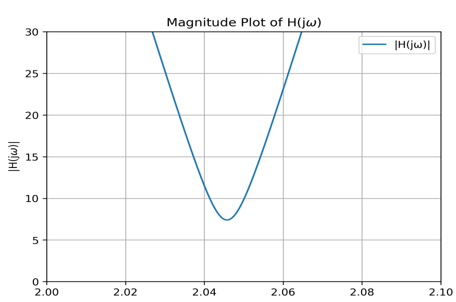

Fig. 3: Impedance vs $\log \omega$