GATE 2022 NM Q24

EE23BTECH11009 - AROSHISH PRADHAN*

Question: If

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

is the Fourier cosine series of the function

$$f(x) = \sin(x), 0 < x < \pi$$

then which of the following are TRUE?

(a) $a_0 + a_1 = \frac{4}{\pi}$
(b) $a_0 = \frac{4}{\pi}$
(c) $a_0 + a_1 = \frac{2}{\pi}$
(d) $a_1 = \frac{2}{\pi}$

Solution:

Symbol	Value	Description
a_0, a_n, b_n		Fourier Series Coefficients
T	π	Time Period
n		Positive Integer

TABLE I: Input Parameters

Fourier series of a function $f(x)$:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega x) + \sum_{n=1}^{\infty} b_n \sin(n\omega x) \tag{1}$$

where,

$$a_0 = \frac{1}{T} \int_T f(x) dx \tag{2}$$

$$a_n = \frac{2}{T} \int_T f(x) \cos(n\omega x) dx \tag{3}$$

$$b_n = \frac{2}{T} \int_T f(x) \sin(n\omega x) dx \tag{4}$$

Calculating for given function:

$$\frac{a_0}{2} = \frac{1}{\pi} \int_0^{\pi} \sin(x) dx \tag{5}$$

$$\implies a_0 = \frac{4}{\pi} \tag{6}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} \sin(x) \cos(nx) dx \tag{7}$$

$$\implies a_1 = \frac{2}{\pi} \int_0^{\pi} \sin(x) \cos(x) dx \tag{8}$$

$$= 0 \tag{9}$$

Calculating general a_n:

$$a_n = \frac{2}{\pi} \int_0^{\pi} \sin(x) \cos(nx) dx \tag{10}$$

$$= \frac{1}{\pi} \int_0^{\pi} (\sin(x + nx) + \sin(x - nx)) dx \tag{11}$$

$$= \frac{1}{\pi} \left[\frac{-\cos((n+1)x)}{n+1} + \frac{-\cos((1-n)x)}{1-n} \right]_0^{\pi} \tag{12}$$

$$= \frac{2(1 + \cos(n\pi))}{\pi(1 - n^2)} \tag{13}$$

From (6) and (9),

$$a_0 + a_1 = \frac{4}{\pi} \tag{14}$$

$$a_0 = 0 \tag{15}$$

$\therefore$ correct options are (a) and (b).