GATE 2023 PH Q37

EE23BTECH11009 - AROSHISH PRADHAN*

Question: An input voltage in the form of a square wave of frequency 1 kHz is given to a circuit, which results in the output shown schematically below. Which one of the following options is the CORRECT representation of the circuit?

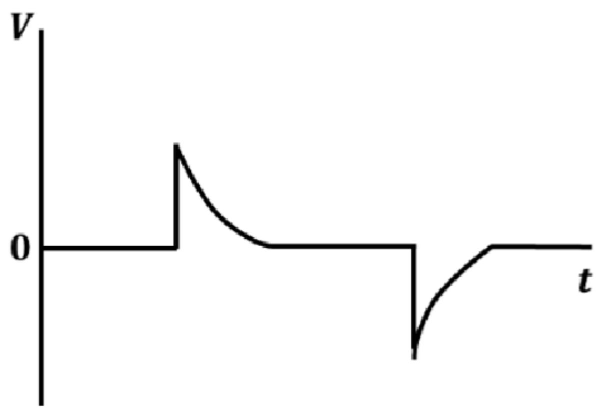
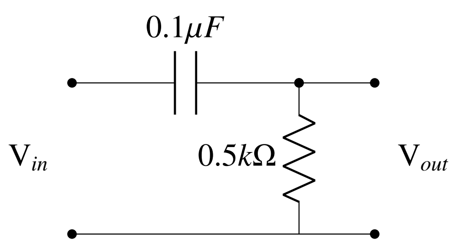
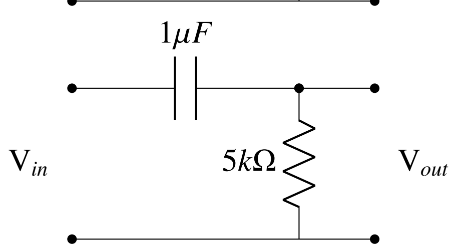


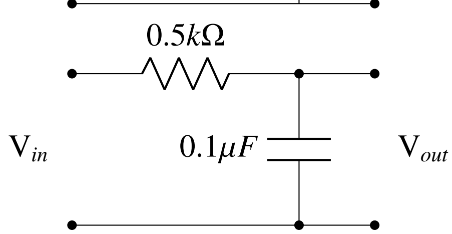
Fig. 1



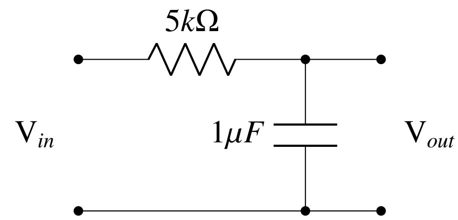
(a)



(b)



(c)



(d)

Solution:

Symbol	Value	Description
$V_{in}(t)$		Input Voltage
$\mathcal{V}_{in}(j\omega)$		Fourier Transform of $V_{in}(t)$
$V_{out}(t)$		Output Voltage
$\mathcal{V}_{out}(j\omega)$		Fourier Transform of $V_{out}(t)$
f	$\frac{\omega}{2\pi} = 1000Hz$	Input Wave Frequency
T	$\frac{2\pi}{\omega} = 10^{-3}s$	Input Wave Time Period
R	(a) $0.5k\Omega$	Resistance
	(b) $5k\Omega$	
C	(a) $0.1\mu F$	Capacitance
	(b) $1\mu F$	
τ	RC	Time Constant
Z	$R + \frac{1}{j\omega C}$	Impedance
$H(j\omega)$	$\frac{V_{out}}{V_{in}}$	General Transfer Function
$H_R(j\omega)$	$\frac{V_{R,out}}{V_{in}}$	Transfer Function for Resistor
$H_C(j\omega)$	$\frac{V_{C,out}}{V_{in}}$	Transfer Function for Capacitor

TABLE I: Given Parameters

Input waveform is a square wave (Fig. 2), so we take its Fourier Transform

$$V_{in}(t) = 2\left(2\left[\frac{\left(t-\frac{T}{4}\right)}{T}\right] - \left[\frac{2\left(t-\frac{T}{4}\right)}{T}\right]\right) + 1 \tag{1}$$

Fourier Series Coefficient:

$$c_k = \frac{1}{T}\int_T V_{in}(t)e^{-jk\omega t}dt \tag{2}$$

As square wave is even, $\sin(k\omega t)$ terms become zero. Cosine coefficients are:

$$a_n = \frac{2}{T}\int_T V_{in}(t)\cos\left(\frac{2\pi nt}{T}\right) \tag{3}$$

$$= \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos(n\pi) \tag{4}$$

Fourier Series of $V_{in}(t)$, visualized in Fig. 2:

$$V_{in}(t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T}\right) \tag{5}$$

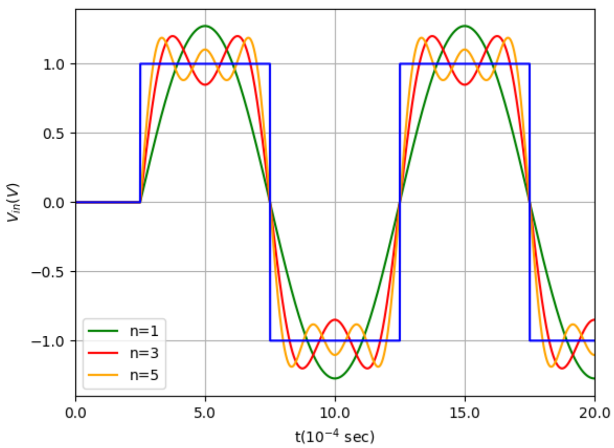


Fig. 2: Input Square Waveform ($V_{in}(t)$)

Taking Fourier Transform of $V_{in}(t)$:

$$V_{in}(t) \xleftrightarrow{\mathcal{F}} \mathcal{V}_{in}(j\omega) \tag{6}$$

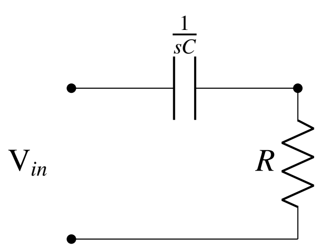


Fig. 3: Series RC Circuit in s-domain

$$s = j\omega \tag{7}$$

$$\implies Z = R + \frac{1}{sC} \tag{8}$$

$$= R + \frac{1}{j\omega C} \tag{9}$$

$\mathcal{V}_{in}(j\omega)$ was input into all four circuits and Inverse Fourier Transform was taken of the response. Transfer Function:

$$H(j\omega) = \frac{V_{out}}{V_{in}} \tag{10}$$

1) Option A

$$H_R(j\omega) = \frac{R}{R + \frac{1}{j\omega C}} \tag{11}$$

$$= \frac{j\omega RC}{1 + j\omega RC} \tag{12}$$

$$= \left(\frac{\omega RC}{\sqrt{1 + (\omega RC)^2}}\right) e^{j\tan^{-1}\left(\frac{1}{\omega RC}\right)} \tag{13}$$

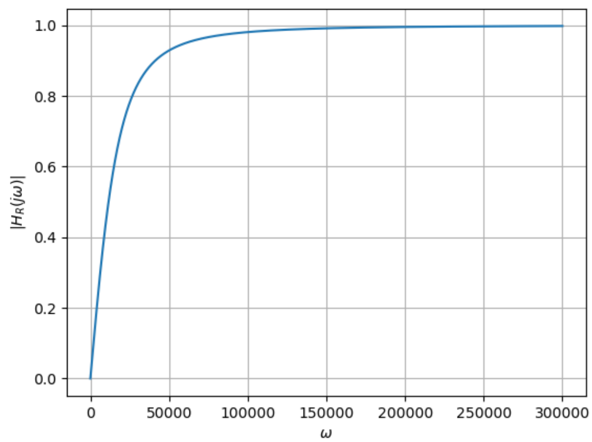


Fig. 4: $|H_R(j\omega)|$ vs ω for $R = 0.5k\Omega$, $C = 0.1\mu F$

$$\implies \mathcal{V}_{out}(j\omega) = H_R(j\omega)\mathcal{V}_{in}(j\omega) \tag{14}$$

$$\implies V_{out}(t) = \mathcal{F}^{-1}\{H_R(j\omega)\mathcal{V}_{in}(j\omega)\} \tag{15}$$

$$\tag{16}$$

Using (5) and (13),

$$V_{out}(t) = \mathcal{F}^{-1}\left(\frac{\omega RC e^{j\tan^{-1}\left(\frac{1}{\omega RC}\right)}}{\sqrt{1 + (\omega RC)^2}}\right) \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos(n\pi) \cos(n\omega t) \tag{17}$$

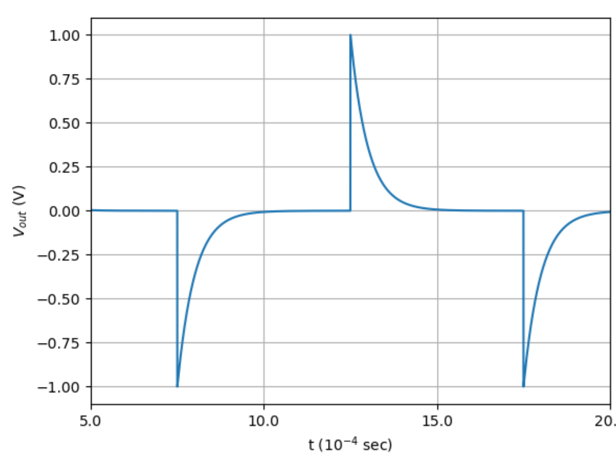


Fig. 5: Opt A: $V_{out}(t)$ vs t

2) Option B

$$H_R(j\omega) = \frac{R}{R + \frac{1}{j\omega C}} \tag{18}$$

$$= \frac{j\omega RC}{1 + j\omega RC} \tag{19}$$

$$= \left(\frac{\omega RC}{\sqrt{1 + (\omega RC)^2}}\right) e^{j\tan^{-1}\left(\frac{1}{\omega RC}\right)} \tag{20}$$

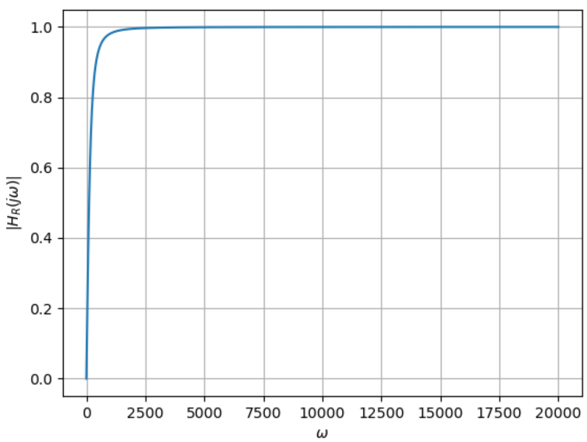


Fig. 6: $|H_R(j\omega)|$ vs ω for $R = 5k\Omega$, $C = 1\mu F$

$$\implies \mathcal{V}_{out}(j\omega) = H_R(j\omega)\mathcal{V}_{in}(j\omega) \tag{21}$$

$$\implies V_{out}(t) = \mathcal{F}^{-1}\{H_R(j\omega)\mathcal{V}_{in}(j\omega)\} \tag{22}$$

Using (5) and (20),

$$V_{out}(t) = \mathcal{F}^{-1}\left(\frac{\omega RC e^{j\tan^{-1}\left(\frac{1}{\omega RC}\right)}}{\sqrt{1 + (\omega RC)^2}}\right)$$

$$\sum_{n=1}^{\infty} \frac{4}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos(n\pi) \cos(n\omega t) \tag{23}$$

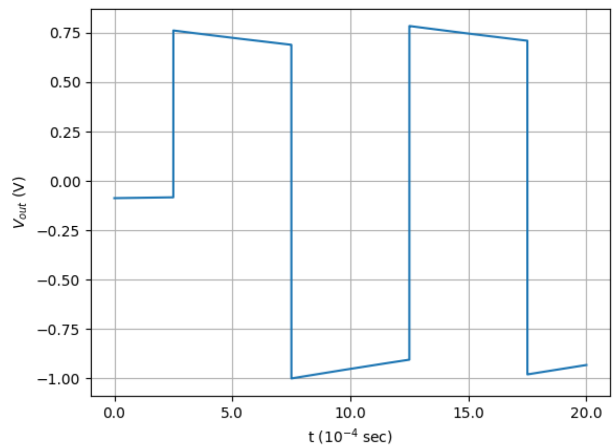


Fig. 7: Opt B: $V_{out}(t)$ vs t

3) Option C

$$H_C(j\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \tag{24}$$

$$= \frac{1}{1 + j\omega RC} \tag{25}$$

$$= \left(\frac{1}{\sqrt{1 + (\omega RC)^2}}\right) e^{-j\tan^{-1}(\omega RC)} \tag{26}$$

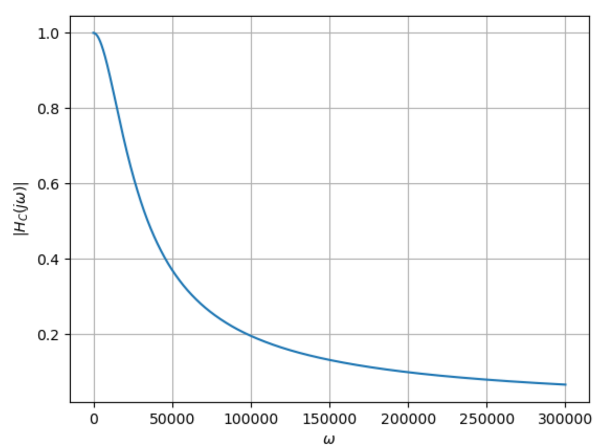


Fig. 8: $|H_C(j\omega)|$ vs ω for $R = 0.5k\Omega$, $C = 0.1\mu F$

$$\implies \mathcal{V}_{out}(j\omega) = H_C(j\omega)\mathcal{V}_{in}(j\omega) \tag{27}$$

$$\implies V_{out}(t) = \mathcal{F}^{-1}\{H_C(j\omega)\mathcal{V}_{in}(j\omega)\} \tag{28}$$

Using (5) and (26),

$$V_{out}(t) = \mathcal{F}^{-1}\left(\frac{e^{j\tan^{-1}\left(\frac{1}{\omega RC}\right)}}{\sqrt{1+(\omega RC)^2}}\right)$$
$$\sum_{n=1}^{\infty}\frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos\left(n\pi\right)\cos\left(n\omega t\right) \quad (29)$$

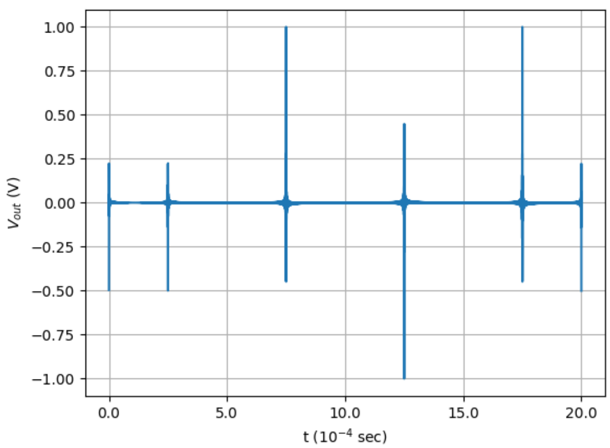


Fig. 9: Opt C: $V_{out}(t)$ vs t

4) Option D

$$H_C(j\omega) = \frac{\frac{1}{j\omega C}}{R+\frac{1}{j\omega C}} \quad (30)$$
$$= \frac{1}{1+j\omega RC} \quad (31)$$
$$= \left(\frac{1}{\sqrt{1+(\omega RC)^2}}\right)e^{-j\tan^{-1}(\omega RC)} \quad (32)$$

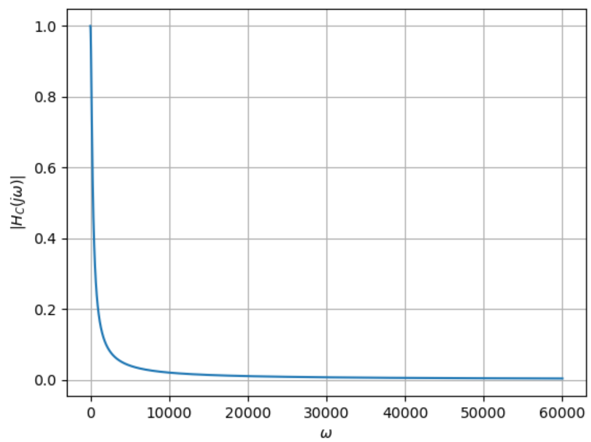


Fig. 10: $|H_C(j\omega)|$ vs ω for $R = 5k\Omega$, $C = 1\mu F$

$$\implies \mathcal{V}_{out}(j\omega) = H_C(j\omega)\mathcal{V}_{in}(j\omega) \quad (33)$$
$$\implies V_{out}(t) = \mathcal{F}^{-1}\left\{H_C(j\omega)\mathcal{V}_{in}(j\omega)\right\} \quad (34)$$

Using (5) and (32),

$$V_{out}(t) = \mathcal{F}^{-1}\left(\frac{e^{j\tan^{-1}\left(\frac{1}{\omega RC}\right)}}{\sqrt{1+(\omega RC)^2}}\right)$$
$$\sum_{n=1}^{\infty}\frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos\left(n\pi\right)\cos\left(n\omega t\right) \quad (35)$$

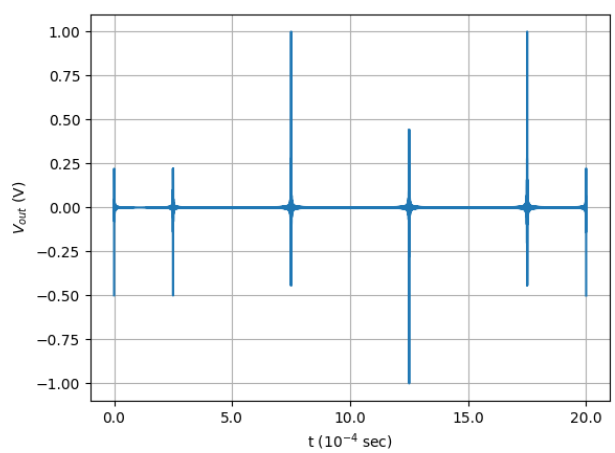


Fig. 11: Opt D: $V_{out}(t)$ vs t