

NCERT Math 11.9.2 Q8

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Question: If the sum of n terms of an AP is $(pn + qn^2)$, where p and q are constants, find the common difference.

Solution:

Symbol	Value	Description
$s(n)$	$(pn + qn^2)$	Sum of n terms
$x(n)$		n^{th} term of AP
d	$x(n+1) - x(n)$	Common Difference

TABLE I: Given Parameters

Sum of n terms, as a discrete signal:

$$s(n) = (pn + qn^2)u(n) \quad (1)$$

Taking the Z-Transform,

1) $\mathcal{Z}\{nu(n)\}$

Using GP summation,

$$\sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}} \quad (2)$$

$$nu(n) \xleftrightarrow{\mathcal{Z}} -zU'(z) \quad (3)$$

$$\Rightarrow \sum_{n=0}^{\infty} nz^{-n} = \frac{z^{-1}}{(1 - z^{-1})^2} \{|z| > 1\} \quad (4)$$

2) $\mathcal{Z}\{n^2u(n)\}$

From (3),

$$n^2u(n) \xleftrightarrow{\mathcal{Z}} -z(\mathcal{Z}\{nu(n)\})' \quad (5)$$

$$\Rightarrow \sum_{n=0}^{\infty} n^2 z^{-n} = \frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3} \{|z| > 1\} \quad (6)$$

Taking the Z-Transform of (1) using (4) and (6)

$$S(z) = p \left(\frac{z^{-1}}{(1 - z^{-1})^2} \right) + q \left(\frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3} \right) \quad (7)$$

Now,

$$s(n) = x(n) * u(n) \quad (8)$$

$$\Rightarrow S(z) = X(z)U(z) \quad (9)$$

$$\Rightarrow X(z) = \frac{S(z)}{U(z)} \quad (10)$$

$$U(z) = \frac{1}{1 - z^{-1}} \quad (11)$$

Using (11) in (10),

$$X(z) = p \left(\frac{z^{-1}}{(1 - z^{-1})} \right) + q \left(\frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^2} \right) \quad (12)$$

Simplifying using partial fractions, we get:

$$X(z) = (q - p) + \frac{p - 3q}{1 - z^{-1}} + \frac{2q}{(1 - z^{-1})^2} \quad (13)$$

$$= (q - p) + \frac{(p - q)}{1 - z^{-1}} + \frac{2qz^{-1}}{(1 - z^{-1})^2} \quad (14)$$

Taking the inverse Z-Transform,

$$x(n) = (q - p)\delta(n) + (p - q)u(n) + 2qnu(n) \quad (15)$$

To simplify, use $n = 0$:

$$s(0) = x(0) = 0 \quad (16)$$

$$\Rightarrow (q - p)\delta(0) + (p - q)u(0) + 2qnu(0) = 0 \quad (17)$$

$$\Rightarrow p = q \quad (18)$$

because $\delta(0) = 1$ and $u(0) = 0$

\therefore rewriting (15):

$$x(n) = 2qnu(n) \quad (19)$$

Common difference is given by:

$$d = x(n+1) - x(n) \quad (20)$$

$$= 2q(n+1)u(n+1) - 2qnu(n) \quad (21)$$

$$= 2q \quad (22)$$

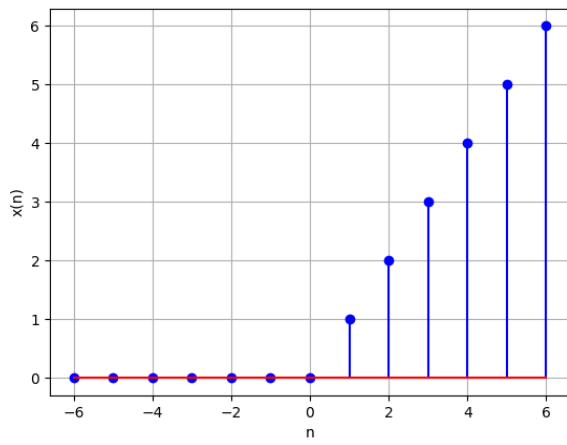


Fig. 1: Plot of $x(n)$ vs n for $p = q = 0.5$