Filter Design

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1 Introduction

We are supposed to design the equivalent FIR and IIR filter realizations for a band-pass filter with pass band: 9.4 kHz and 10.6 kHz.

2 Filter Specifications

The sampling rate for the filter has been specified as $F_s = 48$ kHz.

2.1 The Digital Filter

1. *Tolerances:* The pass-band (δ_1) and stop-band (δ_2) tolerances are given to be equal, so we let $\delta_1 = \delta_2 = \delta = 0.15$.

2. *Pass-band:* The pass-band of the filter is calculated as follows:

$$j = (r - 11000) \bmod \sigma$$

where,

$$\begin{aligned} r &= \text{Roll Number} \\ &= 11009 \\ \sigma &= \text{Sum of digits of roll number} \\ &= 11 \\ \implies j &= 9 \end{aligned}$$

The pass band is then given by:

$$\begin{aligned} &4 + 0.6(j) \,\text{to}\, 4 + 0.6(j+2) \\ \implies &9.4\text{kHz} \,\text{to}\, 10.6\text{kHz} \end{aligned}$$

Hence, the un-normalized discrete time filter pass-band frequencies are:

$$\begin{aligned} F_{p1} &= 10.6\,\text{kHz} \\ F_{p2} &= 9.4\,\text{kHz} \end{aligned}$$

If the un-normalized discrete-time (natural) frequency is F, the corresponding normalized digital filter (angular) frequency is given by $\omega = 2\pi\left(\frac{F}{F_s}\right)$. Therefore,

$$\omega_{p1} = 2\pi\frac{F_{p1}}{F_s} = 0.441\pi$$
$$\omega_{p2} = 2\pi\frac{F_{p2}}{F_s} = 0.391\pi$$

The centre frequency is then given by $\omega_c = \frac{\omega_{p1}+\omega_{p2}}{2} = 0.416\pi$.

3. *Stop-band:* The *transition band* for band-pass filters is $\Delta F = 0.3$ kHz on either side of the pass-band. Hence, the un-normalized *stop-band* frequencies are:

$$F_{s1} = 10.6 + 0.3 = 10.9\,\text{kHz}$$
$$F_{s2} = 9.4 - 0.3 = 9.1\,\text{kHz}$$

The corresponding normalized frequencies are:

$$\omega_{s1} = 0.454\pi$$
$$\omega_{s2} = 0.379\pi$$

The above parameters are summarized in the table below:

Paramter	**Value**	**Description**
F_{p_1}	10.6 kHz	Unnormalized passband upper frequency
F_{p_2}	9.4 kHz	Unnormalized passband lower frequency
ω_{p_1}	0.441π	Normalized passband upper frequency
ω_{p_2}	0.391π	Normalized passband lower frequency
ΔF	0.3kHz	Transition Band
F_{s_1}	10.9kHz	Unnormalized stopband upper frequency
F_{s_2}	9.1kHz	Unnormalized stopband lower frequency
ω_{s_1}	0.454π	Normalized stopband upper frequency
ω_{s_2}	0.379π	Normalized stopband lower frequency

Table 1: Input Paramters

2.2 The Analog filter

In the bilinear transform, the analog filter frequency (Ω) is related to the corresponding digital filter frequency (ω) as $\Omega = \tan\frac{\omega}{2}$. Using this relation, we obtain the analog pass-band and stop-band frequencies as:

$$
\begin{aligned}
\Omega_{p1} &= \tan\left(\frac{\omega_{p_1}}{2}\right) = 0.8298 \\
\Omega_{p2} &= \tan\left(\frac{\omega_{p_2}}{2}\right) = 0.7051 \\
\Omega_{s1} &= \tan\left(\frac{\omega_{s_1}}{2}\right) = 0.8649 \\
\Omega_{s2} &= \tan\left(\frac{\omega_{s_2}}{2}\right) = 0.6772
\end{aligned}
$$

respectively.

3 The IIR Filter Design

Filter Type: We are supposed to design filters whose stop-band is monotonic and pass-band equiripple. Hence, we use the *Chebyschev approximation* to design our band-pass IIR filter.

3.1 The Analog Filter

1. *Low Pass Filter Specifications:* If $H_{a,BP}(j\Omega)$ be the desired analog band pass filter, with the specifications provided in Section 2.2, and $H_{a,LP}(j\Omega_L)$ be the equivalent low pass filter, then

$$
\Omega_L = \frac{\Omega^2 - \Omega_0^2}{B\Omega} \tag{1}
$$

where

$$
\begin{aligned}
\Omega_0 &= \sqrt{\Omega_{p1}\Omega_{p2}} = 0.7649 \\
B &= \Omega_{p1} - \Omega_{p2} = 0.1247 \\
\implies \Omega_{L_{p_1}} &= \frac{\Omega_{p_1}^2 - \Omega_{p_1}\Omega_{p_2}}{(\Omega_{p_1} - \Omega_{p_2})\Omega_{p_1}} = 1
\end{aligned}
$$

The low pass filter has the pass-band edge at $\Omega_{Lp} = 1$ and stop-band edges at

$$
\begin{aligned}
\Omega_{L_{s_1}} &= \frac{\Omega_{s_1}^2 - \Omega_0^2}{B\Omega_{s_1}} = 1.5111 \\
\Omega_{L_{s_2}} &= \frac{\Omega_{s_2}^2 - \Omega_0^2}{B\Omega_{s_2}} = -1.4976
\end{aligned}
$$

We choose the stop-band edge of the analog low pass filter as $\Omega_{Ls} = \min(|\Omega_{Ls_1}|, |\Omega_{Ls_2}|) = 1.4976$.

2. *The Low Pass Chebyschev Filter Parameters:* The magnitude squared of the Chebyschev low pass filter is given by

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \epsilon^2 c_N^2\left(\frac{\Omega_L}{\Omega_{L_p}}\right)} \tag{2}$$

where $c_N(x)$ is the Chebyshev Polynomial of the first kind of order N, given by:

$$c_N(x) = \begin{cases} \cos(N\cos^{-1}(x)) & \text{if } |x| \leq 1 \\ \cosh(N\cosh^{-1}(x)) & \text{if } x \geq 1 \end{cases}$$

where N and ϵ are design parameters. Since $\Omega_{Lp} = 1$, (2) may be rewritten as

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \epsilon^2 c_N^2(\Omega_L)} \tag{3}$$

Also, the design paramters have the following constraints

$$\begin{aligned} \frac{\sqrt{D_2}}{c_N(\Omega_{Ls})} \leq \epsilon \leq \sqrt{D_1}, \\ N \geq \left\lceil \frac{\cosh^{-1}\sqrt{D_2/D_1}}{\cosh^{-1}\Omega_{Ls}} \right\rceil, \end{aligned} \tag{4}$$

where

$$D_1 = \frac{1}{(1-\delta)^2} - 1 = 0.384$$
$$D_2 = \frac{1}{\delta^2} - 1 = 43.444$$

After substituting $\delta = 0.15$ and $\Omega_{L_s} = 1.4976$, we obtain $N \geq 4$ and $0.2897 \leq \epsilon \leq 0.6197$. In Fig. 1, we plot $|H(j\Omega)|$ for a range of values of ϵ, for $N = 4$. We find that for larger values of ϵ, $|H(j\Omega)|$ decreases in the transition band. We choose $\epsilon = 0.4$ for our IIR filter design.

Listing 1: Code for Figure 1

```
wget https://github.com/aroshishp/EE1205/blob/main/Filter_Design/
    codes/low_pass.py
```

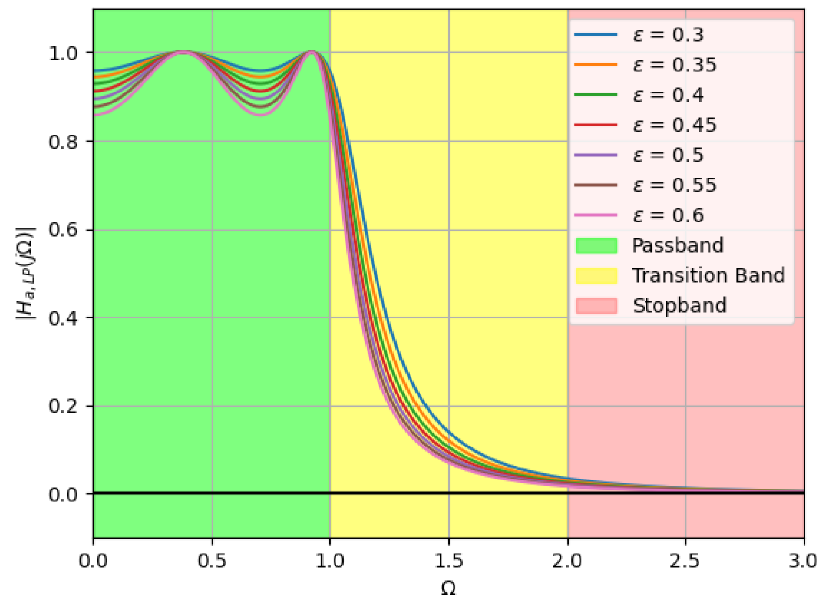

Figure 1: Analog Low-Pass Frequency Response for $0.3 \leq \epsilon \leq 0.6$

3. *The Low Pass Chebyshev Filter:* Thus, we obtain

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + 0.16c_4^2(\Omega_L)} \tag{5}$$

where,

$$c_4(x) = 8x^4 - 8x^2 + 1. \tag{6}$$

The poles of the frequency response in (5) lying on the left half of the Argand Plane are in general obtained as

$$s_k = r_1 \cos(\phi_k) + jr_2 \sin(\phi_k) \tag{7}$$

where

$$\phi_k = \frac{\pi}{2} + \frac{(2k+1)\pi}{2N}, k = 0, 1, \ldots, N-1$$

$$r_1 = \frac{\beta^2 - 1}{2\beta}, r_2 = \frac{\beta^2 + 1}{2\beta}, \beta = \left[\frac{\sqrt{1+\epsilon^2}+1}{\epsilon}\right]^{\frac{1}{N}} \tag{8}$$

Values of s_k using (7) are given in the table below:

s_k	**Value**
s_0	$-0.162 + 1.003j$
s_1	$-0.391 + 0.415j$
s_2	$-0.391 - 0.415j$
s_3	$-0.162 - 1.003j$

Table 2: Poles on left half of complex plane

s_k	**Value**
s_0	$-0.162 + 1.003j$
s_1	$-0.391 + 0.415j$
s_2	$-0.391 - 0.415j$
s_3	$-0.162 - 1.003j$
s_4	$0.162 + 1.003j$
s_5	$0.391 + 0.415j$
s_6	$0.391 - 0.415j$
s_7	$0.162 - 1.003j$

Table 3: Poles of frequency response in (5)

Note that the poles of frequency response in (5) are calculated through the below code separately and plotted in Figure 2.

Listing 2: Code for Figure 2

```
wget https://github.com/aroshishp/EE1205/blob/main/Filter_Design/
    codes/poleplot.py
```

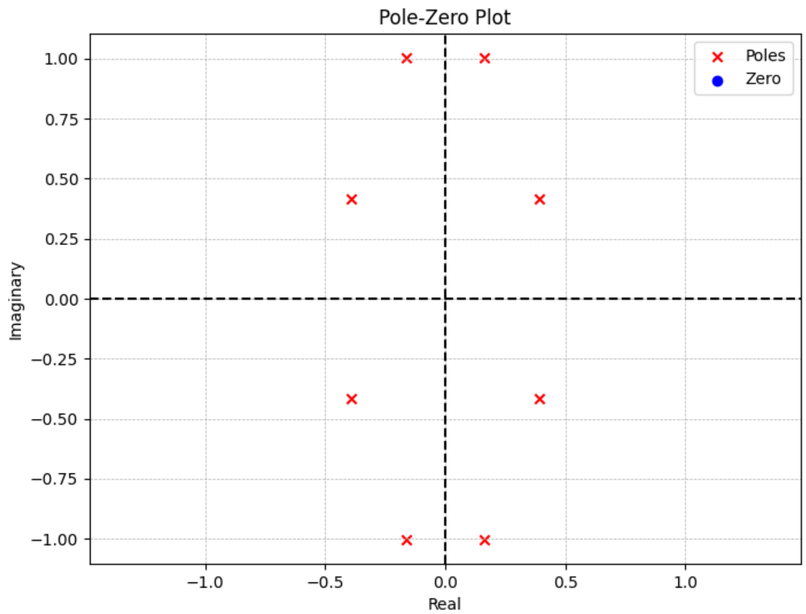

Figure 2: Pole-Zero Plot of frequency response of (5)

Thus, for even N, the low-pass stable Chebyshev filter, with a gain G_{LP} has the form

$$H_{a,LP}(s_L) = \frac{G_{LP}}{\prod_{k=1}^{\frac{N}{2}-1}(s_L^2 - 2r_1\cos\phi_k s_L + r_1^2\cos^2\phi_k + r_2^2\sin^2\phi_k)} \tag{9}$$

$$= \frac{G_{LP}}{(s-s_0)(s-s_1)(s-s_2)(s-s_3)} \tag{10}$$

Substituting $N = 4$, $\epsilon = 0.4$ and $H_{a,LP}(j) = \frac{1}{\sqrt{1+\epsilon^2}}$, from (8) and (9), we obtain

$$H_{a,LP}(s_L) = \frac{0.3125}{s_L^4 + 1.1068s_L^3 + 1.6125s_L^2 + 0.9140s_L + 0.3366} \tag{11}$$

In Figure 3 we plot $|H(j\Omega)|$ using (5) and (11), thereby verifying that our low-pass Chebyschev filter design meets the specifications.

Listing 3: Code for Figure 3

```
wget https://github.com/aroshishp/EE1205/blob/main/Filter_Design/
    codes/verification.py
```

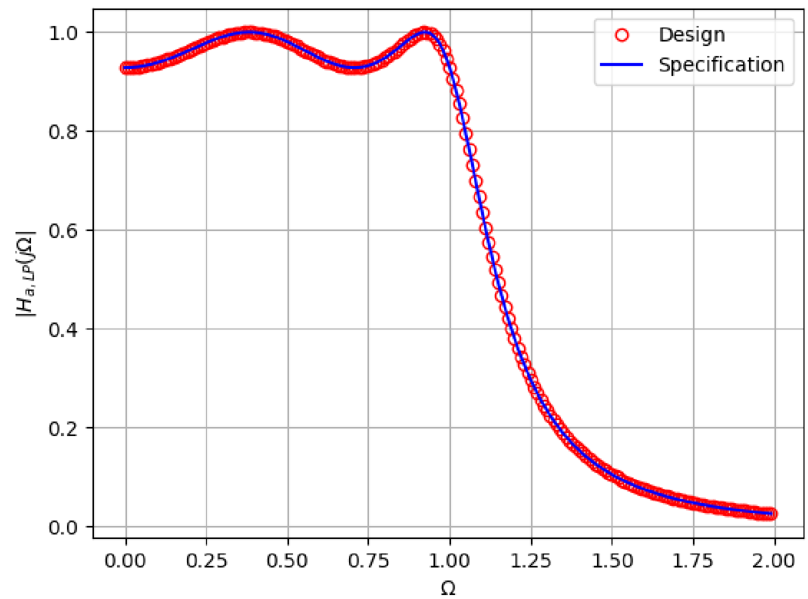

Figure 3: Plot of Frequency Response obtained from specifications in (5) and design in (11)

4. *The Band Pass Chebyschev Filter:* The analog band-pass filter is obtained from (11) as follows. From (1):

$$\Omega_L = \frac{\Omega^2 - \Omega_0^2}{B\Omega} \tag{12}$$

$$\implies j\Omega_L = \frac{\Omega_0^2 - \Omega^2}{Bj\Omega} \tag{13}$$

$$= \frac{\Omega_0^2 + (j\Omega)^2}{Bj\Omega} \tag{14}$$

$$\implies s_L = \frac{s^2 + \Omega_0^2}{Bs} \tag{15}$$

Hence,

$$H_{a,BP}(s) = G_{BP} H_{a,LP}(s_L)\Big|_{s_L = \frac{s^2+\Omega_0^2}{Bs}}, \tag{16}$$

where G_{BP} is the gain of the band-pass filter. The below code evaluates G_{BP} and finds the coefficients of $H_{a,BP}$. We get $G_{BP} = 1.077$ by evaluating gain such that $H_{a,BP}(j\Omega_{p1}) = 1$.

$$H_{a,BP}(s) = \frac{7.55642 \times 10^{-5} s^4}{s^8 + 0.138015s^7 + 2.36536s^6 + 0.244019s^5 + 2.08328s^4 + 0.142769s^3 + 0.809685s^2 + 0.0276411s + 0.117176} \tag{17}$$

In Figure 4, we plot $|H_{a,BP}(j\Omega)|$ as a function of Ω for both positive as well as negative frequencies. We find that the pass-band and stop-band frequencies in the figure match well with those obtained analytically through the bilinear transformation.

Listing 4: Code for Figure 4

```
wget https://github.com/aroshishp/EE1205/blob/main/Filter_Design/
    codes/Hbp.py
```

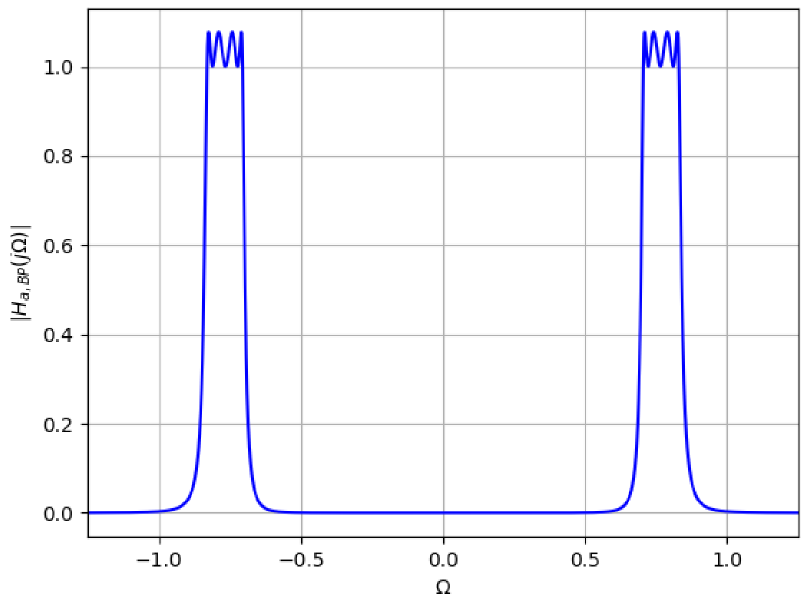

Figure 4: Analog Band-Pass Frequency Response from (17)

3.2 The Digital Filter

From the bilinear transformation, we obtain the digital band-pass filter from the corresponding analog filter as

$$H_{d,BP}(z) = GH_{a,BP}(s)\big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \tag{18}$$

where G is the gain of the digital filter. From (17) and (18), we obtain

$$H_{d,BP}(z) = G\frac{N(z)}{D(z)} \tag{19}$$

All coefficients and G are calculated from the below code.

Listing 5: Code for G, N(z), D(z)

```
wget https://github.com/aroshishp/EE1205/blob/main/Filter_Design/codes/
    substitutor.py
```

We get:

$$G = 7.55642 \times 10^{-5} \tag{20}$$

$$N(z) = 1 - 4z^{-2} + 6z^{-4} - 4z^{-6} + z^{-8} \tag{21}$$

$$\begin{aligned} D(z) = {} & 5.8230569z^{-8} - 12.4205486z^{-7} + 34.1023786z^{-6} - 42.2777094z^{-5} \\ & + 58.95155z^{-4} - 44.1531786z^{-3} + 37.1935974z^{-2} - 14.1500354z^{-1} \\ & + 6.9279451 \end{aligned} \tag{22}$$

The plot of $|H_{d,BP}(z)|$ with respect to the normalized angular freqency (normalizing factor π) is available in Figure 5. Again we find that the pass-band and stop-band frequencies meet the specifications well enough.

Listing 6: Code for Figure 5

```
wget https://github.com/aroshishp/EE1205/blob/main/Filter_Design/codes/
    Hdbp.py
```

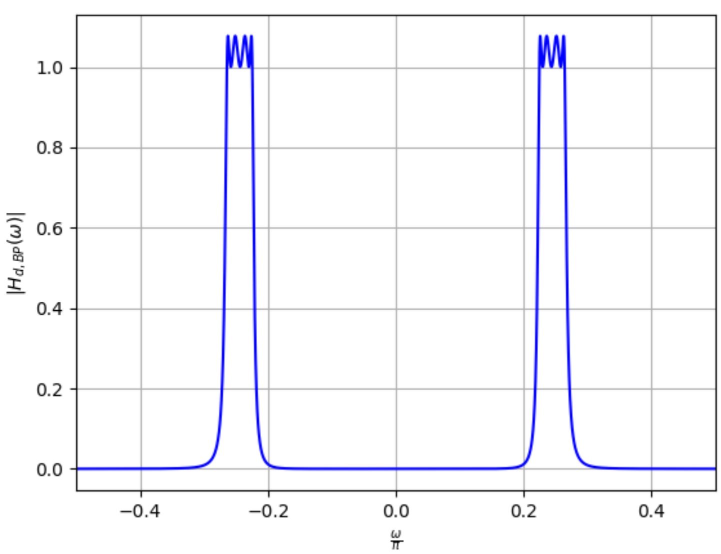

Figure 5: The magnitude response of the band-pass digital filter

4 The FIR Filter

We design the FIR filter by first obtaining the (non-causal) low-pass equivalent using the Kaiser window and then converting it to a causal band-pass filter.

4.1 The Equivalent Low-Pass Filter

The low-pass filter has a pass-band frequency ω_l and transition band $\Delta\omega = 2\pi\frac{\Delta F}{F_s} = 0.0125\pi$. The stop-band tolerance is δ.

1. The *pass-band frequency* ω_l is defined as:

$$\omega_l = \frac{\omega_{p1} - \omega_{p2}}{2} = 0.025\pi \tag{23}$$

   Substituting the values of ω_{p1} and ω_{p2} from section 2.1, we obtain $\omega_l = 0.025\pi$.

2. *The impulse response* $h_{lp}(n)$ of the desired low-pass filter with cutoff frequency ω_l is given by

$$h_l(n) = \frac{\sin(n\omega_l)}{n\pi} w(n), \tag{24}$$

   where $w(n)$ is the Kaiser window obtained from the design specifications.

4.2 The Kaiser Window

The Kaiser window is defined as

$$\begin{aligned} w(n) &= \frac{I_0\left[\beta N\sqrt{1-\left(\frac{n}{N}\right)^2}\right]}{I_0(\beta N)}, \qquad -N \le n \le N, \qquad \beta > 0 \\ &= 0 \qquad\qquad\qquad\qquad \text{otherwise,} \end{aligned} \tag{25}$$

where $I_0(x)$ is the modified Bessel function of the first kind of order zero in x and β and N are the window shaping factors. In the following, we find β and N using the design parameters in section 2.1.

1. N is chosen according to

$$N \ge \frac{A-8}{4.57\Delta\omega}, \tag{26}$$

   where $A = -20\log_{10}\delta$. Substituting the appropriate values from the design specifications, we obtain $A = 16.4782$ and $N \ge 48$.

2. β is chosen according to

$$\beta N = \begin{cases} 0.1102(A-8.7) & A > 50 \\ 0.5849(A-21)^{0.4} + 0.07886(A-21) & 21 \le A \le 50 \\ 0 & A < 21 \end{cases} \tag{27}$$

   In our design, we have $A = 16.4782 < 21$. Hence, from (27) we obtain $\beta = 0$.

3. We choose $N = 100$, to ensure the desired low pass filter response. Substituting in (25) gives us the rectangular window

$$\begin{aligned} w(n) &= 1, -100 \le n \le 100 \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{28}$$

From (24) and (28), we obtain the desired low-pass filter impulse response

$$\begin{aligned} h_{lp}(n) &= \frac{\sin(\frac{n\pi}{40})}{n\pi} \qquad -100 \le n \le 100 \\ &= 0, \qquad \text{otherwise} \end{aligned} \tag{29}$$

Listing 7: Code for Figure 6 and 7

```
wget https://github.com/aroshishp/EE1205/blob/main/Filter_Design/codes/
    fir_hH.py
```

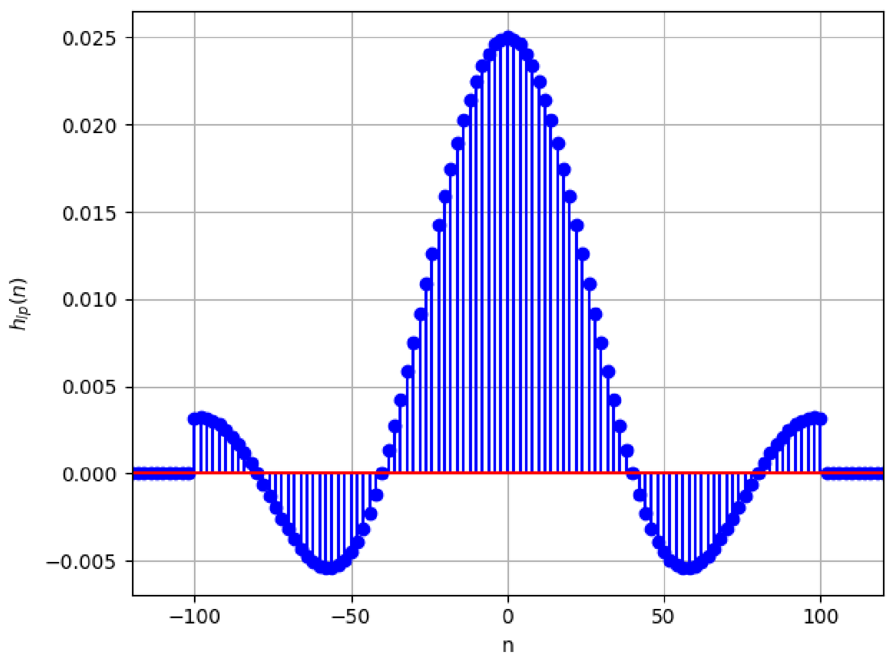

Figure 6: Plot of FIR Low-Pass Filter Impulse Response

The magnitude of the Frequency Response of low-pass filter is plotted in Figure 7.

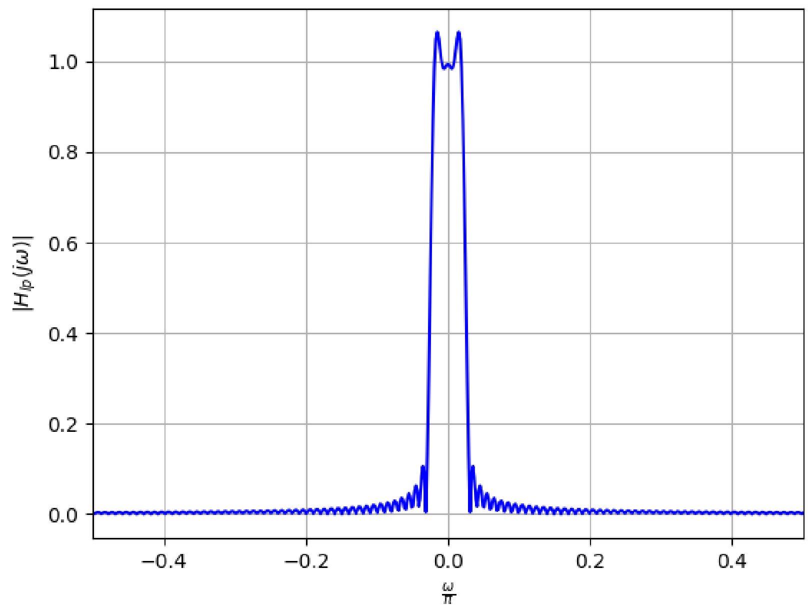

Figure 7: Plot of FIR Low-Pass Filter Frequency Response

4.3 The FIR Band-pass Filter

The centre of the pass-band of the desired band-pass filter was found to be $\omega_c = 0.416\pi$ in Section 2.1. The impulse response of the desired band-pass filter is obtained from the impulse response of the corresponding low-pass filter as

$$h_{bp}(n) = 2h_{lp}(n)cos(n\omega_c) \tag{30}$$

Thus, from (29), we obtain (plotted in Figure 8)

$$\begin{aligned} h_{bp}(n) &= \frac{2\sin(\frac{n\pi}{40})\cos(0.416n\pi)}{n\pi} && -100 \le n \le 100 \\ &= 0, && \text{otherwise} \end{aligned} \tag{31}$$

Listing 8: Code for Figure 8 and 9

```
wget https://github.com/aroshishp/EE1205/blob/main/Filter_Design/codes/
    fir_bp.py
```

The magnitude response of the FIR band-pass filter designed to meet the given specifications is plotted in Figure 9.

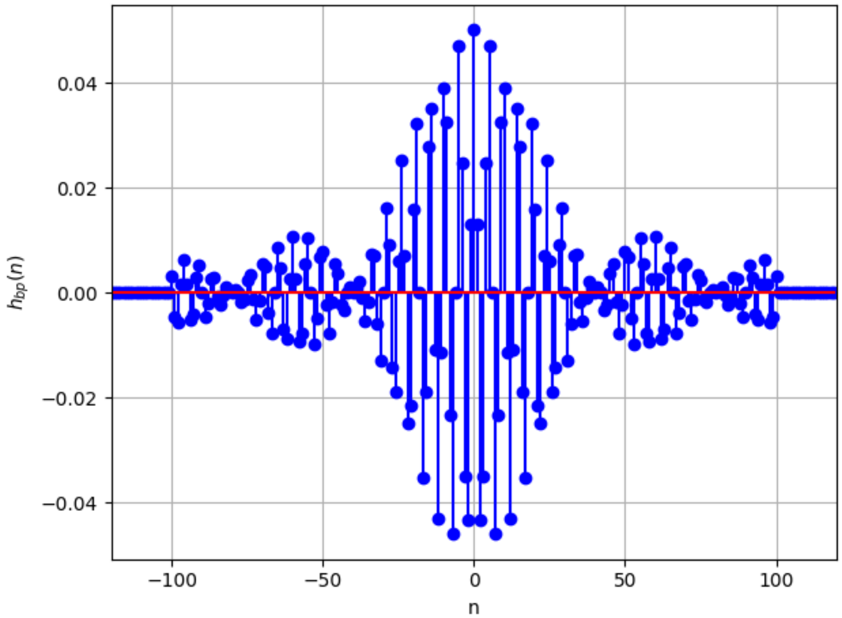

Figure 8: Plot of FIR Band-Pass Filter Impulse Response

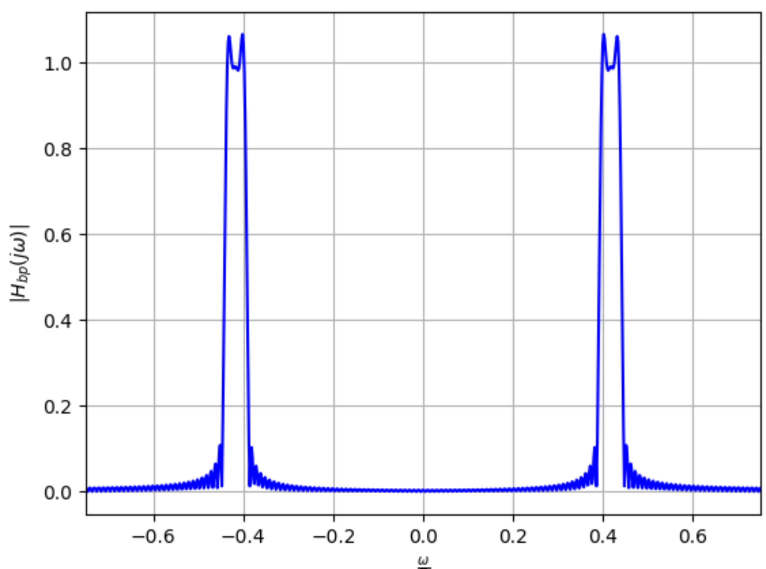

Figure 9: Plot of FIR Band-Pass Filter Frequency Response