NCERT Math 11.9.2 Q8

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Question: If the sum of n terms of an AP is $(pn + qn^2)$, where p and q are constants, find the common difference.

Solution:

Symbol	Value	Description
$s(n)$	$(pn + qn^2)$	Sum of n terms
$x(n)$		n^{th} term of AP
d	$x(n+1) - x(n)$	Common Difference

TABLE I: Given Parameters

Sum of n terms, as a discrete signal:

$$s(n) = (pn + qn^2)u(n) \tag{1}$$

Taking the Z-Transform,

1) $\mathcal{Z}\{nu(n)\}$
   Using GP summation,

$$\sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}} \tag{2}$$

$$nu(n) \overset{\mathcal{Z}}{\longleftrightarrow} -zU'(z) \tag{3}$$

$$\implies \sum_{n=0}^{\infty} nz^{-n} = \frac{z^{-1}}{(1 - z^{-1})^2} \{|z| > 1\} \tag{4}$$

2) $\mathcal{Z}\{n^2u(n)\}$
   From (3),

$$n^2u(n) \overset{\mathcal{Z}}{\longleftrightarrow} -z\left(\mathcal{Z}\{nu(n)\}\right)' \tag{5}$$

$$\implies \sum_{n=0}^{\infty} n^2z^{-n} = \frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3} \{|z| > 1\} \tag{6}$$

Taking the Z-Transform of (1) using (4) and (6)

$$S(z) = p\left(\frac{z^{-1}}{(1 - z^{-1})^2}\right) + q\left(\frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3}\right) \tag{7}$$

Now,

$$s(n) = x(n) * u(n) \tag{8}$$

$$\implies S(z) = X(z)U(z) \tag{9}$$

$$\implies X(z) = \frac{S(z)}{U(z)} \tag{10}$$

where,

$$U(z) = \frac{1}{1 - z^{-1}} \tag{11}$$

Using (11) in (10),

$$X(z) = p\left(\frac{z^{-1}}{(1 - z^{-1})}\right) + q\left(\frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^2}\right) \tag{12}$$

Simplifying using partial fractions, we get:

$$X(z) = (q - p) + \frac{p - 3q}{1 - z^{-1}} + \frac{2q}{(1 - z^{-1})^2} \tag{13}$$

$$= (q - p) + \frac{(p - q)}{1 - z^{-1}} + \frac{2qz^{-1}}{(1 - z^{-1})^2} \tag{14}$$

Taking the inverse Z-Transform,

$$x(n) = (q - p)\delta(n) + (p - q)u(n) + 2qnu(n) \tag{15}$$

To simplify, use first term:

$$s(1) = x(0) \tag{16}$$

$$\implies p + q = (q - p)\delta(0) + (p - q)u(0) + 2qnu(0) \tag{17}$$

$$\implies p = -q \tag{18}$$

because $\delta(0) = 1$ and $u(0) = 1$

$\therefore$ rewriting (15):

$$x(n) = 2q((n - 1)u(n) + \delta(n)) \tag{19}$$

Common difference is given by:

$$d = x(n + 1) - x(n) \tag{20}$$

$$= 2q(nu(n + 1) + \delta(n + 1)) - 2q((n - 1)u(n) + \delta(n)) \tag{21}$$

$$= 2q \tag{22}$$

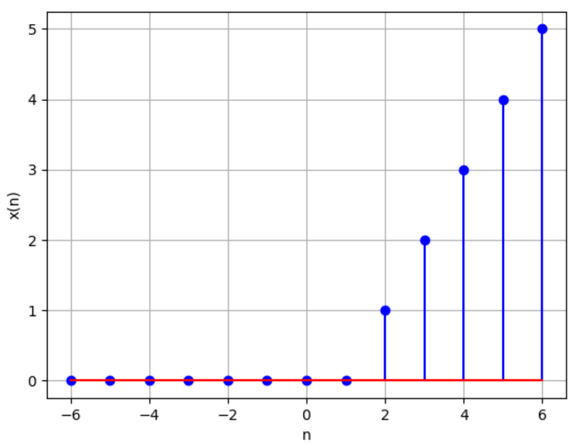

Fig. 1: Plot of x(n) vs n for $q = 0.5$