NCERT Math 11.9.2 Q8

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Question: If the sum of n terms of an AP is $(pn + qn^2)$, where p and q are constants, find the common difference.

Solution:

Symbol	Value	Description
$y(n)$	$(pn + qn^2)$	Sum of n terms
$x(n)$		n^{th} term of AP
d	$x(n+1) - x(n)$	Common Difference

TABLE I: Given Parameters

Sum of n terms, as a discrete signal:

$$y(n) = (pn + qn^2)u(n) \tag{1}$$

Taking the Z-Transform,

1) $\mathcal{Z}\{u(n)\}$

$$u(n) \xleftrightarrow{\mathcal{Z}} \frac{1}{1 - z^{-1}} \{|z| > 1\} \tag{2}$$

2) $\mathcal{Z}\{nu(n)\}$

$$nu(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}}{(1 - z^{-1})^2} \{|z| > 1\} \tag{3}$$

3) $\mathcal{Z}\{n^2 u(n)\}$

$$n^2 u(n) \xleftrightarrow{\mathcal{Z}} \frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3} \{|z| > 1\} \tag{4}$$

Taking the Z-Transform of (1) using (3) and (4)

$$Y(z) = p\left(\frac{z^{-1}}{(1 - z^{-1})^2}\right) + q\left(\frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3}\right) \tag{5}$$

Now,

$$y(n) = x(n) * u(n) \tag{6}$$
$$\implies Y(z) = X(z)U(z) \tag{7}$$
$$\implies X(z) = \frac{Y(z)}{U(z)} \tag{8}$$

Using (2) in (8),

$$X(z) = p\left(\frac{z^{-1}}{(1 - z^{-1})}\right) + q\left(\frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^2}\right) \tag{9}$$

Using contour integration for inverse Z-Transform:

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \tag{10}$$

$$= \frac{1}{2\pi j} \oint_C \left[p\left(\frac{z^{-1}}{(1 - z^{-1})}\right) + q\left(\frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^2}\right) \right] z^{n-1} dz \tag{11}$$

Calculating the residues R_1 and R_2 at pole $z = 1$:

$$R_1 = \frac{1}{0!} \lim_{z \to 1} (z - 1)\left(p\left(\frac{z^{-1}}{1 - z^{-1}}\right)\right) z^{n-1} \tag{12}$$

$$= p \tag{13}$$

$$R_2 = \frac{1}{1!} \lim_{z \to 1} \frac{d}{dz}\left((z-1)^2 q\left(\frac{z^{-1}(1 + z^{-1})}{(1 - z^{-1})^2}\right)\right) z^{n-1} \tag{14}$$

$$= q \lim_{z \to 1} \frac{d}{dz}\left(z^n + z^{n-1}\right) \tag{15}$$

$$= q(2n - 1) \tag{16}$$

$$\implies x(n) = R_1 + R_2 \tag{17}$$

$$= p + q(2n - 1) \tag{18}$$

Writing x(n) as a discrete signal we get:

$$x(n) = (p - q)u(n) + 2qnu(n) \tag{19}$$

To simplify, use $n = 0$:

$$y(0) = x(0) \tag{20}$$
$$\implies 0 = (p - q)u(0) + 2q(0)u(0) \tag{21}$$
$$\implies p = q \tag{22}$$

$\therefore$ (19) an be written as:

$$x(n) = 2qnu(n) \tag{23}$$

Common difference is given by:

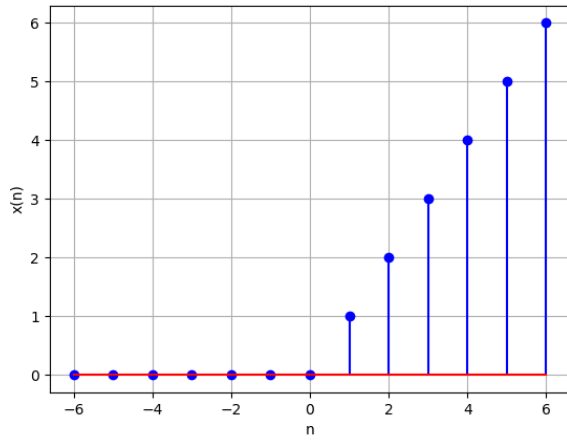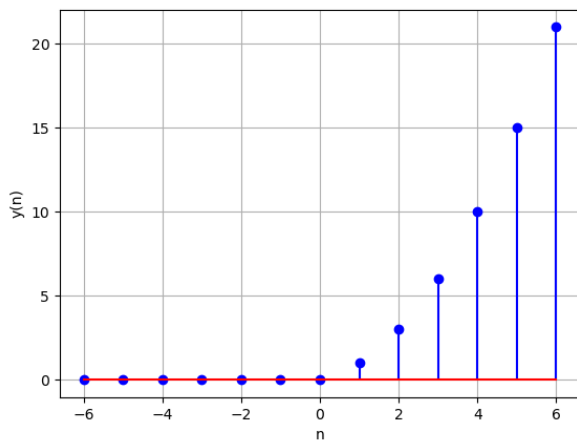$$d = x(n + 1) - x(n) \tag{24}$$
$$= 2q \tag{25}$$

Fig. 1: Plot of x(n) vs n for $p = q = 0.5$

Fig. 2: Plot of y(n) vs n for $p = q = 0.5$