

GATE 2023 PH Q37

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Question: An input voltage in the form of a square wave of frequency 1 kHz is given to a circuit, which results in the output shown schematically below. Which one of the following options is the CORRECT representation of the circuit?

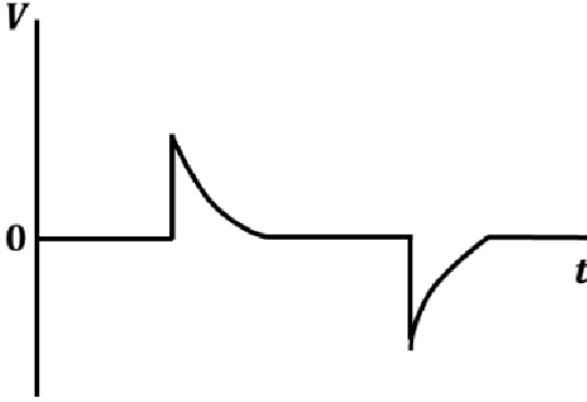
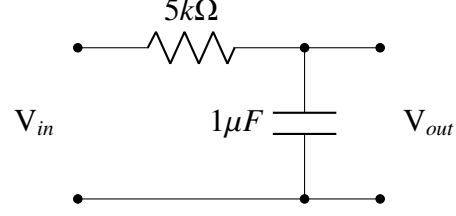


Fig. 1



Solution:

Symbol	Value	Description
$V_{in}(t)$		Input Voltage
$\mathcal{V}_{in}(j\omega)$		Fourier Transform of $V_{in}(t)$
$V_{out}(t)$		Output Voltage
$\mathcal{V}_{out}(j\omega)$		Fourier Transform of $V_{out}(t)$
f	$\frac{\omega}{2\pi} = 1000\text{Hz}$	Input Wave Frequency
T	$\frac{2\pi}{\omega} = 10^{-3}\text{s}$	Input Wave Time Period
R	(a) $0.5\text{k}\Omega$ (b) $5\text{k}\Omega$	Resistance
C	(a) $0.1\mu\text{F}$ (b) $1\mu\text{F}$	Capacitance
τ	RC	Time Constant
Z	$R + \frac{1}{j\omega C}$	Impedance
$H(j\omega)$	$\frac{V_{out}}{V_{in}}$	General Transfer Function
$H_R(j\omega)$	$\frac{V_{R,out}}{V_{in}}$	Transfer Function for Resistor
$H_C(j\omega)$	$\frac{V_{C,out}}{V_{in}}$	Transfer Function for Capacitor

TABLE I: Given Parameters

Input waveform is a square wave (Fig. 2), so we take its Fourier Transform

$$V_{in}(t) = 2 \left(2 \left[\frac{\left(t - \frac{T}{4}\right)}{T} \right] - \left[\frac{2\left(t - \frac{T}{4}\right)}{T} \right] \right) + 1 \quad (1)$$

Fourier Series Coefficient:

$$c_k = \frac{1}{T} \int_T V_{in}(t) e^{-jk\omega t} dt \quad (2)$$

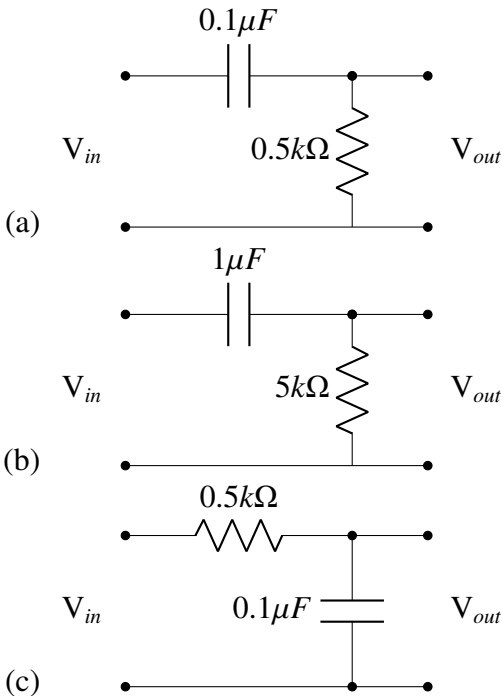
As square wave is even, $\sin(k\omega t)$ terms become zero. Cosine coefficients are:

$$a_n = \frac{2}{T} \int_T V_{in}(t) \cos\left(\frac{2\pi n t}{T}\right) dt \quad (3)$$

$$= \frac{4}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos(n\pi) \quad (4)$$

Fourier Series of $V_{in}(t)$, visualized in Fig. 2:

$$V_{in}(t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T}\right) \quad (5)$$



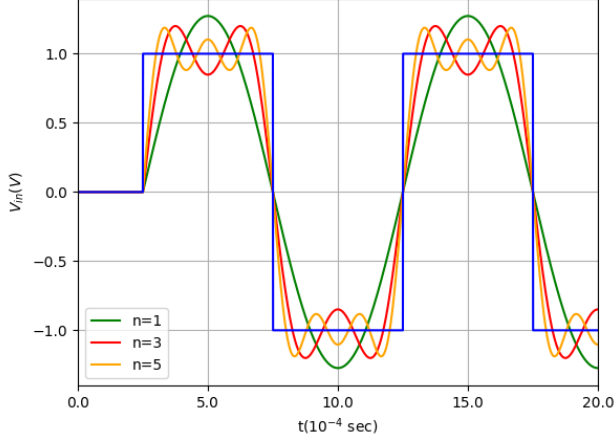


Fig. 2: Input Square Waveform ($V_{in}(t)$)

Taking Fourier Transform of $V_{in}(t)$:

$$V_{in}(t) \xleftrightarrow{\mathcal{F}} \mathcal{V}_{in}(j\omega) \quad (6)$$

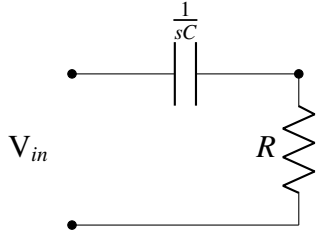


Fig. 3: Series RC Circuit in s-domain

$$s = j\omega \quad (7)$$

$$\Rightarrow Z = R + \frac{1}{sC} \quad (8)$$

$$= R + \frac{1}{j\omega C} \quad (9)$$

$\mathcal{V}_{in}(j\omega)$ was input into all four circuits and Inverse Fourier Transform was taken of the response. Transfer Function:

$$H(j\omega) = \frac{V_{out}}{V_{in}} \quad (10)$$

1) Option A

$$H_R(j\omega) = \frac{R}{R + \frac{1}{j\omega C}} \quad (11)$$

$$= \frac{j\omega RC}{1 + j\omega RC} \quad (12)$$

$$= \left(\frac{\omega RC}{\sqrt{1 + (\omega RC)^2}} \right) e^{j \tan^{-1}(\frac{1}{\omega RC})} \quad (13)$$

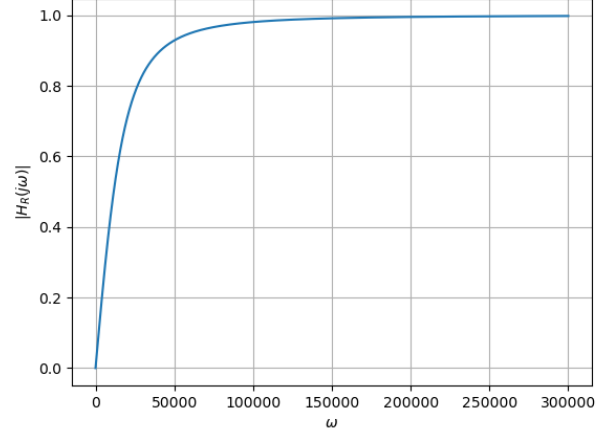


Fig. 4: $|H_R(j\omega)|$ vs ω for $R = 0.5k\Omega$, $C = 0.1\mu F$

$$\Rightarrow \mathcal{V}_{out}(j\omega) = H_R(j\omega) \mathcal{V}_{in}(j\omega) \quad (14)$$

$$\Rightarrow V_{out}(t) = \mathcal{F}^{-1} \{H_R(j\omega) \mathcal{V}_{in}(j\omega)\} \quad (15)$$

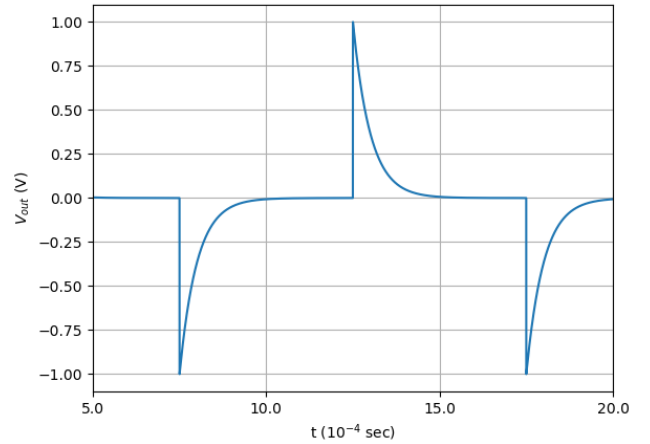


Fig. 5: Opt A: $V_{out}(t)$ vs t

2) Option B

$$H_R(j\omega) = \frac{R}{R + \frac{1}{j\omega C}} \quad (16)$$

$$= \frac{j\omega RC}{1 + j\omega RC} \quad (17)$$

$$= \left(\frac{\omega RC}{\sqrt{1 + (\omega RC)^2}} \right) e^{j \tan^{-1}(\frac{1}{\omega RC})} \quad (18)$$

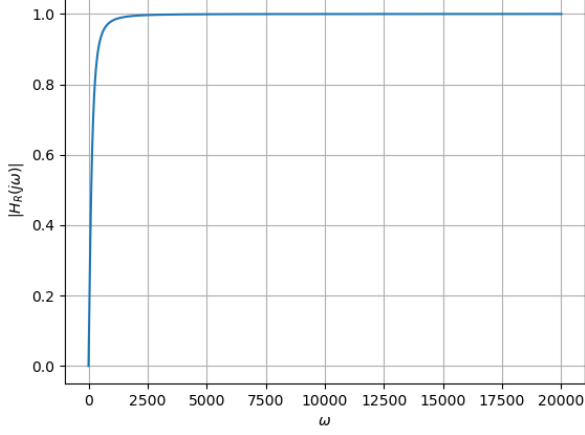


Fig. 6: $|H_R(j\omega)|$ vs ω for $R = 5k\Omega$, $C = 1\mu F$

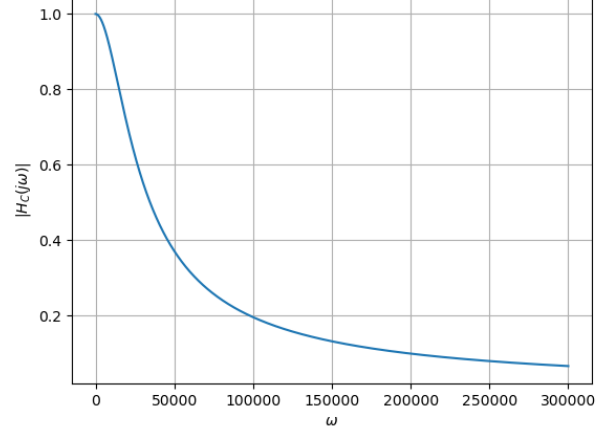


Fig. 8: $|H_C(j\omega)|$ vs ω for $R = 0.5k\Omega$, $C = 0.1\mu F$

$$\Rightarrow \mathcal{V}_{out}(j\omega) = H_R(j\omega)\mathcal{V}_{in}(j\omega) \quad (19)$$

$$\Rightarrow V_{out}(t) = \mathcal{F}^{-1}\{H_R(j\omega)\mathcal{V}_{in}(j\omega)\} \quad (20)$$

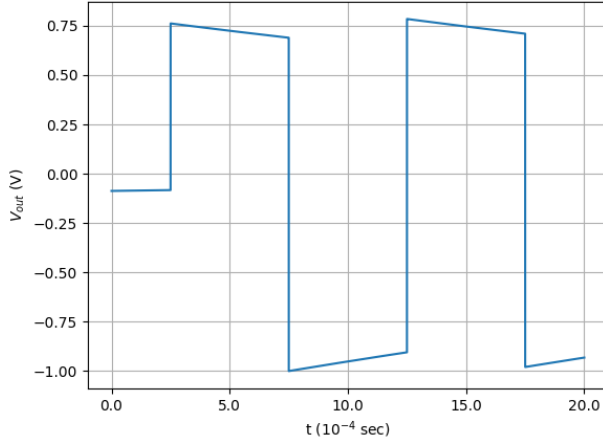


Fig. 7: Opt B: $V_{out}(t)$ vs t

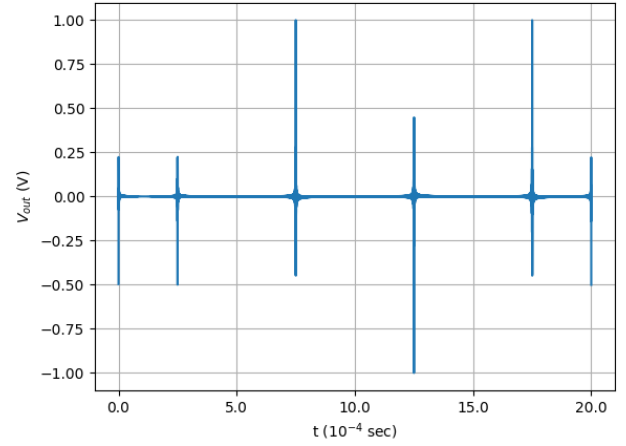


Fig. 9: Opt C: $V_{out}(t)$ vs t

3) Option C

$$H_C(j\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \quad (21)$$

$$= \frac{1}{1 + j\omega RC} \quad (22)$$

$$= \left(\frac{1}{\sqrt{1 + (\omega RC)^2}} \right) e^{-j \tan^{-1}(\omega RC)} \quad (23)$$

$$\Rightarrow \mathcal{V}_{out}(j\omega) = H_C(j\omega)\mathcal{V}_{in}(j\omega) \quad (24)$$

$$\Rightarrow V_{out}(t) = \mathcal{F}^{-1}\{H_C(j\omega)\mathcal{V}_{in}(j\omega)\} \quad (25)$$

4) Option D

$$H_C(j\omega) = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} \quad (26)$$

$$= \frac{1}{1 + j\omega RC} \quad (27)$$

$$= \left(\frac{1}{\sqrt{1 + (\omega RC)^2}} \right) e^{-j \tan^{-1}(\omega RC)} \quad (28)$$

$$\Rightarrow \mathcal{V}_{out}(j\omega) = H_C(j\omega)\mathcal{V}_{in}(j\omega) \quad (29)$$

$$\Rightarrow V_{out}(t) = \mathcal{F}^{-1}\{H_C(j\omega)\mathcal{V}_{in}(j\omega)\} \quad (30)$$

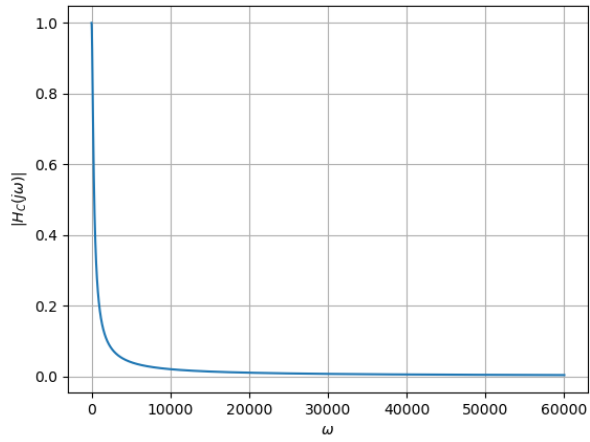


Fig. 10: $|H_C(j\omega)|$ vs ω for $R = 5k\Omega$, $C = 1\mu F$

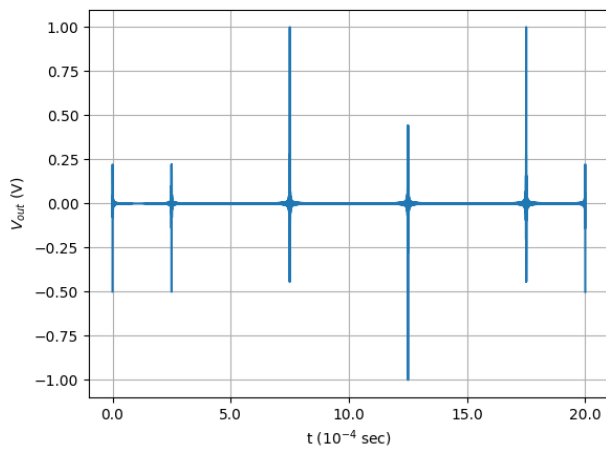


Fig. 11: Opt D: $V_{out}(t)$ vs t