

# Filter Design

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## 1 Introduction

We are supposed to design the equivalent FIR and IIR filter realizations for a band-pass filter with pass band: 9.4 kHz and 10.6 kHz.

## 2 Filter Specifications

The sampling rate for the filter has been specified as  $F_s = 48$  kHz.

### 2.1 The Digital Filter

1. *Tolerances:* The pass-band ( $\delta_1$ ) and stop-band ( $\delta_2$ ) tolerances are given to be equal, so we let  $\delta_1 = \delta_2 = \delta = 0.15$ .
2. *Pass-band:* The pass-band of the filter is calculated as follows:

$$j = (r - 11000) \bmod \sigma$$

where,

$$\begin{aligned} r &= \text{Roll Number} \\ &= 11009 \\ \sigma &= \text{Sum of digits of roll number} \\ &= 11 \\ \implies j &= 9 \end{aligned}$$

The pass band is then given by:

$$\begin{aligned} &4 + 0.6(j) \text{ to } 4 + 0.6(j + 2) \\ \implies &9.4\text{kHz to } 10.6\text{kHz} \end{aligned}$$

Hence, the un-normalized discrete time filter pass-band frequencies are:

$$\begin{aligned} F_{p1} &= 10.6 \text{ kHz} \\ F_{p2} &= 9.4 \text{ kHz} \end{aligned}$$

If the un-normalized discrete-time (natural) frequency is  $F$ , the corresponding normalized digital filter (angular) frequency is given by  $\omega = 2\pi \left( \frac{F}{F_s} \right)$ . Therefore,

$$\omega_{p1} = 2\pi \frac{F_{p1}}{F_s} = 0.441\pi$$

$$\omega_{p2} = 2\pi \frac{F_{p2}}{F_s} = 0.391\pi$$

The centre frequency is then given by  $\omega_c = \frac{\omega_{p1} + \omega_{p2}}{2} = 0.416\pi$ .

3. *Stop-band*: The *transition band* for band-pass filters is  $\Delta F = 0.3$  kHz on either side of the pass-band. Hence, the un-normalized *stop-band* frequencies are:

$$F_{s1} = 10.6 + 0.3 = 10.9 \text{ kHz}$$

$$F_{s2} = 9.4 - 0.3 = 9.1 \text{ kHz}$$

The corresponding normalized frequencies are:

$$\omega_{s1} = 0.454\pi$$

$$\omega_{s2} = 0.379\pi$$

The above parameters are summarized in the table below:

Paramter	Value	Description
$F_{p1}$	10.6 kHz	Unnormalized passband upper frequency
$F_{p2}$	9.4 kHz	Unnormalized passband lower frequency
$\omega_{p1}$	$0.441\pi$	Normalized passband upper frequency
$\omega_{p2}$	$0.391\pi$	Normalized passband lower frequency
$\Delta F$	0.3kHz	Transition Band
$F_{s1}$	10.9kHz	Unnormalized stopband upper frequency
$F_{s2}$	9.1kHz	Unnormalized stopband lower frequency
$\omega_{s1}$	$0.454\pi$	Normalized stopband upper frequency
$\omega_{s2}$	$0.379\pi$	Normalized stopband lower frequency

Table 1: Input Paramters

## 2.2 The Analog filter

In the bilinear transform, the analog filter frequency ( $\Omega$ ) is related to the corresponding digital filter frequency ( $\omega$ ) as  $\Omega = \tan \frac{\omega}{2}$ . Using this relation, we

obtain the analog pass-band and stop-band frequencies as:

$$\Omega_{p1} = \tan\left(\frac{\omega_{p1}}{2}\right) = 0.8298$$

$$\Omega_{p2} = \tan\left(\frac{\omega_{p2}}{2}\right) = 0.7051$$

$$\Omega_{s1} = \tan\left(\frac{\omega_{s1}}{2}\right) = 0.8649$$

$$\Omega_{s2} = \tan\left(\frac{\omega_{s2}}{2}\right) = 0.6772$$

respectively.

### 3 The IIR Filter Design

*Filter Type:* We are supposed to design filters whose stop-band is monotonic and pass-band equiripple. Hence, we use the *Chebyshev approximation* to design our band-pass IIR filter.

#### 3.1 The Analog Filter

1. *Low Pass Filter Specifications:* If  $H_{a,BP}(j\Omega)$  be the desired analog band pass filter, with the specifications provided in Section 2.2, and  $H_{a,LP}(j\Omega_L)$  be the equivalent low pass filter, then

$$\Omega_L = \frac{\Omega^2 - \Omega_0^2}{B\Omega} \quad (1)$$

where

$$\begin{aligned} \Omega_0 &= \sqrt{\Omega_{p1}\Omega_{p2}} = 0.7649 \\ B &= \Omega_{p1} - \Omega_{p2} = 0.1247 \\ \Rightarrow \Omega_{L_{p1}} &= \frac{\Omega_{p1}^2 - \Omega_{p1}}{\Omega_{p2}} (\Omega_{p1} - \Omega_{p2})\Omega_{p1} = 1 \end{aligned}$$

The low pass filter has the pass-band edge at  $\Omega_{Lp} = 1$  and stop-band edges at

$$\begin{aligned} \Omega_{L_{s1}} &= \frac{\Omega_{s1}^2 - \Omega_0^2}{B\Omega_{s1}} = 1.5111 \\ \Omega_{L_{s2}} &= \frac{\Omega_{s2}^2 - \Omega_0^2}{B\Omega_{s2}} = -1.4976 \end{aligned}$$

We choose the stop-band edge of the analog low pass filter as  $\Omega_{Ls} = \min(|\Omega_{L_{s1}}|, |\Omega_{L_{s2}}|) = 1.4976$ .

2. *The Low Pass Chebyshev Filter Parameters:* The magnitude squared of the Chebyshev low pass filter is given by

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \epsilon^2 c_N^2 \left( \frac{\Omega_L}{\Omega_{Lp}} \right)} \quad (2)$$

where  $c_N(x)$  is the Chebyshev Polynomial of the first kind of order  $N$ , given by:

$$c_N(x) = \begin{cases} \cos(N \cos^{-1}(x)) & \text{if } |x| \leq 1 \\ \cosh(N \cosh^{-1}(x)) & \text{if } x \geq 1 \end{cases}$$

where  $N$  and  $\epsilon$  are design parameters. Since  $\Omega_{Lp} = 1$ , (2) may be rewritten as

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \epsilon^2 c_N^2(\Omega_L)} \quad (3)$$

Also, the design parameters have the following constraints

$$\begin{aligned} \frac{\sqrt{D_2}}{c_N(\Omega_{Ls})} &\leq \epsilon \leq \sqrt{D_1}, \\ N &\geq \left\lceil \frac{\cosh^{-1} \sqrt{D_2/D_1}}{\cosh^{-1} \Omega_{Ls}} \right\rceil, \end{aligned} \quad (4)$$

where

$$\begin{aligned} D_1 &= \frac{1}{(1 - \delta)^2} - 1 = 0.384 \\ D_2 &= \frac{1}{\delta^2} - 1 = 43.444 \end{aligned}$$

After substituting  $\delta = 0.15$  and  $\Omega_{Ls} = 1.4976$ , we obtain  $N \geq 4$  and  $0.2897 \leq \epsilon \leq 0.6197$ . In Fig. 1, we plot  $|H(j\Omega)|$  for a range of values of  $\epsilon$ , for  $N = 4$ . We find that for larger values of  $\epsilon$ ,  $|H(j\Omega)|$  decreases in the transition band. We choose  $\epsilon = 0.4$  for our IIR filter design.

Listing 1: Code for Figure 1

```
wget https://github.com/aroshishp/EE1205/blob/main/Filter_Design/
codes/low_pass.py
```

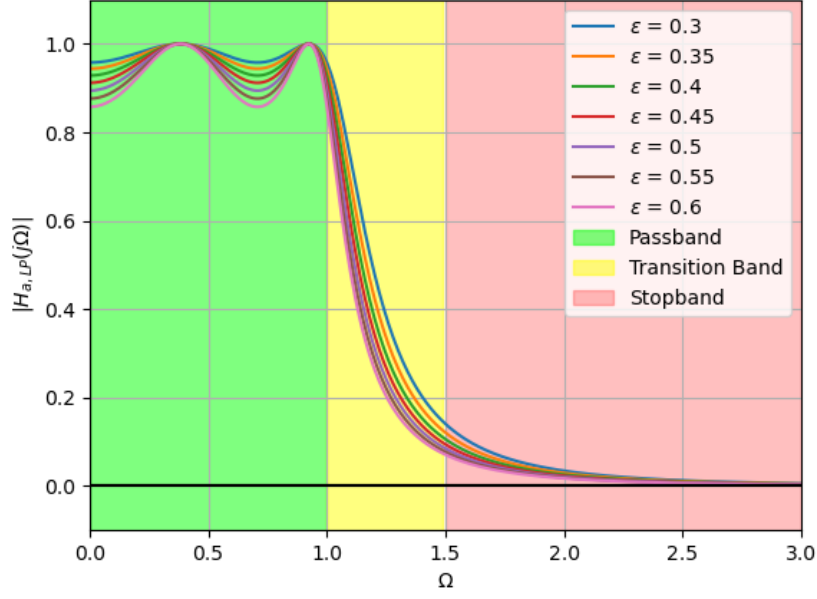


Figure 1: Analog Low-Pass Frequency Response for  $0.3 \leq \epsilon \leq 0.6$

3. *The Low Pass Chebyshev Filter:* Thus, we obtain

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + 0.16c_4^2(\Omega_L)} \quad (5)$$

where,

$$c_4(x) = 8x^4 - 8x^2 + 1. \quad (6)$$

The poles of the frequency response in (5) lying on the left half of the Argand Plane are in general obtained as

$$s_k = r_1 \cos(\phi_k) + jr_2 \sin(\phi_k) \quad (7)$$

where

$$\begin{aligned} \phi_k &= \frac{\pi}{2} + \frac{(2k+1)\pi}{2N}, k = 0, 1, \dots, N-1 \\ r_1 &= \frac{\beta^2 - 1}{2\beta}, r_2 = \frac{\beta^2 + 1}{2\beta}, \beta = \left[ \frac{\sqrt{1 + \epsilon^2} + 1}{\epsilon} \right]^{\frac{1}{N}} \end{aligned} \quad (8)$$

Values of  $s_k$  using (7) are given in the table below:

$s_k$	Value
$s_0$	$-0.162 + 1.003j$
$s_1$	$-0.391 + 0.415j$
$s_2$	$-0.391 - 0.415j$
$s_3$	$-0.162 - 1.003j$

Table 2: Poles on left half of complex plane

$s_k$	Value
$s_0$	$-0.162 + 1.003j$
$s_1$	$-0.391 + 0.415j$
$s_2$	$-0.391 - 0.415j$
$s_3$	$-0.162 - 1.003j$
$s_4$	$0.162 + 1.003j$
$s_5$	$0.391 + 0.415j$
$s_6$	$0.391 - 0.415j$
$s_7$	$0.162 - 1.003j$

Table 3: Poles of frequency response in (5)

Note that the poles of frequency response in (5) are calculated through the below code separately and plotted in Figure 2.

Listing 2: Code for Figure 2

```
wget https://github.com/aroishp/EE1205/blob/main/Filter_Design/
codes/poleplot.py
```

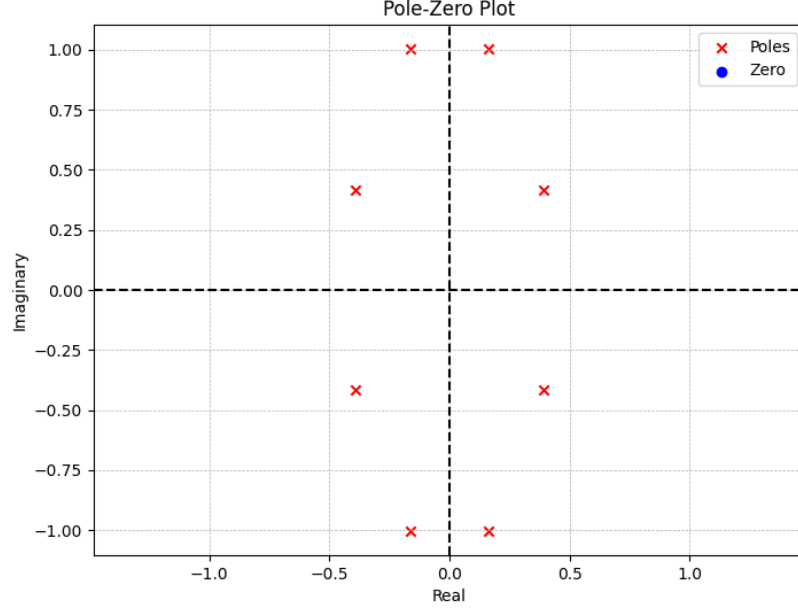


Figure 2: Pole-Zero Plot of frequency response of (5)

Thus, for even  $N$ , the low-pass stable Chebyshev filter, with a gain  $G_{LP}$  has the form

$$H_{a,LP}(s_L) = \frac{G_{LP}}{\prod_{k=1}^{\frac{N}{2}-1} (s_L^2 - 2r_1 \cos \phi_k s_L + r_1^2 \cos^2 \phi_k + r_2^2 \sin^2 \phi_k)} \quad (9)$$

$$= \frac{G_{LP}}{(s - s_0)(s - s_1)(s - s_2)(s - s_3)} \quad (10)$$

Substituting  $N = 4$ ,  $\epsilon = 0.4$  and  $H_{a,LP}(j) = \frac{1}{\sqrt{1+\epsilon^2}}$ , from (8) and (9), we obtain

$$H_{a,LP}(s_L) = \frac{0.3125}{s_L^4 + 1.1068s_L^3 + 1.6125s_L^2 + 0.9140s_L + 0.3366} \quad (11)$$

In Figure 3 we plot  $|H(j\Omega)|$  using (5) and (11), thereby verifying that our low-pass Chebyshev filter design meets the specifications.

Listing 3: Code for Figure 3

```
wget https://github.com/aroshishp/EE1205/blob/main/Filter_Design/
codes/verification.py
```

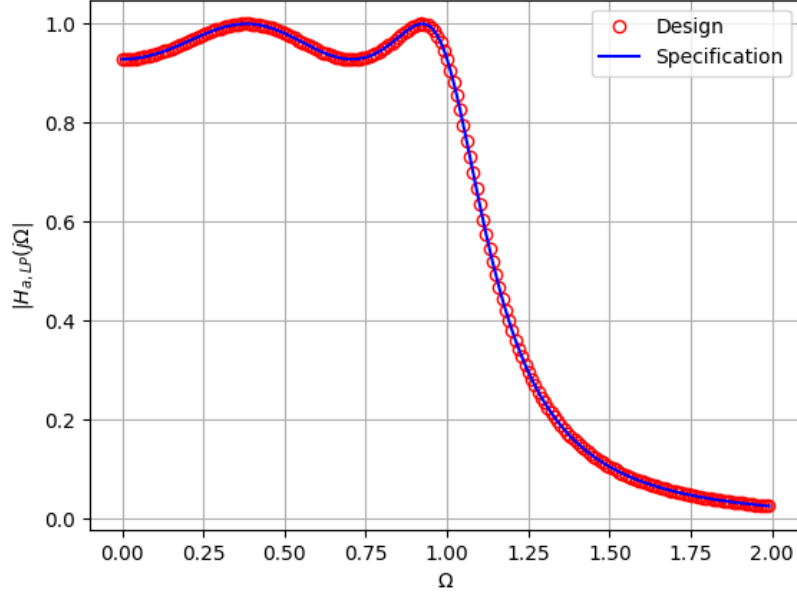


Figure 3: Plot of Frequency Response obtained from specifications in (5) and design in (11)

4. *The Band Pass Chebyshev Filter:* The analog band-pass filter is obtained from (11) as follows. From (1):

$$\Omega_L = \frac{\Omega^2 - \Omega_0^2}{B\Omega} \quad (12)$$

$$\Rightarrow j\Omega_L = \frac{\Omega_0^2 - \Omega^2}{Bj\Omega} \quad (13)$$

$$= \frac{\Omega_0^2 + (j\Omega)^2}{Bj\Omega} \quad (14)$$

$$\Rightarrow s_L = \frac{s^2 + \Omega_0^2}{Bs} \quad (15)$$

Hence,

$$H_{a,BP}(s) = G_{BP} H_{a,LP}(s_L) \Big|_{s_L = \frac{s^2 + \Omega_0^2}{Bs}}, \quad (16)$$

where  $G_{BP}$  is the gain of the band-pass filter. The code in Listing 5 evaluates  $G_{BP}$  and finds the coefficients of  $H_{a,BP}$ . We get  $G_{BP} = 1.077$  by evaluating gain such that  $H_{a,BP}(j\Omega_{p1}) = 1$ .

$$H_{a,BP}(s) = \frac{7.55642 \times 10^{-5} s^4}{s^8 + 0.138015s^7 + 2.36536s^6 + 0.244019s^5 + 2.08328s^4 + 0.142769s^3 + 0.809685s^2 + 0.0276411s + 0.117176} \quad (17)$$



In Figure 4, we plot  $|H_{a,BP}(j\Omega)|$  as a function of  $\Omega$  for both positive as well as negative frequencies. We find that the pass-band and stop-band frequencies in the figure match well with those obtained analytically through the bilinear transformation.

Listing 4: Code for Figure 4

```
wget https://github.com/aroshishp/EE1205/blob/main/Filter_Design/
codes/Hbp.py
```

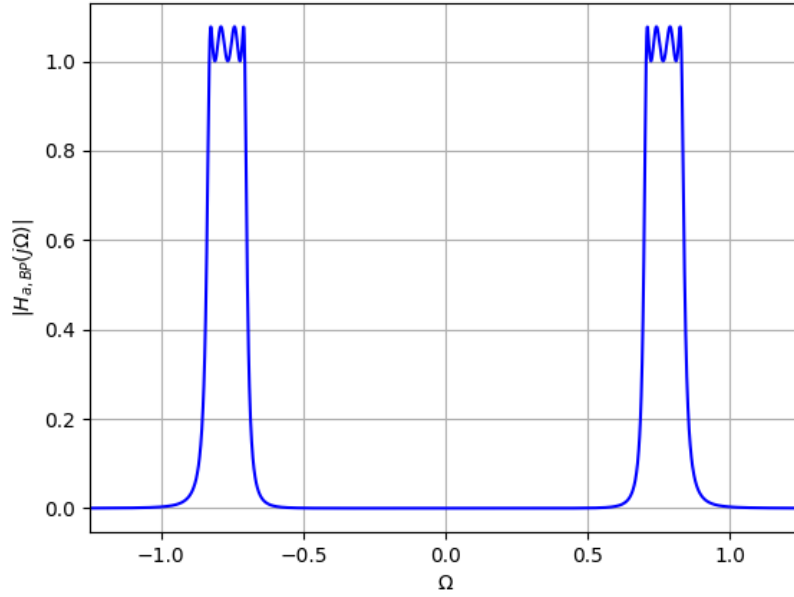


Figure 4: Analog Band-Pass Frequency Response from (17)

### 3.2 The Digital Filter

From the bilinear transformation, we obtain the digital band-pass filter from the corresponding analog filter as

$$H_{d,BP}(z) = GH_{a,BP}(s) \Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \quad (18)$$

where  $G$  is the gain of the digital filter. From (17) and (18), we obtain

$$H_{d,BP}(z) = G \frac{N(z)}{D(z)} \quad (19)$$

All coefficients and  $G$  are calculated from the below code.

Listing 5: Code for  $G$ ,  $N(z)$ ,  $D(z)$

```
wget https://github.com/aroshishp/EE1205/blob/main/Filter_Design/codes/
substitutor.py
```

We get:

$$G = 7.55642 \times 10^{-5} \quad (20)$$

$$N(z) = 1 - 4z^{-2} + 6z^{-4} - 4z^{-6} + z^{-8} \quad (21)$$

$$\begin{aligned} D(z) = & 5.8230569z^{-8} - 12.4205486z^{-7} + 34.1023786z^{-6} - 42.2777094z^{-5} \\ & + 58.95155z^{-4} - 44.1531786z^{-3} + 37.1935974z^{-2} - 14.1500354z^{-1} \\ & + 6.9279451 \end{aligned} \quad (22)$$

The plot of  $|H_{d,BP}(z)|$  with respect to the normalized angular frequency (normalizing factor  $\pi$ ) is available in Figure 5. Again we find that the pass-band and stop-band frequencies meet the specifications well enough.

Listing 6: Code for Figure 5

```
wget https://github.com/aroshishp/EE1205/blob/main/Filter_Design/codes/
Hdbp.py
```

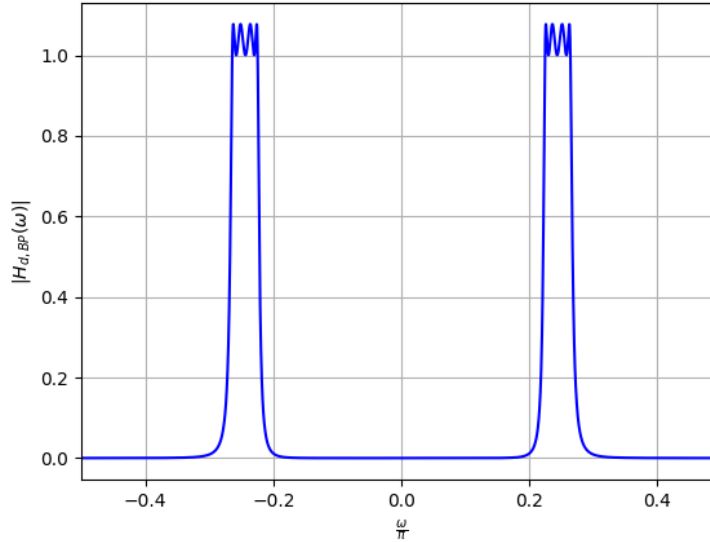


Figure 5: The magnitude response of the band-pass digital filter

## 4 The FIR Filter

We design the FIR filter by first obtaining the (non-causal) low-pass equivalent using the Kaiser window and then converting it to a causal band-pass filter.

### 4.1 The Equivalent Low-Pass Filter

The low-pass filter has a pass-band frequency  $\omega_l$  and transition band  $\Delta\omega = 2\pi \frac{\Delta F}{F_s} = 0.0125\pi$ . The stop-band tolerance is  $\delta$ .

1. The *pass-band frequency*  $\omega_l$  is defined as:

$$\omega_l = \frac{\omega_{p1} - \omega_{p2}}{2} = 0.025\pi \quad (23)$$

Substituting the values of  $\omega_{p1}$  and  $\omega_{p2}$  from section 2.1, we obtain  $\omega_l = 0.025\pi$ .

2. The *impulse response*  $h_{lp}(n)$  of the desired low-pass filter with cutoff frequency  $\omega_l$  is given by

$$h_l(n) = \frac{\sin(n\omega_l)}{n\pi} w(n), \quad (24)$$

where  $w(n)$  is the Kaiser window obtained from the design specifications.

### 4.2 The Kaiser Window

The Kaiser window is defined as

$$\begin{aligned} w(n) &= \frac{I_0 \left[ \beta N \sqrt{1 - \left( \frac{n}{N} \right)^2} \right]}{I_0(\beta N)}, & -N \leq n \leq N, & \beta > 0 \\ &= 0 & \text{otherwise,} & \end{aligned} \quad (25)$$

where  $I_0(x)$  is the modified Bessel function of the first kind of order zero in  $x$  and  $\beta$  and  $N$  are the window shaping factors. In the following, we find  $\beta$  and  $N$  using the design parameters in section 2.1.

1.  $N$  is chosen according to

$$N \geq \frac{A - 8}{4.57\Delta\omega}, \quad (26)$$

where  $A = -20 \log_{10} \delta$ . Substituting the appropriate values from the design specifications, we obtain  $A = 16.4782$  and  $N \geq 48$ .

2.  $\beta$  is chosen according to

$$\beta N = \begin{cases} 0.1102(A - 8.7) & A > 50 \\ 0.5849(A - 21)^{0.4} + 0.07886(A - 21) & 21 \leq A \leq 50 \\ 0 & A < 21 \end{cases} \quad (27)$$

In our design, we have  $A = 16.4782 < 21$ . Hence, from (27) we obtain  $\beta = 0$ .

3. We choose  $N = 100$ , to ensure the desired low pass filter response. Substituting in (25) gives us the rectangular window

$$\begin{aligned} w(n) &= 1, -100 \leq n \leq 100 \\ &= 0 \quad \text{otherwise} \end{aligned} \quad (28)$$

From (24) and (28), we obtain the desired low-pass filter impulse response

$$\begin{aligned} h_{lp}(n) &= \frac{\sin(\frac{n\pi}{40})}{n\pi} \quad -100 \leq n \leq 100 \\ &= 0, \quad \text{otherwise} \end{aligned} \quad (29)$$

Listing 7: Code for Figure 6 and 7

```
wget https://github.com/aroshishp/EE1205/blob/main/Filter_Design/codes/
fir_hH.py
```

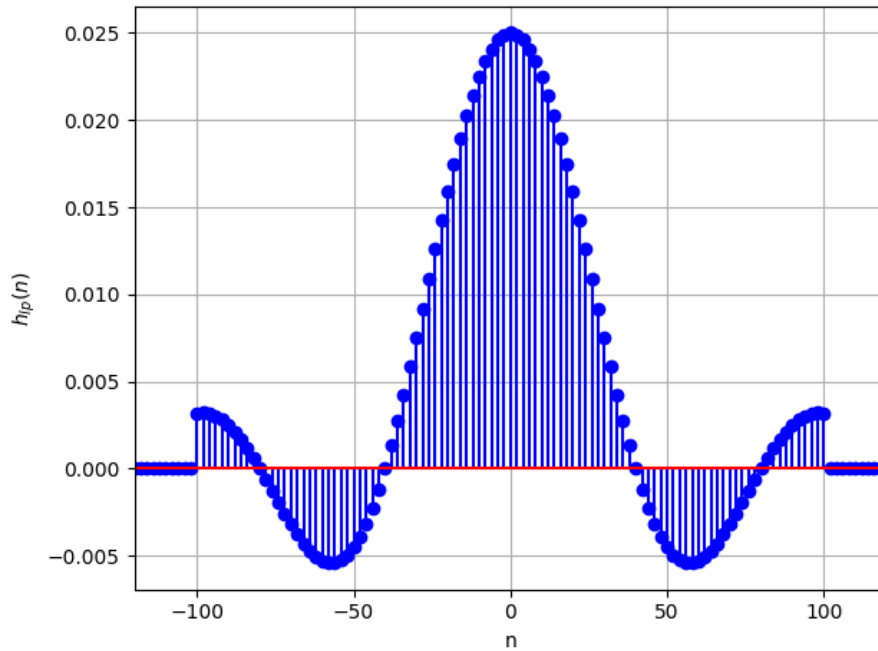


Figure 6: Plot of FIR Low-Pass Filter Impulse Response

The magnitude of the Frequency Response of low-pass filter is plotted in Figure 7.

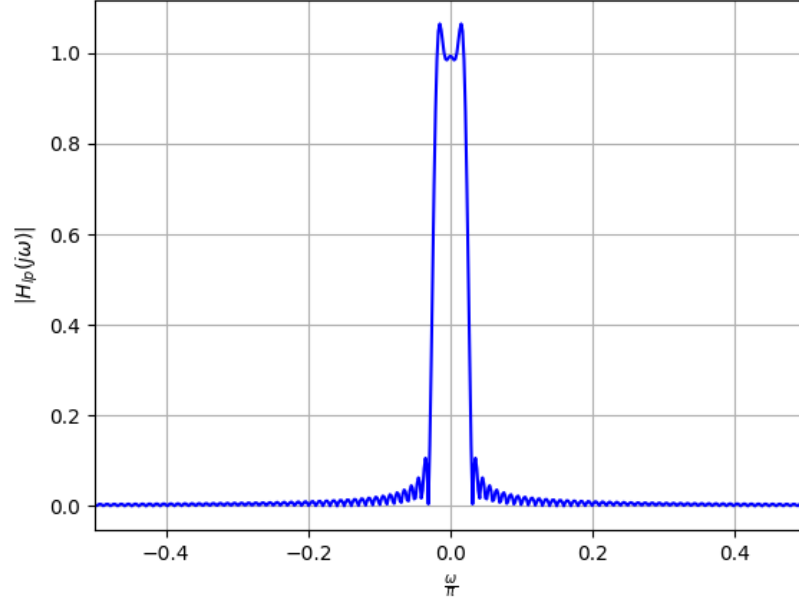


Figure 7: Plot of FIR Low-Pass Filter Frequency Response

### 4.3 The FIR Band-pass Filter

The centre of the pass-band of the desired band-pass filter was found to be  $\omega_c = 0.416\pi$  in Section 2.1. The impulse response of the desired band-pass filter is obtained from the impulse response of the corresponding low-pass filter as

$$h_{bp}(n) = 2h_{lp}(n)\cos(n\omega_c) \quad (30)$$

Thus, from (29), we obtain (plotted in Figure 8)

$$\begin{aligned} h_{bp}(n) &= \frac{2 \sin(\frac{n\pi}{40}) \cos(0.416n\pi)}{n\pi} & -100 \leq n \leq 100 \\ &= 0, & \text{otherwise} \end{aligned} \quad (31)$$

Listing 8: Code for Figure 8 and 9

```
wget https://github.com/aroshishp/EE1205/blob/main/Filter_Design/codes/
fir_bp.py
```

The magnitude response of the FIR band-pass filter designed to meet the given specifications is plotted in Figure 9.

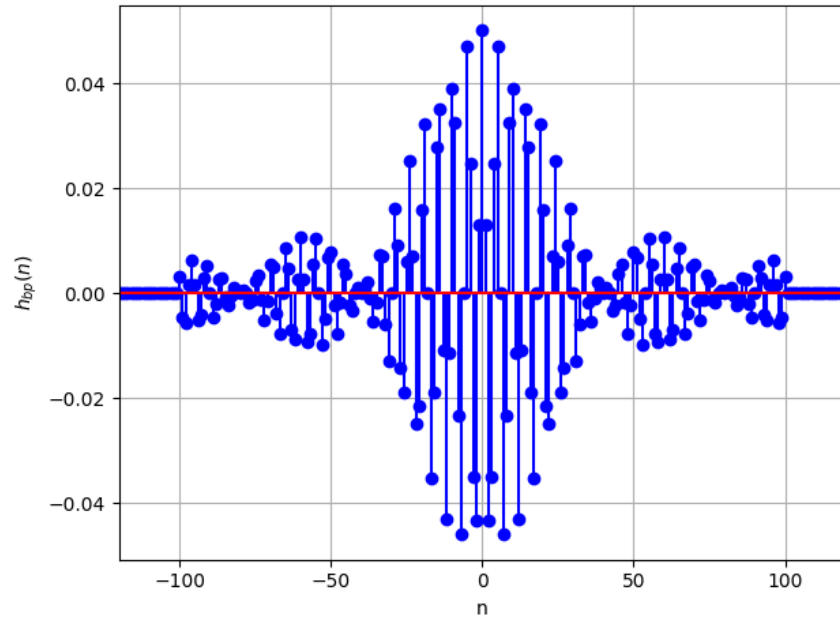


Figure 8: Plot of FIR Band-Pass Filter Impulse Response

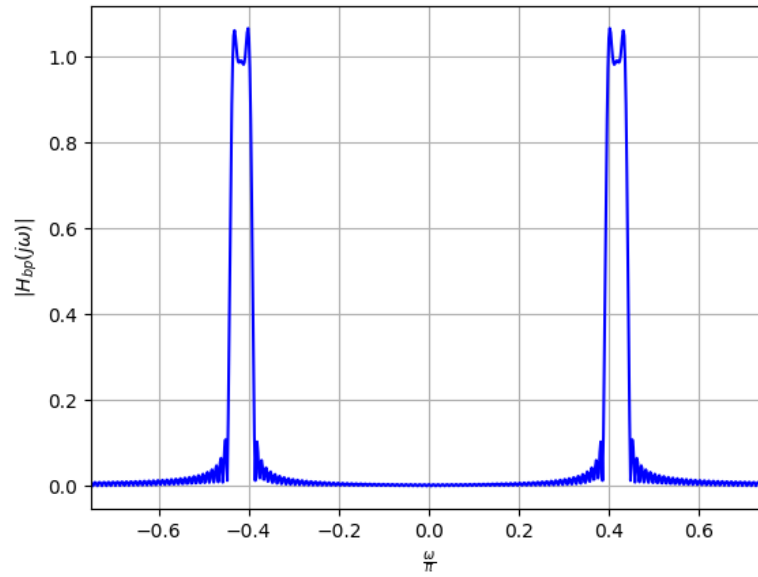


Figure 9: Plot of FIR Band-Pass Filter Frequency Response