Mathematics of Doing, Understanding, Learning, and Educating for Secondary Schools

MODULE(S^2) SAMPLER

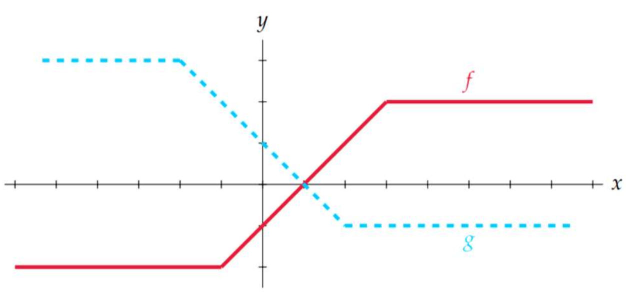
Curriculum Materials for Content Courses for Secondary Mathematics Teachers

Published Version: July 2022

This e-book is meant to be just a sampler of the full sets of free materials.

For more information about the MODULE(S^2) Project and access to the MODULE(S^2) materials please visit www.modules2.com

Algebra



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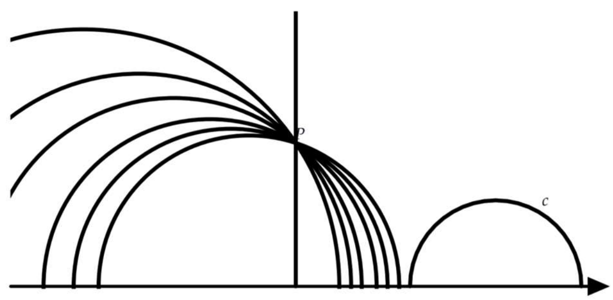
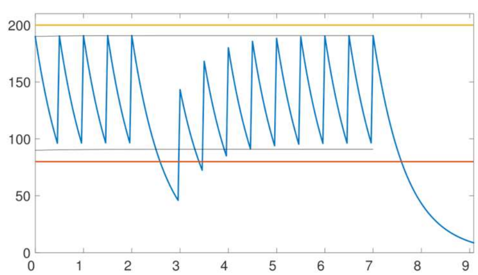
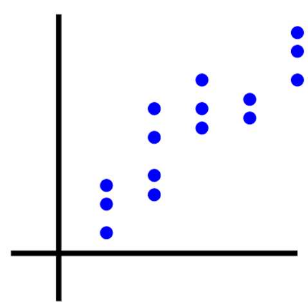


Math Modeling



Statistics



The Mathematics Of Doing, Understanding, Learning, and Educating for Secondary Schools (MODULE(S^2)) project is partially supported by funding from a collaborative grant of the National Science Foundation under Grant Nos. DUE-1726707, 1726804, 1726252, 1726723, 1726744, and 1726098. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

This sampler contains the general instructor preface materials for each topic and module, along with the student view of Lesson 1 for each topic and module. Lesson 1 usually takes about 75-150 minutes in the classroom. In the full materials, each module has about 8 lessons, for a total of about 4 class weeks. All together, the 3 modules for each topic would take approximately one semester for a 3-credit-hour class. Additionally, to the extent possible, each module is independent of other modules, so a college instructor could decide to use 1 module from each of 3 topics to form a capstone course, for example.

The materials are available for free; please see our website www.modules2.com to request access to them.

Sampler Table of Contents (clickable links)

Algebra Module 1: Concepts of Relations and Functions

Algebra Module 2: From Numbers, to Powers, to Logarithms

Algebra Module 3: Fields and Polynomials

Geometry Module 1: Axiomatic Systems

Geometry Module 2: Transformational Geometry

Geometry Module 3: Similarity

Modeling Module 1: The Process and Purpose of Mathematical Modeling

Modeling Module 2: Advancing Competency in Mathematical Modeling

Modeling Module 3: Diverse Perspectives in Mathematical Modeling

Statistics Module 1: Study Design and EDA

Statistics Module 2: Statistical Inference

Statistics Module 3: Statistical Association

Our project's OSF page is https://osf.io/t56em/ or https://doi.org/10.17605/OSF.IO/T56EM

Mathematics Of Doing, Understand, Learning, and Educating Secondary Schools

MODULE(S^2): Algebra for Secondary Mathematics Teaching

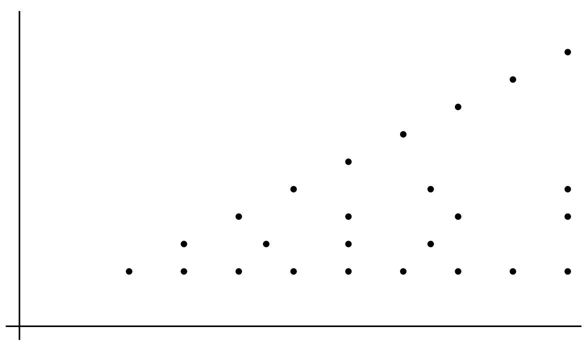
Module 1: Concepts of Relations and Functions

Yvonne Lai

James Hart

Published Version: July 2022

INSTRUCTOR VERSION



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Contents

Overview of MODULE(S^2) Project iii

Overview of Algebra MODULE(S^2) vii

Overview of Module 1: Concepts of Relations and Functions vii

I How We Talk About and Explore Math 1

1 Sets and Conjectures (Week 1) (Length: 2.5 hours) 4

1.1 Overview . . . 4
1.2 Opening inquiry: Number parents . . . 6
1.3 Reasoning with Sets . . . 9
1.4 Proving Conjectures About Sets . . . 13
1.4.1 Subset exploration . . . 13
1.4.2 Sample Proof . . . 15
1.4.3 Features of Good Proof Communication . . . 19
1.4.4 Your Turn to Prove . . . 21
1.5 Back to the opening inquiry . . . 23
1.6 Summary of mathematical practices . . . 27
1.7 Homework for Chapter 1 . . . 28

II Relations and Functions 32

2 Relations (Week 2) (Length: ~3 hours) 32

2.1 Overview . . . 32
2.2 Relations . . . 34
2.3 Inverse of a relation . . . 45
2.4 Composition of relations . . . 50
2.5 Working with graphs of relations . . . 56
2.6 Putting it all together: Investigating graphs of inverses . . . 62
2.7 Summary . . . 68
2.8 Homework for Chapter 2 . . . 69
2.8.1 Simulation of Practice Written Assignment on Intercepts . . . 73
2.8.2 Simulation of Practice Written Assignment on Notion of Inverse Functions (Optional) . . . 75

3 Functions: Teaching Definitions and Partial Inverses (Week 3: Length: ~6 hours) 77

3.1 Overview . . . 77
3.2 Review of key examples . . . 78

3.3 Teaching Definitions and Mathematical "Tests" 80
3.4 Invertible Functions 84
3.5 Homework for Chapter 3 93

4 Covariational view on function (Week 4: Length: ˜6 hours) 97
4.1 Noticing student thinking 100
4.2 Different ways to explain: Correspondence and covariation views 101
4.3 Revisiting a Key Example 104
4.4 Bottle Problem 105
4.5 Summary of mathematical/teaching practices 106
4.6 Homework for Chapter 4 107
4.6.1 Simulation of Practice Video Assignment (with Written component) 107

Overview of MODULE(S^2) Project

The NSF-funded MODULE(S^2) project offers materials to help develop pre-service teachers' mathematical knowledge as it relates to the work of teaching secondary school. We offer 4 topics, each meant for a 3-credit-hour semester-long class:

- Algebra from an Advanced Perspective (including functions, relations, and fields)
- College Geometry
- Mathematical Modeling
- Introductory Statistics

Each set of course materials consists of 3 Modules, each taking about 4 weeks of class time or a total of approximately one semester of instruction. The materials are meant for use in college mathematics/statistics content classes. They are not meant to be used in middle- or high-school classes. To the extent possible, each module is independent of other modules, so a college instructor could decide to use 1 module from each of 3 topics to form a capstone course, for example.

The materials use the term **student** to refer to K-12 students, and **teacher** to refer to both preservice and inservice teachers as well as any college student who is learning from these MODULE(S^2) materials. Those teaching with these materials (usually college/university faculty) are referred to as **instructors**.

The materials use a variety of active-learning methods rather than being traditional textbooks. They also include some discussion of five equity-based mathematics teaching practices (Aguirre, Mayfield-Ingram, & Martin, 2013):

- Going deep with mathematics
- Leveraging multiple mathematical competencies
- Affirming mathematics learners' identities
- Challenging spaces of marginality
- Drawing on multiple resources of knowledge

In writing the Modules, we have tried to cultivate the use of Prioritized Instructional Practices:

PRACTICE 1: Learning about student understanding using their explanations, justifications, and representations

PRACTICE 2: Generating questions and discussion that promote students exploring conjectures and justification

In each Module, teachers are required to complete two Simulations of (teaching) Practice. In one, they must videotape themselves responding as a teacher to hypothetical statements by students. These are called Video Simulations of Practice. In the other, called a Written Simulation of Practice, they are given a task and asked to design and envision a task -based classroom discussion. These are included in the exercise sections and noted as Simulations of Practice. Instructors are encouraged to give more time for teachers to complete the Simulations of Practice than they do for the other homework exercises.

Equitable Teaching Practices in MODULE(S^2) Curriculum

The curriculum produced by the Mathematics of Doing, Understanding, Learning and Educating for Secondary Schools (MODULE(S^2)) Project is an outgrowth of the work of the Mathematics Teacher Education Partnership (MTE-Partnership), a national collaborative working towards improving the number and quality of secondary mathematics teachers prepared in institutions of higher education. The MTE-Partnership is founded on a set of Guiding Principles (MTE-Partnership, 2014) which include a focus on transforming preparation programs so that prospective teachers develop teaching practices that "demonstrate a dedication to equitable pedagogy" (p. 5) and convey views that "mathematics is a living and evolving human endeavor" (p. 4). Equitable practice must include both appropriate equitable teaching practices and the development of learners' mathematical identities.

As such, the MODULE(S^2) curriculum materials include opportunities to engage in the use of teaching practices that support developing content knowledge in equitable ways as well as practices that support developing productive dispositions and identities in mathematics. The Mathematics Teaching Practices (NCTM, 2014) describe quality instructional practices with a focus on developing content knowledge of learners. This core set of eight research-based teaching practices support equitable teaching and are shown in the framework below (Figure 1). When implemented intentionally as interconnected practices in mathematics teaching, these practices provide space for the instructor to view all students as mathematical thinkers and to develop agency among the learners in the classroom (Berry, 2019).

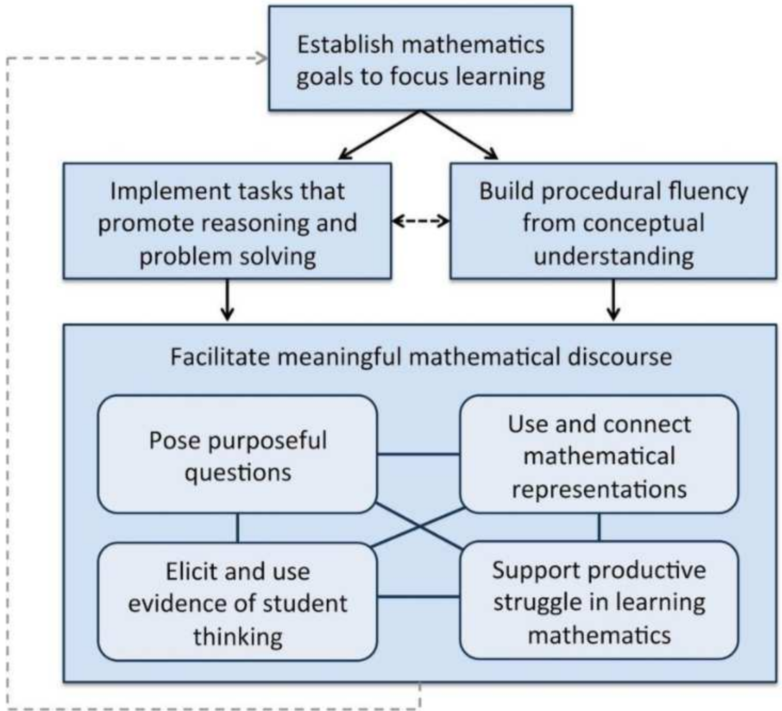

Figure 1: The Mathematics Teaching Framework demonstrating the interconnected nature of teaching practices that support equitable teaching. (Boston, Dillon, Smith, & Miller, 2017, p. 215).

In addition to implementing practices which focus on building content knowledge, the MODULE(S^2) materials are intended to “strengthen mathematical learning and cultivate positive student mathematical identity” (Aguirre, Mayfield-Ingram, & Martin, 2013, p. 43). Throughout the MODULE(S^2) materials, instructors will find opportunities to enact the five equity-based practices described in Table 1.

As instructors engage with the MODULE(S^2) curriculum materials, efforts should be made to utilize equitable teaching practices in order to engage prospective teachers in learning about mathematics and learning about teaching in equitable ways. Through the activities and practices contained in the MODULE(S^2) materials, instructors and students will have opportunities to reflect on the power dynamics inherent in the teaching and learning of mathematics and consider how that reflection might inform their practice. Instructors can find more detailed descriptions of these practices in the first three references below.

Equity Based Practices That Attend to Learners' Identities
Go deep with mathematics. • Support students in analyzing, comparing, justifying, and proving their solutions. • Engage students in frequent debates. • Present tasks that have high cognitive demand and include multiple solution strategies and representations.
Leverage multiple mathematical competencies. • Structure student collaboration to use varying math knowledge and skills to solve complex problems. • Present tasks that offer multiple entry points, allowing students with varying skills, knowledge, and levels of confidence to engage with the problem and make valuable contributions.
Affirm mathematics learners' identities. • Promote student persistence and reasoning during problem solving. • Encourage students to see themselves as confident problem solvers who can make valuable mathematical contributions. • Assume that mistakes and incorrect answers are sources of learning. • Explicitly validate students' knowledge and experiences as math learners. • Recognize mathematical identities as multifaceted, with contributions of various kinds illustrating competence.
Challenge spaces of marginality. • Center student authentic experiences and knowledge as legitimate intellectual spaces for investigation of mathematical ideas. • Position students as sources of expertise for solving complex mathematical problems and generating math-based questions to probe a specific issue or situation. • Distribute mathematics authority and present it as interconnected among students, teacher, and text. • Encourage student-to-student interaction and broad-based participation.
Draw on multiple resources of knowledge (math, culture, language, family, community). • Make intentional connections to multiple knowledge resources to support mathematics learning. • Use previous mathematics knowledge as a bridge to promote new mathematics understanding. • Tap mathematics knowledge and experiences related to students' culture, community, family, and history as resources. • Recognize and strengthen multiple language forms, including connections between math language and everyday language. • Affirm and support multilingualism.

Table 1: Adapted from Aguirre, J., Mayfield-Ingram, K., & Martin, D. B. (2013). *The impact of identity in K–8 mathematics: Rethinking equity-based practices.* Reston, VA: National Council of Teachers of Mathematics.

The MODULE(S^2) project's home page is https://modules2.com/

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The Robert Noyce Teacher Scholarship program is providing co-funding for this project in recognition of its alignment with the broader teacher preparation goals of the Noyce effort. The work is also partly supported by the Mathematics Teacher Education Partnership (MTEP) and the Association of Public and Land-grant Universities (APLU).

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Aguirre, J., Mayfield-Ingram, K., & Martin, D. B. (2013). The impact of identity in K-8 mathematics: Rethinking equity-based practices. Reston, VA: National Council of Teachers of Mathematics.

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National Council of Teachers of Mathematics (2014). Principles to actions: Ensuring mathematical success for all. Reston, VA: Author.

Overview of Algebra MODULE(S^2)

Through the material in the Algebra MODULE(S^2), we address mathematical foundations for teaching algebra at the secondary level. We begin in Module 1 with a treatment of relations and functions from an advanced perspective. This means defining relations and functions in such a way that inverses and graphs of functions are seen as a special case of inverses and graphs of relations, and then addressing covariation and correspondence approaches to describing functions. In Module 2, we address exponentiation, and its structural parallels to number systems more generally. In Module 3, we address links between abstract algebra and high school algebra, particularly through field properties, integer and polynomial arithmetic, and factors and roots of polynomials.

Overview of Module 1: Concepts of Relations and Functions

The theme of this module is functions and relations from an advanced perspective.

In Chapter 1, you will use the "parent relation" as a way to reflect on mathematical practices, learn how to work with each other, and gain intuition for sets, subsets, and negations.

In Chapter 2, you will see how the "parent relation" is an example of a relation. You will learn how to think about relations and functions in a way that connects middle school, high school, and university mathematical ideas. Here we will introduce functions and relations in general terms and then we will quickly specialize to the case where domain and codomain are subsets of real numbers.

In Chapters 3 - 4, you will dig deeper into the concept of inverse relations. You will have an opportunity to practice teaching mathematical procedures to help students understand why they work, and also learning about the difference between correspondence and covariation points of view, and how keeping both in mind when you teach can be very helpful.

Part I

How We Talk About and Explore Math

Welcome! We hope that this module will help give you opportunities to reflect on your own mathematical understandings – and how others may come to understand those ideas in different ways than you may have. In doing so, you will begin the process of thinking about mathematics so that you can understand your students' point of view and how to help them learn more deeply – whether in your future you are tutoring a niece or nephew or your own child, or you are a graduate student who is a TA, or you are a high school teacher, a middle school teacher, or an elementary school teacher.

What this class looks like. One common way that this module is used is as follows:

- At the beginning of each unit, you will work on an "Opening Inquiry". This inquiry will put into play some of the main concepts to be explored in the coming days.
- After reflecting on the ideas that surface from your work on the Opening Inquiry, you will work as a class to understand how these concepts are expressed formally and informally. Here, it can be good to reflect on why it is important to have **both** the formal and informal ways of thinking, not just one or the other, and why having both is important to students and teachers.
- In general, the module focuses on developing ideas through experiencing them in the form of problem solving.
- You are responsible for taking your own notes for the class. While some summary notes may be distributed each week, you should take notes on what is happening each lesson and organize them with any hand outs with in-class tasks so you can find things.
- Homework will be assigned on a weekly basis for most weeks. They will build on the ideas developed in class.

Types of goals. There are three overall kinds of learning goals in each lesson:

- Mathematical concepts – including definitions, theorems, procedures, and why they work.
- Mathematical/Teaching Practices – experiencing practices that cut across many aspects of mathematics or mathematics teaching, such as such as conjecturing, communicating with precision, leading a discussion, or introducing a mathematical procedure.
- Proof/Mathematical Structures – We use the phrase "Proof structures" to mean components needed in a proof of a certain kind of statement. E.g., showing membership in a set or showing that one set is a subset of another. The "components" are typically descriptions of the definition. E.g., "Showing $A \subseteq B$ requires showing that $x \in A$, then $x \in B$." We use the phrase "mathematical structures" to mean components needed to describe or define something. E.g., "To extend the definition of exponentiation from natural positive integers to powers of other real values means finding the definition for those other powers that is logically necessitated by the definition for positive integer powers and the properties of exponentiation."

Key for Instructor Notes.

Notes written in boxes are instructor notes. They are not visible in the student version.

?

A task given with a speech bubble indicates a task given orally. These are not contained in any worksheets, but they are visible in the student version of the text.

A task given inside grey brackets indicates a task that is provided in the *In-Class Resources* worksheets of each section. These worksheets are intended to be printed and distributed to students. There are also miniature versions of these worksheets in the student version.
⟨ *Notes for solutions are often provided within the tasks. They are not visible in the student version.* ⟩

Structure of lessons. Each lesson begins with an opening inquiry, then alternates between discussion of mathematics more formally and more tasks to explore ideas less formally. Each lesson ends with a summary. The reason for this summary is to help teachers process what just happened because sometimes it can be hard for learners to know immediately what it is that they have just experienced.
We have written some thoughts here that some piloters have found helpful:

- (A [Unit of Time] in the Life of Teaching the Algebra Modules) https://docs.google.com/document/d/1KC11eb0sb1pXRhJ4bPFn1qSWQjvj3x9m9tfi6lnZMBc/edit#heading=h.12egiwxwcyzm
- (Types of Learning Objectives in the Algebra Modules) https://docs.google.com/document/d/1Uc2OrjE2T1K89Y9N1ih0itM3dpZqXXW8LLdJEUiMMw0/edit#heading=h.tfkxj324wsda

Things to keep in mind on the first day. This first lesson is an important place to "set norms and expectations". This means communicating, both implicitly and explicitly, what productive conversation, exploration, questioning, and justification look and feel like. For instance, you may want to teach a class where:

- *Students embrace learning from their own individual and each others' work* – they view their own mistakes courageously and with an open mind; they accept that making errors and learning from them is a natural part of the mathematical process; they recognize what is worthwhile about others' reasoning and what needs further thought, and they do so constructively; they celebrate others' ideas.
- *Students view mathematical reasoning as the ultimate mathematical authority* – they have faith in their ability to learn to reason mathematically; they come back to the mathematics rather than to a perceived authority figure such as an instructor or a "smart" student to figure out what works; they seek precision in language while also understanding that going from informal language to precise language may take some time, may not happen right away, but is a valuable goal.
- *Students persist in seeking mathematical questions and answers* – they accept that setbacks are an important part of learning; they can work for an extended amount of time on one problem in productive ways; they celebrate when they do come to an understanding of a mathematical idea, especially one that is hard-won.

If these are values that you see a productive class expressing, there is much that you can do to foster these values beginning the first day. There are many different things you can do and say, and certainly different things may work better or worse for different instructors and different students. Here are some examples of things to do and say that have helped previous MODULE(S^2)instructors:

- *Praise thoughtful errors.* It's easy to spot "right" answers and there can be a temptation to run with the way that some students have found exactly the "right" way to approach a problem. There is also a temptation to respond to "wrong" answers with saying matter-of-factly, "Not quite; what did others get?" But if you respond in these ways, and exclusively so as your form of interacting with students about their thinking, what message does that send to students about the role of mistakes in the process of working through mathematics? It may well send the message that the best work is the work that is correct the first try, or worse, that the most worthy students are those that only do correct mathematics and make no mistakes. Instead, an alternative approach is to look for thoughtful errors – the kind of thinking that is ultimately mathematically incorrect for some reason, but where thinking through the mistake has the potential to really get at something fundamental about the mathematics at hand or in the future. Moves that you can make to acknowledge thoughtful errors might include:

  - "I am so glad that you brought that up, [student name]. Did everyone understand what [student name] said? Can someone say in their own words what they understand of [student name]'s reasoning?" [If someone raises their hand to counter this idea] "Right now we're not interested in whether we agree or disagree with [student name], we are trying to understand what [student name] is thinking. What might they thinking? Why does it make sense to do this?"
  - "Let's see what happens when we follow this reasoning."
  - "We just learned a really important lesson about doing mathematics because of this reasoning. Thank you, [student name], for sharing your idea. This was incredibly helpful. Let's remember the lesson we learned throughout today and also as we move forward in this class."

- *Do not make a big deal when students get a correct answer right away. Focus on the process of getting to the answer, and on understanding the answer, rather than the answer itself.* The Fields Medalist William Thurston (1994) observed of his colleagues, "I thought that what they sought was a collection of powerful proven theorems that might be applied to answer further mathematical questions. But that's only one part of the story. More than the knowledge, people want *personal understanding*." (p. 51, emphasis by Thurston). The same is true of students, or at least we would like to be a truth about students. Moves that emphasize understanding over the answer might include:

  - (As a matter-of-fact first reaction to the correct answer) "You answered X. What was your reasoning for that answer?" ... "What do others think of this reasoning?"
  - "[Student name] arrived at the solution X, and just shared their reasoning. Did anyone else arrive at this solution? Did you have similar reasoning or different reasoning?"
  - "Let's think back on why this answer makes sense."

- *Relinquish your authority to the students and the mathematics.* A common question instructors hear is, "What do you want?" or "Is this what you are looking for?" Sometimes the answer to these questions really does rest with you, the instructor – especially if it is about specific directions that you are setting for your students that can't be derived from mathematical reasoning. However, answering these questions from your authority as an instructor can be less useful if the questions are actually about mathematical reasoning, for instance, if the question is about whether a proof or solution is correct. In these cases, it can be more productive to return the responsibility of these questions to the students and the mathematics:

  - "Can you tell me more about how you arrived at this?"
  - "Tell me about what's here."
  - "How does this help to give a solution to the question we are working on?"
  - "How complete do you think it is?" ... "What about your work are you sure about, and what are you less sure about?"

- *Give students ways to work constructively with each other.* Working with each other on mathematics is not necessarily a natural skill; it is a learned skill. Help your students find ways to talk to each other about their thinking. While students are working, stir the pot (meaning, find ways to provoke productive disagreement and/or discussion).

  - "I see that [Student A] and [Student B] have different answers. It looks like you have something to resolve. [Student A] and [Student B], will you share how you did your work with each other and figure out what's really going on?"
  - "I see that [Student A] and [Student B] have arrived at the same answer, but it looks like you've done it in different ways. Will you compare what you've done and see how they match each other or do not?"
  - "It looks like [Student A] has drawn a graph and [Student B] has used mostly equations. Are you thinking about the same thing? Will you talk to each other about how your thinking matches up or not?"
  - "It looks like [Student A] worked on [Case 1] whereas [Student B] worked on [Case 2]. Are there more cases to consider? Are both cases necessary? You should talk to each other to figure this out."

Opening inquiry: Number parents

Activity 1 — Number Parents

Two numbers a and b are *parents* of another number c if $c = ab$. We will call c the *child* of a and b. Let's assume that a child cannot be one of its own parents.

Q1. What are all the pairs of parents of 12?

Q2. A student says $\{5, \frac{12}{5}\}$ is a pair of parents for 12. How would you respond to this student?

Q3. What numbers can never be parents?

Q4. Are there any numbers that have an infinite number of parent sets?

Q5. If you only want numbers to have a *finite* number of parent sets, what changes would you make in the definition?

Activity 2, Part 1—Refining Number Parents

Look over your work for Activity 1. What questions and comments do you have?

Let's refine our definition by saying:

From now on, number parents and children are always natural numbers.

Discuss:

- What are all the parents of 5?
- What are all the parents of 6?
- What are all the parents of 4?
- What are all the parents of 18?
- What do you *notice* about number parents and children? What mathematical questions do you *wonder* about for number parents and children?

Activity 2, Part 2—What does it mean to find a satisfying answer?

Two natural numbers a and b are *parents* of another natural number c if $c = ab$. We will call c the *child* of a and b. A child cannot be one of its own parents. Parents a and b can equal each other.

Which natural numbers have more than one set of parents?

Q1. Write your thoughts:

In middle and high school, a "factor" is assumed to be a natural number. We will make that assumption for now.

Q2. Here are some possible answers (without explanations) to the last question that prospective teachers in previous courses have given. On a 1 to 5 scale, rank these answers as least *satisfying* (1) to most *satisfying* (5).

A1. Any number with at least three different factors has more than one pair of parents.

A2. 12 has more than one pair of parents.

A3. Any number with at least three different factors (other than itself or 1) has more than one pair of parents. There are no other numbers with more than one pair of parents.

A4. 12, 18, 20, 28, 30, 42, 44 each have more than one pair of parents.

A5. Any number with at least three different factors (other than itself or 1) has more than one pair of parents.

Q3. What criteria did you use in deciding how satisfying each answer is?

Reasoning with Sets

Activity 3, Part 1 — Reasoning with Sets

When we are trying to answer a mathematical question, or when we are trying to improve an answer to a mathematical question, we are engaging in the process of *conjecturing*. A *conjecture* is a statement that arises from this process. A conjecture becomes an *answer* to the mathematical question once we are able to give a satisfying justification for it.

Definition 1.1. A *set* is any collection of objects. We typically use a capital letter like S to represent a set. The objects constituting the set are called *members* or *elements* of the set. If a is an element of a set S, it is common to write $a \in S$; and it is common to write $a \notin S$ when a is *not* an element of S.

Q1. When a set S consists of just a few members, we often define the set simply by listing the members, separated by commas and enclosed in brackets. For example, consider the set $S = \{1, 2, \{3, 4\}, \{5\}\}$ along with the mathematical question "What objects are elements of the set S?" Which of the following conjectures do you think are true?

Circle the true statements, and cross out the false statements. Then give justifications for your decisions.

(a) We have $2 \in S$.

(b) We have $\{3, 4\} \in S$.

(c) We have $3 \in S$.

(d) We have $\{2\} \in S$.

Q2. We say that a set A is a *subset* of a set B provided every element of A is also an element of B. We write $A \subseteq B$ in this case. Consider the set S from Question 1 along with the mathematical question "What sets are subsets of S?" Which of the following conjectures do you think are true?

Circle the true statements, and cross out the false statements. Then give justifications for your decisions.

(a) We have $S \subseteq S$.

(b) We have $1 \subseteq S$.

(c) We have $\{1\} \subseteq S$.

(d) We have $\{5\} \subseteq S$.

(e) We have $\{\} \subseteq S$.

Note: It can be helpful to think of sets as packages. As you know, packages can contain other packages. We need a way to distinguish between what is a package and what is **<u>in</u>** a package. Sometimes sets can be in other sets, just as packages can be in other packages. We can think of the braces (the $\{$ and $\}$) as **<u>permanent</u>** packaging, like gift wrap that doesn't come off. You can't take out what's inside the packaging. You can only hold the whole package. Even if only one thing is wrapped, you still can't hold the thing by itself, you can only hold it with its gift wrap. But if an object is not wrapped, you can hold that object by itself.

The set that contains no elements is called the *empty set*. The empty set can be denoted by $\{\}$; it is also customary to let $\emptyset$ represent this set. In the gift wrap metaphor, the empty set is the packaging that contains nothing. The empty set is not exactly nothing; rather, it is the set containing nothing. This is because a set must have "packaging" to be a set.

Activity 3, Part 2 — Reasoning with Sets, continued

Q1. Think about the set A consisting of all members of this class, along with the mathematical question "Who is an element of the set A?" Is the following conjecture TRUE or FALSE?

- All students in this class who are under 5 years old are also over 100 years old.

Justify your answer.

When a set S contains a lot of elements, we often define the set by providing a generic letter name for its members along with a collection of rules that allow us to determine whether a particular object is an element of S. The object name and the defining rules are separated by a colon or a vertical line, and the whole thing is enclosed in brackets. For example,

$$B = \{x \mid x \in A, \text{ and } x \text{ is less than 5 years old, and } x \text{ is more than 100 years old}\}$$

defines the set of all members of our class who are both less than five years old and greater than 100 years old. This formatting is called *set-builder notation*.

Activity 3, Part 3 — Reasoning with Sets, continued

Q1. We say that a set is *well-defined* when it is possible to determine exactly which objects are members of the set. Are any of the following sets well-defined?

(a) $X = \{y \mid y \text{ is an even, positive integer}\}$

(b) $Y = \{a \mid a \text{ is a person living close to Baltimore}\}$

(c) $Z = \{x \mid x \text{ is not a person}\}$

Q2. We say that a set is *well-founded* provided the set does not have itself as a member. Is the set Z well-founded? Explain your thinking.

Proving Conjectures About Sets

Subset exploration

Activity 4 — Describing Sets

Proof Structure: Showing set membership. To show that $x \in S$ means showing that x satisfies set membership rules for S; to show that $x \notin S$ means showing that x does not satisfy at least one set membership rule of S.

Q1. It is common to let $\mathbb{Q}$ represent the set of rational numbers. Let

$$S = \{x \in \mathbb{Q} : x = \frac{a}{2}, |x| < 2, \text{ and } a \in \mathbb{R}\}.$$

Are the following statements TRUE or FALSE? $0.5 \in S$, $3.5 \in S$, $0.25 \in S$, $-1 \in S$

Q2. Is A a subset of B or vice versa? Complete this table with "yes" or "no" in each cell.

	$A \subseteq B$	$A \subsetneq B$	$A \supseteq B$	$A \supsetneq B$	$A = B$	$A \neq B$	Neither is subset of the other
$A = \{n \in \mathbb{Z} : n \text{ is a multiple of } 3\}$, $B = \{n \in \mathbb{Z} : n \text{ is a multiple of } 6\}$							
$A = \{n \in \mathbb{Z} : n \text{ is a multiple of } 6\}$, $B = \{n \in \mathbb{Z} : n \text{ is a multiple of } 9\}$							
$A = \{n^2 : n \in \mathbb{N}, n \geq 1\}$, $B = \{1 + 3 + \cdots + (2n-1) : n \geq 1\}$							
$A = \{$Functions equivalent to $f(x) = a(x+h)^2 + v$, where $a, h, v \in \mathbb{R}$ and $a \neq 0\}$, $B = \{$Functions equivalent to $g(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ and $a \neq 0\}$							

?

What do you want to remember for when you teach set builder notation to middle or high school students?

Sample Proof

In the previous activity, we found that there were several ways that these questions needed to be clarified:

In Row 1, we asked: what kind of multiples? We decided to consider only integer multiples. In Row 4, we asked: How do we decide whether two functions are equal? It also helped to have students construct example functions from each set and then figure out how to transform the representative from A into a representative from B and vice-versa.

Now we are going to talk about proving conjectures related to the previous activity.

Activity 5 — A Sample Proof, Part 1 (Intro)

In logic, a *statement* is a declarative sentence that is either TRUE of FALSE but not both (or neither).

Definition 1.2. A *conjecture* about a mathematical process is a statement whose truth-value can be tested using the rules we have assumed in the process. Conjectures are conditional (IF-THEN) statements, but they may not always be phrased that way.

We don't always write conjectures explicitly as conditional statements, but with enough care, it is always possible to rephrase a conjecture in the form

IF [Statement 1] THEN [Statement 2].

Written in this form, Statement 1 is called the *hypothesis* and Statement 2 is called the *conclusion* of the conjecture.

We can think of a conjecture as a "promise." We *promise* that *IF* Statement 1 is true, *THEN* Statement 2 must also be true. There is only one situation in which a promise is broken — when Statement 1 is true *and* Statement 2 is false.

When we *prove* a conjecture, we are presenting a logically sound argument which demonstrates that *it is impossible to break the promise*.

Q1. The following statements are conjectures about the set $\mathbb{Z}$ of integers. How could you write these conjectures as IF-THEN statements?

[Statement 1.] Every integer that is a multiple of 4 is also a multiple of 2.

[Statement 2.] There exists an integer that is both even and odd. (It helps to consider the set B of all integers that are both even and odd.)

Activity 5 — A Sample Proof, Part 2 (Proof Structure for Showing Subsets)

Statement 1 above is an example of a *universal* conjecture. Universal conjectures claim one set is a subset of another. (Statement 1 is a claim is that the set of integer multiples of 4 is a subset of the set of integer multiples of 2.)

Statement 2 above is an example of an *existential* conjecture. Existential conjectures claim that a particular set is nonempty. (Statement 2 is a claim is that the set of all integers that are both even and odd is nonempty.)

Let's consider Statement 1 — Every integer that is a multiple of 4 is also a multiple of 2.

If we want to prove this conjecture, then we have to construct a logically sound argument that shows letting "X be the set of all integer multiples of 4 and Y be the set of all integer multiples of 2" *guarantees* that $X \subseteq Y$.

Proof Structure for Showing Proper Subsets

Showing $X \subseteq Y$.

- To show that $X \subseteq Y$, you begin with an arbitrary $a \in X$. You then use the membership rules for X and Y to explain why a is also a member of Y.

Proof. Suppose a is a member of the set X. This tells us that a is an integer multiple of 4. Therefore, we know that $a = 4n$ for some integer n. Now, we know $4 = 2 \cdot 2$, so the fact that integer multiplication is associative lets us conclude

$$a = 4n = (2 \cdot 2) \cdot n = 2 \cdot (2 \cdot n)$$

Of course, $2n$ is also an integer; hence we may conclude that a is an integer multiple of 2. This tells us that a satisfies the criteria for membership in the set Y. Consequently, we know that $a \in Y$.

Since we have shown that an arbitrary member of X is also a member of Y, we may conclude that $X \subseteq Y$, as desired. □

Definition 1.3. We say that a set Y is a *proper* subset of a set X provided every element of Y is a member of X, but there are members of X which are *not* members of Y. It is common to write $Y \subsetneq X$ in this case.

Here is a conjecture we could make using this notion:

- **Conjecture 5.1** If A is the set of all integer multiples of 3 and B is the set of all integer multiples of 6, then $B \subsetneq A$.

Q2. Is this conjecture universal or existential? Justify your thinking.

Activity 5 — A Sample Proof, Part 3 (Applying the Proof Structure)

Q3. Let's think about how we could go about proving **Conjecture 5.1**. If we let A be the set of all integer multiples of 3 and let B be the set of all integer multiples of 6, then we really have two claims to verify. What does each part mean?

Proof Structure for Showing that Something is a Proper Subset

Part 1: Proving that every member of B is a member of A.

Notes on what this means:

Part 2: Proving there exist members of A that are not members of B.

Notes on what this means:

Activity 5 — A Sample Proof, Part 4 (Looking at a sample proof)

Let's look at a student's proof from a previous offering of this course.

Claim. $A = \{3n : n \in \mathbb{Z}\}$, $B = \{6n : n \in \mathbb{Z}\}$ $\Rightarrow B \subsetneq A$.

Proof. Given $A = \{3n : n \in \mathbb{Z}\}$
$B = \{6n : n \in \mathbb{Z}\}$.

① $\boxed{B \subseteq A}$ We show: $x \in B \Rightarrow x \in A$.

Given $x \in B$.

(*) $x = 6k,\ k \in \mathbb{Z}$ — by defn of membership in B

$= 3 \cdot 2k$

$= 3n,\ n \in \mathbb{Z}$. — by closure of mult in $\mathbb{Z}$ ($2, k \in \mathbb{Z} \Rightarrow 2k \in \mathbb{Z}$)

Hence x satisfies membership rules for A

$\Rightarrow x \in A$.

By defn of subset, $B \subseteq A$. $\boxed{1}$

② $\boxed{\exists\, x \in A \text{ s.t } x \notin B}$ We find an elt of A not in B. Observe that all $x \in B$ are even (by defn of even):

$x \in B \Rightarrow x = 3 \cdot 2k = 2 \cdot 3k$ — by (*) and comm of mult in $\mathbb{Z}$

$= 2m,\ m \in \mathbb{Z}$ — by closure of mult in $\mathbb{Z}$ ($3, k \in \mathbb{Z} \Rightarrow 3k \in \mathbb{Z}$)

But there are members of A that are odd, e.g., 3, 6, 15, . . . These members of A are not in B. $\boxed{2}$

B and A satisfy the defn of strict subset $\Rightarrow B \subsetneq A$. □

Q4. Look over this proof carefully.

- Has the student addressed the two steps necessary to verify that B is a proper subset of A?
- What is the students' reasoning for each part?
- What is done well? What could be clearer?

Activity 5 — A Sample Proof, Part 5 (Features of Good Proof Communication)

While you are looking over any proof, think about the following.

Features of communicating proof well

(**Essential features in bold**)

1. **Label the claim.**
2. **State the claim precisely.**
3. **Label the proof beginning.**
4. Begin a proof by reminding yourself and readers of the starting point: the conditions of the claim.
5. **End the proof with where you need to go: the conclusions of the claim.**
6. Summarize your approach to the reader.
7. **Label the proof end.** A traditional way is to use a box.
8. **Write up parts within a proof properly**. **Label when they begin and end.**
   - Give them a name (e.g., Claim A) if it is a proof within a proof
   - **Use labels like** $[\Rightarrow]$ **and** $[\Leftarrow]$ **if doing an if and only if proof.**
9. Diagrams are good only if you explain what you are showing. Give a caption.

Q5. Where are each of these features in the student's proof? Are there any features not present?

Claim. 1 $\left.\begin{array}{l} A = \{3n : n \in \mathbb{Z}\} \\ B = \{6n : n \in \mathbb{Z}\} \end{array}\right] \Rightarrow B \subsetneq A.$ 2

Proof. 3 Given $A = \{3n : n \in \mathbb{Z}\}$
$B = \{6n : n \in \mathbb{Z}\}$. 4

8 1 $\boxed{B \subseteq A}$ We show: $x \in B \Rightarrow x \in A$.

6

Given $x \in B$.

(*) $x = 6k, \ k \in \mathbb{Z}$ — by defn of membership in B

$= 3 \cdot 2k$

$= 3n, \ n \in \mathbb{Z}$. — by closure of mult in $\mathbb{Z}$ $(2, k \in \mathbb{Z} \Rightarrow 2k \in \mathbb{Z})$

Hence x Text satisfies membership rules for A

$\Rightarrow x \in A$.

By defn of subset, $B \subseteq A$. $\boxed{1}$ 8

8 2 $\boxed{\exists x \in A \text{ s.t } x \notin B}$ We find an elt of A not in B. Observe that all $x \in B$ are even (by defn of even):

$x \in B \Rightarrow x = 3 \cdot 2k = 2 \cdot 3k$ — by (*) and comm of mult in $\mathbb{Z}$

$= 2m, \ m \in \mathbb{Z}$ — by closure of mult in $\mathbb{Z}$ $(3, k \in \mathbb{Z} \Rightarrow 3k \in \mathbb{Z})$

But there are members of A that are odd, e.g., 3, 6, 15, …. These members of A are not in B. $\boxed{2}$ 8

B and A satisfy the defn of strict subset $\Rightarrow$ $\boxed{B \subsetneq A}$ 5. $\boxed{\square}$ 7

Activity 6 — Proving a Conjecture

Now that you have had a chance to critique a proof, let's try creating one together. Consider the following conjecture. An outline of the proof for this conjecture is provided. Fill in the missing details.

- If $A = \{f : \mathbb{R} \to \mathbb{R}, | f(x) = a(x-h)^2 + v$ where $a, h, v \in \mathbb{R}\}$ and $B = \{f : \mathbb{R} \to \mathbb{R}, | f(x) = ax^2 + bx + c$ where $a, b, c \in \mathbb{R}\}$, then $A = B$.

Proof. Let A and B be the sets defined in the conjecture.

Q1. *Why* $A \subseteq B$*:* Let $f \in A$. This tells us [Fill this in]... Hence, we may conclude that $f \in B$. $\boxed{1}$

Q2. *Why* $B \subseteq A$*:* Let $g \in B$. This tells us [Fill this in]... Hence, we may conclude $g \in A$. $\boxed{2}$

Q3. Explain why we can now conclude that $A = B$. $\boxed{3}$

□

Activity 6 Bonus

Definition 1.4. Let A and B be sets.

- The *intersection* of A and B is defined to be the set consisting of those elements that are members of A AND are members of B. This special set is denoted by $A \cap B$.
- The *union* of A and B is defined to be the set consisting of those elements that are members of A OR are members of B. This special set is denoted by $A \cup B$.

Consider the following conjecture.

- If A, B, and C are sets, then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Q4. Construct your own proof of this conjecture.

Back to the opening inquiry

Activity 7 — Revisiting the Parent Relation, Part 1

Having devoted several activities to set concepts and proofs, let's take a second look at the Parent Relation from Activity 1.

From now on, let's agree to say that an integer factor p of an integer a is *proper* provided $p \neq \pm 1$ and $p \neq \pm a$. In Activity 2, we made a conjecture, namely:

> If a positive integer has at least three different proper factors, then it has more than one pair of parents. There are no other numbers with more than one pair of parents.

Applying set notation. Using set notation, we can interpret the conjecture in the following way.

Conjecture 1.5 (Number parent conjecture: Take 1)**.** If we let

$$\begin{aligned} S &= \{n \in \mathbb{N} : n \text{ has at least three different non-trivial factors}\} \\ T &= \{n \in \mathbb{N} : n \text{ has more than one pair of parents}\} \end{aligned}$$

then $S = T$.

Q1. How does this way of phrasing the conjecture match up with the original way?

(a) Look up the definition of set equality. What does $S = T$ mean by definition of set equality?

(b) Which part of set equality implies the first sentence ("If a number has least three different proper factors, then it has more than one pair of parents.")?

(c) Which part of set equality implies the second sentence? ("There are no other numbers with more than one pair of parents")

Activity 7 — Revisiting the Parent Relation, Part 2

There is another mathematically equivalent way of phrasing the conjecture.

Conjecture 1.6 (Number parent conjecture, take 2)**.** Let n be a positive integer. The number n has more than one pair of parents *if and only if* n has at least three proper factors.

In mathematics, the phrase "if and only if" indicates the presence of a *biconditional* statement — a kind of "two for one" conditional statement. In particular, if P and Q represent logical statements, then "P if and only if Q represents the compound statement

- (If P then Q) AND (If Q then P).

Mathematicians often use "iff" (or the symbol $\iff$) as an abbreviation for the phrase "if and only if".

Proving an "if and only if" conjecture requires us to verify that *both* conditional statements are unbreakable promises.

Q2. Suppose P and Q are logical statements, and think about the conjecture $P \iff Q$. Explain why proving this conjecture establishes that P and Q are *logically equivalent*; that is, have the same truth value.

Activity 7 — Revisiting the Parent Relation, Part 3

Here are the two equivalent ways of phrasing our number parent conjecture, written side by side.

Proposition 1.7 (Number parent proposition)**.**

If $S = \{n \in \mathbb{N} : n \textit{ has at least three different non-trivial factors}\}$ *and* $T = \{n \in \mathbb{N} : n \textit{ has more than one pair of parents}\}$*, then* $S = T$.	*For all* $n \in \mathbb{N}$*,* n *has more than one pair of parents if and only if* n *has at least three different proper factors.*

Proof. For readability, let P represent the statement "The positive integer n has more than one pair of parents" and let Q represent the statement "The positive integer n has at least three different proper factors."

Part A: Proving the conjecture "If P then Q".

Suppose that P is TRUE. In other words, suppose n is a positive integer that has more than one pair of parents. Let $\{a, a'\}$ and $\{b, b'\}$ be two different pairs of number parents for n. We need to explain why we can conclude that n has at least three different proper factors.

Q3. By assumption, we know $a \cdot a' = n = b \cdot b'$.
Explain why it is not possible to have $a = b$.

Assuming that $\{a, a'\}$ and $\{b, b'\}$ are *different* pairs of number parents leads to several possibilities for us to consider as separate cases. Specifically, the case where neither number parent pair is the same number (i.e. n is not a perfect square) and the case where one number parent pair is the same number (i.e. n is a perfect square).

Case 1: Suppose $a \neq a'$ and $b \neq b'$.

(a) Explain why it is not possible to have $a' = b$.

(b) What can we say about the number of proper factors for n?

Case 2: Suppose that $a = a'$.

(a) Explain why it is not possible to have $b = b'$.

(b) What can we say about the number of proper factors for n?

(c) Explain why we do not need to consider the case where $b = b'$ or the case where $a' = b'$.

We have now established that *IF* a positive integer n has at least two different pairs of number parents, *THEN* n has at least three different proper factors. $\boxed{A}$

Activity 7 — Revisiting the Parent Relation, Part 4

Part B: Proving the conjecture "If Q then P."

Q4. Now, suppose that Q is TRUE; that is, suppose n is a positive integer that has at least three different proper factors. Let a, b, and c be such factors. Either $a \cdot b = n$ OR $a \cdot b \neq n$. Let's consider each case.

(a) If $a \cdot b \neq n$ explain why we must conclude that n has at least two pairs of number parents.

(b) If $a \cdot b = n$, explain why it is not possible to have $a \cdot c = n$ as well.

(c) If $a \cdot b = n$, explain why we know that n has at least two pairs of number parents.

We have now established that *IF* a positive integer n has at least three different proper factors, *THEN* n has at least two different pairs of number parents. $\boxed{\text{B}}$

We have shown both directions, so we have shown that for all $n \in \mathbb{N}$, n has more than one pair of parents if and only if n has at least three different proper factors. $\square$

Summary of mathematical practices

Summary of Mathematical Practices

Clarifying the question

- Make the best sense as you can of the question with what is available.
- Identify what is unambiguous, and then identify what is ambiguous.
- For the ambiguous parts, play around with different possibilities to see what is the most mathematically interesting possibility. Sometimes you may find that there are multiple interesting mathematical possibilities.

Conjecturing and Claim making

- Think of conejctures as "promises".

  If you're making a promise, you would want to make absolutely sure that everyone knows exactly what the promise means, and also that everyone would agree on what evidence would count as showing you have kept your promise!

  The same is true about mathematical statements. A mathematical statement needs to be crystal clear about what it means.
- Mathematical conjectures should either be true or false; if they "depend" on something, this means that there is often a better conjecture that can be made.
- The more general a conjecture, the greater its potential usefulness.

  For instance, "12 has more than one pair of parents" is a true conjecture, but a potentially more useful conjecture is "All numbers with at least three distinct proper factors have more than one pair of parents".
- The more "directions" a conjecture addresses, the more useful it will likely be.

  For instance, "All numbers with at least three distinct factors have more than one pair of parents" is a true conjecture, but "A number has more than one pair of parents if and only if it has at least three distinct factors" provides much more understanding.

Exploring math: Our expectations

- Make a conjecture.
- Try to prove it.
- If you get stuck, consider the negation of the conjecture.
- Try to prove the negation instead.
- Consider the "opposite direction" conjecture. (The "converse" of the conjecture.)
- Try to prove it instead.
- Aim to make the most satisfying conjecture possible.
- Rewrite, rewrite, rewrite! Use the rewriting process to help things get clear for yourself, your future students, and your future self and peers.

Homework for Chapter 1

In this chapter, we learned about:

- Showing that an element is or is not a member of a particular set
- Showing that a set is a subset or strict subset of another set
- The definition of set equality and how to show that two sets are equal
- Number parents and children, in particular, that if $S = \{n \in \mathbb{N} : n \text{ has at least three different non-trivial factors}\}$ and $T = \{n \in \mathbb{N} : n \text{ has more than one pair of parents}\}$, then $S = T$.

Problem 0 Think back to your work in class.

(a) What are you learning about asking and answering mathematical questions? What are you most proud of doing? What is the biggest lesson you want to remember about doing math? What is something you wish you could have done differently? What is something you want to learn more about?

Problem 1 Let $S = \{x \in \mathbb{Q} : x \text{ can be written as a fraction with denominator 2 and } |x| < 2\}$.
Let $B = \{1 + 3 + \cdots + (2n-1) : n \in \mathbb{N}\}$.
Let C be the set of numbers of the form $27a$.
Let D be the set of numbers with more than 5 parents.
Suppose you are teaching high school students about set builder notation and what it means for something to be an element or not of a set. For each of the following, explain how you know, in a way that is understandable by high school students. You are not expected to write a complete proof. You are expected to write 1-2 sentences at most.

(a) Is "$0.33 \in S$" a true statement?

(b) Is "$1 \in S$" a true statement?

(c) Is 25 an element of B?

(d) Is 24 an element of B?

(e) When $a \in \mathbb{Z}$, is 3×5 contained in C?

(f) When $a \in \mathbb{Q}$, is 3×5 contained in C?

(g) Find a number that is a member of both B and D.

Problem 2 (a) Complete this table with "True" or "False".

	$A \subseteq B$	$A \subsetneq B$	$A \supseteq B$	$A \supsetneq B$	$A = B$	$A \neq B$	Neither is subset of the other
$A =$ integer multiples of 14, $B =$ integer multiples of 21							
$A = \{n^2 : n \in \mathbb{N}, n \geq 1\}$, $B = \{1 + 3 + \cdots + (2n-1) : n \geq 1\}$							
$A =$ numbers of the form $27a$ $B =$ numbers of the form $3a$, $a \in \mathbb{Z}$							
$A =$ numbers of the form $27a$ $B =$ numbers of the form $3a$, $a \in \mathbb{Q}$							

In your responses to the following, articulate clearly:

- Whether you are showing that the element is or is not contained in the set;
- How you used the definition of set membership to determine "yes" or "no"; and
- Any definitions or ideas that you need in your reasoning.

Use the Features of Good Proof Communication (p. 13).

(b) Prove your "true" responses to Row 1.

(c) Prove your "true" responses to Row 2.

(d) Prove your "true" responses to Row 3.

(e) Prove your "true" responses to Row 4.

Problem 3 Factoring and whole number multiplication is emphasized in upper elementary through middle school, because it is a foundation for high school mathematics, and because it presents opportunities for students to explore patterns and develop mathematical reasoning. In some high school textbooks, reasoning with numbers is used to introduce the idea of "proof" to high school geometry students.

Suppose that you are using the number parent/child idea as a way to introduce the concept of "proof" to high school students. You and your students agree that:

> Two natural numbers a and b are *parents* of another natural number c if $c = ab$. We will call c the *child* of a and b. A child cannot be its own parent. Parents a and b can equal each other.

You ask your students to explore patterns in parents and children, specifically:

> **Exploration 1.** Which natural numbers have NO parents?
>
> **Exploration 2.** Which natural numbers have exactly ONE parent?

(a) Write an ideal proof to Exploration 1 that demonstrates your understanding of how to communicate proof well, and that you could use with students to show them how to write a proof.

(b) Write an ideal proof to Exploration 2 that demonstrates your understanding of how to communicate proof well, and that you could use with students to show them how to write a proof.

(c) Label the parts of your responses to Exploration 1 and Exploration 2 with (1) through (7) according to our Features of Good Proof Communication (p. 13).

(d) How would you use one or both of these explorations to explain to students what it means to give a satisfying answer to a question?

(e) How would you use one or both of these explorations to explain to students what it means to prove something?

Problem 4 Read over the section containing Conjectures 1.5 and 1.6 and Proposition 1.7.

(a) Write the following statements in your own words: Conjecture 1.5 and Conjecture 1.6. Then explain why they are mathematically equivalent.

(b) Walk through the steps of the entire proof of Proposition 1.7 using the example of $n = 12$.

(c) Early in the proof of Proposition 1.7, we encounter the statement, "Let $\{a, a'\}$ and $\{b, b'\}$ be two different pairs of number parents for n." Why is it legitimate to say these two pairs exist?

(d) Consider the cases in this proof.

  i. What is the negation of the statement "$a = a'$ or $b = b'$"?

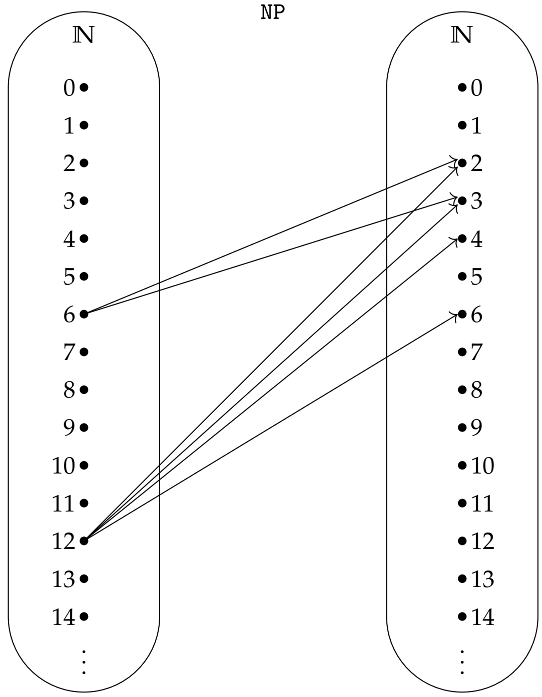
  ii. Give an example of n such that "$a = a'$ or $b = b'$" is true.

  iii. Give an example of n such that "$a = a'$ or $b = b'$" is false.

(e) The proof of Proposition 1.7 focuses on the rightmost formulation of the proposition. Consider the leftmost formulation. What part of the presented proof shows that $S \subseteq T$? What part shows that $T \subseteq S$?

Problem 5 (a) The diagram below shows `NP`, a collection of arrows from a natural number to its parents. Some arrows below have been filled in. For example, P assigns 6 to 2 and 3, and assigns 12 to $2, 3, 4, 6$. Draw in three more arrows from a natural number to its parents.

NP



(b) Here is a grid representation for NP. How would you complete it to be consistent with part (a)?

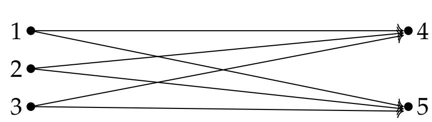
14															
13															
12															
11															
10															
9															
8															
7															
6													✓		
5															
4													✓		
3							✓						✓		
2							✓						✓		
1															
0															
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14

Problem 6 Let D and R be sets. The **Cartesian product** of D and R, which we will work with more next time, is defined as the set of ordered pairs $\{\overline{(x,y) : x \in D, y \in R}\}$.

Example. If $D = \{1,2,3\}$ and $R = \{4,5\}$, then $D \times R$ is the set

$$\{(1,4),(2,4),(3,4),(1,5),(2,5),(3,5)\}.$$

One way to think about it is that it is all the pairs you can get by drawing all possible arrows from elements of D to elements of R when you draw the elements lined up in parallel to each other:



Each arrow represents an ordered pair, with the starting point of the arrow being the first coordinate of the ordered pair and the ending point of the arrow being the second coordinate of the ordered pair.

Your turn. Let $A = \{5, 6, 10\}$ and $B = \{-1, -2, -3\}$. Draw an arrow representation of the Cartesian product $A \times B$ and list its elements as ordered pairs.

Mathematics Of Doing, Understand, Learning, and Educating Secondary Schools

MODULE(S^2): Algebra for Secondary Mathematics Teaching

Module 2: From Numbers, to Powers, to Logarithms

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Contents

I From Numbers to Powers 1

0 Introduction 1

1 Natural, Integer, and Rational Numbers and Powers (Lesson 1) (Length: ~3 hrs) 3
1.1 Overview 3
1.2 Opening Inquiry: Double Sunglasses / Stacking Sunglasses 4
1.3 Exponential Expressions and Power Properties in Teaching 7
1.4 Positive Natural Powers 10
1.5 Positive Rational Powers 12
1.6 Integer Powers 18
1.7 Positive and Negative Rational Powers 21
1.8 Homework 24
1.9 Simulation of Practice Video Assignment: Responding to Student Thinking about Power Properties 26
1.10 Simulation of Practice Written Assignment: Explaining Power Properties 28

2 Real Numbers and Powers (Lesson 2, Part 1) (Length: ~1 hour) 30
2.1 Overview 30
2.2 Defining Real Numbers and Powers 32
2.3 Definitions of Powers and Their Properties, with Common Restrictions 34
2.4 Homework 35

3 Appendix 37
3.1 Digging Deeper: Rational Powers and Music 37
3.2 Digging Deeper: Proofs that Power Properties Extend 38
3.2.1 Extending to Natural Powers 38
3.2.2 Extending to Integer Powers 40
3.2.3 Extending to Rational Powers 41
3.2.4 Extending to Real Powers 44
3.3 Digging Deeper: Not Every Number Is Rational 44

II From Powers to Logs 46

4 Exploring Powers, Introducing Logarithms (Lesson 2, Part 2) (Length: ~2 hours) 46
4.1 Working with Powers from a Correspondence Perspective 47
4.2 Setting Up Existence of Logarithms 49
4.3 Properties of G, the Inverse of $F(x) = 5^x$ 52
4.4 Logarithm Definition and Properties 56
4.5 Homework 57

5 (Optional) Covariation View: Exponential Growth is Defined by Constant Change Factors (Length: ~2 hours) 59

5.1 Introducing Constant Change Factors . . . 59
5.2 Constant Change Factors Are a Key Property of Exponential Growth . . . 61
5.3 Consequences of Constant Change Factors . . . 64
5.4 Constant Change Factors Implies Exponential, and Vice Versa . . . 66
5.5 Homework . . . 69

III Topics in Exponential Expressions and Functions **71**

6 Inequalities, Percent Change, and Powers of 2 (Lesson 3) (Length ~90 minutes) **71**
6.1 Inequalities . . . 72
6.2 Exponential Functions from a Covariation Perspective . . . 76
6.3 Relating Constant Percent Change and Constant Change Factor . . . 80
6.3.1 Quick Warm-up: Factoring to Get to One Term . . . 80
6.3.2 Savings Account Task: Getting from Percent Change to Growth Factor . . . 81
6.3.3 Power of Two: Relating Percent Change, Growth Factor, and Doubling and Halving Time . . . 82
6.4 Homework . . . 85

7 Powers and Numbers Together: Base Ten Multiplication **88**
7.1 Shabakh, Gelosia, Venetian Squares, Lattice Method . . . 88
7.2 Homework . . . 90

Part I

From Numbers to Powers

Key concepts of Part I:

- Explanations for powers and properties can be patterned after ones that students saw in middle school.
- **We can use power properties to extend the definition of powers from positive natural powers to 0, integer, rational, and real powers.**
- In defining powers, here and elsewhere, the scope for possible bases depend on well-definedness of the value of the power.

0 Introduction

Notation is one of the unsung heroes of mathematics. Good notation allows us to organize our thinking and opens the door to connections between ideas that we might not see otherwise. Classical "power" notation is one of the best examples. Originally introduced as a simple bookkeeping method in the 1600's, power notation opened up a new way of thinking as our understanding of real numbers grew; and this new thinking ultimately led to powerful classes of functions that are essential tools in much of modern science.

Let a be a number, and let n be a positive integer. We use the special symbol "a^n" to represent the number we reach using "repeated multiplication by a." In particular, the symbol a^n is bookkeeping notation, simply telling us how many factors of a appear in the repeated multiplication:

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \textit{ times}}$$

The superscript indicates the number of times a appears in the multiplication and is known as the *exponent*, from the Latin phrase *expo ponere* which means "out of place." (Michael Stifel is credited with coining the term in the 1500's.) The number a is now called the *base* of the expression.

The expression a^n is referred to as a *power* of a. Use of this term to mean repeated multiplication by a number seems to be Greek in origin. Euclid is known to have used this term (*dunamis* in Greek), and scholars attribute its first use to Hippocrates of Chios.

In the expression a^n, we say the **base** is a, the **exponent** is n. The entire expression a^n is called a **power**.

When $n \in \mathbb{N}^+$, we define

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \textit{ times}}$$

with repeated multiplication by a.

When $n \in \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, we need to extend the definition of a^n. We will do so in this chapter.

Using superscripts to indicate repeated multiplication is an ancient idea, having first been introduced in very limited form by Diophantus nearly 2,000 years ago in his so-called *syncopated* algebra. With the exception of using superscripts, Diophantus' notation bears little resemblance to our modern idea of exponents. (For example, he used Δ^Y to represent "the second power of an unknown" and K^Y to represent "the cube of an unknown.")

The brilliant twelfth-century Muslim mathematician Ibn Yaha al-Maghribi Al-Samawa'al came close to developing modern power notation in his algebra text *al-Bahir fi'l-jabr* (The Brilliant in Algebra). In many respects, this text was centuries ahead of its time, but it found little traction outside of the Islamic world. Some of the ideas appearing in this work (such as polynomial division and considering $1/a$ to be a power of a) would not be systematically tackled in the West until the 1600's.

The fourteenth-century mathematician Nicolas Oresme is credited as the first Western mathematician to associate numbers with powers of a, although he did not use superscripts. Significantly, he is also the first individual to consider non-integer powers; he did so in the context of music. (We will have more to say about this later.)

Modern power notation seems to have originated (at least in the West) in the fifteenth century through the work of Nicolas Chuquet, although it was not widely used until Renee Descartes popularized it in the 1600's.

Now that we have introduced the expression a^n as a bookkeeping symbol for repeated multiplication, we can think about productive ways to *extend* the meaning of this symbol. Mathematicians often extend the meaning of a symbol using analogy. Let's look at an example.

Let "a" be a number. We know that $(-1)a$ represents the translation from "a" to "0"; that is, we know that $a + (-1)a = 0$. In additive thinking, *multiplying* a real number by -1 produces the additive inverse for a.

Now, let's think about multiplication instead of addition. The multiplicative inverse for any nonzero number a is defined to be $1/a$; that is, we know $a \cdot (1/a) = 1$.

So, proceeding by analogy, we could *define* a^{-1} as a symbol that stands for $1/a$. By doing this, we would be letting a^{-1} serve a role in *multiplication* that is analogous to the role $-a$ plays in *addition*.

To see whether an extension is useful, we sometimes check whether it satisfies properties that the original use satisfies. For powers, we would want to make sure that this new definition satisfies power properties such as the power of a power property ($(a^{x_1})^{x_2}$) or product of power property ($a^{x_1}a^{x_2} = a^{x_1+x_2}$).

In Section 1, we:

- Review power properties
- Give a working definition for positive rational powers and integer powers
- Motivate why we need to be careful about our definitions for powers, especially regarding the possible values of the base and exponent.

These ideas set up in Section 2, where we:

- Revisit the definitions
- Define real powers.

1 Natural, Integer, and Rational Numbers and Powers (Lesson 1) (Length: ~3 hrs)

For the ideas in this chapter to work, we are implicitly assuming that $a \geq 0$. For this and other reasons, definitions are labelled as drafts. We will discuss this later, but put this out for now.

Overview

The key ideas presented in this chapter are as follows.

Content Goals

Natural Powers. Definition of positive natural powers as repeated multiplication, in parallel with positive natural multiplication as repeated addition. Introduce this parallel.

Positive Rational Powers. Building the definition for $a^{p/q}$ (for p, q, positive natural numbers) from power properties. Making sense of issues that can come up with notation.

Integer powers. Building definitions for a^0 and a^{-1}, and other integers.

Rational powers. Put together all the ideas from above to build definition for all rational powers.

Opening Inquiry: Double Sunglasses / Stacking Sunglasses

Activity 1 — Double Sunglasses

This activity uses a Three Act Math Task to motivate the material in this chapter. Math Three Acts use mathematical storytelling to motivate learning. Basically, Act I is meant to activate mathematical curiosity and set up a conflict / problem to solve, Act II develops tools for solving the conflict / problem, and Act III resolves the mathematical conflict / problem and sets up a sequel! You can learn more about Dan Meyer's Three Acts at this link:
https://blog.mrmeyer.com/2011/the-three-acts-of-a-mathematical-story/

Q1. Watch the Act I video. Use the space below to write questions that come to mind.

Q2. Act II will be presented by your instructor.

Q3. Watch the Act III video. Why does that answer make sense?

Q4. Watch the Sequel video.

Activity 2 — Stacking Sunglasses

Based on our work so far, we can say that...

A Person wearing ______ sunglasses, each at 5% tint	Can see ______% of the light in front of them
Single	$0.95 = 95\%$
Quintuple	$(0.95)^5 = 77.4\%$
Hextuple	$(0.95)^6 = 73.5\%$
Septuple	$(0.95)^7 = 69.8\%$

Using the expressions in the table, find **three different ways** to compute the answer to this question:

A person wears 18 sunglasses, each at 5% tint. How much of the color around them can they see?

Exponential Expressions and Power Properties in Teaching

In the sunglasses tasks, we used:

Definition 1.1 (Exponential Expression)**.** In expressions such as $(0.95)^n$, we refer to 0.95 as the base and n as the exponent. The expression $(0.95)^n$ is called a power.

- This use of 'power' and 'exponent' is used in K-12 texts. It distinguishes between components of the expression and the whole expression.
- It allows us to say key properties of exponentiation more clearly, for instance: 'multiplying powers means adding exponents', 'logarithms map powers to exponents', 'exponentiation maps exponents to powers'.

Property 1.2 (Power Properties: Draft)**.** When n is a positive natural number, exponential expressions satisfy the following properties:

- (1st power) $a^1 = a$ (take this as definition)
- (product of powers property) $a^{x_1+x_2} = a^{x_2+x_1} = a^{x_1}a^{x_2}$
- (power of a power property) $(a^{x_1})^{x_2} = (a^{x_2})^{x_1} = a^{x_1 x_2}$
- (power of a product property)$(ab)^x = a^x b^x = b^x a^x$

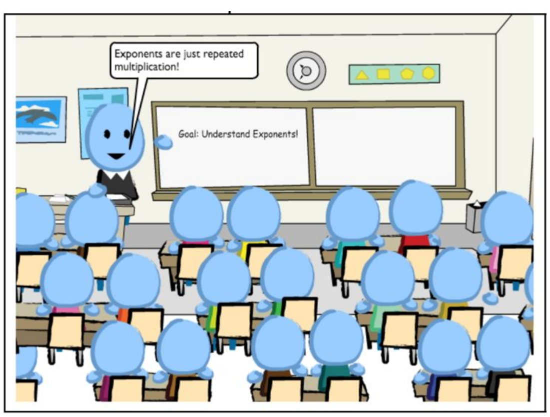
Note: The first three Power Properties allow us to say that exponent addition and multiplication are commutative, associative, and satisfy the distributive property.

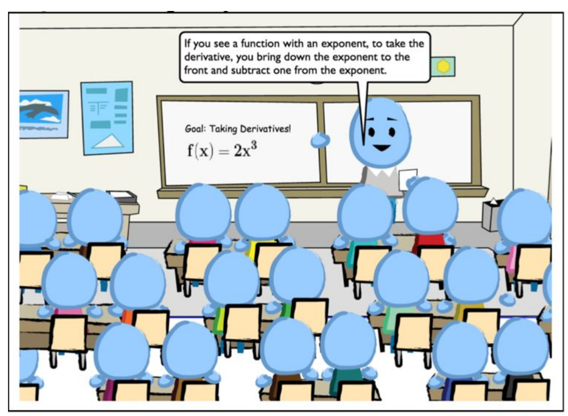
This is a rough draft for a few reasons. One is that we don't specify anything about the domains of a, x_1, x_2. These will come later.

Activity 3 — Exponential Expressions and Power Properties in Teaching

Mathematics teachers find themselves in teaching situations where they must speak about powers and exponents in many different ways. This activity will give you an opportunity to reflect on these sorts of situations.

Classroom Scenario: Mr. Ryan teaches everything from Pre-Algebra to Calculus. The following two scenes are snapshots from his classes at different times during the year.





Read the scenario aloud with a partner. What are some initial thoughts you have?

Activity 3 — Exponential Expressions and Power Properties in Teaching, Part 2

Write down your thoughts on the following questions and save them. We will come back to these.

Q1. How would you evaluate the pedagogical quality of these two explanations: (a) about exponentiation and (b) about taking derivatives?

Q2. If your evaluation depends on the context of the statement, provide some sense of why and when your evaluation might change.

Activity 3 — Exponential Expressions and Power Properties in Teaching, Part 3

Now, we consider an additional explanation. Here is one for the power of power property:

> "Let's think about $(b^{x_1})^{x_2} = b^{x_1 x_2}$ with a specific example. Take $(2^3)^5$)
> $(2^3)^5$) is five copies of (2^3)
> $(2^3)^5 = (2^3)(2^3)(2^3)(2^3)(2^3)$
> So, the rule above says that this just equals $2^{3+3+3+3+3} = 2^{3\times 5}$."

What numbers and/or number sets does **this explanation** work for? Circle what works

This explanation works when x_1 is: $0 \quad -1 \quad 1/3 \quad \mathbb{N} \quad \mathbb{Z} \quad \mathbb{Q} \quad \mathbb{R}$

This explanation works when x_2 is: $0 \quad -1 \quad 1/3 \quad \mathbb{N} \quad \mathbb{Z} \quad \mathbb{Q} \quad \mathbb{R}$

This explanation works when b is: $0 \quad -1 \quad 1/3 \quad \mathbb{N} \quad \mathbb{Z} \quad \mathbb{Q} \quad \mathbb{R}$

Let's return to Mr. Ryan and the context for the two scenes we saw earlier.

- **Scene 1** was from when the class was studying the graphs of exponential functions, more specifically, $y = 3^x$. As students were plugging in values to find specific points on the graph, Mr. Ryan said, "Remember, exponents just mean repeated multiplication. Like when $x = 4, 3^4$ means 3 multiplied by itself four times.'
- **Scene 2** was from when his Calculus class was having an end-of-year review about some of the most important ideas and results about derivatives: "If you see a function with an exponent, like $f(x) = 2x^3$ to take the derivative, you bring down the exponent to the front and subtract one from the exponent.

Activity 3 — Exponential Expressions and Power Properties in Teaching, Part 4

Q1. Given these contexts, give Mr. Ryan feedback about his explanations: What applies well? What could help students think more generally?

Mr. Ryan asked students to submit questions to him for homework. Some students emailed him these questions. He asks you for your ideas to respond.

Q2. One student asks: "Hey Mr. R, Here's my question. Why do $\sqrt{a}$ and $a^{(1/2)}$ mean the same thing? If it's a square root, why does it have to be $a^{(1/2)}$ and not $a^{(-2)}$ or a^2 or something else with 2?"
What ideas do you have for how Mr. Ryan should respond?

Q3. A second student asks: "Yo Mr. R, are $-(3^x)$ and $(-3)^x$ the same thing? They seem like they should be different, but they also seem like they should the same."
What ideas do you have for how Mr. Ryan should respond?

So far in this module, we:

1. Identified Exponential Expressions and Power Properties
2. Saw how these things can be used in rich problems (e.g. Double Sunglasses)
3. Thought about how explanations generalize, and what might be tricky to explain

Be sure to save your your explanations and responses from the above activity. We will revisit them later, after a few more activities.

Positive Natural Powers

In this section, we further explore explaining power properties—providing opportunities for teachers to give explanations that build on ideas that students learn before high school. Mathematicians often provide explanations by way of analogy, and we use this tactic repeatedly in the remainder of this chapter. We make explicit the structure of multiplication as repeated addition, and leverage the analogous structure of exponentiation as repeated multiplication.

Activity 4 — From Numbers to Powers: Positive Natural Numbers

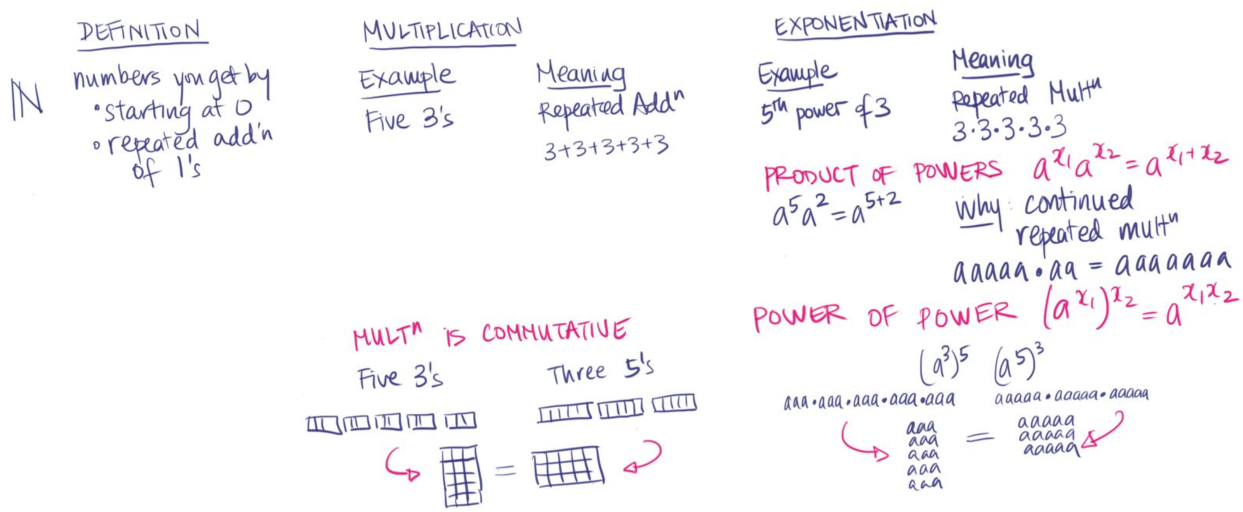
In this activity and the ones to follow, we explore different types of numbers, and how multiplication and exponentiation behave for those number types. We start with positive natural numbers.

Definition 1.3. When $n \in \mathbb{N}^+$, we define

$$a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ times}}$$

with repeated multiplication by a.

To see why the power properties hold for positive natural powers, we can make an analogy to familiar properties of multiplication of positive natural numbers. We will make graphical summaries of possible explanations based on the analogous structures we notice.



Q1. Your turn! In the space below, create two visual displays:

(a) One visual for explaining the distributive property

(b) A second visual for explaining the power of product property

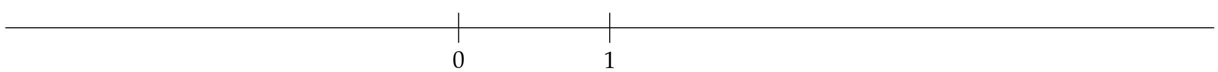
After completing the activities in this section, we saw a few things. First, making a chart of definitions and visual representations for multiplication and exponentiation along with their properties can be useful for organizing our thinking. Second, explanations for powers and properties can be patterned after ones that students saw in middle school.
In the next section, we will make charts for other number systems in the order: $\mathbb{Q}^+$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$.
We will proceed by:

- Defining numbers in that system
- Defining powers with exponents in that system
- Explaining multiplication properties
- Explaining power properties

Positive Rational Powers

Activity 5—Number Line: Locating Numbers

In this activity, we carefully explore how we talk about numbers. We do so by considering the numberline below:



Indicate your answers to the following questions by pointing.

Q1. Where is 5? How do you know this is where it is?

Q2. Where is -3? How do you know this is where it is?

Q3. Where is $1/3$? How do you know this is where it is?

Q4. Where is $7/3$? How do you know this is where it is?

Let's dig a little deeper on answers to Q3 and Q4.

Q5. How many solutions are there to $3x = 1$? How do you know?

Q6. What are some different ways to explain $7/3$?

Based on our progress in the previous activity, we are now ready for the following definition.

Definition 1.4. Any natural number n can be thought of as location and movement, where, given an origin 0 and a unit distance 1 on the number line,

- "0" is the movement of staying still, and its location is the origin.
- "1" is the movement of translating 1 unit to the right, and the location of moving 1 unit to the right when starting from 0.
- "n", for $n > 0$, is the movement of n repeated translations of 1, and the location of moving n units to the right when starting from 0.

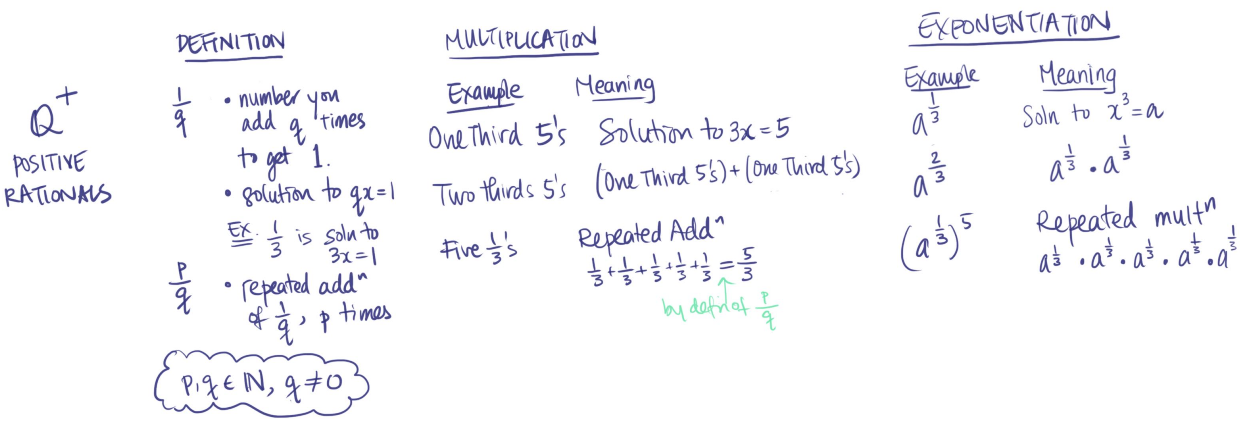
A rational number p/q, where $p, q \in \mathbb{N}$ and $q \neq 0$, can be thought of as location and movement as follows:

- $1/q$ is the movement that you need to take q times to get from 0 to 1; in other words, it is a solution to the equation $qx = 1$. It is the location you get by moving this much to the right when starting from 0.
- p/q is the movement of p repeated translations of $1/q$, and the location of moving this much to the right when starting from 0.

With this definition in hand, we can see how it has power to inform the way we define and explain multiplication, exponentiation, and their properties when using positive rationals.

Activity 6—Positive Rationals

Q1. Explain how this chart conveys **definitions** of multiplication and exponentiation with positive rational numbers.



Q2. Because we think of positive rational powers as solutions to equations, consider:

- How many solutions are there for x in the equation $x^2 = 1$?
- How many solutions are there for x in the equation $x^3 = 1$?
- How many solutions are there for x in the equation $x^q = 1$ where q is a natural number?

Q3. How does our above thinking help us to explain what $10^{1/3}$ is. Explain. Also, without the use of a calculator, explain where would you place it on the number line?

We are now ready for the following definitions:

Definition 1.5 (Rational Powers: Draft)**.** When $p/q \in \mathbb{Q}^+$, where $p, q \in \mathbb{N}^+$, we define

- $a^{1/q}$ as the positive number that you multiply q times to get a.
  Equivalently, it is a positive or zero solution to $x^q = a$. This is also known as a **q-th root of a**.
- $a^{p/q}$ as $(a^{1/q})^p$, which we take to be equivalent to $(a^p)^{1/q}$.

Note: Such a solution, when it does exist, is unique. When such a solution does not exist, we say that the $1/q$-th power does not exist.

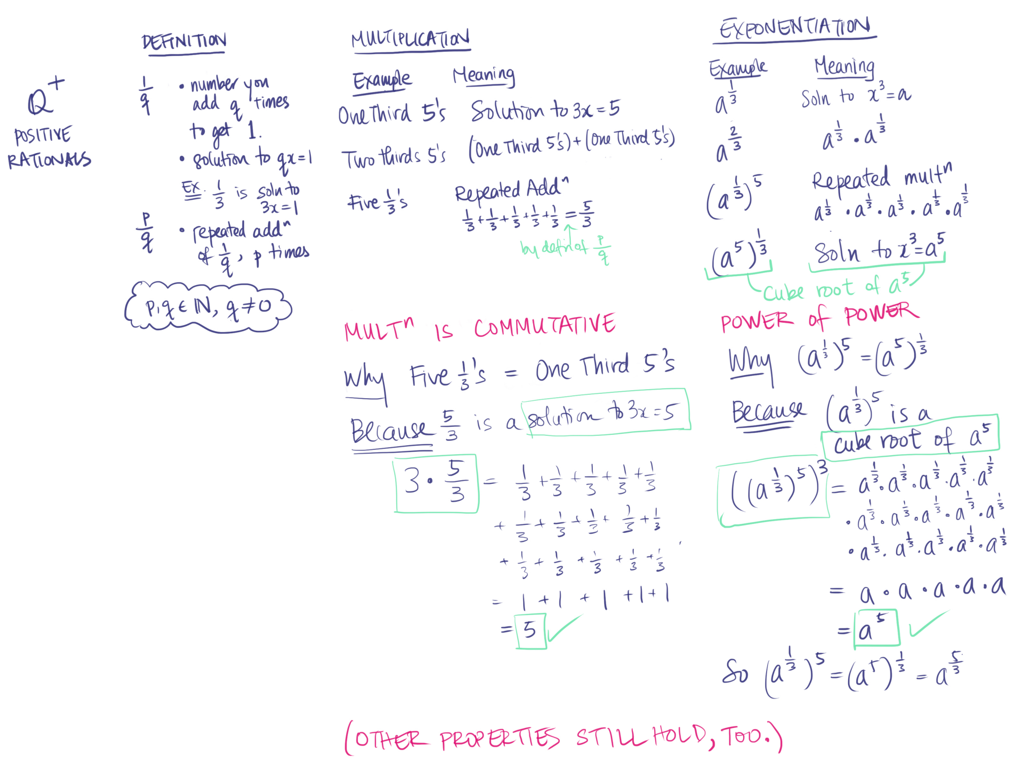
Definition 1.6 (q-th root: Draft)**.** The notation $\sqrt[q]{a}$ means a positive or zero q-th root of a, so it is another way to write $a^{1/q}$. These notations mean the same thing.

Historically, exponential notation a^x was used for natural number exponents. Euclid used the term "power" to mean the square of a line segment, and Archimedes used the idea of powers to work with base 10 arithmetic. Radical notation $\sqrt[q]{a}$ used for square roots, cube roots, etc. Eventually, some mathematicians realized they could unify the notation, using power properties. A q-th root of a, by definition, is a solution to $x^q = a$. Therefore, applying power properties to the notation $a^{1/q}$ yields $(a^{1/q})^q = a^{\frac{1}{q}\cdot q} = a^1 = a$.

Thus power properties are used to create the definition of positive rational number powers. At the same, we can also explain why these properties should hold for positive rational number powers, in analogy to properties of multiplication of positive rational numbers. The following is a graphical summary of a possible explanation of one property. Other properties hold as well, though their graphics are a bit messier.

Activity 7—Positive Rationals in Teaching

Explain how this chart conveys **properties** of multiplication and exponentiation with positive rational numbers.



Activity 7—Positive Rationals in Teaching, continued

Suppose you are teaching about rational powers. Think about each of the below ideas and write down your thoughts on how to respond.

Q1. You ask students the question ”Where is $(\sqrt[6]{27})^2$ on the number line?”
What are three different strategies that students might understand and use to figure out this problem?

Q2. You assign this problem to your students:
”Which of the following shows a cube root of 100,000? Circle them.

$$(\sqrt[3]{10})^5 \qquad 10^{3/5} \qquad \sqrt[3]{10^5} \qquad 10^{15} \qquad (100,000)^{1/3} \qquad 100$$

Also, what are other ways to right the same number?”
First, solve the problem. Then write down different ways you can explain the first part (i.e., the circle the answers part).

Q3. A student asks ”What if you wrote other numbers for the root? Like, if you wrote $\sqrt[-3]{27}$ or $\sqrt[1/3]{27}$, what would these mean?”
What are reasonable values for these? Why?

Integer Powers

Activity 8—Integer Powers

We now turn to thinking of integer powers that are not positive. The case of 0 and -1 often pique students' curiosity, as the following scenario depicts.

Q1. In Ms. Swain's Algebra I class, a student says: "I don't know why $x^0 = 1$. Is it just because someone said so? Like a convention? To me, it seems like it should be 0 because anything times 0 is 0."
What could Ms. Swain say to address these ideas?

Q2. Later, a student asks: "So if a^3 is repeated multiplication, can we think of a^{-3} like repeated division? Then a^{-3} would be a divided by a divided by a. So for 10, then 10^{-3} would be 10 divided by 10 divided by 10 which is 1/10."
What could Ms. Swain say to address these ideas?

Q3. How do conversations similar to those in Ms. Swain's Algebra I class reveal the parallel between multiplication and exponentiation with integers and natural numbers as summarized in the table below?

Multiplication by $\pm\mathbb{N}$	Exponentiation $a^{\pm\mathbb{N}}$	Notes
Repeated addition/subtraction starting from 0	Repeated multiplication / division starting from 1	0 is the additive identity (0+something = something) 1 is the multiplicative identity (1$\times$ something = something)

Q4. Take out your responses to the questions that Mr. Ryan's students asked in their "homework" (Activity 3 Questions 7 and 8). Revisit your responses. Based on our discussion so far, take a moment to make any changes that seem appropriate, or to write more about why your initial reasoning holds.

Definition 1.7. An integer $\pm n$, where $n \in \mathbb{N}$, can be thought of as location and movement, where, given an origin 0 and a unit distance 1 on the number line,

- "0" is the movement of staying still, and its location is the origin. So 0 is the **additive identity**, for $x + 0 = 0$ for any $x \in \mathbb{Z}$.
- "1" is the movement of translating 1 unit to the right, and the location of moving 1 unit to the right when starting from 0.
- "-1" is the movement of translating 1 unit to the left, and moving this much when starting from 0. It is also the solution to $x + 1 = 0$, and so is the **additive inverse of** 1.
- "n", for $n > 0$, is the movement of n repeated translations of 1, and the location of moving n units to the right when starting from 0.
- "$-n$", for $n > 0$, is the movement of n repeated translations of -1, and the location of moving n units to the left when starting from 0. It is also the solution to $x + n = 0$, and so is the **additive inverse of** n.

Analogously,

Definition 1.8. We define, for $n \in \mathbb{Z}$,

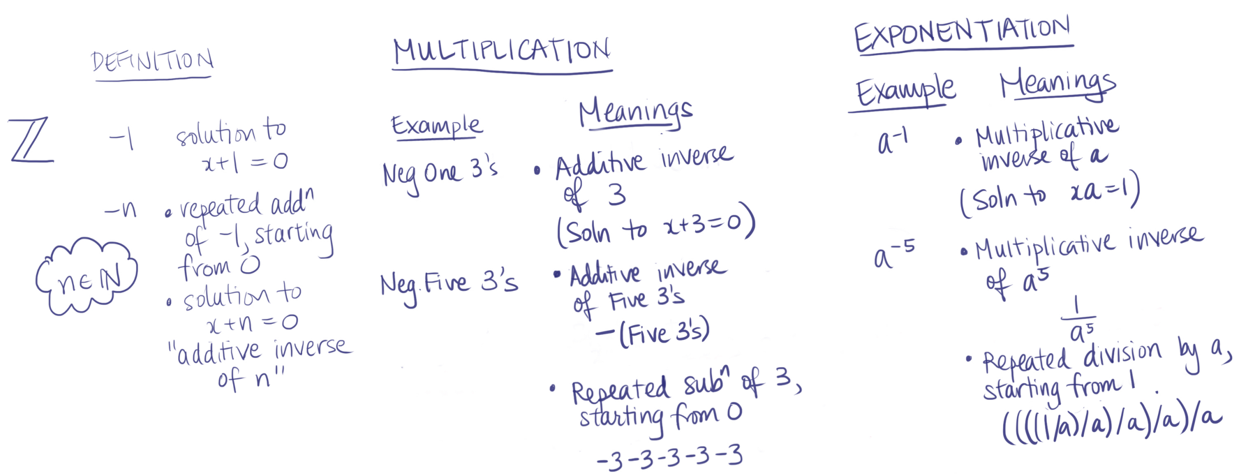
- a^0 is the **multiplicative identity**, so $a^0 a^x = a^x$ for any x. This means $a^0 = 1$.
- a^{-1} as a positive or zero solution to $ax = 1$. This means $a^{-1} = \frac{1}{a}$, for $a \neq 0$.
- a^{-n} as a positive or zero solution to $a^n x = 1$. This means $a^{-n} = \frac{1}{a^n}$, or equivalently, $(\frac{1}{a})^n$, for $a \neq 0$.

Note: Such solutions, when they exist, are unique. When such a solution does not exist, we say that the value of the power does not exist.

Note that another way to extend the analogy, for $n \in \mathbb{N}$, is as follows:

- $-n$ can be thought of repeated subtraction by 1, starting from the additive identity 0.
- a^{-n} can be thought of as repeated division by a, starting from the multiplicative identity 1.

Here is a graphical summary:



The power properties can be shown in analogy to properties of multiplication and addition; however, the proofs get very nitty gritty, so we do not go into them.

Positive and Negative Rational Powers

Activity 9 —Putting together definitions and properties

Let's think more about what our progress thus far means when considering all types of possible rational number powers. The only case we have left to consider are the negative rationals.

Which equations do $10^{-1/3}$ solve? How do you know?

Q1. $x^3 = 10$

Q2. $x^{-3} = 10$

Q3. $1/(x^3) = 10$

Q4. $(1/x)^3 = 10$

Q5. $(-x)^3 = 10$

Q6. $-(x^3) = 10$

Which equations do $10^{-2/3}$ solve? How do you know?

Q7. $x^3 = 10^{-2}$

Q8. $x^{-3} = 10^2$

Q9. $1/(x^3) = 10^{-2}$

Q10. $(1/x)^3 = 10^2$

Q11. $x^{3/2} = 10$

Back to Mr. Ryan's class. Take out your responses to Activity 3, Part 3. Revisit your answers for x_1 and x_2. Remember these are about the **explanation**, rather than the rule.

Q12. Do you still agree with yourself? Are you uncertain for any?

Q13. Revisit your responses. Based on our discussion so far, take a moment to make any changes that seem appropriate, or to write more about why your initial reasoning holds.

To finish defining rational powers, we need to define negative rational powers. We use the same idea as previously for negative integers, namely, to define in analogy to additive inverses.

Definition 1.9 (Negative Rational Powers: Draft)**.** We define, for $-p/q \in \mathbb{Q}$, where $p, q \in \mathbb{N}$, and $q \neq 0$,

- $a^{-1/q}$ as a positive or zero solution to $a^{1/q}x = 1$. This means $a^{-1/q} = \frac{1}{a^{1/q}}$, or equivalently, $(\frac{1}{a})^{1/q}$, for $a \neq 0$.

Note: Such solutions, when they exist, are unique. When such a solution does not exist, we say that the value of the power does not exist.

In summary:

Working definition for powers

For the below to work, we must assume that $a \geq 0$. We will discuss this later.

When $n \in \mathbb{N}^+$, we define:

- $a^n = \underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ times}}$ as repeated multiplication by a.

When $-n \in \mathbb{Z}$, (so $n \in \mathbb{N}$),

- a^0 is the **multiplicative identity**, so $a^0 a^x = a^x$ for any x. This means $a^0 = 1$.
- a^{-1} as a positive or zero solution to $ax = 1$. This means $a^{-1} = \frac{1}{a}$, for $a \neq 0$.
- a^{-n} as a positive or zero solution to $a^n x = 1$. This means $a^{-n} = \frac{1}{a^n}$, or equivalently, $(\frac{1}{a})^n$, for $a \neq 0$.

When $p/q \in \mathbb{Q}^+$, where $p, q \in \mathbb{N}^+$, we define

- $a^{1/q}$ as a positive or zero solution to $x^q = a$. This is also known as a **q-th root of a**.
- $a^{p/q} = (a^{1/q})^p$, which we take to be equivalent to $(a^p)^{1/q}$.

When $-p/q \in \mathbb{Q}$, where $p, q \in \mathbb{N}$, and $q \neq 0$,

- $a^{-1/q}$ as a positive or zero solution to $a^{1/q}x = 1$. This means $a^{-1/q} = \frac{1}{a^{1/q}}$, or equivalently, $(\frac{1}{a})^{1/q}$, for $a \neq 0$.
- $a^{-p/q} = (a^{-1})^{p/q}$, which we take to be equivalent to $(a^{p/q})^{-1}$

Note: Such solutions, when they exist, are unique. When such a solution does not exist, we say that the value of the power does not exist.

What we have not done so far includes:

- Discussing the possible values for the base a
- Whether there are values where we will run into any trouble
- Real powers.

The homework will prepare you for discussing these, which we do next, in Section 2.

Homework

1. Read Chapters 1-2 of *Radical Equations: Civil Rights from Mississippi to the Algebra Project* . Do some google or other searching on terms that you have not seen before, as well as on the author of the book, Robert (Bob) Moses. Think about and be ready to discuss:
   - Who is Bob Moses?
   - What is the Algebra Project?
   - What does Bob Moses believe that Algebra and Civil Rights are tied together, especially for Blacks, Hispanic, and poor White populations?
   - How does the subway system give rise to talking about numbers, and what was Bob Moses' goal in doing these activities and discussions?
   - What lessons do you want to remember from this reading?

2. Something we discussed in this section is explaining numbers using location and movement. Some middle school teachers have found it fun and helpful to visualize the movement as skateboarding and to think about recording someone skateboarding. Here:
   - 1 is like a video of someone skateboarding forward from 1 unit.
   - n is like a video of someone skateboarding forward from n units.
   - -1 is like a video of someone skateboarding backwards 1 unit.
   - $-n$ is like a video of someone skateboarding backwards n units.
   - Multiplying something by -1 is like playing the video of something, backwards!

   Using this idea, how would you explain why it makes sense that -1×-1 should equal 1? Complete this dialogue.

   You: "Think about what it means to take -1×5 and -1×-1." Student 1: "So if you play a video of me going forward 5 units, but you play the video backwards, then it looks like I am going backwards 5! And that's -5." Student 2: "Okay, so for the other one. -1×-1. Wait, so if you play the video backwards, but let's say I was going backward in the video ... My brain hurts. What does it mean to play a video backwards of someone skating backwards?" You: "Let's think about this together. ...."

3. Suppose that you know $a^n = \underbrace{a \cdot a \cdot \dots \cdot a}_{n \text{ times}}$, for $n \in \mathbb{N}^+$, and you take for granted that the power properties hold.

   Use these ideas to explain the values of:

   (a) a^0

   (b) a^{-1}.

   You can assume $a \neq 0$.

   In your work, say exactly what power properties you are using for each step of your explanation.

4. Suppose that you are teaching powers, and you want to help students use power properties and the definition of powers. What are three different ways to understand the value of $(\sqrt[10]{32})^2$? In each of your explanations, be specific about what definitions or power properties you are using.

To get ready for next time:

5. (a) What equation does $a^{1/q}$ solve?

   (b) Based on this equation, what should $(-8)^{\frac{1}{3}}$ equal?

   (c) Find three different ways to simplify the expression $(-8)^{2/6}$. (Hmm ...)

6. (a) Does this sequence converge? If so, to what? If not, why not? $1^0, \quad (\frac{1}{2})^0, \quad (\frac{1}{3})^0, \quad (\frac{1}{4})^0, \dots$

   (b) Based on what you have found, what should 0^0 equal?

7. (a) Does this sequence converge? If so, to what? If not, why not? $0^1, \quad 0^{\frac{1}{2}}, \quad 0^{\frac{1}{3}}, \quad 0^{\frac{1}{4}}, \dots$

   (b) Based on what you have found, what should 0^0 equal?

Mathematics Of Doing, Understand, Learning, and Educating Secondary Schools

MODULE(S^2): Algebra for Secondary Mathematics Teaching

Module 3: Fields and Polynomials

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Published Version: July 2022

INSTRUCTOR VERSION

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Contents

Overview of MODULE(S^2) Project — iii

Overview of Algebra MODULE(S^2) — 1

Overview of Module 3: Fields and Polynomials — 1

1 Defining Fields — **2**
- 1.1 Solving equations — 2
- 1.2 Analyzing the steps — 4
- 1.3 What number systems can you solve an equation over? — 7
- 1.4 Digging Deeper: Attending to precision — 9
- 1.5 Suggested homework — 10

2 Fundamental Properties of Fields — **11**
- 2.1 The field axioms — 11
- 2.2 Elementary properties of fields — 13
- 2.3 Digging Deeper: Reflecting on field properties — 17
- 2.4 Suggested Homework — 19

3 Simulation of Practice: Using Algebraic Principles to Unpack Procedures — **21**

4 Integer and Polynomial Arithmetic — **24**
- 4.1 Algebraic properties of functions — 24
- 4.2 Polynomial addition — 27
- 4.3 Polynomial multiplication — 31
- 4.4 Digging Deeper: A connection to geometry — 35
- 4.5 Suggested Homework — 36

5 Integer Division and Polynomial Division — **37**
- 5.1 Exploring integer division — 37
- 5.2 Exploring polynomial division — 41
- 5.3 Adapting whole number division to polynomials — 44
- 5.4 Polynomial division in different ways — 47
- 5.5 Making long division shorter — 51
- 5.6 Suggested Homework — 54

6 Factors and Roots of Polynomials — **55**
- 6.1 Name that polynomial — 55
- 6.2 What does synthetic division tell us about roots? — 57
- 6.3 The Remainder Theorem and Factor Theorem — 59

6.4 How many roots can a polynomial have? 61
6.5 How many roots can a polynomial have? (Revisited!) 63
6.6 Suggested Homework 65

7 Simulation of Practice: Analyzing Solutions by Factoring 66

8 Finding Roots of Polynomials 70
8.1 Integer roots of polynomials 70
8.2 The rational root theorem 72
8.3 Solving quadratic equations by factoring 74
8.4 The quadratic equation 76
8.5 Completing the square 78
8.6 Suggested Homework 80

Overview of Algebra MODULE(S^2)

Through the material in the Algebra MODULE(S^2), we address mathematical foundations for teaching algebra at the secondary level. We begin in Module 1 with a treatment of relations and functions from an advanced perspective. This means defining relations and functions in such a way that inverses and graphs of functions are seen as a special case of inverses and graphs of relations, and then addressing covariation and correspondence approaches to describing functions. In Module 2, we address exponentiation, and its structural parallels to number systems more generally. In Module 3, we address links between abstract algebra and high school algebra, particularly through field properties, integer and polynomial arithmetic, and factors and roots of polynomials.

Overview of Module 3: Fields and Polynomials

This module focuses on polynomials. We chose to focus on polynomials because they are used a lot in Algebra 1 and Algebra 2 in high school mathematics. Our goal here is to make connections between this material and "higher" mathematics.

Instructor note. We know pre-service teachers have seen much of this material and they know the basic mechanics to a varying degree.
In designing and writing this material, we have attempted to adhere to the following design principles:

1. We're always keeping an eye one whether the mathematical work we have them do has some structural similarity to the mathematics they will encounter in their teaching.
2. Where there is an opportunity to assist pre-service teachers in discovering something, we do that. We have organized the material so that they can discover and justify important mathematical ideas.
3. We build up from examples, and only ask for extensions to proof when those proofs are not so much of a step up, that they would be uncharacteristic of the work of teaching.
4. We aim for conceptual coherence, so while abstract definitions and proofs are not emphasized, such things do make an appearance.
5. We aim to show through examples and activities that algorithms for manipulating and solving polynomials come out of creative and inquisitive human thought processes.

Instructor note. The time estimates given with each activity are *estimates*. The actual time taken will depend on the particular circumstances of your class. As a result, students in your class may not complete all of the tasks in a particular activity or they may complete them more quickly than expected. As your class progresses through this module, it is to be expected that you will not have time to allow students to thoroughly explore every part of every activity. The later tasks in the activities make excellent homework problems! They are typically thought-provoking problems, and planning to assign some tasks as homework will allow you time in class to more thoroughly explore crucial parts of the activities and allow students to more fully engage with this challenging material. An ideal homework assignment will include parts of the activities together with some of the suggested homework problems.

1 Defining Fields

Solving equations

Activity 1

1. Find all solutions to $3x + 8 = 14$.
   - What does "solution" mean?
   - How do you know you found them all?
2. Suppose you assign the task of solving $3x + 8 = 14$ to high school students. What are three different approaches to solving this equation that students might come up with?

Analyzing the steps

Activity 2

In the preceding activity, we asked you to solve the equation

$$3x + 8 = 14.$$

In this activity, we are going to think very carefully about what is involved in "solving" this equation.

1. What exactly does the "equal sign" mean in this equation?

2. In other mathematical contexts, what are other meanings the "equal sign" could have?

Two equations in x are *equivalent* if they can be manipulated into each other with valid mathematical operations. When we solve equations, we want to maintain equivalence over steps, or be very aware when we are not and how this might lead to missing or extraneous solutions. (This is unlikely to come up when solving linear equations, but it does come up when high school or middle school students solve equations with radical or quadratic expressions.)

Here is a detailed solution to the problem. It may seem like we are writing more steps than a typical high school student might write. The reason why we are writing it this way is to unpack all the ideas that we are drawing on. What are these ideas? For each step in the solution, write down, as best you can, justifications for that step.

$$\begin{aligned}
3x + 8 &= 14 \\
(3x + 8) + (-8) &= 14 + (-8) \\
3x + (8 - 8) &= 6 \\
3x + 0 &= 6 \\
3x &= 6 \\
\left(\frac{1}{3} \cdot 3\right) x &= 2 \\
1x &= 2 \\
x &= 2
\end{aligned}$$

3. Observe the solution and explain how each line "preserves equivalence" to the original equation.

4. Suppose you have a student who writes the following solution steps.

$$\begin{aligned}
3x + 8 &= 14 \\
(3x + 8) + (-8) &= 14 \\
3x &= 14
\end{aligned}$$

   What questions would you ask to probe the student's thinking? How would you respond?

5. Take a different approach to solving the equation $3x + 8 = 14$ and break it down as we did above.

6. (a) Suppose you are using the task of solving $3x + 8 = 14$ to review properties of number systems, operations, and equations. What are some properties that you could review using the various solutions that could come up?

   (b) Suppose you assign the task of solving $3x + 8 = 14$ to fourth or fifth grade students. Would all the approaches that make sense at the high school level also make sense to fourth or fifth grade students? Why or why not?

What number systems can you solve an equation over?

Activity 3

In this activity and throughout these notes, we will let $\mathbb{R}$ denote the *system* of real numbers with real number multiplication and addition, $\mathbb{Q}$ denote the system of rational numbers with rational number multiplication and addition, $\mathbb{Z}$ denote the system of integers, and $\mathbb{N}$ denote the system of natural numbers, both with integer multiplication and addition. (Note: To be consistent with Modules 1 and 2, we will assume $0 \in \mathbb{N}$; that is, we will equate the set of natural numbers with the set of *nonnegative integers*.)

1. Jeremy's class is going to visit the aquarium on a field trip. He must divide his class of ten students into separate groups for the tour, and each group can contain no more than two students. Three of the students are easily overwhelmed and must go on a tour alone with a guide.
   (a) If we let x represent the number of student pairs Jeremy must make, explain why the equation $2x + 3 = 10$ tells us how many such pairs are needed.
   (b) Is it correct to say that this equation has no solution? Explain your thinking.

   Mathematicians sometimes say that a particular equation has solutions "over" some specified system of numbers.

2. Define what it means for an equation in x to "have a solution over the integers" and to "have a solution over the rationals". Define what it means for an equation in x to "have a solution over a set $\mathbb{S}$."
3. Which of these number systems can the equation $3x + 10 = 1$ be solved over?

$$\mathbb{Z} \qquad \mathbb{N} \qquad \mathbb{Q}$$

4. How do you know there's a solution?
5. How do you know that there is only one solution?
6. Come up with equations that that CAN be solved in one system and CANNOT be solved in another system.
7. Does having a solution in $\mathbb{N}$ mean having a solution in $\mathbb{Q}$? Does having a solution in $\mathbb{Q}$ mean having a solution in $\mathbb{N}$? What other of these kinds of relationships can you find?
8. Does every linear equation with integer coefficients have a solution over the integers?
9. Does every linear equation with rational coefficients have a solution over the rationals?

Digging Deeper: Attending to precision

Digging Deeper: Attending to precision

To end this section, we consider one of the Common Core Standards for Mathematical Practice, namely MP6: Attend to Precision:

> Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately.

and later

> In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

Here are two scenarios where this practice can arise.

Suppose that you have assigned students in your high school class the following task:

> Find all solutions to each of these equations. How do you know that you have found them all?

$$x^2 + 5x + 8 = 2$$
$$x^2 = -1$$

1. Solve these equations, then discuss the following question with a neighbor: What are some ways that students might not attend to precision? How would you use this task to help students understand what it means to attend to precision?
2. In a small group, discuss why it is important for high school students to learn and use definitions. Why is it important for teachers?

Suggested homework

1. Find three different approaches to solving the equation $4x - 3 = 9$.

2. Notice that $4x - 3 = 9$ has solution $x = 3$. Write down one very detailed, step-by-step solution. Then verify that each line in your solution is true when you plug in $x = 3$.

3. We often encourage students to check their work after solving an equation. How might a student do this?

4. Give three examples of logical or algebraic errors that a student might make when trying to solve this equation. For each error, explain the mistake and how you would help the student understand and correct the error.

5. How would you use a number line representation to show that addition is commutative? That multiplication is commutative?
6. How would you use an area model representation to show that multiplication is commutative and associative?
7. How would you use these explanations to model for students how to attend to precision?

2 Fundamental Properties of Fields

The field axioms

Activity 1

In this activity, we will introduce a mathematical entity that helps us make explicit the arithmetic processes used to solve linear equations.

Throughout these notes, we will assume the word ”operation” on a set X refers to a well-defined process that assigns a unique element of X to each pair of elements from X. For example, we would consider ”integer subtraction” to be an operation on the set $\mathbb{Z}$ of integers but not an operation on the set $\mathbb{Z}^+$ of positive integers.

Definition 2.1. Definition of field A field is a set $\mathbb{F}$, together with an addition operation $+$, a multiplication operation $\cdot$, an additive identity element 0, and a multiplicative identity element 1 with the following properties:

F1 (Addition is commutative) For all $x, y \in \mathbb{F}$, $x + y = y + x$.

F2 (Addition is associative) For all $x, y, z \in \mathbb{F}$, $(x + y) + z = x + (y + z)$.

F3 (Additive identity) For all $x \in \mathbb{F}$, $x + 0 = x$.

F4 (Additive inverses) For all $x \in \mathbb{F}$ there exists an element $y \in \mathbb{F}$ so that $x + y = 0$.

F5 (Multiplication is commutative) For all $x, y \in \mathbb{F}$, $x \cdot y = y \cdot x$.

F6 (Multiplication is associative) For all $x, y, z \in \mathbb{F}$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

F7 (Multiplicative identity) For all $x \in \mathbb{F}$, $x \cdot 1 = x$.

F8 (Multiplicative inverses) For all $x \in \mathbb{F}$ such that $x \neq 0$, there exists an element $y \in \mathbb{F}$ so that $x \cdot y = 1$.

F9 (Distributive Property) For all $x, y, z \in \mathbb{F}$, $x \cdot (y + z) = x \cdot y + x \cdot z$.

F10 (Non-triviality) $0 \neq 1$.

1. Think about the system $\mathbb{N}$ of natural numbers under integer addition and multiplication. Does this system form a field? Explain your thinking.

2. Think about the system $\mathbb{Q}$ of rational numbers under rational number addition and multiplication. Does this system form a field? Explain your thinking.

3. Think back over some of your other undergraduate classes. Did you encounter any additional examples of fields? Share any examples with a neighbor.

Our number system fields have other arithmetic processes like subtraction and division that are not mentioned at all in our general definition. When it is necessary to talk about these processes, it is often convenient to express them using only the field operations and properties.

4. Suppose x and y are elements of a field $\mathbb{F}$. How do you think we could define the *difference* $x - y$ in this field?

5. Suppose x and y are elements of a field $\mathbb{F}$. How do you think we could define the *quotient* $\frac{x}{y}$ in this field?

Elementary properties of fields

Activity 2

In Activity 1, we codified the arithmetic properties of a field as a list of ten axioms. Most of these axioms present arithmetic processes and properties that seem self-evident (at least for real numbers). Why were these *particular* ones chosen as axioms?

There are many other arithmetic processes and properties that the system of real numbers possess (like subtraction and division by nonzero elements) that were not included in the list. In Activity 1, we showed that subtraction and division can be defined in arbitrary fields using only the field axioms listed.

This points out a key feature of our field axioms. They are *not* an exhaustive list of every possible arithmetic property or process we may encounter when working with these systems. Rather, we *hope* this list is complete enough to allow us to recast these other processes (like subtraction) in terms of those in the list, and we *hope* this list is enough to allow us to *deduce* any other properties we may need.

For example, what about these familiar properties or real number multiplication?

1. For all real numbers x, we have $0 \cdot x = 0$.
2. For all real numbers x, we have $-x = (-1) \cdot x$.

Can we *prove* that these familiar properties also hold for arbitrary fields just by using our field axioms? It turns out that we can. Consider the following proof.

Theorem 2.2. *In any field $\mathbb{F}$, if $x \in \mathbb{F}$, then $0 \cdot x = 0$.*

Proof. We begin by observing that $0 = 0 + 0$. Then multiplying both sides on the right by x, we obtain

$$0x = (0 + 0)x.$$

Applying the distributive property to the right hand side, we obtain

$$0x = 0x + 0x.$$

Now we add the additive inverse of $0x$ to both sides. This gives

$$0x + -(0x) = (0x + 0x) + -(0x).$$

By definition of $-(0x)$, we have that on the left hand side, $0x + -(0x) = 0$. Applying the associative law to the right hand side then gives,

$$0 = 0x + (0x + -(0x)).$$

This becomes $0 = 0x + 0$. So we have $0 = 0x + 0$. By definition of the additive identity, 0, the right hand side is $0x + 0 = 0x$. So we finally have our conclusion,

$$0 = 0x.$$

□

Now, consider the following conjecture.

Theorem 2.3. *In any field $\mathbb{F}$, if $x \in \mathbb{F}$, then $-1 \cdot x = -x$.*

A proof for this conjecture based solely on the field axioms is provided below, but the steps have been jumbled. In your small group, order these steps so that each step provides justification for the next by the field axioms. Also, explain why each step is true.

L1 $(-1 + 1) \cdot x = 0$

L2 $-1 \cdot x + 1 \cdot x = 0$

L3 $0 \cdot x = 0$

L4 $-1 \cdot x + (x + -x) = -x$

L5 $-1 \cdot x = -x$

L6 $-1 \cdot x + 0 = -x$

L7 $(-1 \cdot x + x) + -x = 0 + -x$

L8 $-1 \cdot x + x = 0$

Most of the familiar algebraic properties of real numbers can be deduced for arbitrary fields directly from the field axioms. The following is a (by no means exhaustive) list of such properties.

Theorem 2.4. *Let $\mathbb{F}$ be a field. The following statements are true.*

1. *If $x \in \mathbb{F}$ and a and b are additive inverses for x, then $a = b$.*
2. *If $x \in \mathbb{F}$ and a and b are multiplicative inverses for x, then $a = b$.*
3. *If $x \in \mathbb{F}$, then $-(-x) = x$.*
4. *If $x, y \in \mathbb{F}$, then $-(x + y) = -x + (-y)$. (You can "distribute" a negative sign over a sum of two terms.)*
5. *If $x, y \in \mathbb{F}$, then $(-x) \cdot y = x \cdot (-y) = -(x \cdot y)$ (A negative sign in a product can be "attached" to either factor, or to the product of the two factors.)*
6. *If $x, y \in \mathbb{F}$, then $(-x)(-y) = xy$. (The product of two "negatives" is a "positive."*
7. *If $a, b, x, y \in \mathbb{F}$, then $(a + b)(x + y) = ax + ay + bx + by$. (You can multiply two binomials by multiplying each term of one binomial by each term of the other, and adding the products.)*
8. *The multiplicative inverse of 1 is 1; the multiplicative inverse of -1 is -1.*
9. *If $x, y, z \in \mathbb{F}$, then $x - (y - z) = x - y + z$ (Subtraction inside a binomial that is being subtracted leads to addition.)*
10. *If $x, y \in \mathbb{F}$ are nonzero, then $(xy)^{-1} = x^{-1}y^{-1}$ (The reciprocal of a product is the product of the reciprocals.)*
11. *If $a, x, y \in \mathbb{F}$ and a and y are nonzero, then $\frac{ax}{ay} = \frac{x}{y}$ (Fractions with common factors can be simplified.)*
12. *If $x, y \in \mathbb{F}$ and y is nonzero, then $\frac{-x}{y} = \frac{x}{-y} = -\frac{x}{y}$ (If a fraction contains a negative sign, then this sign can be placed in the numerator, the denominator, or in front of the entire fraction.)*
13. *If $x, y, a \in \mathbb{F}$ and $a \neq 0$, then $\frac{x}{a} + \frac{y}{a} = \frac{x+y}{a}$. (Fractions with the same denominator are added simply by adding the numerators.)*
14. *If $x, y, a, b \in \mathbb{F}$ and $a, b \neq 0$, then $\frac{x}{a} + \frac{y}{b} = \frac{bx+ya}{ab}$. (Mixed fractions are added by finding a "common" denominator.)*
15. *If $x, y, \in \mathbb{F}$ and $y \neq 0$, then $(\frac{x}{y})^{-1} = \frac{y}{x}$. (The multiplicative inverse of a fraction is its "reciprocal".)*
16. *If $x, y, a, b \in \mathbb{F}$ and $a, b, y \neq 0$, then $\frac{x/a}{y/b} = \frac{x}{a} \cdot \frac{b}{y}$. (Dividing by a fraction is the same as multiplying by its "reciprocal".)*
17. *If $x, y \in \mathbb{F}$ such that $xy = 0$, then $x = 0$ or $y = 0$. (The only way for a product of two numbers to be zero is for at least one of the numbers to be zero – the* **zero product property**.*)*

Your instructor will assign all or some of these properties for you to prove using the field axioms.

Optional Question

If you worked through Module 2, then you developed many of the properties of rational number arithmetic using the "movement" model for integer addition and multiplication. Compare this approach to the field-axiom approach presented in Theorem 2.4. What advantages and disadvantages do both approaches have? Which do you prefer and why?

Digging Deeper: Reflecting on Field Properties

Think about the properties in the preceding activity that are valid, and answer the following questions.

1. Where are these properties used in middle and high school mathematics (aside from just simplifying algebraic expressions)? Give examples for three different properties on the list.
2. How did you first learn about these properties? What challenges did you face as you learned to use these properties to solve algebra problems?
3. When these properties were presented in your middle or high school mathematics classes, do you remember being presented with reasons why these properties are valid, and other ones are not?
4. What challenges or obstacles to students' learning of algebra might occur if students learn the above properties without learning about the underlying reasoning?

Suggested Homework

1. Prove the properties from Activity 2 above assigned by your instructor.
2. Consider the collection of all 2×2 matrices whose entries are real numbers. Write down the additive identity of this set.
3. How would you define the general notion of a **multiplicative identity**? What is a multiplicative identity in $\mathbb{Q}$?
4. Is there a multiplicative identity for the set of all 2×2 matrices with real entries?
5. Recall that a rational function is a function of the form $r(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials. The set of rational functions is sometimes called $\mathbb{Q}(x)$. What do you think is the multiplicative identity in $\mathbb{Q}(x)$?

We are interested in mathematical structures with operations of addition and multiplication that satisfy the field axioms. Let $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ and define operations of addition $\boxplus_5$and multiplication $\boxdot_5$ according to the following rules.

- $a \boxplus_5 b$ is the remainder obtained when $a + b$ is divided by 5.
- $a \boxdot_5 b$ is the remainder obtained when ab is divided by 5.

We call $\boxplus_5$ and $\boxdot_5$ addition and multiplication *modulo* 5. (This word is derived from the Latin word for "measure.")

For example, since we obtain a remainder of 2 when 7 is divided by 5, we let $3 \boxplus_5 4 = 2$. Fill in the following table.

$\boxplus_5$	0	1	2	3	4
0					
1					
2					
3					2
4					

Use the definition of $\boxplus_5$ or your table to help answer the following questions.

6. Is $\boxplus_5$ a commutative operation? How did you decide?
7. Is there an additive identity? How did you decide?
8. Does every member of $\mathbb{Z}_5$ have an additive inverse? How did you decide?

We follow the analogous process for multiplication. For example, $4 \cdot 4 = 16$. When we divide 16 by 5 we get 3 with a remainder of 1. So we conclude that in $\mathbb{Z}_5$, $4 \boxdot_5 4 = 1$. Fill in the following multiplication table:

$\boxdot_5$	0	1	2	3	4
0					
1					
2					
3					
4					1

Use the definition of $\boxdot_5$ or your table to answer the following questions.

9. Is $\boxdot_5$ a commutative operation? How did you decide?
10. Is there a multiplicative identity? How did you decide?

11. Does every nonzero member of $\mathbb{Z}_5$ have an additive inverse? How did you decide?

Now, let's consider the set $\mathbb{Z}_7 = \{0,1,2,3,4,5,6\}$ along with addition and multiplication defined "modulo 7" like we defined them for $\mathbb{Z}_5$.

12. How should the operations $\boxplus_7$ and $\boxdot_7$ be defined? Write your definitions carefully.

The addition and multiplication tables for $\mathbb{Z}_7$ under addition and multiplication modulo 7 are large, so this time, let's answer some questions *without* using tables.

13. Do you think $\boxplus_7$ and $\boxdot_7$ are commutative? Justify your answer.
14. Does $\mathbb{Z}_7$ have an additive identity under these operations? Does it have a multiplicative identity? Justify your answer.
15. Does every member of $\mathbb{Z}_7$ have an additive inverse? Justify your answer.
16. Does every nonzero member of $\mathbb{Z}_7$ have a multiplicative inverse? Justify your answer.
17. Verify that in $\mathbb{Z}_7$ the following number sentences are true. Be prepared to share your work with the class.
    - $3 \boxdot_7 (1 \boxplus_7 4) = (3 \boxdot_7 1) \boxplus_7 (3 \boxdot_7)4$
    - $5 \boxdot_7 (2 \boxplus_7 6) = (5 \boxdot_7 2) \boxplus_7 (5 \boxdot_7 6)$
18. Make up two number sentences with the same form using numbers $0 - 6$ and verify that they are true in $\mathbb{Z}_7$.
19. Do you believe that the distributive law holds in $\mathbb{Z}_7$? What justification would you give for your answer?
20. Verify that in $\mathbb{Z}_7$ the following number sentences are true. Be prepared to share your work with the class.
    - $3 \boxplus_7 (1 \boxplus_7 4) = (3 \boxplus_7 1) \boxplus_7 4$ and $5 \boxplus_7 (2 \boxplus_7 6) = (5 \boxplus_7 2) \boxplus_7 6$.
    - $3 \boxdot_7 (1 \boxdot_7 4) = (3 \boxdot_7 1) \boxdot_7 4$ and $5 \boxdot_7 (2 \boxdot_7 6) = (5 \boxdot_7 2) \boxdot_7 6$.
21. Do you believe that the associative laws hold in $\mathbb{Z}_7$? What justification would you give for your answer?
22. Based on your work above, does $\mathbb{Z}_7$ satisfy the field properties? Is it a field?
23. Finally, let's think about the set $\mathbb{Z}_4 = \{0,1,2,3\}$ along with operations of addition and multiplication "modulo" 4.
    (a). Create addition and multiplication tables for $\mathbb{Z}_4 = \{0,1,2,3\}$.
    (b). Based on your tables, is $\mathbb{Z}_4$ a field under addition and multiplication "modulo" 4? Justify your answer based on the field properties.
24. Form a conjecture for which values of n is $\mathbb{Z}_n$ a field under addition and multiplication "modulo" n. Choose another value of n and test your conjecture. Be prepared to discuss your conjecture with the class.

Mathematics Of Doing, Understanding, Learning, and Educating for Secondary Schools

MODULE(S^2): Geometry for Secondary Mathematics Teaching

Module 1: Axiomatic Systems

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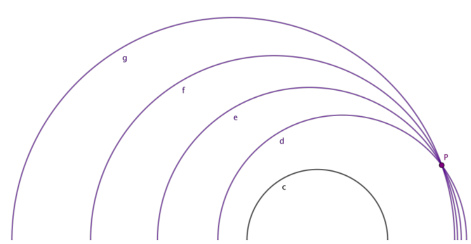
INSTRUCTOR VERSION



For more information about the MODULE(S^2) Project and other MODULE(S^2) materials please visit www.modules2.com

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Contents

Overview of MODULE(S^2) Project iii

Overview of Geometry MODULE(S^2) vii

Overview of Module 1: Axiomatic Development of Geometry viii

1 Where Should We Live? 1
- Overview . . . 1
- Lesson Description . . . 2
  - Activity: Choosing a Place to Live . . . 5
  - Homework: Introduction Survey . . . 7

2 What is Geometry? 9
- Overview . . . 9
- Lesson Description . . . 10
  - Activity: The Building Blocks . . . 14
  - Homework: Shortest Paths . . . 16

3 Where do the differences come from? 19
- Overview . . . 19
- Lesson Description . . . 21
  - Activity: Euclid's Assumptions . . . 28
  - Homework: Constructions . . . 29
  - Optional: All the Points . . . 33
  - Optional: Distances on a Sphere . . . 34

4 Sets of Axioms 37
- Overview . . . 37
- Lesson Description . . . 38
  - Activity: Incidence Geometry . . . 42
  - Handout: Hilbert's Axioms . . . 44
  - Homework: Alternative Set of Axioms . . . 46
  - Homework: Angle Proofs . . . 48

5 Consequences of our Choices 51
- Overview . . . 51
- Lesson Description . . . 53
  - Activity: Right Angles . . . 63
  - Activity: Parallel Lines . . . 65
  - Activity: Sum of the Angles in a Triangle . . . 67

Simulation of Practice Video Assignment: Axiomatic Systems 69
Optional: Triangle Exploration 70
Optional: Triangle Exploration 72

6 What Do Students Understand About Axiomatic Systems? 73
Overview 73
Lesson Description 74
Activity: How Do Students Understand Axiomatic Systems? 77
Simulation of Practice Written Assignment: Axiomatic Systems 79
Midterm Exam 80

Overview of Geometry MODULE(S^2)

In order to teach geometry one needs to understand its place in the wider mathematical context, its development and possible approaches to its study. There is much that can be done with and learned about geometry, but our path for this course will be focused on the needs of our students as future teachers.
We are used to the Euclidean approach to developing geometry involving an axiomatic approach and sometimes that of straightedge and compass constructions. Although we understand the need for our students to understand the axiomatic structure of geometry and to develop their proving skills, we also see competing needs for the time in a geometry class for preservice teachers. For many of our teachers, the geometry course they will be teaching necessitates understanding of transformations that are not always included in a traditional Euclidean geometry class.

For this reason we chose to split the course into three modules hoping to address content relevant to the practice of teaching mathematics.

- Module 1: Axiomatic Development
- Module 2: Transformational Geometry
- Module 3: Similarity

The work of a teacher consists of more than just possessing the knowledge of mathematics. Teachers are supposed to help others attain knowledge they are required to attain. For example, the teacher is tasked with creating, selecting, and modifying tasks; a skill not often addressed in the mathematics classroom. Another relevant skill is ability to understand, find flaws, and help students find errors in their arguments. In this course we are providing opportunities for students to develop and practice these skills in addition to learning geometry.

Instructional Notes and Expectations

In an effort to create a learning environment that is aligned with current recommendations for teaching and learning mathematics in K-12 settings, the following practices are suggested for instruction in this course:

- Use of Technology: We encourage the investigative use of dynamic geometry programs for the exploration of ideas generated in class and through assignments. In addition, there are places where students will be directed to particular internet sites to read about the concepts of study. However, we discourage students from general internet searches when completing homework and instead encourage them to problem solve and communicate with peers in order to work as mathematicians work when pursuing new ideas. Reminder of this policy is often necessary throughout the course.
- Note-taking Assignment: As there is no textbook for this course, we recommend assigning note-taking as part of the regular work of the course. Each day, one student should be assigned as the official note-taker, who then submits notes completed in a template you provide. These notes can be used as part of a homework or participation grade for the course and posted in your learning management system for all students to have access. If you use a template for this activity, it is easily combined into a book authored by your class at the end of the semester.
- Handouts: In-class activities for students and homework assignments are listed as handouts. However, most can be easily incorporated into a digital display for the class or shared electronically through your learning management system.
- Homework: In many cases, homework assignments are structured so that they generate discussion for the next or an upcoming lesson. Assigning homework so that it can be submitted through a learning management system prior to the class in which it will be discussed provides the instructor an opportunity to peruse the work and adequately anticipate questions for the following class discussions. In some cases, considering how students respond to questions in homework prior to a class session can aid in assigning students to groups that will then move forward in their thinking based on shared ideas.
- Video Assignments: Two of the culminating assignments require access to online videos. In order to engage with video animations of mathematics classrooms you will need to invite students to open accounts at specified websites.

Overview of Module 1: Axiomatic Development of Geometry

In Module 1 our goal is to develop appreciation for and understanding of the structure of mathematics in general and geometry in particular. Most of the time, people go about it through a careful development of Euclidean geometry, starting with a set of axioms, often small, then proving everything one possibly can, then expanding the set of axioms. We have opted against this approach since it is a time consuming endeavor, and there are several topics of which we felt pre-service teachers need to have a good grasp.

We will try to lead students through some of the interesting proofs, but this will be a shaky territory for both them and us. Namely, we haven't developed the machinery very carefully, so there could be some discomfort on our side with deciding what to demand be proven and what to leave out. On the students' side, it may be frustrating not to know what is allowed and what isn't, what is known, and what isn't. You may want to leave it to students to decide which set of axioms they prefer to work with once we discuss several of them. We will try to indicate where it may be necessary to allow students to get away without proof as to not to burden the discussion unnecessarily. On the other hand, it is important to provide some opportunities for students to practice their proof writing skills, so we will take some time to do that as well. We like to think that it is much more important for students to develop a sense when their argument is flawed or incomplete and to say things like ”If I knew (blank), then I can prove (blank)”, and we will work toward that end.

- **Big ideas** In this module we will help students understand the structure of axiomatic systems. We will consider the impact different axioms have on the structure of geometry with a culminating experience involving the parallel postulate so that students understand the diversity of alternatives to Euclidean geometry.
- **Goals for studying the topic**:
  - Understand how geometry is built.
  - Become aware of and appreciate geometries other than Euclidean.
  - Be able to articulate differences between neutral, Euclidean, spherical, taxicab, and hyperbolic geometries.
  - Develop skills for writing proofs and understand the role of axioms and rules of logic in proofs.
  - Develop ability to make sense of others' reasoning.
- **Rationale** A common complaint of high school students is that they are asked to prove things that are entirely obvious to anyone who is willing to look. In other words, they do not understand the need for proof. Part of the reason is that they have no sense that alternatives are possible. In this module, we introduce student to non-Euclidean geometries both through considerations of axioms, but also more intuitively with a goal of easier transfer to the K-12 classroom. While we believe that it is important for students to understand the structure of axiomatic systems and what constitutes mathematical reasoning, we also believe it is important for them to develop intuition and familiarity with different geometries.
- **Connections to Secondary Mathematics** In this module the student will start to understand the axiomatic system the Common Core State Standards advocates using in 6-12 curriculum: a transformational approach. Unfortunately, as far as we know the writers of CCSS have not developed such a system, so we cannot choose it to help students develop the knowledge of curriculum. In order to remedy that, we will consider variety of systems in order to help them realize that while the choice impacts what results we can obtain, it is possible to develop sets of axioms which yield the same geometry.
- **Overview of content**
  - Lesson 1 Where Should We Live?: In this lesson, students are introduced to the notion that Euclidean geometry isn't the only valuable one. Urban geography gives rise to taxicab geometry as an alternative system to the one students are used to. Through this module students will get acquainted with other geometries, most notably spherical and hyperbolic. Students will also encounter modeling which is an important strand in the standards for high school geometry.
  - Lesson 2 What is Geometry?: Here, students are asked to reflect on and consider the building blocks of geometry. Through this discussion we will start talking about definitions and undefined terms, and begin developing our skills in writing definitions. Additionally, we will discuss proving and what constitutes a

good proof. As we start exploring models of different axiomatic systems without being very formal about it, students will talk about lines and notion of straightness on the plane (with both Euclidean and taxicab distance), as well as on a sphere.

  - Lesson 3 Where do the differences come from?: We continue to challenge students' notions of what geometry is. We further identify the differences between models of different geometries, this time through talking about circles. We leave some space for possible exploration of distances on a sphere, and thinking about the ratio of circumference to diameter of a circle in these different geometries. We include in this lesson examination of Euclid's assumptions and first proposition.
  - Lesson 4 Sets of Axioms: We finally discuss axiomatic systems in order to more formally explain where all these differences are coming from. We discuss them from the mathematical point of view, but also from the point of the view of the user. This lesson introduces Hilbert's Axioms and neutral geometry.
  - Lesson 5 Consequences of our Choices: We elucidate further differences by discussing parallel lines. We begin with considerations of angle relationships that can be proved in neutral geometry and then discuss the parallel postulate, its historical and mathematical significance, and prove some important results in Euclidean geometry. This lesson concludes with an examination of ways the triangle angle sum theorem might be introduced and proved in K-12 mathematics classrooms.
  - Lesson 6 What Do Students Understand About Axiomatic Systems?: We conclude the Module by considering a representation of practice and engaging students in developing their own classroom discussion as it relates to axiomatic systems.

  We should note that when we first taught the course we used heavily both Nathaniel Miller's course notes as well as the Math812T Geometry for Geometry Teachers course notes from the University of Nebraska - Lincoln[1]

  In the intervening years, the order as well as the assignments have been modified, but the flavor is distinctly recognizable, and several problems remain unchanged.

- **Overview of mathematical and teaching practices** While we hope that all of our lessons will be in line with the *Standards of Mathematical Practice* as outlined in the Common Core State Standards, in this module we will particularly pay attention to:

SMP 2 Reason abstractly and quantitatively

SMP 3 Construct viable arguments and critique the reasoning of others

SMP 4 Model with mathematics

SMP 6 Attend to precision

SMP 7 Look for and make use of structure

- **Expectations for assignments**

  - Students can be tasked with writing a course textbook. After each lesson, a student is asked to write out the accomplishments of the day. The goal is to leave a reference for all students, give everyone an opportunity to practice writing, and provide feedback to each other.
  - Homework Assignments are generally a preparation for class or planning information for the instructor. The instructor may choose whether to review and provide feedback.
  - Instructors may choose to have a class discussion board where questions can be posted and encouraged. The following prompt may serve well to encourage questioning and discussions throughout the course: "An important part of doing mathematics is to learn to ask one's own questions. You might not be able to answer them, but you can always learn more through seeking the answer. Which questions has this work made you ask? Record all the questions that you'd like to investigate on the class website."

[1] Miller and UNL Source

Lesson	Projected Length	Geometric Content	In-Class Activities	Homework	Connections and Notes
1: Where Should We Live?	90 min	Equidistance, perpendicular bisectors, taxicab Geometry introduction, mathematical modeling	Activity: Choosing a Place to Live	Homework: Introduction Survey	• Choosing a Place to Live will be used to generate discussion in the lesson 2 and 3 activity discussions • Introductory Survey Question 6 will be revisited in Lesson 3 Homework: Constructions • Introductory Survey Question 7 will be used for discussion of the activity in Lesson 2 and 3
2: What is Geometry?	90 min	Foundations of a Geometry (undefined terms), shortest paths (straightness), Spherical Geometry	Activity: The Building Blocks	Homework: Shortest Paths	• Shortest Paths Question 1c will be used to generate discussion in Lesson 3 • Shortest Paths Question 2 will be used to generate discussion in Lesson 3
3: Where do the differences come from?	180 min	Circles, attending to precision, definitions, Euclids Postulates, interpretation and model	Activity: Euclid's Assumptions	Homework: Constructions Optional: Distances on a Sphere (optional) Optional: All the Points (optional)	• Constructions Question 2 and 3 will be revisited in Lesson 6
4: Sets of Axioms	180 min	Incidence Axioms, Hilberts Axioms, proof, Hyperbolic Geometry	Activity: Incidence Geometry Handout: Hilbert's Axioms	Homework: Alternative Set of Axioms Homework: Angle Proofs	• You may choose to assign Incidence Geometry #5 as homework • An Alternative Set of Axioms cannot be assigned until after students have completed Incidence Geometry and started discussion of Hilberts Axioms
5: Consequences of our Choices	180 min	Differences between neutral, Euclidean, hyperbolic, and spherical geometries; definitions, right angles, parallel lines, angles formed by transversals	Activity: Right Angles Activity: Parallel Lines Activity: Sum of the Angles in a Triangle	Simulation of Practice Video Assignment: Axiomatic Systems VanHiele Readings Optional: Triangle Exploration (optional)	• Video Simulation of Practice can be assigned during Lesson 5 or at the end • You may choose to assign the Sum of the Angles in a Triangle as homework
6: What Do Students Understand About Axiomatic Systems?	90 min	Summary of different axiomatic systems, examples of axiomatic systems in secondary classrooms	Activity: How Do Students Understand Axiomatic Systems?	Simulation of Practice Written Assignment: Axiomatic Systems Midterm Exam	

1 Where Should We Live?

Summary

In this lesson, we will connect the work we do in geometry to the world around us. We will see how the modeling and problem solving processes can play out in a geometrical context. We consider a situation many of us are familiar with: choosing a place to live. Such decisions come with a myriad of constraints, and we will attempt to solve this problem in the most general case.

Goals

- Students will engage in the modeling and problem solving processes.
- Students will explore the differences between a plane equipped with the Euclidean distance and a plane equipped with the taxicab distance.
- Students will be introduced to the construction of lines, perpendicular bisectors, and circles.

Materials

- Compasses and Rulers (optional)
- Activity: Choosing a Place to Live
- Homework: Introduction Survey

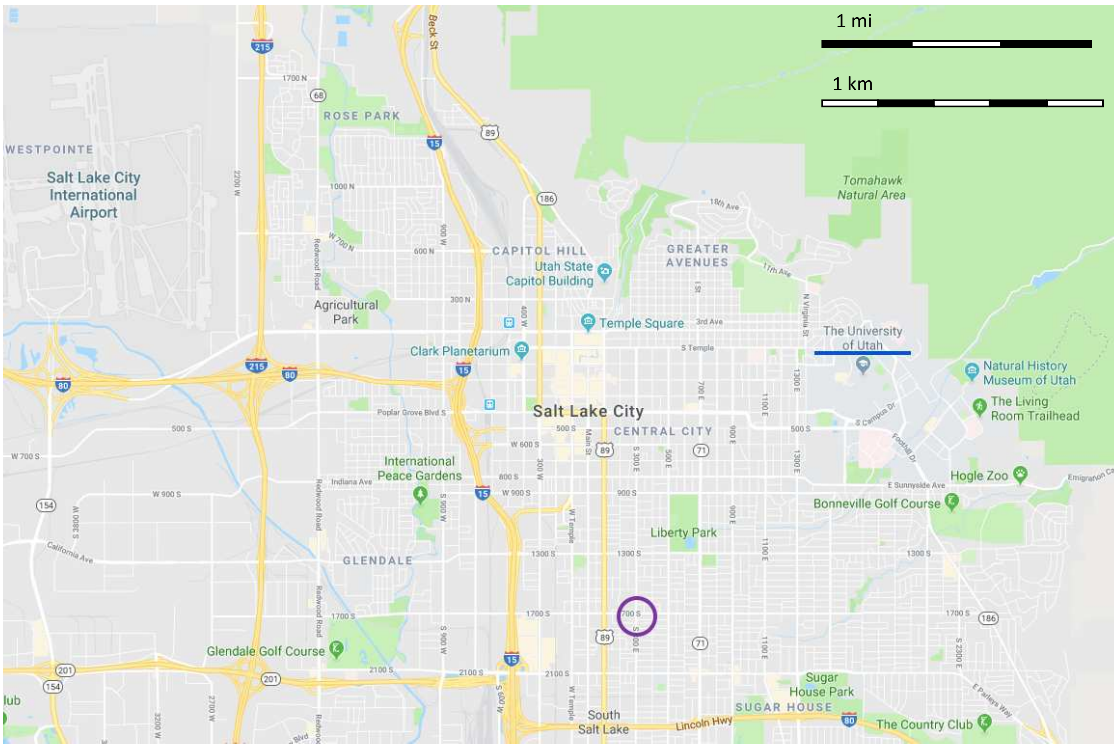
Preparation for the Next Lesson

- Students
  - Introduction survey
  - Obtain Geogebra

ACTIVITY: CHOOSING A PLACE TO LIVE

Shefali and Vanya, two friends who recently graduated college, are planning to rent an apartment together.

1. Shefali works at the University of Utah (underlined in blue), while Vanya works southwest of Liberty Park (location circled in purple). They started considering where they should search for a place to live if they wanted their commutes to be roughly equal. What are some possible locations that would be equidistant from both of their places of work?



2. Both Shefali and Vanya commute on their bicycles. Turns out Shefali is far better cyclist than Vanya. She can usually ride the same distance in about two thirds the time Vanya takes. What would be a fair place for them to look for the house considering this newly available information?

3. (optional) What isn't necessarily clear from the map is that Shefali's place of work is about 400 feet higher in elevation than Vanya's. What are your recommendations now?

HOMEWORK: INTRODUCTION SURVEY

1. Your name
2. Email address - this should be an address you regularly check and one that is associated with your Google Docs and Google Sites account.
3. I have a laptop I can easily bring to class every day
   - Yes, no problem
   - Yes, but I prefer not to bring it every day
   - I do not own one.
4. I have necessary software for the class.
   There is only one correct answer to this question. Visit GeoGebra and LaTeX (or Another) to obtain necessary materials. If you are using public computers for this work, you can use GeoGebra's web app and save your work to a google drive, or GeoGebra account. As for Latex, Overleaf is a lovely web based option.
   - Yes, I have it!
5. Your thoughts:
   - Any questions or comments?
   - Any concerns?
   - Any requests?
6. Open your newly acquired GeoGebra and construct a quadrilateral. Make sure to use the *Polygon tool* from the menu instead of constructing four independent segments and connecting them at their endpoints. Use the *Midpoint tool* to construct the midpoints of each side, then construct a quadrilateral whose vertices are the consecutive midpoints. Save this file as FirstName_LastName_MidpointQuad. Select the pointer, then drag the vertices of your original quadrilateral around. What do you notice? What do you wonder? Are there any properties of the quadrilateral that stay the same throughout your manipulation of the vertices? Make as many conjectures as you can and be prepared to share them with your partners. Is there any way you can be certain that your conjectures are true?
7. Before reading further, take a moment to answer the following prompt: What do you think geometry is?

   The word "geometry" comes from the Greek for "earth measurement", so it seems reasonable that Greeks thought that geometry was a true and absolute description of the physical world. Because they were studying physical objects in the real world, the measurements came with a certain degree of imprecision, and the relationships one finds are only as reliable as the measurements one makes. This is always the case when we describe the physical world: Our knowledge of the world around us is only as good as the observations or measurements we can make about it.
   Despite the origins of its name, geometry is not, in fact, a method for describing the world around us. When studying geometry, we do not try to make claims that can be verified or falsified through measurement; instead, we make claims that can be verified through logical reasoning or falsified through counterexamples. Geometry is not an empirical science, it is a deductive science. We decide whether statements are true or false GIVEN CERTAIN HYPOTHESES.We make hypotheses and work to discover their consequences. If we can demonstrate the consequences to be true through accepted mathematical arguments, we call those conclusions **theorems**.
   Poincaré said:

   > If geometry were an experimental science, it would not be an exact science. It would be subject to continual revision. The geometrical axioms are therefore neither synthetic *a priori* intuitions nor experimental facts. They are conventions. Our choice among all possible conventions is guided by experimental facts, but it remains free, and is only limited by necessity of avoiding every contradiction. What then are we to think of the question: Is Euclidean geometry true? It has no meaning. We might as well ask if the metric system is true and if the old weights and measurements are false, if Cartesian coordinates are true and polar coordinates false. One geometry cannot be more true than another: it can only be more convenient.

   What questions do you have after reading these paragraphs? Write at least three you would like to discuss with your colleagues.

2 What is Geometry?

Summary

In this lesson, we will begin to articulate what geometry is. We will begin to define the objects of study and try to reconcile some of the issues that the previous lesson brought up. Further, we will start thinking about the role of axiomatic systems in geometry, starting with definitions and assumptions. This lesson and the next will also serve as a brief historical overview of Euclid and his work.

Goals

- Students will determine possible building blocks of geometry.
- Students will begin thinking about geometric definitions, their structure and uses.
- Students will be introduced to alternatives to Euclidean geometry.

Materials

- Activity: The Building Blocks
- Homework: Shortest Paths
- Lénárt spheres, if you have them. Globes, beach balls or tennis balls will work also.
- String
- Markers

Preparation for the Next Lesson

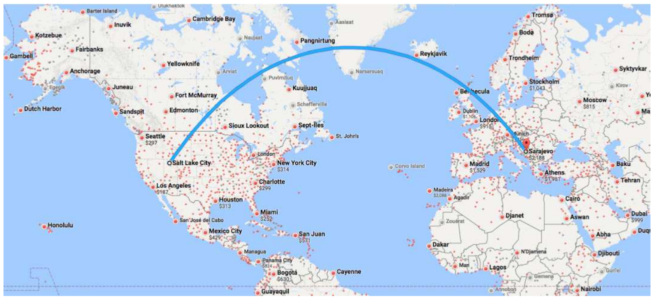
- Students:
  - Homework: Shortest Paths

Activity: The Building Blocks

1. What geometric objects did you work with in Activity: Choosing a Place to Live? Give a definition for each of the objects:

2. There are other objects that are essential to geometry work that may not have been a part of Activity: Choosing a Place to Live. List some additional geometric objects that are important when ″doing geometry.″

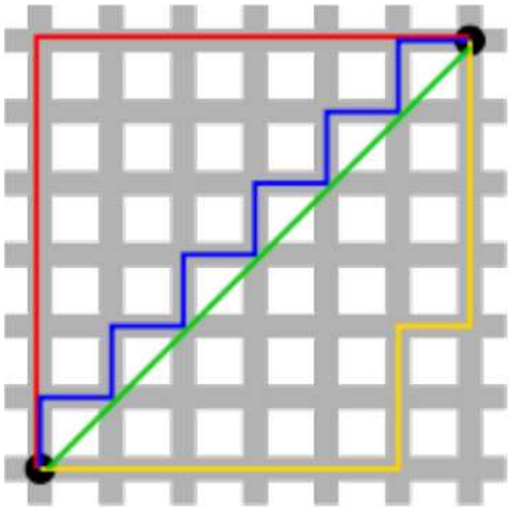
3. A passenger was settling into their flight from Salt Lake City, UT, to Sarajevo, BIH. The flight path looked strange. Would you call it a line? What consideration might have gone into determining this flight plan?



4. Take a moment to study some geometry definitions given by Euclid: `https://goo.gl/axf87H`. What do you think about these definitions? Are there definitions you understand? Which ones? Are there definitions that do not communicate precisely what the object is, or for which you need to know what the geometric object is before the definitions make sense? Which ones? Does the definition of a line that we have considered fit into Euclid's definition?

HOMEWORK: SHORTEST PATHS

1. You have already had some experience with taxicab geometry, which is a geometry anyone living in a town built on a grid is very familiar with. In most parts of town, youre confined to traveling along the grid, just like a cab. Lets look at an example:



   (a) How far would you travel from one black point to the other if you took the red, blue or yellow path? Are there any paths that are shorter than these? Are there any paths that are longer than these? How many shortest paths are there between the two black points?

   (b) Use graph paper to make a coordinate system. Imagine you are standing at point $(4, 9)$. Find all the points that are exactly 5 units away from you in taxicab geometry. Also, find all the points that are 5 units away from $(4, 9)$ in Euclidean geometry. For each case, describe the collection of points that constitutes the answer and/or the geometric object you drew.

2. Consider that the Earth is a sphere with radius 6371 km. What is the shortest distance between each pair of cities listed below? As you work on this problem, it may be helpful to use GeoGebra 3D to represent the Earth and visualize various cities under consideration.

   (a) from Quito, Ecuador (0° N, 78°W) to Kampala, Uganda (0°N, 32°E)?

   (b) from Saint Petersburg, Russia (60°N, 30°E) to Anchorage, Alaska (60°N, 150°W). Additionally, what is the distance if traveling due east? If you continue traveling due east from Anchorage until you reach St. Petersburg from the west, what path will you have traversed?

   (c) (optional practice) from Paris (48°N, 2°E) to Seattle (48°N, 122°W).

   (d) (extension) from Lincoln, NE (40°N, 96°W) to Sydney, Australia (34°S, 151°E)?

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Module 2: Transformational Geometry

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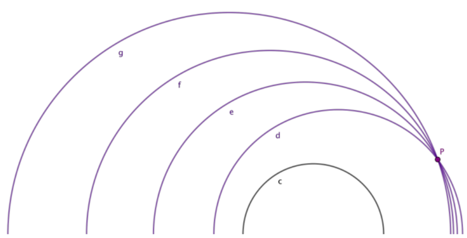

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Contents

Overview of MODULE(S^2) Project iii

Overview of Geometry MODULE(S^2) vii

Overview of Module 2: Transformational Geometry viii

1 Introduction to Transformations 4
- Overview . . . 4
- Lesson Description . . . 6
  - Activity: What is the Shortest Way? . . . 14
  - Alternative Context Activity: What is the Shortest Way? . . . 15
  - Handout: CCSS Mathematical Practice 1 . . . 17
  - Homework: Visualizing Functions . . . 18
  - Handout: Standards for Mathematical Practice . . . 23
  - Homework: Writing Definitions . . . 26
  - Optional: Building Bridges . . . 29

2 Distance-preserving Transformations 30
- Overview . . . 30
- Lesson Description . . . 31
  - Activity: Isometries Preserve Distance . . . 36
  - Handout: Core Congruence . . . 40
  - Handout: UCSMP Axioms . . . 41
  - Formal Writing Assignment: Composing Isometries . . . 43

3 Rotations and Reflections 47
- Overview . . . 47
- Lesson Description . . . 48
  - Activity: Reflections . . . 57
  - Homework: Simulation of Practice Written Assignment . . . 62
  - Activity: Student Thinking about Rotations . . . 65
  - Activity: Rotations . . . 66
  - Homework: Simulation of Practice Video Assignment . . . 70

4 Transformations and Congruence 72
- Overview . . . 72
- Lesson Description . . . 73
  - Activity: How to Get From Here to There . . . 78
  - Homework: When Does Order Not Matter? . . . 83

Instructor Notes: Symmetries 85
Optional Homework: Symmetries 86

5 Fixed Points **90**
Overview 90
Lesson Description 91
Activity: Fixed Points Theorems Proof Outlines 94

6 Triangle Congruence **95**
Overview 95
Lesson Description 96
Activity: Construct This Triangle 101
Activity: Triangle Congruence Theorems 104
Optional: Triangle Congruence Proofs 105
Formal Writing Assignment: Transformations 106

7 Optional Explorations **110**
Overview 110
Exploration 1: Coordinate Geometry 112
Exploration 2: Graph Transformations 113
Exploration 3: Slopes 114
Project: Presenting Geometric Topics 115
Project: Presenting Geometric Topics 116
Optional Handout: Notes on Using GeoGebra to Visualize General Transformations of the Plane 118

Overview of Geometry MODULE(S^2)

In order to teach geometry one needs to understand its place in the wider mathematical context, its development and possible approaches to its study. There is much that can be done with and learned about geometry, but our path for this course will be focused on the needs of our students as future teachers.
We are used to the Euclidean approach to developing geometry involving an axiomatic approach and sometimes that of straightedge and compass constructions. Although we understand the need for our students to understand the axiomatic structure of geometry and to develop their proving skills, we also see competing needs for the time in a geometry class for preservice teachers. For many of our teachers, the geometry course they will be teaching necessitates understanding of transformations that are not always included in a traditional Euclidean geometry class.

For this reason we chose to split the course into three modules hoping to address content relevant to the practice of teaching mathematics.

- Module 1: Axiomatic Development
- Module 2: Transformational Geometry
- Module 3: Similarity

The work of a teacher consists of more than just possessing the knowledge of mathematics. Teachers are supposed to help others attain knowledge they are required to attain. For example, the teacher is tasked with creating, selecting, and modifying tasks; a skill not often addressed in the mathematics classroom. Another relevant skill is ability to understand, find flaws, and help students find errors in their arguments. In this course we are providing opportunities for students to develop and practice these skills in addition to learning geometry.

Instructional Notes and Expectations

In an effort to create a learning environment that is aligned with current recommendations for teaching and learning mathematics in K-12 settings, the following practices are suggested for instruction in this course:

- Use of Technology: We encourage the investigative use of dynamic geometry programs for the exploration of ideas generated in class and through assignments. In addition, there are places where students will be directed to particular internet sites to read about the concepts of study. However, we discourage students from general internet searches when completing homework and instead encourage them to problem solve and communicate with peers in order to work as mathematicians work when pursuing new ideas. Reminder of this policy is often necessary throughout the course.
- Note-taking Assignment: As there is no textbook for this course, we recommend assigning note-taking as part of the regular work of the course. Each day, one student should be assigned as the official note-taker, who then submits notes completed in a template you provide. These notes can be used as part of a homework or participation grade for the course and posted in your learning management system for all students to have access. If you use a template for this activity, it is easily combined into a book authored by your class at the end of the semester.
- Handouts: In-class activities for students and homework assignments are listed as handouts. However, most can be easily incorporated into a digital display for the class or shared electronically through your learning management system.
- Homework: In many cases, homework assignments are structured so that they generate discussion for the next or an upcoming lesson. Assigning homework so that it can be submitted through a learning management system prior to the class in which it will be discussed provides the instructor an opportunity to peruse the work and adequately anticipate questions for the following class discussions. In some cases, considering how students respond to questions in homework prior to a class session can aid in assigning students to groups that will then move forward in their thinking based on shared ideas.
- Video Assignments: Two of the culminating assignments require access to online videos. In order to engage with video animations of mathematics classrooms you will need to invite students to open accounts at specified websites.

Overview of Module 2: Transformational Geometry

In Module 2 our goal is to develop a solid understanding of distance-preserving transformations. The sequence of activities is organized so that the students are led to construct definitions, then use those definitions to develop properties of transformations, all the while taking their distance-preserving properties as axioms. In the lessons students will encounter common conceptions learners have about transformations and discuss ways of helping learners understand the nature of isometries. The transformational approach offers an opportunity to engage in cross-curricular development; computer science and coding fits well with geometry work. This gives another compelling reason to take this approach in secondary classrooms.
In this module we will have more opportunities to engage students with proof. We will take the time to prove that reflections generate the group of Euclidean isometries. We will classify isometries based on their fixed points and use this classification to find the generating set for the group of isometries. We complete the modules with a conversation about triangle congruence, which has for decades been the cornerstone of high school geometry.

- **Big ideas** distance-preserving transformations of the Euclidean plane come in three flavors: reflections, translations, and rotations, although we could easily work only with reflection as every isometry can be expressed as a composite of reflections.
- **Goals for studying the topic**:
  - Know the definitions of isometries (rigid transformations) and be able to use coordinates to describe them.
  - Be able to describe axioms in which we postulate that reflections, rotations, and translations are distance-preserving and prove theorems using this set of axioms.
  - Be able to describe the compositions and decomopsitions of isometries.
  - Know that reflections generate the group of isometries of the plane. Further, be able to explain that any isometry can be expressed as a composition of at most three reflections.
  - Know the definition of congruent triangles using isometries.
  - Be able to describe symmetries of plane figures.
  - Be able to prove SAS, SSS, ASA.
- **Rationale** The CCSS have changed the emphasis from the Euclidean approach, often using the SMSG set of axioms, to the transformational approach. At this point, the students have very little experience with transformations, especially in the more formal, axiomatic way. In this module, students will develop deeper understanding of isometric transformations, their properties, and their relationship to each other. They will see how the congruence theorems follow from the set of axioms built on transformations.
- **Connections to Secondary Mathematics** Transformations have become the building block of geometric reasoning in the secondary schools. The transformations are used to define congruence between geometric figures and students should be able to describe how the congruence theorems and corollary (SSS, SAS, ASA, AAS) follow from this definition.
- **Overview of content**
  - Lesson Introduction to Transformations: In this lesson, the students are asked to find a shortest path between two points given certain constraints. There are many different approaches to the solutions available, but one that is particularly fruitful involves transformations.
  - Lesson Distance-preserving Transformations: In this lesson the students will develop further their skill to write definitions. They will develop definitions for reflections, rotation, and translations, and discuss their usability. We will take the time to introduce and discuss the choices made by the writers of CCSS for mathematics in the way they are suggesting geometry is developed, and consider some historical developments of geometry curriculum.
  - Lesson Rotations and Reflections: Students take the time to perform rotations and reflections using definitions they developed. They analyze learners' thinking and work on developing their ability to assist learners in advancing their understanding of mathematical concepts.

- Lesson Transformations and Congruence: In this lesson we define congruence. For two congruent shapes, we find an isometry (a sequence or rigid motions) taking one to the other.
- Lesson Fixed Points: In this lesson, we further develop the understanding of congruence from a transformational perspective by first conjecturing and then proving properties about fixed points in isometries. This lays the foundation for understanding that the isometries are a group generated by reflections.
- Lesson Triangle Congruence: The culminating lesson in this module provides an opportunity for students to develop intuitions about triangle congruence, and then prove the triangle congruence theorems.
- Lesson Optional Explorations: Additional optional explorations are provided that delve into coordinate geometry, graph transformations, and slopes as related to transformational geometry. We also provide options for a topic presentation project that may serve as a culminating project for the course.

- **Overview of mathematical and teaching practices** While we hope that all of our lessons will be in line with the *Standards of Mathematical Practice* as outlined in the CCSS, in this module we will particularly pay attention to:

SMP 1 Make sense of problems and persevere in solving them

SMP 3 Construct viable arguments and critique the reasoning of others

SMP 4 Model with mathematics

SMP 5 Use appropriate tools strategically

SMP 6 Attend to precision

SMP 7 Look for and make use of structure

- **Expectations for assignments**
  - The students can be tasked with writing a course textbook. After each lesson, a student is asked to write out the accomplishments of the day. The goal is to leave a reference for all students, give everyone an opportunity to practice writing, and provide feedback to each other.
  - Writing Assignments are to be graded and feedback provided. These assignments may be commented on before a final version is submitted for evaluation.
  - Homework Assignments are generally a preparation for class or planning information for the instructor. The instructor may choose whether to review and provide feedback.
  - Instructors may choose to have a class discussion board where questions can be posted and encouraged. The following prompt may serve well to encourage questioning and discussions throughout the course: "An important part of doing mathematics is to learn to ask one's own questions. You might not be able to answer them, but you can always learn more through seeking the answer to your questions. Which questions has this work made you ask? Record all the questions that you'd like to investigate on the class website."

Lesson	Projected Length	Geometric Content	In-Class Activities	Homework	Connections and Notes
1: Introduction to Transformations	180 min	Mathematical modeling, Transformations as functions	Activity: What is the Shortest Way? Visualizing Functions HW Discussion Handout: Standards for Mathematical Practice	Homework: Visualizing Functions Homework: Writing Definitions	• We provide an alternate context for the Activity: What is the Shortest Way? task that allows for discussions of equity and social justice. • Homework: Visualizing Functions assignment must be completed so that students can engage in the second class session discussion. • Homework: Writing Definitions is used to begin the discussion for Lesson 2. Work must be collected from students prior to the start of Lesson 2.
2: Distance-preserving Transformations	90 min	Definition of reflection, rotation, and translation, introduction to geometry from a transformational perspective	Activity: Isometries Preserve Distance Handout: UCSMP Axioms Handout: Core Congruence	Formal Writing Assignment: Composing Isometries	• Student responses from Lesson 1 homework will be used to generate class discussion • Students will begin work on a project that is best completed with a partner (Formal Writing Assignment: Composing Isometries) and will seed discussion in Lesson 4
3: Rotations and Reflections	180 min	Constructing reflections and rotations, understanding student thinking, Introduction to CCSS Transformational Geometry	Activity: Reflections Activity: Student Thinking about Rotations Activity: Rotations	Homework: Simulation of Practice Written Assignment Homework: Simulation of Practice Video Assignment	• The in-class activities provide introduction to the two homework assignments. • Students should complete the Formal Writing Assignment: Composing Isometries prior to the next lesson. Adjust deadlines for these homework assignments appropriately around the project.
4: Transformations and Congruence	180 min	Transformational definition of congruence, Sequence of transformations mapping congruent objects, Isometric transformations as a group	Activity: How to Get From Here to There	Homework: When Does Order Not Matter? Optional Homework: Symmetries	• You can choose what to emphasize (constructions, proofs, etc.) and split the class sessions as appropriate. • The optional homework (Optional Homework: Symmetries) is the only place symmetry is treated in the modules so you may choose to assign it instead of Homework: When Does Order Not Matter?. • It may be appropriate to assign Project: Presenting Geometric Topics as a culminating activity for the course at this time. Examples are provided in the final section of the module.
5: Fixed Points	90 min	Properties of isometries (conjecturing and proving), isometries as a group generated by reflections	?? Activity: Fixed Points Theorems Proof Outlines		• Students may be working on Project: Presenting Geometric Topics at this time. • Homework from Lesson 4 may be carried through Lesson 5 if needed. • The opening investigation found in Lesson 6, Activity: Construct This Triangle, may be assigned as homework to support the introduction to the next lesson.
6: Triangle Congruence	90 min	Development and proof of triangle congruence theorems in transformational geometry	Activity: Construct This Triangle Activity: Triangle Congruence Theorems Optional: Triangle Congruence Proofs	Formal Writing Assignment: Transformations	• Formal Writing Assignment: Transformations provides scaffolding for the proof activity if needed for your students. • You will need to prepare measurement cards for each student for Activity: Construct This Triangle.
7: Optional Explorations	Adjustable Length	Coordinate Geometry, Graph Transformations, Slopes		Exploration 1: Coordinate Geometry Exploration 2: Graph Transformations Exploration 3: Slopes Project: Presenting Geometric Topics	• These explorations are offered as options for delving more deeply into topics closely related to the study of transformational geometry. You may choose to use these as whole class activities or individual homework assignments. We also provide Project: Presenting Geometric Topics description for use as a culminating course project should you choose to do so.

1 Introduction to Transformations

Summary

In Module 1, we explored the meaning of the word "straight" in several different ways, one of which is the shortest path between two points. In this introductory lesson to Transformational Geometry we will attempt to solve some problems in which the goal is to find a shortest path even though the constraints given preclude straight lines as a solution.

Goals

- Students will model with mathematics to solve problems.
- Students will explore various transformations in order to expand their ideas about transformations.
- Students will use mathematical characteristics of functions to describe transformations.
- Students will develop definitions of transformation and isometric transformation.

Materials

- Dynamic geometry software
- Compass, ruler, and protractor
- Graph paper
- Activity: What is the Shortest Way?
- Alternative Context Activity: What is the Shortest Way?
- Handout: CCSS Mathematical Practice 1
- Handout: Standards for Mathematical Practice
- Homework: Visualizing Functions
- Homework: Writing Definitions
- Optional: Building Bridges

Preparation for the Next Lesson

- Students:
  - Homework: Writing Definitions

Activity: What is the Shortest Way?

Sam is on her way back to her tent from a long hike when she realizes that she was supposed to gather a few rocks from the river for use at her family's campsite. She figures she better run to the river to retrieve the rocks so she can get them back to her family before they think she neglected her chore. What path should Sam take to get the rocks and to get them back to her family's tent?

Notes:

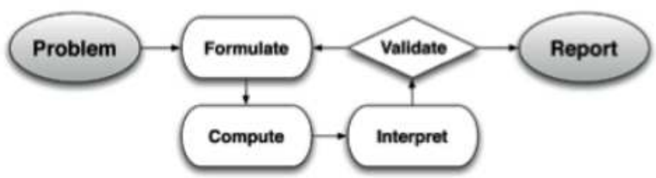
- Locations are not specified in the problem, so your group should consider a variety of options for where Sam is, where the river is, and where the tent is. Consider how your answer changes depending on the particular arrangement of Sam, the river, and the tent.
- Your group will likely need to make some assumptions when solving this problem. Be sure to make a list of the assumptions you make.
- Consider what tools you / your group can use to conduct this investigation and utilize some/all of those tools.

HANDOUT: CCSS MATHEMATICAL PRACTICE 1

CCSS.MATH.PRACTICE.MP1 - Make sense of problems and persevere in solving them

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, “Does this make sense?” They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

MODELING:



The basic modeling cycle is summarized in the diagram. It involves (1) identifying variables in the situation and selecting those that represent essential features, (2) formulating a model by creating and selecting geometric, graphical, tabular, algebraic rules/equations, and/or statistical representations that describe relationships between the variables, (3) analyzing and performing operations on these relationships to draw conclusions, (4) interpreting the results of the mathematics in terms of the original situation, (5) validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable, (6) reporting on the conclusions and the reasoning behind them.

HOMEWORK: VISUALIZING FUNCTIONS

We can use coordinates to describe functions on the plane. For each of the given functions, describe their effect on points of the plane, coordinate axes, and characteristics that they leave invariant. Make predictions about what will happen before you "graph" these functions.[2]

Function	Prediction	Diagram
$f(x,y) = (x^3, y^3)$		
$g(x,y) = (2x, 3y)$		
$h(x,y) = (x+3y, y)$		
$i(x,y) = (\cos x, \sin x)$		
$j(x,y) = (-x, x+3)$		
$l(x,y) = (x^3 - x, y)$		

[2]You might consider a discussion concerning the use of the term "graph" in this description.

Function	Prediction	Diagram
$m(x, y) = (3y, x + 2)$		
$n(x, y) = (x + 2, y - 3)$		
$p(x, y) = (0.6x - 0.8y, 0.8x + 0.6y)$		

If you were to group these functions into two disjoint groups, what would your classification be?

- Group 1:

- Group 2:

What is your rationale for this classification?

HANDOUT: STANDARDS FOR MATHEMATICAL PRACTICE

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

1. **Make sense of problems and persevere in solving them**

   Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identifying correspondences between different approaches.

2. **Reason abstractly and quantitatively**

   Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize - to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents - and the ability to contextualize - to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand, considering the units involved, attending to the meaning of quantities (not just how to compute them), and knowing and flexibly using different properties of operations and objects.

3. **Construct viable arguments and critique the reasoning of others**

   Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and - if there is a flaw in an argument - explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4. **Model with mathematics**

   Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5. **Use appropriate tools strategically**

   Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6. **Attend to precision**

   Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7. **Look for and make use of structure**

   Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

8. **Look for and express regularity in repeated reasoning**

   Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through $(1, 2)$ with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Connecting the Standards for Mathematical Practice to the Standards for Mathematical Content

The Standards for Mathematical Practice describe ways in which developing student practitioners of the discipline of mathematics increasingly ought to engage with the subject matter as they grow in mathematical maturity and expertise throughout the elementary, middle and high school years. Designers of curricula, assessments, and professional development should all attend to the need to connect the mathematical practices to mathematical content in mathematics instruction.

The Standards for Mathematical Content are a balanced combination of procedure and understanding. Expectations that begin with the word "understand" are often especially good opportunities to connect the practices to the content.

Students who lack understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems, represent problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from a known procedure to find a shortcut. In short, a lack of understanding effectively prevents a student from engaging in the mathematical practices.

In this respect, those content standards which set an expectation of understanding are potential "points of intersection" between the Standards for Mathematical Content and the Standards for Mathematical Practice. These points of intersection are intended to be weighted toward central and generative concepts in the school mathematics curriculum that most merit the time, resources, innovative energies, and focus necessary to qualitatively improve the curriculum, instruction, assessment, professional development, and student achievement in mathematics.

HOMEWORK: WRITING DEFINITIONS

We defined an isometry as:

Definition 1.2

A transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ of the plane is called an isometry if it preserves distance between any two given points: for any A, B

$$d(f(A), f(B)) = d(A, B)$$

1. If you ask a student for examples of these transformations, they are likely to tell you: slide, flip, and turn. What do they mean by that? If you ask them for a definition of these transformations, what might they say?
2. Give precise definitions of those transformations.
3. Use dynamic geometry software to model and explore these transformations. For instance, you may want to draw an object, such as a polygon, and then use the available tools to transform the object. Pay attention what information the software requires in order to complete the required task.
4. In connection with the exploration you conducted, reflect on your definitions and modify them if necessary. Explain why you made changes.
5. Look up the definitions of the basic isometric transformations in a high school textbook of your choosing and list them here. How are these definitions similar/different from yours?

Problem 2 Two cities, Yankeetown and Stanley, are separated by a river. They'd like to build a road and bridge between the two for the most efficient travel and they've asked you for help. Where would you build the road and bridge? Include in your solution all the assumptions, simplifications, and modifications you decide to use in your problem.

Mathematics Of Doing, Understanding, Learning, and Educating for Secondary Schools

MODULE(S^2):
Geometry for Secondary Mathematics Teaching

Module 3: Similarity

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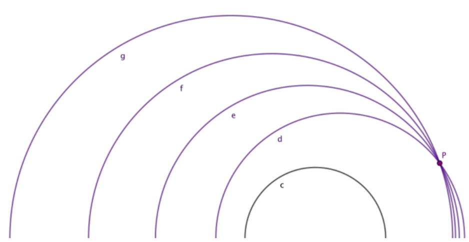

For more information about the MODULE(S^2) Project and other MODULE(S^2) materials please visit
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Contents

Overview of MODULE(S^2) Project iii

Overview of Geometry MODULE(S^2) vii

Overview of Module 3: Similarity viii

1 Changing Size 1
- Overview . . . 1
- Lesson Description . . . 3
  - Activity: Viewing Tubes . . . 8
  - Activity: Make it Bigger . . . 10
  - Homework: Dilations and ratios . . . 12

2 Understanding Dilations 14
- Overview . . . 14
- Lesson Description . . . 15
  - Homework: Find the Center . . . 19
  - Homework: Simulation of Practice Video Assignment: Dilations . . . 23

3 Similarity 25
- Overview . . . 25
- Lesson Description . . . 26
  - Activity: Are We Similar? . . . 31
  - Activity: Triangle Similarity Theorems . . . 33
  - Activity: Scaffolded Triangle Similarity Theorems . . . 34
  - Homework: Transforming Graphs . . . 36

4 Area 39
- Overview . . . 39
- Lesson Description . . . 40
  - Activity: Finding Areas . . . 44
  - Activity: Developing Area Formulas . . . 45
  - Homework: Final Topic Presentation Project . . . 49

5 Distance 50
- Overview . . . 50
- Lesson Description . . . 51
  - Activity: Find all the squares . . . 55
  - Activity: Simplifying radicals . . . 58
  - Homework: Pythagorean Theorem . . . 62

Homework: Simulation of Practice Written Assignment 65

6 Trigonometry **67**

Overview 67

Lesson Description 68

Activity: Trigonometry 71

7 Applications **75**

Overview 75

Lesson Description 76

Activity: The Farmer Problem 78

Activity: The Rotating Squares Problem 79

Activity: Area of Spherical Triangles 80

Overview of Geometry MODULE(S^2)

In order to teach geometry one needs to understand its place in the wider mathematical context, its development and possible approaches to its study. There is much that can be done with and learned about geometry, but our path for this course will be focused on the needs of our students as future teachers.
We are used to the Euclidean approach to developing geometry involving an axiomatic approach and sometimes that of straightedge and compass constructions. Although we understand the need for our students to understand the axiomatic structure of geometry and to develop their proving skills, we also see competing needs for the time in a geometry class for preservice teachers. For many of our teachers, the geometry course they will be teaching necessitates understanding of transformations that are not always included in a traditional Euclidean geometry class.

For this reason we chose to split the course into three modules hoping to address content relevant to the practice of teaching mathematics.

- Module 1: Axiomatic Development
- Module 2: Transformational Geometry
- Module 3: Similarity

The work of a teacher consists of more than just possessing the knowledge of mathematics. Teachers are supposed to help others attain knowledge they are required to attain. For example, the teacher is tasked with creating, selecting, and modifying tasks; a skill not often addressed in the mathematics classroom. Another relevant skill is ability to understand, find flaws, and help students find errors in their arguments. In this course we are providing opportunities for students to develop and practice these skills in addition to learning geometry.

Instructional Notes and Expectations

In an effort to create a learning environment that is aligned with current recommendations for teaching and learning mathematics in K-12 settings, the following practices are suggested for instruction in this course:

- Use of Technology: We encourage the investigative use of dynamic geometry programs for the exploration of ideas generated in class and through assignments. In addition, there are places where students will be directed to particular internet sites to read about the concepts of study. However, we discourage students from general internet searches when completing homework and instead encourage them to problem solve and communicate with peers in order to work as mathematicians work when pursuing new ideas. Reminder of this policy is often necessary throughout the course.
- Note-taking Assignment: As there is no textbook for this course, we recommend assigning note-taking as part of the regular work of the course. Each day, one student should be assigned as the official note-taker, who then submits notes completed in a template you provide. These notes can be used as part of a homework or participation grade for the course and posted in your learning management system for all students to have access. If you use a template for this activity, it is easily combined into a book authored by your class at the end of the semester.
- Handouts: In-class activities for students and homework assignments are listed as handouts. However, most can be easily incorporated into a digital display for the class or shared electronically through your learning management system.
- Homework: In many cases, homework assignments are structured so that they generate discussion for the next or an upcoming lesson. Assigning homework so that it can be submitted through a learning management system prior to the class in which it will be discussed provides the instructor an opportunity to peruse the work and adequately anticipate questions for the following class discussions. In some cases, considering how students respond to questions in homework prior to a class session can aid in assigning students to groups that will then move forward in their thinking based on shared ideas.
- Video Assignments: Two of the culminating assignments require access to online videos. In order to engage with video animations of mathematics classrooms you will need to invite students to open accounts at specified websites.

Overview of Module 3: Similarity

The idea of scaling is introduced to students in elementary grades, and students continue working with those ideas through middle school grades, often from the perspective of ratios and proportions. Throughout their high school career students learn about trigonometry, and frequently they come to associate it with pre-calculus and calculus through graphing trigonometric functions and later solving a myriad of integral problems. Somehow, in this transition through the curriculum the connection between similarity and trigonometry often gets lost. Our goal in this module is to develop connections between transformations and similarity as well as similarity and trigonometry.

- **Big Idea** There is another family of transformations of the plane that does not preserve lengths, but behaves extremely nicely: it preserves angles but rather than preserving lengths it preserves ratios of lengths. It allows us to make observations about individual figures in the family of figures related by the transformation.
- **Goals for studying the topic**:
    - Develop understanding of similarity transformations.
    - Know how to perform dilations, as well as locate the center of a given dilation and the scale factor.
    - Understand that all circles and parabolas are similar.
    - Be able to explain how dilations impact lengths, areas, and volumes.
    - Make connections between the need for measures of lengths and the Pythagorean Theorem.
    - Understand that trigonometric ratios describe similar right triangles.
- **Rationale**: This module connects geometry to algebra and calculus and will provide pre-service teachers with a deeper understanding of those connections. Additionally, it will provide an appreciation for the value that understanding geometry brings to one's ability to solve problems that appear both in mathematics and the real world.
- **Connections to Secondary Mathematics**: Similarity is a primary component of high school curriculum. Understanding of similarity as a transformation is critical to comparing ideas of congruence and similarity. In addition, a strong understanding of similarity situates the study of trigonometry, another central component of upper-level high school mathematics, within prior knowledge.
- **Overview of Content**
    - Lesson Changing Size: In this lesson students explore dilations through a viewing tubes activity prior to formally defining dilation. The viewing tube activity grounds the study of similarity in a real-world modeling problem.
    - Lesson Understanding Dilations: In this lesson, we situate dilations within an axiomatic system and lead them through several proofs related to dilations. This allows us to finally prove that dilations map segments to parallel segments, somethign that many students simply assume.
    - Lesson Similarity: Next we build on our knowledge of dilations and extend dilations to be a part of similarity transformations, thus establishing a distinction between dilations and similar figures. We also now have the necessary tools to prove the triangle similarity theorems and do so in this lesson.
    - Lesson Area: Thus far, we have operated without the need for measures. In this lesson, we explore area as part of an axiomatic system and make sense of what area actually is prior to discussing distance.
    - Lesson Distance: In this lesson, we extend the discussion of area to distance and develop the need for the Pythagorean Theorem to help us find lengths of sides of squares of given areas. Students will prove both the Pythagorean Theorem and its converse as well as make sense of simplifying radicals geometrically.
    - Lesson Trigonometry: Having developed the Pythagorean Theorem, we extend our discussion of right triangles by considering similar right triangles as the basis for defining trigonometric functions. Along with developing sine and cosine, we consider all six trigonometric functions as related to similar triangles connected to a unit circle.
    - Lesson Applications: We conclude this module with three application problems that provide opportunity to apply knowledge from various parts of the three modules. One problem will extend area discussions to Spherical Geometry.

- **Overview of mathematical and teaching practices** While we hope that all of our lessons will be in line with the Standards of Mathematical Practice as outlined in the CCSS, in this module we will particularly pay attention to:

SMP 1 Make sense of problems and persevere in solving them

SMP 2 Reason abstractly and quantitatively

SMP 3 Construct viable arguments and critique the reasoning of others

SMP 4 Model with mathematics

SMP 5 Use appropriate tools strategically

SMP 6 Attend to precision

SMP 7 Look for and make use of structure

- **Expectations for assignments**
  - If the instructor so chooses, the students can be tasked with writing a course textbook. After each lesson, a student is asked to write out the accomplishments of the day. The goal is to leave a reference for all students, give everyone an opportunity to practice writing, and provide feedback to each other.
  - Writing Assignments are to be graded and feedback provided. These assignments may be commented on before a final version is submitted for evaluation.
  - Homework Assignments are generally a preparation for class or planning information for the instructor. The instructor may choose whether to review and provide feedback.
  - Instructors may choose to have a class discussion board where questions can be posted and encouraged. The following prompt may serve well to encourage questioning and discussions throughout the course: "An important part of doing mathematics is to learn to ask one's own questions. You might not be able to answer them, but you can always learn more through seeking the answer to your questions. Which questions has this work made you ask? Record all the questions that you'd like to investigate on the class website."
  - Instructors may choose to include a reflective portfolio for the course. We find it helpful to have students reflect on their work across all three modules and produce a portfolio that demonstrates their growth in thinking throughout the semester. This can be completed either electronically or in hard copy form. We recommend providing some prompts of types of assignments to include (i.e., a homework assignment that changed their thinking, a use of technology that provided access to a different way of working, an assignment in which they felt particularly successful, etc.). Students should write a brief statement explaining the importance of each item included.

Lesson	Projected Length	Geometric Content	In-Class Activities	Homework	Connections and Notes
1: Changing Size	150 min	Mathematical modeling, Creating dilations, defining dilation	Activity: Viewing Tubes Activity: Make it Bigger	Homework: Dilations and ratios Formal Write-Up of Activity: Viewing Tubes (Optional)	• The results from Homework: Dilations and ratios will seed the discussion at the beginning of Lesson 2. • Requiring a formal write-up of the Activity: Viewing Tubes is optional but would be a nice addition to a culminating portfolio for the course. This activity will be revisited in Lesson 4. • There are moments in this lesson when referring back to the UCSMP Axioms and definitions developed in Module 2 will be useful.
2: Understanding Dilations	75 min	Properties of dilations: parallel projection, parallels, side-splitting theorem	Proofs of Theorems related to Properties of Dilations	Homework: Find the Center Homework: Simulation of Practice Video Assignment: Dilations	• Homework: Find the Center must be completed prior to the start of Lesson 3 as it seeds the opening discussion. • Video Simulation of Practice: Dilations can be assigned at the end of this lesson or you can hold off until the next. Prepare a way for students to submit video for this assignment.
3: Similarity	150 min	Defining similarity, similarity of functions, triangle similarity theorems	Activity: Are We Similar? Activity: Triangle Similarity Theorems Activity: Scaffolded Triangle Similarity Theorems (Optional)	Homework: Transforming Graphs	• We provide two versions of the triangle similarity proof activity, one with more scaffolding than the other. Choose the one that best fits the needs of your students. • If you did not assign the Video Simulation of Practice assignment in Lesson 2, do so during this lesson.
4: Area	150 min	Area of similar figures, area axioms, area formulas	Activity: Finding Areas Activity: Developing Area Formulas	Geogebra Area Formula Exploration (links provided in Lesson Description) Topic Presentation Project (from Module 2)	• Depending on the depth to which you wish to explore the derivation of area formulas, you can adjust this lesson to be a 75 minute lesson and complete it in one class session. • We provide a Geogebra Exploration as a potential homework assignment. You will need to create a Geogebra group for your class in order to assign this. • You may choose to have students working on a culminating Topic Presentation project at this time.
5: Distance	150 min	Distance, Pythagorean Theorem, Converse of the Pythagorean Theorem, Simplifying Radicals	Activity: Find all the squares Activity: Simplifying radicals	Homework: Pythagorean Theorem Homework: Simulation of Practice Written Assignment	• Students may be working on Topic Presentation Projects at this time. • There are options to shorten the length of time needed in class for this lesson by assigning parts of the in-class activities as homework. • An alternative to the Pythagorean Theorem Homework using Geogebra and Flipgrid submissions is described in the lesson.
6: Trigonometry	75 min	Trigonometric ratios developed from similar triangles, relationships between six trigonometric functions and the unit circle through similar triangles	Activity: Trigonometry	Homework: Simulation of Practice Written Assignment (If not already submitted)	• At this point students may be working on culminating projects and assignments for the course. • Plan for how final presentations may be worked into the timing in these last class sessions.
7: Applications	Adjustable Length	Problem solving applications	Activity: The Farmer Problem Activity: The Rotating Squares Problem Activity: Area of Spherical Triangles		• The three problems in this lesson bring together multiple ideas from the entire course. Choose how you want to engage your class with these problems.

1 Changing Size

Summary

The goal of this lesson is to introduce the ideas of similarity, and introduce the transformation that similarity is built upon. We continue to work on the modeling strands of the Core Standards through an activity in which the field of vision for viewing tubes is explored.

Goals

- Students will engage in modeling and problem solving processes with viewing tubes.
- Students will construct dilations using informal methods.
- Students will develop a definition for dilations.
- Students will explore properties of dilations.

Materials

- Paper/cardboard tubes of various lengths and diameters - different sized papers to fold them into the tubes should suffice
- Tape measure or meter/yard sticks
- Activity: Viewing Tubes
- Rubber bands (a variety of sizes but fairly new: tightness will aid in precision)
- Activity: Make it Bigger
- Dynamic Geometry Software (Optional)
- Homework: Dilations and ratios

Preparation for the next lesson

- Student: Homework: Dilations and ratios

ACTIVITY: VIEWING TUBES

In your group you should use the tubes provided to gather data about the size of the field of vision for tubes. After you have gathered the data, you should find a meaningful way to present the data in the form of a table and graph. Be sure to collect enough data to show relationships among the variables, diameter of tube, length of tube and distance from wall. Once you have done this, you should investigate the data for a pattern as well as find a geometrical justification for the pattern. Once you have done this, you should be able to answer the following questions:

1. What did you do to mitigate error in your data collection? What could you have done?
2. How does your model take units into account? Is it necessary to convert the units of the tube into the same units as the distance?
3. If you had a tube with diameter 2 inches and length 14 inches, how far would you need to stand from the wall to view a 3 x 4 foot painting on the wall? Is the problem making an assumption on where the painting is hanging?
4. For a given tube of diameter d and length l, at what distance from the wall would your formula for the size of field of vision break down because the ground came into view if a person's eye level was h? Create a piece-wise function that accounts for this phenomenon.
5. How does the problem change if the tube is not held level, but inclined at an angle θ?

Formal Write-Up of Experimental Results

Explain what you did, what you found, the conjectures that you made, and whatever evidence you have that they are correct. You should try to write your explanation so that it would convince one of your peers that your conclusions are correct.
In order to be considered complete, a formal write up must contain:

- a reasonable description of the method of collecting data;
- a chart of the data that includes a sufficient number of data points;
- a graph of the data;
- a correct formula for computing the viewing diameter;
- a complete proof of why the formula must always work that proceeds from reasonable assumptions that are explicit (e.g., if triangles are claimed to be similar, this must be justified by referring explicitly to the angle-angle triangle similarity criterion); and
- an analysis of how closely the measured values and predicted values agree (for example by computing percentage error for each measurement).

ACTIVITY: MAKE IT BIGGER

Draw your favorite doodle in the square provided below:

Your goal is to double the size of your favorite doodle.

To save you some trouble, I already googled the solution to this problem for you, and here are the instructions I found:

Take two rubber bands of equal length and tie them together so that there are equally sized bands on either side of the knot. Pick an anchor point somewhere on the paper and pin the end of rubber band to the anchor point with your finger. On the opposite side of the other band, place a pen. Trace a new object while keeping the knot consistently on top of the figure you are trying to enlarge.

Your task is to:

1. Follow directions as accurately as you can.
2. Explain what in this procedure caused the shape to be twice the ”size” of the original one.
3. How can you perform the same process if instead of rubber bands you used a ruler?
4. List as many different things you notice that
   (a) stayed the same
   (b) changed

   in this process.

If you’d like a nice clear page, re-draw your doodle inside the square on the next page.

HOMEWORK: DILATIONS AND RATIOS

Use Geogebra for the following activities.

1. Construct two lines ℓ and ℓ', and their transversal p. Place three points, A, B, and C on ℓ so that they are equally far apart from each other: $\overline{AB} \cong \overline{BC}$.

   Construct lines through A, B, and C parallel to p and call the intersections of those lines with ℓ' A', B', and C', respectively. What can you say about the segments $\overline{A'B'}$ and $\overline{B'C'}$?

   Include a copy of your diagram here, and provide a convincing argument for your claim.

   Repeat your diagram, except now let that point C be anywhere on the line ℓ. What can you say now about about the segments $\overline{A'B'}$ and $\overline{B'C'}$?

Mathematics Of Doing, Understanding, Learning, and Educating for Secondary Schools

MODULE(S^2): Mathematical Modeling

Module 1: The Process and Purpose of Mathematical Modeling

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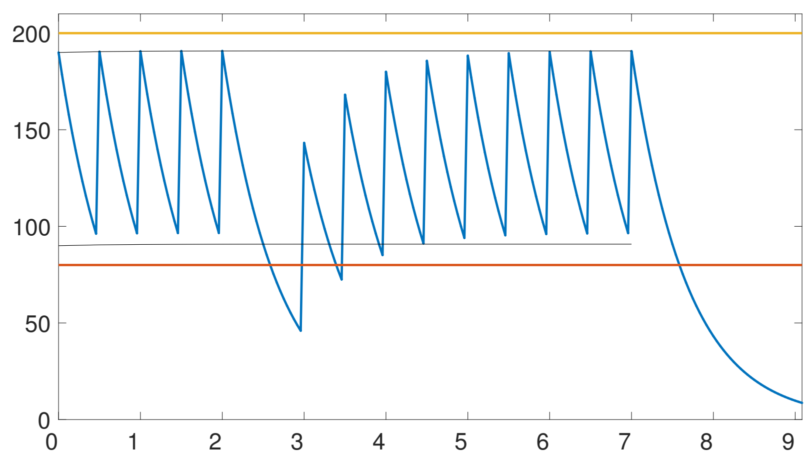
INSTRUCTOR VERSION



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Contents

Overview of MODULE(S^2) Project iii

Overview of Mathematical Modeling MODULE(S^2) vii

Overview of Module 1: The Process and Purpose of Mathematical Modeling viii

1 Developing Ways of Thinking for Mathematical Modeling 1
1.1 Overview 1
1.2 Description 2
1.2.1 Crowd-size Estimation 6
1.2.2 Crowd-size Estimation Historical Version 6
1.3 In-Class Resources 9
1.3.1 Assignment: Video Introduction 9
1.3.2 Handout 1: Mathematical Modeling Written Survey 10
1.3.3 Handout 2: Crowd Size Estimation 11
1.3.4 Assignment: Current Event with Mathematical Analysis 14
1.3.5 Handout 3: Standards and their Connections to this Lesson 15

2 Fighting Floods with Sandbags 16
2.1 Overview 16
2.2 Description 17
2.3 Sample Approach & Possible Model 19
2.4 In-Class Resources 25
2.4.1 Handout 1: US Army Corps of Engineers Sandbag Article 25
2.4.2 Handout 2: Missouri Department of Natural Resources Sandbag Article 26
2.4.3 Handout 3: Fighting Floods with Sandbags 27
2.4.4 Handout 4: Possible Model 28
2.4.5 Handout 5: Standards and their Connections to this Lesson 32
2.4.6 Assignment: Sandbags Problem Reflection 33

3 Elements of the Mathematical Modeling Process 34
3.1 Overview 34
3.2 Description 35
3.2.1 General Approach for Mathematical Modeling 40
3.3 In-Class Resources 42
3.3.1 Handout 1: Modeling Cycle - Bare 42
3.3.2 Handout 2: Modeling Cycle - Described 43
3.3.3 Handout 3: CCSSM Modeling Standards - Portioned for Group Work 44
3.3.4 Handout 4: Standards and their Connections to this Lesson 49
3.3.5 Assignment: Readings about Mathematical Modeling. 50

4 Predicting the Evolution of STDs in the USA 51
4.1 Overview 51
4.2 Description 53
4.3 Commentary, Sample Approach & Possible Model 57
4.4 In-Class Resources 67
4.4.1 Handout 1: Live Science Article and Data 67
4.4.2 Handout 2: Predicting the Evolution of STDs in the USA 68
4.4.3 Handout 3: US Census data 69
4.4.4 Handout 4: Guidelines for Preparing a Mathematical Modeling Report 70
4.4.5 Handout 5: Standards and their Connections to this Lesson 71

4.4.6 STD Revised Model . . . 73
4.4.7 STD Modeling Report . . . 74
4.4.8 Rubric . . . 75

5 Water Conservation: Shower v. Bath **76**
5.1 Overview . . . 76
5.2 Description . . . 78
5.3 Commentary, Sample Approach & Possible Model . . . 79
5.4 In-Class Resources . . . 83
5.4.1 Handout 1: USEPA Water Conservation Plan Guidelines . . . 83
5.4.2 Handout 2: Indoor Water Efficiency Fact Sheet - Portland Water Bureau . . . 84
5.4.3 Handout 3: Water Conservation Problem and Guidelines for Preparing a Mathematical Modeling Report . . . 85
5.4.4 Handout 4: Standards and their Connections to this Lesson . . . 86
5.4.5 Water Conservation Report . . . 88
5.4.6 Rubric . . . 89
5.5 MODULE 1 Simulation of Practice (SoP) - Written . . . 90

6 Analyzing Modeling Tasks: Rolling Cups **92**
6.1 Overview . . . 92
6.2 Description . . . 93
6.3 Commentary, Sample Approach & Possible Model . . . 96
6.4 In-Class Resources . . . 98
6.4.1 Handout 1: Student Work from MARS Resource . . . 98
6.4.2 Handout 2: Standards and their Connections to this Lesson . . . 99
6.4.3 Rolling Cups Reflection . . . 101

7 Critical Reading of Mathematical Models: Muffin Sale Task **102**
7.1 Overview . . . 102
7.2 Description . . . 104
7.3 Commentary, Sample Approach & Possible Model . . . 105
7.4 In-Class Resources . . . 109
7.4.1 Handout 1: Muffins Article . . . 109
7.4.2 Handout 2: Student Work . . . 110
7.4.3 Handout 3: Baseball Article . . . 112
7.4.4 Handout 4: Standards and their Connections to this Lesson . . . 113
7.5 MODULE 1 Simulation of Practice (SoP) - Video . . . 115
7.5.1 Critical Reading - Baseball Article . . . 117
7.5.2 Midterm Questions . . . 118

A Module 1 Assessment and Synthesis **119**
A.1 Reflection on the Mathematical Modeling Process . . . 119

B Video Recording Instructions **120**

Overview of Modeling MODULE(S^2)

This course is designed as a semester-long course on mathematical modeling for future secondary school mathematics teachers. While several assumptions have molded the structure and design of these materials, we are aware that mathematics teacher preparation programs vary widely, and thus lessons are intended to be adapted to fit the needs of many instructors at various institutions. We have assumed a 15-week semester, and created these materials in 3 Modules such that each one takes approximately 5 weeks. We have further assumed that the course meets twice per week in 75-minute class sessions, but adaptation into other formats is expected. The titles and aims of each module are summarized in the table below. Each module contains seven lessons, and is further described in an introductory section prior to the table of contents.

Our general philosophy is that future mathematics teachers will learn the appropriate knowledge, skills, and dispositions for teaching mathematical modeling by engaging in mathematical modeling activities themselves, and reflecting on the process and skills required as they develop their own modeling competencies.

Module 1 The Process and Purpose of Mathematical Modeling
• Gain appreciation for mathematical modeling as an approach to gaining understanding of real world issues, current events, problems, and questions of all kinds. • Create models with attention to units, dependent and independent variables, and informally analyze parameter sensitivity. • Analyze the cyclical process of mathematical modeling and tasks from the K-12 curriculum. • Develop strategies for selecting tasks/topics with attention to student motivation, opportunities to address mathematics in the K-12 curriculum, and potential for addressing important scientific and social issues.
Module 2 Advancing Competency in Mathematical Modeling
• Recognize that the modeling process requires careful analysis of model assumptions and revision. • Derive, solve, and interpret first order differential equations. • Collect and use real data for model parameterization and validation, and develop a deep understanding of parameter fitting algorithms. • Analyze classic models including Newton's Law of Cooling, the Torricelli Model for fluid flow, and the pendulum equation.
Module 3 Diverse Perspectives in Mathematical Modeling
• Use mathematical modeling to address social justice and environmental issues. • Appreciate how models have evolved over time with contributions from diverse cultures and individuals. • Study compartment models including SIR models of disease transmission. • Conduct an independent investigation through a self-chosen course project.

Overview of Module 1: The Process and Purpose of Mathematical Modeling

The central goal of Module 1 is to lead learners to understand that mathematical modeling (MM) problems are developed in many ways – from observations of the world around us, claims made by authorities, scientific discoveries, desire for predictions, etc. Every mathematical model should have a purpose. In contrast with prepackaged math "application" problems from typical school textbooks, observations from the real world (referred to as situations) must be transformed into a math problem (problem posing), solved, and then reconnected with the original situation to determine if the purpose has been achieved. This module will lead learners to address the roles of variables and parameters in the mathematical models they develop. Through metacognitive reflection on their own work, they will gain experience with the elements of MM. MM practices include becoming comfortable making assumptions that are known to be false but can help gain initial insight into the situation knowing that the false assumption will need to be revised and the modeling process iterated.

This module aims to expose participants to the process of mathematical modeling as a way to describe, explain, understand, or predict situations arising in everyday life; connect everyday experiences with classroom mathematics; explore several frameworks used to illustrate the mathematical modeling process; develop an understanding of modeling goals in the secondary mathematics curriculum; and help prepare mathematics teachers to incorporate mathematical modeling activities into their curriculum.

In this module, course participants will:

- Gain appreciation for mathematical modeling as an approach to gaining understanding of real world issues, current events, problems, and questions of all kinds.
- Create models with attention to units, dependent and independent variables, and informally analyze parameter sensitivity.
- Analyze the cyclical process of mathematical modeling and tasks from the K-12 curriculum.
- Develop strategies for selecting tasks/topics with attention to student motivation, opportunities to address mathematics in the K-12 curriculum, and potential for addressing important scientific and social issues.

Overview of course content, approximate pacing, and assigned homework:

- **Lesson 1 Developing Ways of Thinking for Mathematical Modeling**
  Length: 1.5 Class Meetings, ˜ (75+40) minutes
  - Video Introduction
- **Lesson 2 Fighting Floods with Sandbags**
  Length: 1.5 Class Meetings, ˜ (35+75) minutes
  - Sandbags Problem Reflection
  - Current Event with Mathematical Reflection
- **Lesson 3 Elements of the Mathematical Modeling Process**
  Length: 1.5 Class Meetings, ˜ (75+35) minutes
  - Readings About Mathematical Modeling

- **Lesson 4 Predicting the Evolution of STDs in the USA**
  Length: 2.5 Class Meetings, ˜ (40+75+75) minutes
  - STD Revised Model or STD Model Draft
  - STD Modeling Report
- **Lesson 5 Modeling for Water Conservation**
  Length: 1 Class Meeting, ˜ (75) minutes
  - Water Conservation Report
  - *Simulation of Practice - Written*
- **Lesson 6 Analyzing Modeling Tasks: Rolling Cups**
  Length: 1.5 Class Meetings, ˜ (75+40) minutes
  - Rolling Cups Reflection
- **Lesson 7 Critical Reading of Mathematical Models: Muffin Sale Task**
  Length: 1.5 Class Meetings, ˜ (35+75) minutes
  - *Simulation of Practice - Video*
  - Critical Reading - Baseball Article

These lessons are each described briefly in the table below, along with the emphasized mathematical knowledge for teaching (MKT) or the main mathematical ideas and teaching practices that arise in the lesson.

Module 1 Lesson	**Description**	**Essential Mathematics**
1 Developing Ways of Thinking for Mathematical Modeling	Introduction to several mathematical modeling examples from the newspaper and everyday life. Real life situations can be more deeply explored through the application of mathematics.	The essential mathematical knowledge for teaching (MKT) here is to raise awareness of the many situations all around that can be viewed mathematically, quantitatively, systematically, logically. A specific sub-focus is to pay attention to the variables and units that arise in equations/relations representing aspects of these situations. What are the quantities, and what are their units?
2 Fighting Floods with Sandbags	This lesson stems from an authentic situation, and leads to developing functions from number sequences to explain a discrepancy in published documents regarding the number of sandbags required to build a levee. Spoiler alert: a difference in parameter values can explain the results.	Derive the formula for the triangular numbers, and use it to create other related formulas. Distinguish and discuss parameters versus variables in a model. How do model results depend on parameters?
3 Elements of the Mathematical Modeling Process	Learners reflect on the process they went through to solve the sandbags problem, and describe their work in terms of the mathematical modeling cycle in CCSSM. They are introduced to modeling standards across the various conceptual categories of high school mathematics, and the expectation that teachers integrate modeling experiences into the subject-matter they teach.	The essential MKT in this lesson is to learn that modeling in school mathematics is pervasive across content strands and grade levels. Pay attention to the expectations for students to develop modeling proficiency in the various branches of mathematics content.

Module 1 Lesson	Description	Essential Mathematics
4 Predicting the Evolution of STDs in the USA	Learners are prompted to develop a model and predict the future regarding infections of sexually transmitted diseases based on an authentic published report. In the second day of the lesson they extend and revise their models and discuss essential components of reports about mathematical models.	Learners develop both discrete and continuous functions from number sequences. Continue analysis of parameters by comparing model predictions based on different strategies for reducing disease.
5 Modeling for Water Conservation	Learners explore the question of whether a shower or a bath is more efficient for conservation of water resources.	Learners utilize variable ranges to make reasonable predictions, working with linear functions and inequalities. This lesson requires students to consider flow rates and volumes and conduct a comparative analysis.
6 Analyzing Modeling Tasks: Rolling Cups	Learners review and apply their knowledge about the mathematical modeling process as they analyze the Rolling Cups task from the MARS (Mathematics Assessment Resource Service) project. This allows them to reflect on curricular materials in light of the modeling process, and apply geometry to develop a function for making a prediction. The same geometry could be extended or applied to determine turning radii on skateboards (and other vehicles) or eggs (which may have biological relevance).	Learners analyze the mathematical work of secondary school students, identify errors, and suggest ways to improve work. Similar triangles and proportional reasoning are sufficient for determining the maximum roll radius of a given cup (frustum of a cone). Learners often start with trigonometry and other mathematics more complicated than necessary, so simplifying the problem is an advantage.
7 Critical Reading of Mathematical Models: Muffin Sale Task	Here learners are prompted to critically review materials prepared for teachers that deal with mathematical modeling. The muffins task is an engaging problem that prompts school students to analyze data that is quadratic in nature and find a maximum.	Learners will critically read the mathematical presentation for understanding - noting subtleties that arise with discrete versus continuous functions. The discrete vertex differs from the continuous vertex of the quadratic.

1 Developing Ways of Thinking for Mathematical Modeling

Length: 1.5 Class Meetings, ~ 115 minutes

Overview

Summary

This lesson sets the stage for the class. The problems may be different from what students are used to encountering. Students will raise mathematical questions with connections to several current events, serving as an introduction to an understanding of mathematical modeling.

Goals

- Generate motivation for seeing mathematics in everyday life.
- Orient participants to an applied mathematician's way of thinking.

CCSSM Standard	**Connection to Lesson**
MP4: Model with mathematics	Learners participate in introductory modeling experiences. This will help prepare them for the rest of the course. Learners use mathematics to approximate how many people attend an event based on photographs or other observations.
A-CED.3 Represent constraints by systems of equations, and interpret solutions as viable or non-viable options in a modeling context.*	Learners use two linear equations to represent the number of sales and rentals of "The Interview" based on data from a *New York Times* article. Additionally, learners interpret these results and briefly discuss why they are useful.
A-REI.6 Solve systems of linear equations exactly and approximately	Using the system of two linear equations to represent how many copies of the "The Interview" are distributed and how much money is made, learners solve for the number of sales and rentals.
7-RP.1 Compute unit rates associated with ratios of fractions including quantities measured in like or different units.	Learners use units, rates, ratios, scaling, etc. to address the crowd-size estimation problem.
N-Q.1 Use units as a way to understand problems and to guide the solution of multi-step problems.*	To find a solution to the problems presented, learners determine the units and make conversions.
Concepts Beyond CCSSM	**Connection to Lesson**
Solving systems of equations with linear algebra	Comes up in discussion of "The Interview" article.

Materials

- Modeling Course Module 1 Slides
- Assignment: Video Introduction
- Handout 1: Mathematical Modeling Written Survey
- Handout 2: Crowd-Size Estimation
- Assignment: Current Event with Mathematical Reflection
- Handout 3: Standards and their Connections to this Lesson

> **Note.** Various articles included as handouts in this lesson and other lessons do not necessarily need to be printed and distributed. Most can be viewed online, or perhaps through a course reader that the participants can access. The course slides for this lesson provide images, prompts, and questions to help spark problem solving and guide class discussions. Feel free to customize.

Description

Lesson Overview

With the need to cover the necessary course logistics, this introductory lesson spans the first week of the course. The learners take a brief survey about mathematical modeling and then begin to engage in MM tasks in pairs and small groups during class meetings. As the two homework assignments are introduced, one video and one written, the instructor can go over homework expectations generally.

> **Note.** Detailed information is also included in the Google slides presentation. Feel free to reference the slides to guide your class as you teach.

Day 1:

1. Begin by introducing yourself and your syllabus however you like.
2. Discuss/establish class norms.

> **Pedagogical Note.** You might begin with the following list (included the slides) and discuss the rationale for these norms.
>
> - Rely on your reasoning
> - Keep track of your ideas and your process
> - Listen carefully to others ideas
> - Read critically and be skeptical
> - Ask questions
>
> Then ask students for additional input on the norms. For example, a student in the course might ask “Can we have freedom to select topics that interest us and pursue our own ideas and approaches?” The answer might be “Yes,” and you may add something to the class norms reflecting this idea, such as “pursue extensions to assigned problems based on your own interests.”

3. Introduce the *Video Introduction* assignment. This assignment is to be completed via video recording outside of class. The purpose is for the instructor to get to know class participants, and to provide students with initial practice of making a video of their instructional practice. The information gathered can be used to select topics of interest in the course.

   Video Introduction Instructions: *Create a short video of yourself (2-5 minutes) as you might introduce yourself to a class. Please address the following with as much (or as little) detail as you'd like.*

   (a) Your name, what youd like to be called by your students

   (b) Your background (where youre from, languages you speak, something about your family or community, influential life experiences)

   (c) A brief statement of your interests in and outside of academics

   > **Tool Tip.** This assignment serves as a trial run of the technology for learners to record and submit videos of their teaching practice. If needed, instructions about how to use video recording technology are provided in Appendix B. Video files can be large, so plan to gather electronic submissions accordingly.

4. Ask learners to write their responses to the initial *Written Survey* on mathematical modeling in class to stimulate thoughts about modeling. Collect before proceeding with the lesson.

   Written Survey Instructions: *Answer the following questions with detail.*

   (a) One of the standards for mathematical practice is "Model with mathematics." Explain what this means to you.

   (b) Are modeling with mathematics and solving word problems related? Explain.

   (c) How can teachers understand and prepare to teach modeling at the middle school and high school levels?

   (d) What role do you suppose that "real-life" contexts play in modeling problems?

5. Raise mathematical questions with connections to current events. Included is an example about "The Interview." View the slides for specific details regarding this example, and you may choose to insert a more resent news headline concerning North Korea prior to the 2014 New York Times article featured.

   (a) Introduce the New York Times article regarding film sales of "The Interview" at the end of 2014. This film is a fictional comedy about American entertainers and North Korea's leader Kim Jong-Un. The film was condemned by North Korea, and generated some controversy when released. (Citation: Cieply, M. (2014, December 28). 'The Interview' brings in $15 million on web. *The New York Times*, retrieved from `www.nytimes.com`)

SECTIONS HOME SEARCH The New York Times

MEDIA

'The Interview' Brings In $15 Million on Web

By MICHAEL CIEPLY DEC. 28, 2014

Email Share Tweet Save More

LOS ANGELES — "The Interview" generated roughly $15 million in online sales and rentals during its first four days of availability, Sony Pictures said on Sunday.

Sony did not say how much of that total represented $6 digital rentals versus $15 sales. The studio said there were about two million transactions over all.

"The Interview," a farce that depicts the killing of the North Korean leader Kim Jong-un, was withdrawn from a planned theatrical release after major exhibitors declined to show it because of a terror threat. Small theater chains revived the movie in several hundred theaters, while Sony and its business partners simultaneously offered the film online.

(b) This problem serves as an example of how math comes up in real life. It is almost a textbook word problem, but the writer didn't notice.

LOS ANGELES — "The Interview" generated roughly $15 million in online sales and rentals during its first four days of availability, Sony Pictures said on Sunday.

Sony did not say how much of that total represented $6 digital rentals versus $15 sales. The studio said there were about two million transactions over all.

(c) As a class, guide students to set up and solve the system of linear equations.

- (definitions) $R =$ Rentals, $S =$ Sales
- (transactions) $R + S = 2,000,000$
- (dollars) $6 \cdot R + 15 \cdot S = 15,000,000$
- Solution: $R = 1.67$ million, $S = 0.33$ million

6. Introduce another introductory example: the **Crowd-Size Estimation** problem (Handout 2). President Trump's inauguration is referred to in the Module 1 Slides. Use this example if you like, but feel free to update or replace with relevant recent events. Choose from the generic image, or the historical version with reference to the March on Washington in August 1963, where Dr. Martin Luther King Jr. gave his influential "I have a Dream" speech. Introduce the task, pose the problem, and let students begin their work with a partner near by. Give students adequate time to collaborate and work toward finding a solution.

Pedagogical Note. While they are working, move through the classroom to gather ideas about what different students are working on. Their discussions will help you know what to focus on when you bring the class back together to dissect their work.

> **Note.** Included in most lessons is a "Sample Approach and Possible Model" section. These are ***not*** intended to be used as a lecture. Rather, they are meant to be background knowledge and context for the instructor. After students work and report on these problems, you could highlight certain parts from the Possible Model. Throughout this course, learners should be encouraged to try new things, create different models, and engage fully in the modeling process. As the instructor, you are responsible for helping them succeed in this, not showing them solutions.

7. Discuss solutions, implications, ideas, etc. from the Crowd-size Estimation Problem. Draw attention to the variables and units that arise in equations/relations representing aspects of the situation. What are the quantities, and what are their units?

8. Following the students' discussion, read and comment on the following article, How Do the Media and Police Estimate Crowd Sizes? Discuss how their approaches were similar and different from the estimation techniques presented in the article.

> **Pedagogical Note.** As the instructor you may choose how you want to run these discussions. If possible have some students share their ideas with a document camera.

9. Introduce *Current Event with Mathematical Reflection*. Instructions: *For this assignment, locate an article from a news outlet and begin thinking and working quantitatively with the article content. Your write-up should be 5 paragraphs. Answer the following questions:*

   (a) What is the article and source?

   (b) Why did you choose this article? What is the main issue? (1 paragraph)

   (c) What kind of mathematics is included by the author? (1 paragraph)

   (d) Critique the quantitative presentation. What questions do you still have? (1 paragraph)

   (e) Be creative and expand. What mathematics might further enhance the news story? Develop some ideas and present your thinking. (2 paragraphs)

> **Note.** The purpose of Lesson 1 is to lead mathematics teacher candidates to realize the quantitative nature of every day news. An intended side effect of the lesson is to raise awareness about what's going on in the world. We are not trying to just use real world contexts to bring up mathematics problems, rather the other way around. We are raising relevant questions to real world issues and then seeing the mathematics.

Day 2:

1. Begin class by reviewing *Current Event with Mathematical Reflection*. Have several students share what articles they found, the mathematics that goes along with it, and what (if any) research they did to learn more about the specific situation.

2. Introduce the task from Fighting Floods with Sandbags. Have there been recent floods in your community? Has anyone had experience helping out by creating levees with sandbags? There is no need to give too much away at this point.

3. Introduce the articles from Handout 1 and Handout 2 of Fighting Floods with Sandbags. Learners should read through them to prepare for the Sandbags problem. Instructions: *Familiarize yourself with the data from both information sheets. Look for sandbagging techniques, discrepancies in estimates, and any other useful data from the articles.*

Overview of Modeling MODULE(S^2)

This course is designed as a semester-long course on mathematical modeling for future secondary school mathematics teachers. While several assumptions have molded the structure and design of these materials, we are aware that mathematics teacher preparation programs vary widely, and thus lessons are intended to be adapted to fit the needs of many instructors at various institutions. We have assumed a 15-week semester, and created these materials in 3 Modules such that each one takes approximately 5 weeks. We have further assumed that the course meets twice per week in 75-minute class sessions, but adaptation into other formats is expected. The titles and aims of each module are summarized in the table below. Each module contains seven lessons, and is further described in an introductory section prior to the table of contents.

Our general philosophy is that future mathematics teachers will learn the appropriate knowledge, skills, and dispositions for teaching mathematical modeling by engaging in mathematical modeling activities themselves, and reflecting on the process and skills required as they develop their own modeling competencies.

Module 1 The Process and Purpose of Mathematical Modeling
• Gain appreciation for mathematical modeling as an approach to gaining understanding of real world issues, current events, problems, and questions of all kinds. • Create models with attention to units, dependent and independent variables, and informally analyze parameter sensitivity. • Analyze the cyclical process of mathematical modeling and tasks from the K-12 curriculum. • Develop strategies for selecting tasks/topics with attention to student motivation, opportunities to address mathematics in the K-12 curriculum, and potential for addressing important scientific and social issues.
Module 2 Advancing Competency in Mathematical Modeling
• Recognize that the modeling process requires careful analysis of model assumptions and revision. • Derive, solve, and interpret first order differential equations. • Collect and use real data for model parameterization and validation, and develop a deep understanding of parameter fitting algorithms. • Analyze classic models including Newton's Law of Cooling, the Torricelli Model for fluid flow, and the pendulum equation.
Module 3 Diverse Perspectives in Mathematical Modeling
• Use mathematical modeling to address social justice and environmental issues. • Appreciate how models have evolved over time with contributions from diverse cultures and individuals. • Study compartment models including SIR models of disease transmission. • Conduct an independent investigation through a self-chosen course project.

Overview of Module 2: Advancing Competency in Mathematical Modeling

The lessons in Module 2 familiarize participants with classic mathematical models and modeling approaches seen in various scientific disciplines and quantitative courses. We aim to deepen learner's understanding of these classic models while leading them to gain proficiency with visualizing functions and data and strengthen connections between college courses and the secondary curriculum. Your students will use differential equations and their solutions to serve as models of real phenomena, write algorithms to determine parameters, and use technology to make plots. Various technologies may be useful for them in this module – for some lessons we've provided code in MATLAB, however, feel free to use the software of your choice. We recommend Excel and Desmos because these are also available in most secondary schools. The major theme of this module is focused on model validation from the modeling cycle.

The module is organized to follow the mathematical structures in the classic examples. After a first introduction to parameter fitting with standard function families, we address first order ordinary differential equations in multiple lessons, including linear and nonlinear examples. The module concludes with a pendulum lesson requiring analysis of the corresponding nonlinear second order differential equation and its linearization.

In this module, course participants will

- Realize that the modeling process requires careful analysis of model assumptions and, more often than not, revision.
- Derive, solve, and interpret first order differential equations.
- Collect and use real data for model parameterization and validation, and develop a deep understanding of parameter fitting algorithms.
- Analyze classic models including Newtons Law of Cooling, the Torricelli Model for fluid flow, and the pendulum equation.

Overview of course content and homework assignments:

- Lesson 1 The Area of Tree Leaves
  - Tree Leaf Report
- Lesson 2 Cooling Coffee
  - Discovering Newton's Law of Cooling
  - Linear Algebra and Least Squares (Optional)
- Lesson 3 Memorization
  - Memorization Report
  - **Simulation of Practice - Written**
- Lesson 4 Pain Medication
  - Pain Medication Draft

- Pain Medication Report
- Lesson 5 Leaky Bucket
  - Leaky Bucket Report
- Lesson 6 The Lost Cell Phone
  - Lost Cell Phone Report
  - **Simulation of Practice - Video**
- Lesson 7 The Trapeze and the Pendulum
  - Pendulum Report

Mathematics Of Doing, Understanding, Learning, and Educating for Secondary Schools

MODULE(S^2): Mathematical Modeling

Module 2: Advancing Competency in Mathematical Modeling

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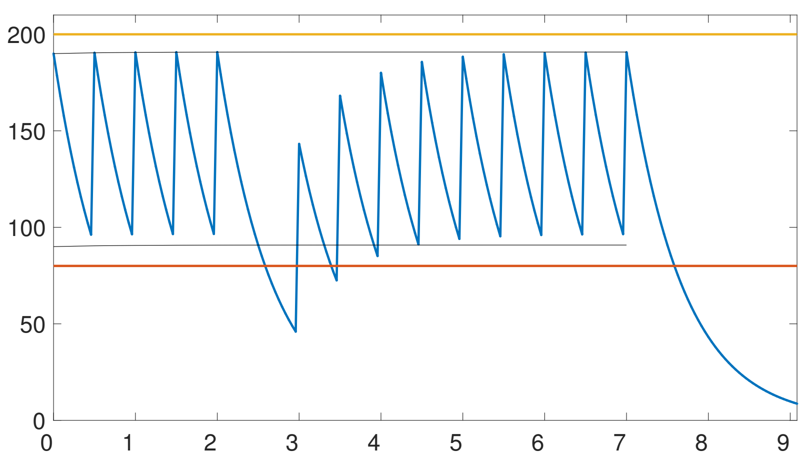
STUDENT VERSION



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Contents

1 The Area of Tree Leaves **1**
1.1 Overview . . . 1
1.1.1 Handout 1: Tree Leaf Data and Problem Description . . . 2
1.1.2 Handout 2: Tree Leaves from Various Species . . . 3
1.1.3 Tree Leaf Report . . . 4

2 Cooling Coffee **5**
2.1 Overview . . . 5
2.1.1 Handout 1: Cooling Coffee Guided Notes . . . 6
2.1.2 Discovering Newton's Law of Cooling . . . 8
2.1.3 Linear Algebra and Least Squares . . . 11

3 Memorization **12**
3.1 Overview . . . 12
3.1.1 Handout 1: Memorization Data Collection Worksheet . . . 13
3.1.2 Memorization Report . . . 15
3.2 MODULE 2: Simulation of Practice (SoP) - Written . . . 16

4 Modeling Pain Medication **18**
4.1 Overview . . . 18
4.1.1 Handout 1: Pain Medication Problem Description . . . 19
4.1.2 Pain Medication Draft . . . 20
4.1.3 Pain Medication Report . . . 21

5 Leaky Bucket **22**
5.1 Overview . . . 22
5.1.1 Handout 1: Leaky Bucket Task Sheet . . . 23
5.1.2 Leaky Bucket Report . . . 26

6 The Lost Cell Phone **27**
6.1 Overview . . . 27
6.1.1 Handout 1: The Lost Cell Phone Task . . . 28
6.1.2 Lost Cell Phone Report . . . 29
6.2 MODULE 2: Simulation of Practice (SoP) - Video . . . 30

7 The Trapeze and the Pendulum **32**
7.1 Overview . . . 32
7.1.1 Handout 1: The Trapeze and the Pendulum Problem Description . . . 33
7.1.2 Pendulum Report . . . 35

A Module 2 Assessment and Synthesis **36**
A.1 Reflection on the Mathematical Modeling Process . . . 36

B Guidelines for Preparing a Mathematical Modeling Report **37**

C Plan for Group Presentations **38**

1 The Area of Tree Leaves

Overview

Summary

The modeling prompt for this lesson comes from research pertaining to the Suradan Hardwood tree. The research paper provides 3 data points that relate tree age with leaf size. Learner's geometrically approximate the area of the leaf and create functions to relate the two variables of tree age and leaf area. They experience the modeling cycle, fit parameters to various functional forms, validate different models by evaluating the utility and accuracy of different models. The limited data set available in this situation stimulates discussion about model validation and evaluation.

Goals

- Learners experience the modeling cycle, fit parameters to various functional forms, validate different models and evaluate the utility and accuracy of different models.
- Learners have limited data, which stimulates discussion about model validation and evaluation.
- Reflection on what makes one model better than another model is a key component of the lesson.

Materials

- Handout 1: Tree Leaves Data and Problem Description
- Handout 2: Tree Leaves from Various Species
- Handout 3: Standards and their Connections to this Lesson
- Modeling Course Module 2 Slides
- Tree Leaf Report

HANDOUT 1: TREE LEAF DATA AND PROBLEM DESCRIPTION

The leaves of a tree are crucial to the life of the tree and our lives (photosynthesis). For this reason, it is important to understand how the size of tree leaves change during the life of the tree. An important dimension is the area of a typical leaf because the amount of sunlight a leaf can capture is proportional to its area. The goal of this project is to determine a process to estimate the area of a typical leaf of the Suradan Hardwood tree (which grows in Costa Rica) depending on the age of the tree. According to a study conducted in 2004, the typical size of the leaves of this tree gets smaller as the tree gets older. Figure 1 below shows the shape and size of typical leaves of the Suradan Hardwood from 1-year-old, 3-year-old, and 11-year-old trees (left to right). Notice that the leaves get smaller as the trees get older. The length of the leaves, illustrated in Figure 2 below, is measured from the bottom to the tip of the leaf along the centerline.

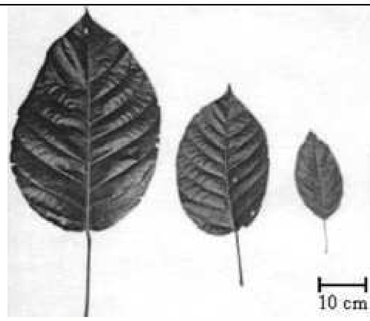


FIGURE 1. Pictures of leaves from the Suradan Hardwood tree. The pictures from left to right are from a 1-year-old tree, a 3-year-old tree, and an 11-year-old tree.

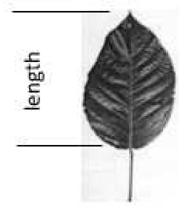


FIGURE 2. Measurements of the average size of leaves from the Suradan Hardwood tree.

The typical leaf lengths for 1-year-old, 3-year-old and 11-year old trees have been measured and collected in Table 1. Using the data provided, determine a process to estimate how the area of a typical leaf changes as the tree gets older. Use any function you know to predict the typical size of the leaves for trees that are 2 years old and 4-10 years old (for the years that we do not have data).

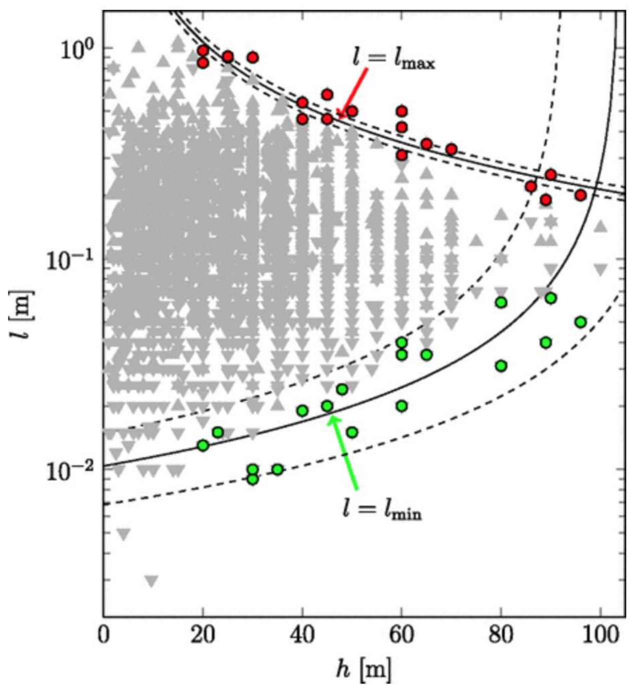
Tree age (in years)	Maximum length of typical leaf
1 year old	45 cm
3 years old	30 cm
11 years old	15 cm

TABLE 1. Measurements of the average size of leaves from the Suradan Hardwood tree.

The data were taken from Reich, A., Holbrook, N. M., & Ewel, J. J. (2005). Developmental and physiological correlates of leaf size in Hyeronima alchorneoides (Euphorbiaceae). *American Journal of Botany*, *91*(4), 582-589.

Handout 2: Tree Leaves from Various Species

Although we have dealt with the Suradan Hardwood tree, scientists have examined the size of tree leaves for many species. A research article published in 2013 showed the variation in leaf size (laminar leaf length l) with tree height (h) based on botanical data covering 1,925 tree species. Using mathematical models, the authors argued that physical limitations result in well-defined curves to describe the minimum and maximum leaf size depending on the tree heights.



Gather Data from a Local Tree

- Pick a Tree (preferably a tall one)
- Estimate the height of the tree using any mathematical method. Consider the following questions:
  - What method did you use to calculate the height?
  - How accurate do you think your method was?
  - What other methods could you have used?
- Measure several leaves and calculate a laminar leaf length average.
- How the leaf size of this tree compare with other trees in the figure above?

Physical Limations of Leaf Size and Height of Tree

Citation: Jensen, K. H & Zwieniecki, M. A. (2013). Physical limitations of leaf size and height of tree. *Physical Review Letters, 110*(1). `https://doi.org/10.1103/PhysRevLett.110.018104`.

TREE LEAF REPORT

Write a report. Your modeling report should include a description of the model(s) that you've selected to approximate the typical leaf area of a Suradan hardwood tree at various ages. Include the following sections when preparing to report out your mathematical model for this task. Your report should contain:

- Title and names of group members
- The statement of the problem
- Information researched
- Your assumptions with justification
- The derivation of your model and solution (show how you arrive at mathematical results). Use various ways to show results (include equations, graphs, tables, diagrams, etc.)
- Computations and interpretations (What does the solution mean?)
- An explanation of conclusions from your model and validation of those conclusions (Is the solution reasonable, and does it make sense? What conclusions can be drawn based on the models?)
- One or more improvements to your model (this means not only mention improvements but actually carry them out to get a revised model)
- Recommendations
- References

I will be especially looking for:

- Assumptions with adequate justification. No justification will receive no credit. Unsupported justification will receive little credit. Justification supported by evidence can receive full credit. If an unsupported assumption is made temporarily to develop an initial model with the intent of replacing the assumption with a realistic (justifiable) one, please indicate so.
- A model that follows from the assumptions and data. If an assumption was not used, then it was unnecessary and should not be listed.
- A solution (show how you arrive at mathematical results). If possible use various ways to show results (graphs, tables, etc)
- An explanation of conclusions from your model and validation of those conclusions
- The improvements to your initial model
- What do these conclusions mean in terms of the context of this situation?

2 Cooling Coffee

Overview

Summary

In this lesson, learners develop their understanding of Newton's Law of Cooling, which describes the transfer of heat from an object in an ambient temperature environment. Learners will also collect data, solve the differential equation, and explore methods of fitting the model parameters to their collected data.

Goals

- Understand Newton's Law of Cooling as a model of heat transfer.
- Collect data and grapple with the messiness of real-world measurements and errors.
- Investigate multiple ways to estimate parameters for exponential functions and linear functions.

Materials

- Coffee or other hot beverage
- Thermometers and a timer (a cell phone works)
- Handout 1: Cooling Coffee Guided Notes
- Handout 2: Standards and their Connections to this Lesson
- Discovering Newton's Law of Cooling
- Linear Algebra and Least Squares
- Modeling Course Module 2 Slides

Handout 1: Cooling Coffee Guided Notes

Newton's Model of Cooling states that the rate of heat loss of a body is directly proportional to the difference in the temperatures between the body and its surroundings.

- Mathematize Newton's Model of Cooling: (Write the statement in mathematical symbols)
- Initial Conditions:
- Solve the Differential Equation:
- Collected Data:

Room Temperature: _____ °C

Elapsed Time (min)	Temperature (degrees C)

- Parameterizing the Model:
  - Measurements known:
  - Parameters to estimate:
  - Choose 2 points and find the equation that goes through both points exactly.
  - Use many points and fit the model parameter(s). Explain your method. Plot the data and the model on the same axes and describe your prediction for the coffee temperature after 60 minutes.
- Prediction:
- Evaluation: *How well did the model do for predicting the temperature? What were possible sources of error?*

DISCOVERING NEWTON'S LAW OF COOLING

This worksheet was designed by Kevin Taylor and distributed to NCTM members through the myNCTM email newsletter on April 28, 2019.

1. Sketch the following basic transformations of $y = ab^x, b > 1$. Label the values of a, b, and c on your sketch.

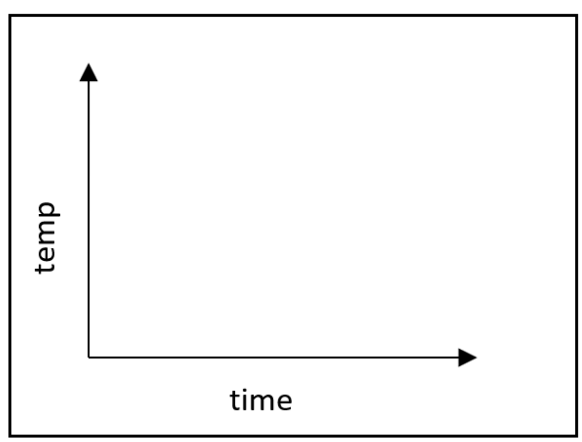
   $y = ab^x,\ b > 1$ $\qquad$ $y = ab^{-x},\ b > 1$

   $y = ab^x + c,\ 0 < b < 1$ $\qquad$ $y = ab^{-x} + c,\ b > 1$

2. Your group will have a beaker of hot water in a moment. Make a prediction and describe how the temperature of the water will change with time.

3. Sketch a graph of how the temperature will change over time. Let the x-axis represent a temperature of 0°C.

4. Which sketch(es) from the parent functions are similar to this story?



5. Grab a thermometer and determine the temperature of the room. Room temp = __________

6. Start a stopwatch and record the temperature of the water at $t = 0$. Don't stop the stopwatch until after you have finished #12. Initial temp = __________

7. Starting with $y = ab^x + c$, use transformations of the parent function to obtain a graph like the one you sketched for the water.

   (a) What is the vertical shift? What does this represent in context?

   (b) If the room temperature stays constant, how much can the temperature of the water actually drop (in degrees)? What value can this represent in $y = ab^x$?

   (c) With b still unknown, write an equation in terms of b and x in the form of $y = ab^x$ that matches the sketch above. In your equation, is $b > 1$ or $0 < b < 1$?

8. To find b, we need to know more information. Record another temperature.
   At t = __________ minutes, the temperature is __________.

9. Use this new information to determine the value of b in your generic equation.

10. Write an equation that models the temp of the water for any give time x (in minutes).

11. Use your model to predict the temperature of the water about 2 minutes from now. (Don't round early!)

12. The predicted temp at __________ min was __________. The actual temp of the water was __________.

1 Native American Reservation Land

Overview

Summary

This lesson provides students with a historical perspectives on the land designated to the Indian Reservations in the U.S. and how over time, the reservation land has shrunk. Learners will look at this social justice issue of the Sioux reservation shrinking. Learners can create functions to approximate the boarder and use integration to estimate the area of the reservation. Learners will, in doing so, develop a method for approximating the area of the Sioux Reservation area in 1851 and 1876.

Goals

In this lesson, Learners:

- Explore the disenfranchisement the Sioux tribes through the repeated reductions in the size of Sioux tribes' reservation land throughout history.
- Develop and improve upon a mathematical method for finding the area of complex planar regions.
- Produce a method for obtaining a reasonable estimate of the error introduced from the area-finding method developed.
- Make pedagogical determinations as part of a simulation of practice in regard to facilitating a productive discussion on a variety of approaches and results for approaching this area-fin

Materials

- Handout 1: New York Times article "*New Lands For Settlers: The Great Sioux Reservation In Southern Dakota To Be Thrown Open.*"
- Handout 2: Images of Historical Reservation Maps
- Handout 3: Mathematical historical remark on approximating areas of irregular regions.
- Handout 4: Standards and their Connections to this Lesson
- Area of Sioux Reservation Land Report
- **MODULE 3: Simulation of Practice - Written**

HANDOUT 2: THE SHRINKING AREA OF THE SIOUX INDIAN RESERVATION

Based on the two maps of the Great Sioux Reservation in 1851 and 1876, develop a procedure that can be used to approximate the area of the Great Sioux Reservation and use it to calculate the percentage in area reduction between 1851 and 1876.

- Describe your method for estimating the area based on the map images.

- Estimate the accuracy of your solution and describe changes you would make to improve the accuracy.

- Develop an improved procedure based on your initial solution.

The following maps were taken from National Geographic (2012).

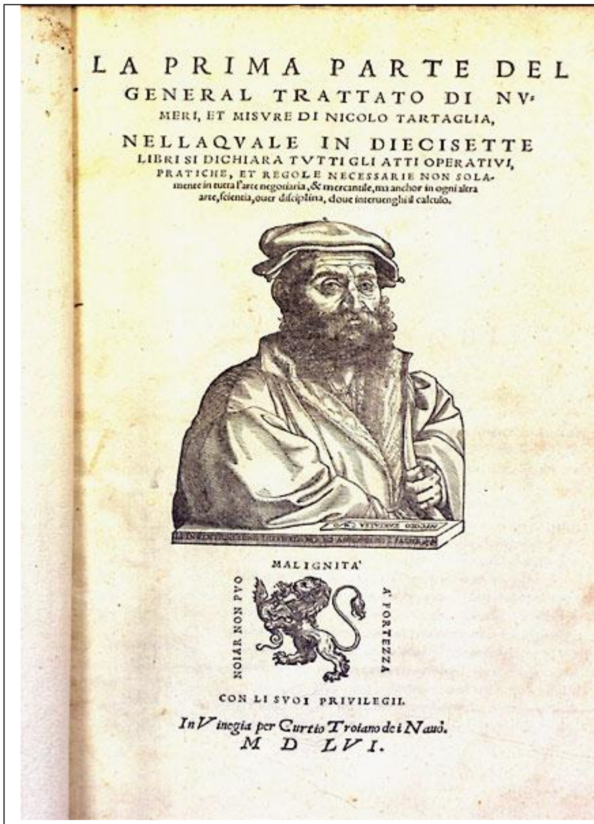
Citation:
Fuller, A. (2012, August). In the shadow of Wounded Knee. *National Geographic*, retrieved from `www.nationalgeographic.com`.

Map of the Great Sioux Reservation in 1851

National Geographic 2012

Map of the Great Sioux Reservation in 1876

National Geographic 2012



LA PRIMA PARTE DEL GENERAL TRATTATO DI NVMERI, ET MISVRE DI NICOLO TARTAGLIA, NELLAQVALE IN DIECISETTE LIBRI SI DICHIARA TVTTI GLI ATTI OPERATIVI, PRATICHE, ET REGOLE NECESSARIE NON SOLAmente in tutta l'arte negotiaria, & mercantile, ma anchor in ogni altra arte, scientia, ouer disciplina, doue interuenghi il calculo.

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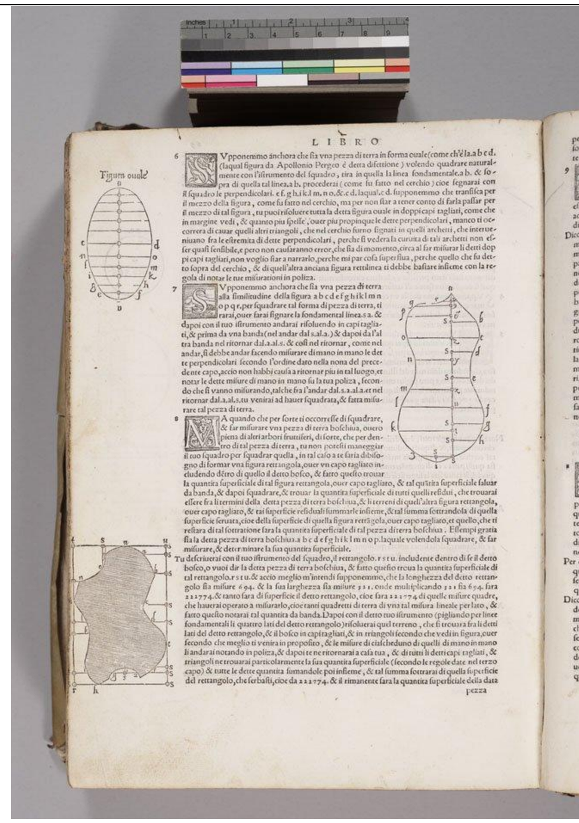


(Left) Title page of part I of the *General Trattato di Numeri (General Treatise on Number and Measure)* (1556) of Nicolo Tartaglia (1500-1557).	(Right) Folio 29 of the General Treatise where the author shows how to determine the area of an irregular curved shape. MAA Link
This is an extensive work on elementary mathematics that was popular in Italy for several decades after its publication.	

Citation:

Swetz, F. J. & Katz, V. J. (2011, January). Mathematical treasures - Nicolo Tartaglia's General Trattato di Numeri et Misure. *Mathematical Association of America*, retrieved from `www.maa.org`.

AREA OF SIOUX RESERVATION REPORT

Include the following sections when preparing to report out your mathematical model for this task. Your report should contain:

- Title and names of group members
- The statement of the problem
- Information researched
- Your assumptions with justification
- The derivation of your model and solution (show how you arrive at mathematical results). Use various ways to show results (include equations, graphs, tables, diagrams, etc.)
- Computations and interpretations (What does the solution mean?)
- An explanation of conclusions from your model and validation of those conclusions (Is the solution reasonable, and does it make sense? What conclusions can be drawn based on the models?)
- One or more improvements to your model (this means not only mention improvements but actually carry them out to get a revised model)
- Recommendations
- References

I will be especially looking for:

- Assumptions with adequate justification. No justification will receive no credit. Unsupported justification will receive little credit. Justification supported by evidence can receive full credit. If an unsupported assumption is made temporarily to develop an initial model with the intent of replacing the assumption with a realistic (justifiable) one, please indicate so.
- A model that follows from the assumptions and data. If an assumption was not used, then it was unnecessary and should not be listed.
- A solution (show how you arrive at mathematical results). If possible use various ways to show results (graphs, tables, etc)
- An explanation of conclusions from your model and validation of those conclusions
- The improvements to your initial model
- What do these conclusions mean in terms of the context of this situation?

Mathematics Of Doing, Understanding, Learning, and Educating for Secondary Schools

MODULE(S^2): Mathematical Modeling

Module 3: Diverse Perspectives in Mathematical Modeling

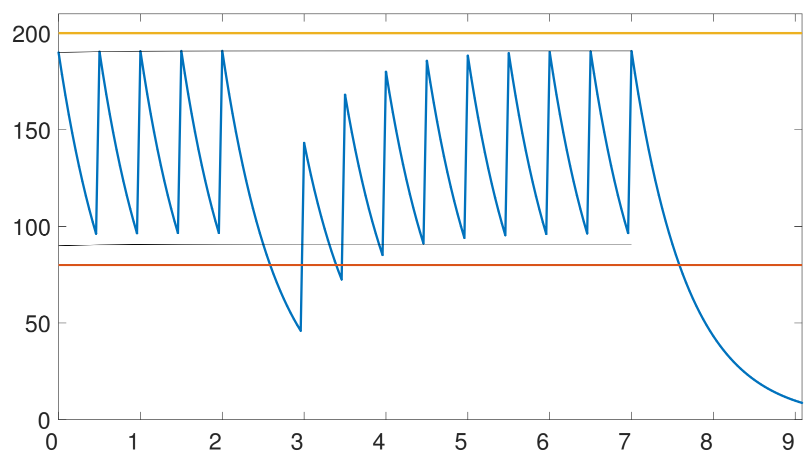
Cynthia Anhalt

Ricardo Cortez

Brynja Kohler

Published Version: July 2022

INSTRUCTOR VERSION



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Contents

Overview of MODULE(S^2) Project iii

Overview of Mathematical Modeling MODULE(S^2) vii

Overview of Module 3:Diverse Perspectives in Mathematical Modeling viii

1 Native American Reservation Land 1
- 1.1 Overview . . . 1
- 1.2 Description . . . 2
- 1.3 In-Class Resources . . . 8
  - 1.3.1 Handout 1: Historical NYT Article from 1883 . . . 8
  - 1.3.2 Handout 2: The Shrinking Area of the Sioux Indian Reservation . . . 9
  - 1.3.3 Handout 3: Historical Mathematical Method for Determining the Area of Irregular Shapes . . . 11
  - 1.3.4 Handout 4: Standards and their Connections to this Lesson . . . 12
  - 1.3.5 Area of Sioux Reservation Report . . . 13
- 1.4 MODULE 3: Simulation of Practice (SoP) - WRITTEN . . . 14

2 Thermoclines and Air Pollution 16
- 2.1 Overview . . . 16
- 2.2 Description . . . 19
- 2.3 Commentary, Sample Approach & Possible Model . . . 21
  - 2.3.1 Sample Approach & Possible Model . . . 21
- 2.4 In-Class Resources . . . 28
  - 2.4.1 Handout #1 . . . 28
  - 2.4.2 Handout 2: Standards and their Connections to this Lesson . . . 30
  - 2.4.3 Thermoclines and Air Pollution Report . . . 32
- 2.5 Resources for Teachers . . . 33
- 2.6 Lab Experiment Set Up . . . 34
  - 2.6.1 Supplementary Handout: Air Pollution, Temperature Inversions, and Human Health . . . 36

3 Energy Saving Lightbulbs 38
- 3.1 Overview . . . 38
- 3.2 Description . . . 39
- 3.3 Commentary, Sample Approach & Possible Model . . . 43
- 3.4 In-Class Resources . . . 47
  - 3.4.1 Handout 1 - Brenda's Lightbulb Problem . . . 47
  - 3.4.2 Handout 2: Standards and their Connections to this Lesson . . . 49

4 Flint Water Crisis 50
- 4.1 Overview . . . 50
- 4.2 Description . . . 52
- 4.3 Commentary, Sample Approach & Possible Model . . . 56
- 4.4 In-Class Resources . . . 61
  - 4.4.1 Handout 1: Water Donations in Flint, Michigan . . . 61
  - 4.4.2 Video - CNN Student News . . . 61
  - 4.4.3 Handout 2: The Flint Water Crisis Tasks A and B . . . 62
  - 4.4.4 TASK A: Flint Water Task: "Is this Enough Water for Flint, MI Schoolchildren?" . . . 63
  - 4.4.5 TASK B: Flint Plastic Waste Task: "How Large is the Environmental Problem?" . . . 64
  - 4.4.6 Handout 3: Standards and their Connections to this Lesson . . . 65
- 4.5 MODULE 3: Simulation of Practice (SoP) - Video . . . 66

5 SIR Disease Transmission Models **68**
5.1 Overview . . . 68
5.2 Description . . . 69
5.3 In-Class Resources . . . 70
5.3.1 Handout 1: Introductory Activity to Flow Diagrams . . . 70
5.3.2 Handout 2: Equations and Units . . . 73
5.3.3 Handout 3: Epidemic Modeling . . . 75

6 Student Projects **76**
6.1 Overview . . . 76
6.2 Description . . . 77
6.3 Rubrics . . . 79

7 The Modeling Process Revisited **82**
7.1 Overview . . . 82
7.2 Description . . . 83
7.3 In-Class Resources . . . 84
7.3.1 Handout 1 - Questionnaire . . . 84
7.3.2 Handout 2 - Effective Mathematics Teaching Practices . . . 87
7.3.3 Handout 3 - Mathematical Modelers' Habits of Mind . . . 87
7.3.4 Handout 4: Standards and their Connections to this Lesson . . . 88
7.3.5 Diagram of the Modeling Cycle . . . 89
7.4 Additional Articles and Resources . . . 90

A Module 3 Assessment and Synthesis **91**
A.1 Reflection on the Mathematical Modeling Process . . . 91

B Menu of Student Project Options **92**

Overview of Modeling MODULE(S^2)

This course is designed as a semester-long course on mathematical modeling for future secondary school mathematics teachers. While several assumptions have molded the structure and design of these materials, we are aware that mathematics teacher preparation programs vary widely, and thus lessons are intended to be adapted to fit the needs of many instructors at various institutions. We have assumed a 15-week semester, and created these materials in 3 Modules such that each one takes approximately 5 weeks. We have further assumed that the course meets twice per week in 75-minute class sessions, but adaptation into other formats is expected. The titles and aims of each module are summarized in the table below. Each module contains seven lessons, and is further described in an introductory section.

Our general philosophy is that future mathematics teachers will learn the appropriate knowledge, skills, and dispositions for teaching mathematical modeling by engaging in mathematical modeling activities themselves, and reflecting on the process and skills required as they develop their own modeling competencies.

Module 1 The Process and Purpose of Mathematical Modeling
• Gain appreciation for mathematical modeling as an approach to gaining understanding of real world issues, current events, problems, and questions of all kinds. • Create models with attention to units, dependent and independent variables, and informally analyze parameter sensitivity. • Analyze the cyclical process of mathematical modeling and tasks from the K-12 curriculum. • Develop strategies for selecting tasks/topics with attention to student motivation, opportunities to address mathematics in the K-12 curriculum, and potential for addressing important scientific and social issues.
Module 2 Advancing Competency in Mathematical Modeling
• Recognize that the modeling process requires careful analysis of model assumptions and revision. • Derive, solve, and interpret first order differential equations. • Collect and use real data for model parameterization and validation, and develop a deep understanding of parameter fitting algorithms. • Analyze classic models including Newton's Law of Cooling, the Torricelli Model for fluid flow, and the pendulum equation.
Module 3 Diverse Perspectives in Mathematical Modeling
• Use mathematical modeling to address social justice and environmental issues. • Appreciate how models have evolved over time with contributions from diverse cultures and individuals. • Study compartment models including SIR models of disease transmission. • Conduct an independent investigation through a self-chosen course project.

Overview of Module 3: Diverse Perspectives in Mathematical Modeling

In this module, course participants will

- Use mathematical modeling to address social justice and environmental issues.
- Appreciate how models have evolved over time with contributions from diverse cultures and individuals.
- Study compartment models including SIR models of disease transmission.
- Conduct an independent investigation through a self-chosen course project.

Overview of course content and homework assignments:

- Lesson 1 Native American Reservation Land
  - Area of Sioux Reservation Report
  - **MODULE 3: Simulation of Practice - Written**
- Lesson 2 Thermoclines and Air Pollution
  - Thermoclines and Air Pollution Report
- Lesson 3 Energy Saving Lightbulbs
- Lesson 4 Flint Water Crisis
  - **MODULE 3: Simulation of Practice - Video**
- Lesson 5 SIR Disease Transmission Models
- Final Projects (Menu Options)
- Reflection on the Mathematical Modeling Process
  - Diagram of the Modeling Cycle

These lessons are each described briefly in the table below, along with the emphasized MKT or the main mathematical ideas and teaching practices that arise in the lesson.

Module 3 Lesson	**Description**	**Essential Mathematics**
1 Native American Reservation Land	Examining some resources from historical documents, learners investigate the size of the Sioux Reservation at different times. They create models to determine areas, and engage in discussions about the treatment of Native Americans in American history.	Learners model areas with various approaches. They may divide the region into geometric shapes, and possi their models with finer mesh. functions and integrals to determine areas. This presents an opportunity to compare sums to integrals and double integrals for area estimation. Stochastic approaches are also possible. This lesson raises awareness about injustice in America.
2 Thermoclines and Air Pollution	Temperature inversions can lead to unsightly smog, and an accumulation of dangerous levels of pollutants that endanger the health of people who live in valleys. To gain a better understanding of this type of atmospheric phenomenon, learners construct a layered system of milk and coffee and measure temperatures at various heights and times in the system. This is used to introduce multivariable functions and fitting surfaces to data. The project can be extended to 3-dimensional heat transfer models.	We discuss partial derivatives and gradients, with a function of two independent variables. Using computational software, we visualize the function as a surface and with level curves. Learners gain appreciation for modeling approaches for environmental science.
3 Energy Saving Lightbulbs	Learners reflect on a high school student's perspective on the difference between typical textbook application' problems and authentic problem solving that involves mathematical modeling. They read a vignette, solve a textbook problem, and engage in further modeling to examine energy usage and ways to reduce atmospheric carbon to address the crisis of climate change.	The learners review student work involving unit analysis, rates, linear functions, and the textbook problem which is a direct application of the law of sines. Then learners create models to determine energy savings through LED lightbulbs to motivate action on the environment in a personally meaningful way.

Module 3 Lesson	**Description**	**Essential Mathematics**
4 Flint Water Crisis	Learners address the water crisis in Flint, Michigan through two modeling tasks. The first task was introduced in the introductory lesson regarding corporate plans to provide safe water for school children, but this time, they consider the problem in more detail. The second task requires thinking about the ecological effects of the plastic bottles.	The main point is to experience how mathematical modeling tasks are stimulating for addressing issues of social justice. Learners need to be thoughtful about selecting tasks and orchestrating them in a way that promotes inclusivity of diverse perspectives and challenges oppression.
5 SIR Disease Transmission Models	This lesson leads learners to develop compartment ODE models by considering flow balance diagrams. The activities build on the STD lesson from Module 1.	This lesson makes the SIR disease transmission models, commonly used in epidemiological modeling and research, understandable and also reveals the principles upon which many mathematical models are based.
6 Final Projects	Here we have a list of various modeling prompts. Individual students or small teams develop models to address the questions raised in each prompt, and prepare a presentation.	Learners conduct an independent investigation.
7 Reflection on the Mathematical Modeling Process	This lesson gives learners a more in-depth look at mathematics education research regarding mathematical modeling. They will begin with a questionnaire to reflect on the process and their beliefs about modeling after their work on the various tasks throughout the course. They learn about habits of mind and mathematical modeling competencies, and create their own modeling diagram.	Learners read more research on mathematical modeling and reflect on the mathematical practices that students exhibit.

1 Native American Reservation Land

Length: 1 Class Meetings, ~75 minutes

Overview

Summary

This lesson provides students with a historical perspectives on the land designated to the Indian Reservations in the U.S. and how over time, the reservation land has shrunk. Learners will look at this social justice issue of the Sioux reservation shrinking. Learners can create functions to approximate the boarder and use integration to estimate the area of the reservation. Learners will, in doing so, develop a method for approximating the area of the Sioux Reservation area in 1851 and 1876.

Goals

In this lesson, Learners:

- Explore the disenfranchisement the Sioux tribes through the repeated reductions in the size of Sioux tribes' reservation land throughout history.
- Develop and improve upon a mathematical method for finding the area of complex planar regions.
- Produce a method for obtaining a reasonable estimate of the error introduced from the area-finding method developed.
- Make pedagogical determinations as part of a simulation of practice in regard to facilitating a productive discussion on a variety of approaches and results for approaching this area-fin

CCSSM Standard	Connection to Lesson
MP4: Model with Mathematics	A large part of the task is about approximating the area of region in a map based on an image that includes a scale for reference.
7G.6: Solve real-life and mathematical problems involving angle measure, area, surface area, and volume of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.	The reservation can be described as a polygon and learners can use the existing shapes inside the polygon to estimate the area.
G-GPE.7: Use coordinates to compute perimeters of polygons and areas of triangles and rectangles, e.g., using the distance formula	Learners can use the coordinates to construct functions to aid in determining the area of the reservation.

A Module 1 Assessment and Synthesis **50**
A.1 Reflection on the Mathematical Modeling Process 50

B Video Recording Instructions **51**

1 Developing Ways of Thinking for Mathematical Modeling

Overview

Summary

This lesson sets the stage for the class. The problems may be different from what you are used to encountering. We raise and address mathematical questions with connections to several current events, serving as an introduction to an understanding of mathematical modeling. The aim is to lead learners to realize that mathematics is more than what many students experience in their classes. We can address social, political, or scientific issues using mathematical ideas to understand, analyze, or predict happenings in the world around us.

Goals

- Generate motivation for seeing mathematics in everyday life.
- Orient participants to an applied mathematician's way of thinking.

Materials

- Modeling Course Module 1 Slides
- Assignment: Video Introduction
- Handout 1: Mathematical Modeling Written Survey
- Handout 2: Crowd-Size Estimation
- Assignment: Current Event with Mathematical Reflection
- Handout 3: Standards and their Connections to this Lesson

Assignment: Video Introduction

This assignment is to be completed via video recording outside of class.

Instructions: Create a short video of yourself (2-5 minutes) as you might introduce yourself to a class. Please address the following with as much (or as little) detail as you'd like.

1. Your name, what you'd like to be called by your students
2. Your background (where you're from, languages you speak, something about your family or community, influential life experiences)
3. A brief statement of your interests in and outside of academics

Handout 1: Mathematical Modeling Written Survey

Instructions: Respond to the following with detail.

1. One of the standards for mathematical practice is "Model with mathematics." Explain what this means to you.

2. Are modeling with mathematics and solving word problems related? Explain.

3. How can teachers understand and prepare to teach modeling at the middle school and high school levels?

4. What role do you suppose that "real-life" contexts play in modeling problems?

Endorsements of the MODULE(S2) Statistics materials

As the advisor to the distinguished development team of the MODULE(S2) statistics materials, I was impressed and grateful for the dedication shown by the team with their vision, commitment to detail and precision of the content, and for the outstanding support provided to the users of the MODULE(S2) materials. I strongly recommend and encourage the use of the MODULE(S2) statistics materials as valued, peer reviewed resources for preparing secondary mathematics teachers.

Christine Franklin
American Statistical Association K-12 Ambassador,
UGA Emerita Faculty in Statistics

The Statistics resources developed by the MODULE(S2) team are an amazing example of curricular resources for preservice (and practicing!) secondary mathematics teachers because they advance both learning and dispositions across multiple domains. Teacher candidates learn statistical content and reasoning in a technology-rich, active-learning environment; they learn how powerful statistics can be to better understand and act upon real-world and personally meaningful contexts, especially those that may involve injustice; and they learn how they can be effective justice-oriented statistics teachers. Through engagement in the tasks, modeling by the instructor, and reflection on instructional practices, teacher candidates are likely to begin to develop an identity as a critically conscious mathematics educator.

Brian Lawler
Leadership Team – Mathematics Teacher Education Partnership (MTEP) Equity and Social Justice Working Group,
Associate Professor of Mathematics Education at Kennesaw State University

Far too many prospective secondary mathematics teachers do not have the experiences they need to successfully teach statistics in the manner called for in state and national recommendations. The MODULE(S2) Statistics for Secondary Mathematics Teaching materials are designed to address that critical need. I am impressed with the careful design of the materials, including meaningful connections to the secondary mathematics classroom. They have a proven record of success, and based on my personal use of some of the modules, they are a lot of fun to teach!

Gary Martin
Leadership Team member of the Mathematics Teacher Education Partnership (MTEP),
Emily R. and Gerald S. Leischuck Endowed Professor in the Department of Curriculum and Teaching at Auburn University

The Mathematics Of Doing, Understanding, Learning, and Educating for Secondary Schools (MODULE(S^2)) project is partially supported by funding from a collaborative grant of the National Science Foundation under Grant Nos. DUE-1726707, 1726804, 1726252, 1726723, 1726744, and 1726098. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

Mathematics of Doing, Understanding, Learning, and Educating for Secondary Schools

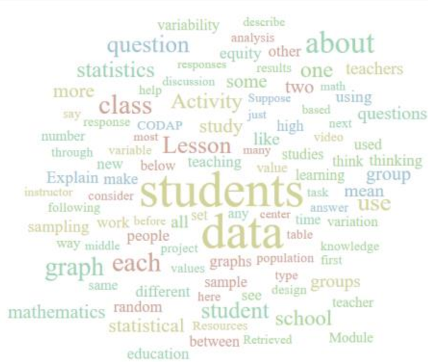
MODULE(S^2):

Statistics for Secondary Mathematics Teaching

Module 1: Study Design and Exploratory Data Analysis

Stephanie Casey*
Andrew Ross*
Samantha Maddox
Melody Wilson

Published Version: July 2022



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For more information about the MODULE(S^2) Project and other MODULE(S^2) materials please visit www.modules2.com

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Contents

Overview of MODULE(S²) Project 4

Overview of Statistics MODULE(S²) 10

Overview of Module 1: Study Design and Exploratory Data Analysis 17

Overview Table for Module 1 18

Lesson 1: What Is Statistics? (Length: ~90 minutes) 21

Overview 21

In-Class Resources for Lesson 1 28

Activity 1: The Statistical Investigation Cycle 28

Activity 2: Asking Good Statistical Questions 33

Exercises 45

Lesson 2: Study Design (Length: ~90 minutes) 51

Overview 51

In-Class Resources for Lesson 2 55

Activity 3: Random Sampling & Common Forms of Bias 55

Activity 4: Types of Studies 63

Exercises 72

Additional Activity for Activity 4: Purpose of Random Assignment 79

Lesson 3: Equity Conversations and Data Visualizations (Length: ~90 minutes) 81

Overview 81

In-Class Resources for Lesson 3 91

Activity 5: Preparing to have Courageous Conversations about Equity 91

Activity 6: Modern Multivariate Data Visualizations 96

Exercises 102

Lesson 4: Multivariate Exploratory Data Analysis with CODAP (Length: ~90 minutes) 108

Overview 108

In-Class Resources for Lesson 4 111

Activity 7: Finding A Story in the Data (I) ... 111
Activity 8: Finding A Story in the Data (II) ... 118
Exercises ... 123
Lesson 5: Teaching Statistics for Social Justice (Length: ~90 minutes) ... 126
Overview ... 126
In-Class Resources for Lesson 5 ... 128
Activity 9: Advocating for Change with Existing Data Sets ... 128
Activity 10: Student Projects and Social Justice ... 131
Exercises ... 137
Lesson 6: Interpreting Graphs (Length: ~90 minutes) ... 142
Overview ... 142
In-Class Resources for Lesson 6 ... 145
Activity 11: Students' Graph Comprehension ... 145
Activity 12: Displaying Aggregated Data ... 150
Exercises ... 162
Lesson 7: Interpreting and Responding to Student Thinking (Length: ~90 minutes) ... 177
Overview ... 177
In-Class Resources for Lesson 7 ... 179
Activity 13: Student Thinking ABOUT Univariate Categorical Graphs ... 179
Activity 14: Responding to Student Thinking ... 182
Exercises ... 191
Lesson 8: Characteristics of Distributions (Length: ~120 minutes) ... 198
Overview ... 198
In-Class Resources for Lesson 8 ... 201
Activity 15: Measures of Center & Variability ... 201
Activity 16: Wrapping Up Shape, Center, and Variability ... 210
Exercises ... 222

Overview of Statistics MODULE(S²)

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The Statistics Modules are:

1) Study Design and Exploratory Data Analysis (EDA)
2) Inference (confidence intervals and hypothesis tests)
3) Association (linear regression, two-way categorical tables, and chi-squared analysis)

Each Module consists of roughly 8 Lessons, each meant to take 90 minutes of class time. Most Lessons are split into 2 Activities.

The modules are aligned with the Common Core State Standards to some extent, but we go beyond them as well to cover college-level material such as formal confidence intervals and hypothesis tests. We take a simulation-based inference (SBI) approach, using randomization/bootstrapping first to develop ideas like p-value and margin of error, since that is the approach that the Common Core takes. Later, we present the traditional formula-driven approach as well (z-tests, but not t-tests, to save time). There are also some CCSS standards for probability and statistics that we do not have time to address.

Statistics classes are a natural place to look at data from the real world. We have chosen many of our data sets to focus on equity and social justice, often in contexts that are important to the US education system. This requires some classroom preparation, in order to have successful courageous conversations. We strongly recommend that you read Module 1 Lesson 3 to get a sense of the preparation needed for this. If you are in the habit of recording class sessions, you should strongly consider not recording these conversations, so students feel a bit more free to speak up. We are also including an overview of equity and social justice topics later in this introduction.

Key Concepts

Each module is organized around a Key Concept:

KEY CONCEPT for Module 1 (Study Design/EDA): Design of statistical studies and analysis should consider the effects of variability and of variables not controlled by the study. Making sense of data through visualizations and statistical analysis can give insights into the data's story.

KEY CONCEPT for Module 2 (Inference): Inferential statistics enables us to infer, though with uncertainty, beyond the data we have to a broader set of individuals or circumstances.

KEY CONCEPT for Module 3 (Association): Association means that information about one variable changes our idea about what happened with the other variable, but does not necessarily establish a causation relationship.

Practical Considerations

Regarding technology requirements, Module 1 uses the free online statistical visualization package CODAP (http://codap.concord.org). Module 2 uses the free online simulation-based inference apps at http://Lock5stat.com/StatKey , with gratitude to the Lock family for their gracious permission. Module 3 uses CODAP again, with a few references to spreadsheets. Our goals in choosing technology were to use only free tools, only tools that could run on a tablet or Chromebook (since that is what many high school students have available), and to be forward-looking regarding analyzing "large" data sets. Thus, we do not refer to handheld calculators at all.

Two versions of the Modules files are available: the instructor version, which has an instructor-oriented introduction to each lesson, and comments on the side using the comments feature of Microsoft Word, and the coursepack version (PDF only), which has those things removed. The comments are a mix of answers for the questions and exercises, Pedagogical Tips for the college instructors using these materials, explanations of why we are doing this, etc. Please do not distribute the instructor version to college students, with the possible exception of solution keys for the exercises showing the answers in the Comments.

We recommend supplying an electronic version of the coursepack PDF to your teachers, so they can more easily click on links during class and when doing the homework, rather than retyping links. Don't worry about them reading ahead and seeing "spoilers"—we've written the materials so that is not a concern.

If any of the links are found to be dead, we suggest going to https://archive.org/web/ and pasting the dead URL into the "WayBack Machine"—sometimes it will pull up a saved copy of the page. Please let the authors know about any links that are dead. Also, we might have a private backup copy of some things.

In general, teachers do not have to finish all of the homework assignment from one Lesson before starting the next Lesson. For example, allowing 5 or 7 days to do the exercises has worked well. However, sometimes there are "Pre-Work" or "Pre-Readings" that should be done before the next Lesson, and these are the first exercises listed in the Exercises section. For example, Exercises 1-1 and 1-2 are pre-work for Lesson 2.

Because discussion is an important aspect of the course, we recommend having engagement and attendance count toward the grade. The system we used (that seemed to work well) was 2 points per day for engaged attendance, versus 1 point per day for attendance without being engaged in the class. Excused absences were allowed. Then all those points were added up and weighted to be 5% or 10% of the total class grade.

A meaningful project for persons learning statistics is to design and carry out a statistical investigation cycle to investigate a topic of personal interest that that choose. We have included information about doing such a project in Module 2 Activity 16, but instructors may choose to start the project much earlier in the semester (including determining a statistical investigation question and design of data collection).

One of the piloters of these materials asked their teachers to teach the last few lessons of each module in order to give them experience teaching statistics in the way supported by these materials. They found this to be a very meaningful experience for their teachers, so that is something for you to consider doing based on your own context.

Please check our project website, or email the authors, to see if there are any errata or supplementary documents.

Course Topics

These 3 Statistics Modules are meant to be algebra-based rather than calculus-based, since they are aimed at a "Stat 101" level. The "Statistical Education of Teachers" document does not call for pre-service teachers to have a calculus-based intro stat course, and of course the Common Core statistics standards are not calculus-based. Various writers have emphasized the need to avoid mathematizing intro statistics, since it distracts from statistical thinking (as opposed to classical mathematical thinking). So, for example, in later modules we do not present the calculus-based derivation of the slope for the Least Squares line of best fit, and we do not present the formulas for the density functions of various distributions like the Normal or Student's *t* or Chi-squared. To save time, we do not even discuss Student's *t* distribution at all. Other MODULES produced by this NSF grant perhaps might require some calculus.

These 3 Statistics Modules also try to take a statistics-first rather than probability-first approach. To keep the materials short enough, we do not include topics such as: Axioms of probability, Probability rules (intersections, unions), Bayes Theorem (sensitivity, specificity, prevalence), the Binomial distribution, or Combinatorics (permutations and combinations), even though some of these are in the Common Core.

We also do not cover the following topics that are commonly in "Stat 101":

- Formula-based confidence intervals
- Student's t distribution (we only use Normal distributions and z-scores)
- 2-sample hypothesis tests for difference-of-proportions, though one could use a chi-squared test for that
- Simpson's Paradox
- The "Nominal, Ordinal, Interval, Ratio" data classification system
- Cluster and Stratified sampling methods

These materials are not meant to be enough to enable someone to teach AP Statistics. There are even some CCSS statistics and probability standards that are not directly addressed in these materials. An alignment table of CCSS statistics standards with each Lesson is available; please email the authors to request it if you are interested.

The "Statistical Education of Teachers" document calls for future high school math teachers to take 3 courses in statistics. Our materials include the underlined topics, though some we have marked as optional:

1) An introductory course that emphasizes a modern data-analytic approach to statistical thinking, a simulation-based introduction to inference using appropriate technologies, and an introduction to formal inference (confidence intervals and tests of significance).

2) A second course in statistical methods that builds on the first course and includes both randomization and classical procedures for comparing two parameters based on both independent and dependent samples (small and large), the basic principles of the design and analysis of sample surveys and experiments, inference in the simple linear regression model, and tests of independence/homogeneity for categorical data.

3) A statistical modeling course based on multiple regression techniques, including both categorical and numerical explanatory variables, exponential and power models (through data transformations), models for analyzing designed experiments [ANOVA], and logistic regression models.

These materials are not meant as a complete "Methods of Teaching Math/Stat" course. We are only addressing methods implicitly (by trying to model them.) For example:

- Balancing group work, individual work, whole-class discussions, & teacher explanations
- Ways to have a whole-class discussion (e.g. class debate on a scenario where class must choose plan A vs. B; have students present work on behalf of their groups, etc.)
- Choosing group-worthy tasks
- Choosing relevant tasks; letting students' interests lead the choice of tasks
- Assessment philosophy and practices
- Highlighting valuable strengths of students who are not traditionally considered "smart at math"
- Attending to which student voices are being heard
- Positioning students as doers and thinkers, not just absorbers of information
- Pushing procedure-focused students to justify their answers
- Bridging everyday language to discipline-specific vocabulary
- Addressing student struggles productively
- Classroom management

Overview of Equity and Social Justice Strand

The Association of Mathematics Teacher Educators' Standards for Preparing Teachers of Mathematics (SPTM) (2017) begin with a fundamental assumption: "Ensuring the success of each and every learner requires a deep, integrated focus on equity in every program that prepares teachers of mathematics" (p. 1) We have integrated a focus on equity into these materials by starting a strand of thinking in Module 1 that continues through the rest of the modules: what are some potential causes of economic inequality by race in the US, and are they due to Black culture/behavior according to stereotypes, or are they aspects of the way that US society is structured? We are aiming to build **equity literacy** (see Gorski 2013; 2018) and **critical statistical literacy** (Bailey, 2019; Weiland, 2017) in preservice teachers. Critical statistical literacy is the practice of analytically examining and assessing sociopolitical statistical content to inform action or change. For more information about these literacies and how our materials develop them, see the 2022 article "Developing Equity-Literacy and Critical Statistical Literacy in Secondary Mathematics Pre-Service Teachers" by Stephanie Casey and Andrew Ross in the journal *Mathematics Teacher Educator*.

Major highlights of the equity and social justice strand are:

- Module 1
    - Conceptions of equity
    - Income mobility by race, conditioned on parents' socioeconomic status
    - State graduation rates, separated by race and income, along with information on teacher salaries, overall funding, racial segregation, and economic segregation
    - School district funding inequalities in Pennsylvania with possible explanatory variables, including income and race
    - Air pollution in a low-income neighborhood, working to change the laws
    - Tracking: Racial demographics of U.S. high school mathematics and science courses
- Module 2
    - Study of effects of class size on student learning (we see later in Module 3 that class sizes often differ according to race)
    - Teachers' perceptions of families' beliefs in and support of education, based on families' demographic features
    - Intervention study to address stereotype threat regarding "girls aren't as good at math as boys"
- Module 3
    - Association of car insurance prices and racial composition by zip code
    - Association between percent of students who receive free lunch and average math test score in each district in a state

- Americans' opinions about the importance of education, by race and income (focusing on the stereotype that Black families don't value education as much as White families)
- Demographics of U.S. mathematics teachers
- Class size inequities in Michigan

Throughout the activities, there is a strand considering the potential causes of income inequality in America and links with racism, including consideration of potential causes that (according to stereotypes) originate from within the group(s) that ends up less wealthy and potential causes that originate outside of the group(s). We chose this topic since it encompasses central themes in U.S. society, both now and throughout history. It is also aligned with the *SPTM* call for examining issues of equity, in education and in general, since educational opportunities are linked with race and income in U.S. society. As teachers work through the activities in this strand, they investigate data that speaks to a number of these potential causes (e.g., differing values of education, K-12 class sizes) and keep returning to the question of what are real causes of differences in incomes of different racial groups in America.

Instructor Preparation/ Orientation Materials

In general, it would be good to read the following documents from professional societies before teaching a class with these modules, but it is not a prerequisite for using them. We also do not require that the teachers in your classes read them.

1) NCTM's Catalyzing Change https://www.nctm.org/catalyzing/ and an executive summary at https://www.nctm.org/uploadedFiles/Standards_and_Positions/executive%20summary.pdf
2) GAISE II K-12 https://www.amstat.org/asa/files/pdfs/GAISE/GAISEIIPreK-12_Full.pdf
3) GAISE College 2016 http://www.amstat.org/asa/files/pdfs/GAISE/GaiseCollege_Full.pdf
4) The Statistical Education of Teachers http://www.amstat.org/asa/files/pdfs/EDU-SET.pdf
5) The AMTE standards, especially for their thoughts on equity: https://amte.net/standards
6) The Common Core State Standards in Mathematics (CCSSM): http://www.corestandards.org/Math/
7) In particular, 6th, 7th, and 8th grade Statistics and Probability standards: http://www.corestandards.org/Math/Content/SP/ and High school Statistics and Probability: http://www.corestandards.org/Math/Content/HSS/introduction/
8) "The ASA's statement on p-values: context, process, and purpose", https://amstat.tandfonline.com/doi/full/10.1080/00031305.2016.1154108
   And the related editorial that advocates not using the concept "statistically significant", and not having a cut-off for p-values like 0.05:
   Ronald L. Wasserstein, Allen L. Schirm & Nicole A. Lazar (2019) Moving to a World Beyond "$p < 0.05$", The American Statistician, 73:sup1, 1-19, DOI:

10.1080/00031305.2019.1583913
https://www.tandfonline.com/doi/full/10.1080/00031305.2019.1583913

9) Any sample standardized test questions, such as:
   - The PSAT: it now has statistics questions involving bias, margin of error, two-way tables, and trendlines, not just probability or mean/median/mode
   - Smarter Balanced and PARCC tests for CCSSM
   - Teacher praxis exams in your state

Links to Free Online Statistics Books, etc.

Statistics and Data Science for Teachers, https://www.statisticsteacher.org/2021/10/21/new-book-statistics-and-data-science-for-teachers/ by Anna Bargagliotti & Christine Franklin

Focus on Statistics: Investigations for the Integration of Statistics into Grades 9–12 Mathematics Classrooms, https://www.statisticsteacher.org/2022/03/25/focus-on-statistics-now-available-for-free-download/ by Sara Brown , Patrick Hopfensperger, Henry Kranendonk.

OpenIntro Statistics: https://www.openintro.org/stat/textbook.php ; includes a version that does simulation/randomization first, rather than formula-based work like z-tests and t-tests first, much like our Module 2 does.

OnlineStatBook from Rice University: http://onlinestatbook.com/

Collaborative Statistics: https://cnx.org/contents/MBiUQmmY@23.31:2T34_25K@14/Introduction and http://cnx.org/content/col10522/latest/

CK-12 Advanced Probability and Statistics: https://www.ck12.org/book/CK-12-Probability-and-Statistics-Advanced-Second-Edition/

https://www.fi.ncsu.edu/pages/dice-probability-resources/ is a collection of resources for supporting grades 6-8 probability

Overview of Module 1: Study Design and Exploratory Data Analysis

The key concept for Module 1 is *Design of statistical studies and analysis should consider the effects of variability and of variables not controlled by the study. Making sense of data through visualizations and statistical analysis can give insights into the data's story.*

An overall theme of Module 1 is to see the big picture of statistics: how it's different than mathematics, and how we can use the Statistical Investigation Cycle to better understand what is going on the world, informing action or change on important issues. We encourage teachers to consider structuring their future classrooms using projects, and especially projects that the future students bring up based on their community knowledge. This interweaves with important study design considerations (such as random selection to avoid sampling bias, and random assignment to avoid impacts of lurking variables), and with data exploration and ways of presenting and summarizing data. We include experiences with multivariate datasets and multivariate data displays, since they are often encountered and needed in society (such as the effects of racial discrimination and poverty on people); this is aligned with the GAISE report's call for more focus on multivariate thinking. This also involves using modern statistical software that can handle large datasets, rather than pencil-and-paper and handheld-calculator methods; we use CODAP for this because, in addition to being highly capable, it is free and we can use it in a web browser without downloading or installing anything. By the end of module, teachers should be comfortable using CODAP to analyze large multivariate datasets.

Module 1 starts up a major strand that continues through all 3 statistics Modules: investigating the potential causes of racial inequality in the U.S., including common stereotypes that teachers might have about their future students and factors that teachers might not yet have thought of. This is aligned with the AMTE's *Standards for Preparing Teachers of Mathematics*' call for a deep, integrated focus on issues of equity. Learn more in the previous section: Overview of the Statistics Materials. Teachers may not be accustomed to discussing social issues in content courses, so helping them see the value and utility of it in the context of statistical investigations is important.

In Module 1, we raise the issue of sample-to-sample variability in summary measures, but leave methods to deal with it (hypothesis tests and confidence intervals) for Module 2. Similarly, we do preliminary thinking about statistical association (scatterplots for quantitative variables; segmented bar graphs for categorical variables), but leave a detailed exploration for Module 3, where we introduce topics like regression and conditional relative frequencies.

Teachers have, in theory, seen many of the statistical topics in this module as they have progressed through middle and high school themselves. We build on that understanding (which is often procedural) to develop their conceptual understanding and pedagogical content knowledge. For instance, we assume teachers have some experience with data displays such as histograms, but then we build on that to focus on students' conceptions about them and learning progressions for data displays. Also, teachers' prior experiences with statistics likely were focused on the analysis part of the Statistical Investigation Cycle (computing means and standard deviations, for example); in this Module we are trying to broaden their horizons to include the whole cycle, both in their own thinking and with their future students.

Note on Technology. In Module 1, we primarily use the free online statistical visualization package CODAP (http://codap.concord.org). It is based on the statistical programs *Tinkerplots* and *Fathom*, and is designed based on research about how students learn statistics.

Note on Homework Exercises. The first number in each exercise number shows which Activity it most relates to, but some offer opportunities to mix knowledge from various activities.

Two types of homework exercises are noted in the table below: pre-work and Simulations of Practice. They are also noted in the exercise headers themselves.

Overview Table for Module 1

Lesson	Projected Length	Content	In-Class Activities	Homework Notes	Connections and Notes
1: What Is Statistics?	90 min	Sources of variation; Confounding; Statistical investigation cycle	Activity 1: The Statistical investigation Cycle Activity 2: Asking Good Statistical Questions	There is pre-work before Activities 1 and 2. Exercises 1-1 and 1-2 are pre-work for Lesson 2.	The Statistical Habits of Mind can be related to the CCSS-M Standards for Mathematical Practice.
2: Study Design	90 min	Differences between study design types; Avoiding common statistical biases.	Activity 3: Random Sampling & Common Forms of Bias Activity 4: Types of Studies	Exercise 3-1 is pre-work for Lesson 3 Activity 5.	First look at confounding and lurking variables; discussed again in Module 3.

3: Equity Conversations and Data Visualizations	90 min	Definitions of equity; Avoiding stereotypes; Modern data visualizations used in media	Activity 5: Preparing to have Courageous Conversations about Equity Activity 6: Modern Multivariate Data Visualizations	Exercises 5-1 and 5-2 are pre-work for Lesson 4 Activity 7.	Activity 5 also helps prepare for conversations that will take place when teaching Lessons 5 and 6.
4: Multivariate Exploratory Data Analysis with CODAP	90 min	Using CODAP; Data visualizations in CODAP; Critical statistical literacy.	Activity 7: Finding a Story in the Data (Part I) Activity 8: Finding a Story in the Data (Part II)	Exercises 7-1 and 7-2 are pre-work for Lesson 5 Activity 10.	Analysis of association of variables connects with Module 3. Consideration of paired vs. unpaired data connects with Module 2.
5: Teaching Statistics for Social Justice	90 min	Developing critical statistical literacy; Interpreting multivariate data visualizations in CODAP; Design of statistical studies	Activity 9: Advocating for Change with Existing Data Sets Activity 10: Student Projects and Social Justice	Exercises 9-1 and 9-2 are pre-work for Lesson 6 Activity 11. Exercise 10-4 is a Video Simulation of Practice.	Emphasizes using statistics to investigate a social situation of interest and using the results to advocate for change. Highlights the advantages of using community-based projects in classes.
6: Interpreting Graphs	90 min	Frameworks regarding graph comprehension; Graphs for univariate data	Activity 11: Students' Graph Comprehension Activity 12: Displaying Aggregated Data	Exercise 11-1 is pre-work for Lesson 7 Activity 14.	Presumes some basic knowledge of histograms and boxplots, but also gives opportunities to deepen understanding.

7: Interpreting and Responding to Student Thinking	90 min	Common student approaches to analyzing univariate data; Professional noticing of students' thinking	Activity 13: Student Thinking about Univariate Categorical Graphs Activity 14: Responding to Student Thinking		Activities and exercises make connections with the graph comprehension frameworks from Lesson 6.
8: Characteristics of Distributions	120 min	Conceptual understanding of measures of center and variability; Z-scores; relative effect size measurement;	Activity 15: Measures of Center & Variability Activity 16: Wrapping Up Shape, Center, & Variability	The exercise set include some synthesizing, cumulative review exercises. Exercise 16-22 is the Written Simulation of Practice.	Activity 16 consists of 4 mini-activities. Mini-activity 4 is a prelude to Module 2 Lesson 1. Z-scores, introduced in mini-activity 2, are used again in Module 2 Lessons 6 & 7, as well as Module 3 Lesson 2. Effect size measurement introduced here in mini-activity 3 is used in Module 2 Lessons 3 & 7.

Responding to Student Thinking:

What Makes for a Good Teacher Response?

- Asking questions is often better than making statements.
- It can be helpful to give an example of the student's reasoning but taken to extremes (either numerical extremes or context extremes where the answer should be clear).
- Simply telling the student the right way to think is often not as helpful as one might hope—they need to see why their reasoning isn't correct.
- High quality responses to student work:
    - Move students toward the student learning objective;
    - Draw on and are consistent with the student thinking presented and research on students' mathematical development; and
    - Leave space for student's thinking (i.e., the teacher should not do all of the thinking; the student needs to be prompted to think about the concept).

In-Class Resources for Lesson 1

ACTIVITY 1: THE STATISTICAL INVESTIGATION CYCLE

A student approaches you and says: "I'm working on a science project. I wanted to know whether caffeine affects heart rate. I got two of the classes I'm in to help me collect data.

When each class started, we each took our own pulse, then drank a can of soda pop. We used the same brand of cola for everyone, but some people got the no-caffeine version. Then at the end of the class period, we each took our own pulse again. Here's my data table. Pulse is beats per minute. If the athlete column says 'n' that means No or not an athlete."

IDnumber	Group	year	athlete	class	teacher	pulse1	pulse2
2482	caf	jr	varsity	9am	A	71	76
7137	caf	sr	varsity	9am	A	58	61
9014	caf	jr	jv	9am	A	58	62
5971	caf	jr	jv	9am	A	74	77
1448	caf	jr	varsity	9am	A	58	68
9392	caf	jr	n	9am	A	81	85
7505	caf	jr	varsity	9am	A	63	70
2560	caf	jr	varsity	9am	A	66	69
9354	caf	sr	varsity	9am	A	66	73
5622	caf	sr	n	9am	A	80	85
3706	caf	jr	jv	9am	A	66	76
1106	caf	jr	n	9am	A	86	94
6265	non	sr	varsity	11am	B	65	64
6997	non	sr	n	11am	B	71	72
9643	non	sr	jv	11am	B	59	60
4734	non	sr	n	11am	B	84	87
5703	non	jr	jv	11am	B	73	77
6456	non	sr	n	11am	B	85	83
8644	non	jr	jv	11am	B	72	67
5864	non	sr	n	11am	B	79	84

In a table of data like this, each row is called a "case" (the way a doctor might say "I saw 2 cases of the flu today"), and each column is called a "variable" or "attribute".

Question 1-a Based on this study, do you think caffeine affects heart rate? Explain how you are analyzing the data. You might use any technology you're familiar with and have handy, or just a pencil would be fine too. If you like, you can download the data from https://bit.ly/CaffeineStudy01_txt or https://bit.ly/CaffeineStudy01_csv [1].

Question 1-b What are the positive aspects of this study's design?

Question 1-c What are the negative aspects of this study's design?

Question 1-d What will you say to the student about the study's design? Remember that asking them questions which probe and push their thinking is often a better approach than simply making statements.

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[1] https://emunix.emich.edu/~aross15/data/CaffeineStudy.txt or https://emunix.emich.edu/~aross15/data/CaffeineStudy.csv

As a mathematics teacher, it will be your job to help students learn statistics. Part of that is helping them explore data. Another part is helping them learn to design a study to generate data that actually answers their study's questions. This Module is about both of these things: Study Design and Exploratory Data Analysis. Module 2 is about more formal methods for analyzing data, usually on 2 groups, and Module 3 is about looking for relationships between variables. In all of the statistics Modules, you will be both learning statistics and learning things relevant to teaching statistics (like how students tend to think, and what the usual curriculum includes).

What is Statistics?

Question 1-e What are some ways that statistics gets used in the real world that you think would interest or motivate students?

Question 1-f What would you say the field of statistics ***is***, and why should students learn statistics?

The American Statistical Association (ASA) says "Statistics is the science of learning from data and of measuring, controlling, and communicating uncertainty." One set of learning standards for high school says "Statistics provides tools for describing variability in data and for making informed decisions that take it into account."

Question 1-g Does anything in these definitions surprise you? How do the two definitions compare to each other? How do they compare to your definition?

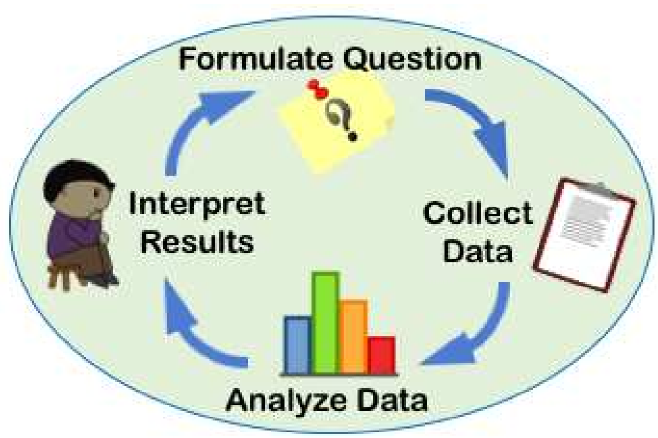
Question 1-h Think back to the introductory activity about caffeine and heart rate. How do its various aspects relate to these definitions?

The Statistical Investigation Cycle

Many people think of statistics as crunching numbers, but simply analyzing the numerical results misses the important aspects of how the data was collected and what conclusions we can or cannot draw from it. This was brought out in the caffeine and heart rate study. Modern statistics teaching aims to include all aspects of doing a study, not just the number-crunching part. The ASA has written a report called Pre-K-12 Guidelines for Assessment and Instruction in Statistics Education: A Framework for Statistics and Data Science Education (GAISE II) that talks about the importance of students learning to carry out the **Statistical Investigation Cycle** (also called the Statistical Problem-Solving Process):

I. Formulate Statistical Investigative Questions
   a. clarify the problem at hand
   b. formulate question(s) that can be answered with data & anticipate variability

II. Collect/Consider the Data
   a. design a plan to collect new data/consider previously-collected appropriate data
   b. employ the plan to collect data (if needed)
   c. interrogate the data



III. Analyze the Data
  a. select appropriate graphical and numerical methods
  b. use these methods to analyze the data
IV. Interpret the Results
  a. interpret the analysis
  b. relate the interpretation to the original question

GAISE II reminds us that constant questioning at each step of the Cycle is important. Formulating an investigative question can include critiquing the framing or point of view of a question—is it helpful? does it focus on the things that are the most important, and avoid being limited by stereotypes? who are the stakeholders? who has been left out? who might be unfairly benefitting from things being the way they are? Similarly, collecting or considering data can include critiquing the ways that things will be or were measured or recorded. For example, are standardized multiple-choice tests the best way to measure what we really value in classrooms? Or, do demographic questions give a full range of options as possible answers? It is also important to question whether the data collection & analysis plan will be helpful in answering the investigative questions.

The Common Core State Standards are also built around the Statistical Investigation Cycle; they emphasize that students should learn about and enact the whole cycle.

Question 1-i Think back on the education you have received about statistics this far in your life. Which phases of the Statistical Investigation Cycle did you learn well? Which were most lacking?

Question 1-j Is it valuable for all high school graduates to know about the entire Statistical Investigation Cycle, when the vast majority of them will never set up and run a statistical study in their adult lives? Explain.

ACTIVITY 2: ASKING GOOD STATISTICAL QUESTIONS

How Are Statistics and Mathematics Questions Different?

Use of Context

"In mathematics, context obscures structure. In data analysis, context provides meaning." (Cobb and Moore, 1997) Students (and teachers) sometimes become frustrated with learning statistics because they are not used to using context and judgment in a math-class-like setting. A student doing math without context might round 0.000000027 down to zero, because it seems small (possibly even a rounding error). However, when measurements of the concentration of the chemical element lead in the drinking water in Flint, Michigan showed a concentration of 2.7×10^{-8}, or 27 parts-per-billion, "it's five times as high as the level of concern, and nearly twice as high as the EPA's already-generous guidelines" (Washington Post 2016).

Question 2-a In the caffeine and heart rate study, describe at least two ways that an understanding of the context resulted in different conclusions than saying "the caffeine group saw more of an increase in pulse than the other group did"?

Statisticians are fond of saying "Statistics is not Mathematics." One of the reasons for this comment is that math typically deals with certainty, while statistics is all about uncertainty and variability. Classically, math is about deductive reasoning: starting with axioms or assumptions and proceeding to conclusions that are guaranteed to be true, such as the Pythagorean Theorem. Statistics, though, is about inductive reasoning: designing a study and seeing a data set and trying to say something more general about the people or items that aren't in the data, while acknowledging that your conclusions aren't perfectly precise or certain, and quantifying how far off you might be.

Asking Good Statistical Investigative Questions

In both statistics and mathematics, it's important to ask good questions, but the kinds of questions differ between the two subjects. For now, we will focus on the investigative questions used in each field to start the problem-solving process. Mathematical investigative questions usually expect the answer to be a single value or yes/no decision and expect that if you worked on the question again you would get exactly the same answer. Statistical investigative questions, though, anticipate that people or items or trials are different from each other, and that we will need data that varies to answer the question.

Statistical investigative question: a query that motivates the need to collect/consider data and launches the statistical investigation cycle.

Key Characteristics:

1) Anticipates needing data that varies to answer it
2) Identifies the population of interest
3) Identifies the variable of interest

Question 2-b Below is a list of 10 questions, some of which are statistical investigative questions and some of which are not. Sort them into the two categories, using the table below to note the letters labeling the questions in each category, and be prepared to explain your decisions.

A. What is the length of my dad's right foot?
B. What is the length of the right foot of American males?

C. How long did it take me to get to school this morning?

D. What is the mean of the following numbers: 2, 5, 46, 2, 8, 32?

E. How much do you like Statistics? Rate on a scale of 1 to 5 (1 means "do not like at all," 5 means "absolutely love it").

F. How do gas prices in Lakeville compare to gas prices in its surrounding suburbs?

G. How much do students at my college like Statistics?

H. If Sunshine gas costs $0.08 per gallon less than Speedy gas, how much can I save on one 12-gallon fill-up at Sunshine?

I. How long does it take kids at my school to get to school each morning?

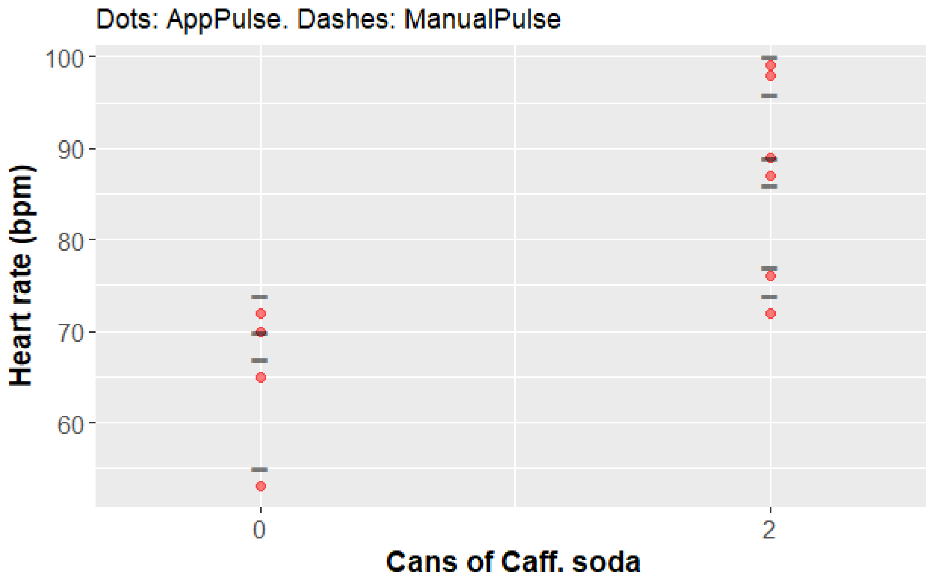
J. Do elementary students in small classes score better than those in large classes on a standardized math test?

Statistical Investigative Questions	Non-Statistical Investigative Questions

Sources of Variation in Data Values

If a statistical question is supposed to anticipate variation, what kinds of variation should we be in the habit of considering? To consider this, let's suppose that another class did a caffeine-and-heart-rate study, with their class split into 2 groups: some people drank 2 cans of caffeinated soda and some drank 2 non-caffeinated cans. The "#Caff. Cans." column records how many cans (out of the 2) had caffeine. Also, they didn't think to record their heart rates at the start, but they recorded their pulse at the end of class in two different ways during a 60-second time interval: they used a pulse-measuring app on their smartphones, and also measured their pulse the usual manual way. They got this data table and graph:

ID Number	#Caff. Cans	AppPulse	ManualPulse
927	0	65	67
473	0	53	55
840	0	70	70
245	0	72	74
366	2	98	96
756	2	76	77
913	2	99	100
422	2	89	89
468	2	72	74
392	2	87	86



Here are some common types of variation in data values that we think about:

Natural variation: people (or items) are naturally not guaranteed to behave the same way, even if they experience the same circumstances.

Measurement variation: our measurements don't always give the same result even if we are measuring the same thing repeatedly.

Induced variation: people (or items) often behave differently when they are treated differently (either the study itself *induced* the variation with different treatments, or some other factor outside the study's control induced variation .)

Question 2-c Circle a set of numbers in the table that shows natural variation, and highlight some part(s?) of the graph that show natural variation. Label it as "natural variation".

Question 2-d Circle a set of numbers in the table that shows measurement variation, and similarly in the graph. Also label it.

Question 2-e Circle a set of numbers in the table that shows induced variation, and similarly on the graph. Also label it.

Question 2-f Did this new caffeine-and-heart-rate study find any effect? Focus just on AppPulse, ignoring ManualPulse, for the rest of this Lesson. Discuss your reasoning.

Variation in Summaries of Data: Sampling Variation

Question 2-g In the previous question, did anyone in your class compare the mean pulse of the no-caffeine group to the mean pulse of the caffeine group? What does comparing those two means tell you? Do you think it's a good comparison to make? Why?

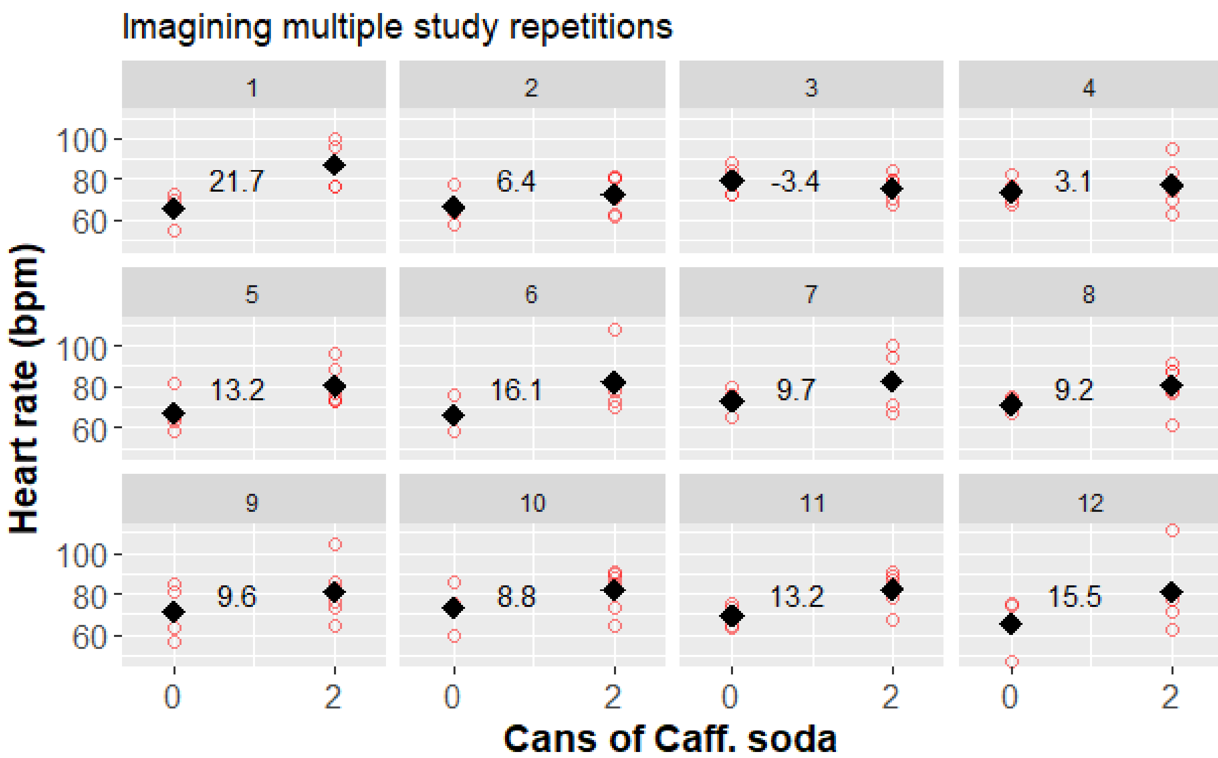
Question 2-h If we ran the whole study again after sampling/choosing different people, would the difference of the two group means (caf. mean pulse – noncaf. mean pulse) be the same as it was in the graph above? How much do you think it might vary? How certain are you that the caf. group mean pulse would still be larger than the noncaf. group mean pulse?

This brings us to our next definitions:

Statistic: a number that is computed from data, or could be computed from data. Examples include a mean, median, proportion, standard deviation, or difference of means for 2 groups.

Sampling variation: how much a statistic would vary from one study repetition to another. Sometimes confusingly called "sampling error".

This graph illustrates what we might see if we had time to do 12 more of the caffeine-and-heart-rate study. Individual data values are shown as open circles, the mean of each group is shown as a filled diamond, and the numerical difference of the means is also shown:



You can see an animation of it (with more than just 12 study repetitions) at http://bit.ly/CaffStudiesAnim [2].

Question 2-i How would you describe the behavior of the differences-of-means? How sure are you that the caf. group mean pulse would be above the noncaf. group mean pulse, if we could sample everyone in the world?

In real studies, we only get to see one set of data, rather than repeated versions of a study. So, how do we deal with sampling variation if we can't see it? That is such a big question that it takes all of Module 2 (and some of Module 3) to answer. Just realizing and anticipating that data summaries have variation is a big learning goal as students learn statistics.

[2] or http://emunix.emich.edu/~aross15/anim/multiple_trials.gif

The Common Core State Standards start statistical topics in 6th grade, with students anticipating variability in data:

> CCSS.MATH.CONTENT.6.SP.A.1 Recognize a statistical question as one that anticipates variability in the data related to the question and accounts for it in the answers.

Then in 7th grade, they think about what the sample says about the population, and think about sampling variability:

> CCSS.MATH.CONTENT.7.SP.A.2 Use data from a random sample to draw inferences about a population with an unknown characteristic of interest. Generate multiple samples (or simulated samples) of the same size to gauge the variation in estimates or predictions.

They also compare two groups, but focus only on the sample data rather than making statements about the population. Two-group comparisons that account for sampling variation of the mean wait until high school.

Student Conceptions of Sampling Variation

With your class, look back at your answers to the "Candy Questions" from the pre-work.

Question 2-j For Candy Question A,

i. What types of answers did students in your class give? How common were the various types?

ii. Can your class agree on the best answer? What percent of your class gave that answer?

Question 2-k On Candy Question B,

i. What types of answers did the students in your class give? How common were the various types?

ii. Can your class agree on the best answer? What percent of your class gave that answer?

iii. Can you think of answers that students might give that you haven't listed so far, and what thinking might be behind them?

Question 2-l To what extent is Candy Question A a "math question" versus a "statistics question"? How about Candy Question B?

Question 2-m Which Candy Question got more answers from your class that included some sort of variability? If you, as a teacher of statistics, are trying to get your class to think statistically, which phrasing is better?

The "Candy Questions" were based on studies done by Shaughnessy and colleagues (Shaughnessy, Canada, & Ciancetta, 2003; Watson, Kelly, Callingham, & Shaughnessy, 2003). They asked those questions of middle- and high-school students in the US and Australia, to get a sense of how students understand sampling variation. The studies found that on Candy Question A, only 6% of the middle school students, and 1.6% of the high school students, gave an answer that had some variability, such as "5 to 7". Most of the students said simply "6". Almost 18% of the middle school students, and 13% of the high school students, gave an incorrect answer.

On Candy Question B, 29% of the middle school students gave a response that was incorrect in some way: showing no variation at all, or too-narrow variation, or too-wide variation, or all values less than 6, or all values greater than 6. High school students did even worse, at 49% incorrect. The authors hypothesized that the older students' previous classroom experience with mathematical probability (rather than statistics) made them think less about variability in sampling results.

Lesson 1 Wrap-Up: Statistical Habits of Mind

So far in this lesson, we have focused on two things: keep context in mind, and statistics is about variation. These are good habits of mind; let's explore them more deeply, in a way that works both when we are the ones doing the investigation and when we are reading about someone else's investigation. It would be nice for us (and our future students) to have a list of ways to think about studies that works in both circumstances. Lee and Tran (2016) have made a list of 7 "Statistics Habits of Mind":

1) Always consider the context of the data
2) Ensure the best measure of an attribute of interest
3) Anticipate, look for, and describe variation
4) Attend to sampling issues
5) Embrace uncertainty, but build confidence in interpretations
6) Use several visual and numerical representations to make sense of data
7) Be a skeptic throughout an investigation

Question 2-n Suppose that a "university scorecard" created by the government looks at each university's impact on students via the incomes of its alumni 20 years after graduation. What do you think of this method? Which of the 7 habits is it related to?

Question 2-o It seems that we often hear news stories that some food is now considered bad for you even though previous studies thought it was good, or vice versa. What is going on? Can we really trust scientific studies? Which of the 7 habits is this related to?

Question 2-p How much do you trust a study about some product that is funded by the company that makes the product? Can you think of some historical examples where this has turned out badly? Which of the 7 habits is this related to?

Question 2-q A 5-hour midterm exam would give a more precise answer about a student's ability than a 1-hour exam. Why don't we always do 5-hour midterms? Which of the 7 habits is this related to?

Question 2-r Some people demand that we know exactly what is going on and exactly what will happen before enacting some public policy change. Can you name a few examples of current debates that are relevant here? Which of the 7 habits is this related to?

Question 2-s In habit #7, does "skeptic" mean someone who never believes anything? What are some ways you can be skeptical during a study? Explain.

Question 2-t Consider the caffeine and heart rate activity/study that we have discussed. Describe how some of these 7 Statistics Habits of Mind would have helped improve the study.

Question 2-u Which of these Habits would you say you are best at? Which are you most in need of practice with? Explain why.

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Shaughnessy, J. M., Canada, D., & Ciancetta, M. (2003). Middle school students' thinking about variability in repeated trials: A cross-task comparison. In *Proceedings of the 27th International Group for the Psychology of Mathematics Education Conference Held Jointly with the 25th PME-NA Conference* (Vol. 4, pp. 159–166). Retrieved from https://eric.ed.gov/?id=ED501118

Watson, J. M., Kelly, B. A., Callingham, R. A., & Shaughnessy, J. M. (2003). The measurement of school students' understanding of statistical variation. *International Journal of Mathematical Education in Science and Technology*, *34*(1), 1–29. https://doi.org/10.1080/0020739021000018791

Additional Resources

Petty, Nicola (2014) "Variation and Sampling Error", Dr. Nic's Maths and Stats, https://www.youtube.com/watch?v=y3A0lUkpAko

Wild, Chris (2011) "Animations of Sampling Variation", https://www.stat.auckland.ac.nz/~wild/WPRH/

If you like the idea of habits of mind and want to read more, try these lists-of-lists:

Lin, Kien (2013) "A collection of lists of mathematical habits of mind", http://www.math.utep.edu/Faculty/kienlim/HoM_Collection.pdf

Wild, Chris, "Statistical Thinking Diagrams and Models", https://www.stat.auckland.ac.nz/~wild/StatThink/index.html

How is the statistical investigation cycle similar to doing science? You can read the full set of science standards at:

Next Generation Science Standards (2013) www.nextgenscience.org/

Exercises

Exercise numbers usually start with the number of the activity that they are most related to. For example, Exercise 2-1, 2-2, 2-3, etc. are mostly related to Activity 2. However, some exercises will draw on previous Activities as well, perhaps without mentioning it.

Also, some Lessons assign pre-reading or other activities that should be done before the start of the next class session, as opposed to having a more flexible due date. These will usually be the first exercises listed in the homework, and they will be marked to indicate that they are pre-work for the next Lesson.

Exercise 1-1 Due before next class: Find an article that describes a statistical study. It could be an academic article from a research journal, or a popular-press article from a newspaper or magazine or website. Write a 1-page summary of the article. Bring a printout of your summary to class. It wouldn't hurt to bring a printout of the article, either. Your instructor might also require you to submit an electronic copy. Your summary should include your name, and the article's citation data with URL if possible.

Exercise 1-2 Due before next class: Complete the survey, "Preparing to Discuss Equity Issues," at the link your instructor has provided on your class web page or by email.

Exercise 1-3 Create a short video of yourself (2-5 minutes) as you might introduce yourself to a K-12 class. Please address the following questions with as much (or as little) detail as you'd like. This information will also be used to select topics of interest to you in this course, and as a test of the technology for making a video of your teaching practice.

- ***(a)*** Your name, what you'd like to be called by your students
- ***(b)*** A brief statement of your interests in and outside of school
- ***(c)*** Where you're from, languages you speak, places you've traveled, what jobs you've had/have
- ***(d)*** Something about your family: large/small, occupations of parents or grandparents or siblings [only select topics that are relevant to you]

Exercise 1-4 To what extent does the statistical investigation cycle match, or not match, various step-by-step descriptions of the scientific method? For example, Wikipedia gives this step-by-step list for the scientific method:

1) Define a question
2) Gather information and resources (observe)
3) Form an explanatory hypothesis
4) Test the hypothesis by performing an experiment and collecting data in a reproducible manner
5) Analyze the data

6) Interpret the data and draw conclusions that serve as a starting point for new hypothesis
7) Publish results
8) Retest (frequently done by other scientists)

Exercise 1-5 If your class has done a Modeling unit from the MODULE(S^2) project: how does the statistical investigation process relate to the modeling cycle?

Exercise 2-1 For each of these items, suppose a student emailed you a project thought for a statistics project. Determine which of these are statistical investigative questions. For one that is, write a response back to the student concerning things they should consider when designing the study.

- ***(a)*** "I want to find the best combination of strength, stamina, special weapons ability, and healing for a character in my favorite video game."
- ***(b)*** "For my project I want to see how far away my dog can usually hear me from"
- ***(c)*** "I want to see if it's more cost-effective for my family to rent a car for a long road trip vs. using our old beat-up one."
- ***(d)*** "In physics we found a way to say how far a projectile would go based on its initial angle and initial height above the ground. I want to find a formula that tells the best angle given the height, for maximum distance."
- ***(e)*** "In my computer science class we timed different programs that solve the same problem in different ways, but we only ran one trial per program. I want to run 100 trials per program and see if the timings are reliable."
- ***(f)*** "I want to see if women authors are equally represented in the kids' picture books section of my local library."

Exercise 2-2 Many stats classes use examples like peoples' weight and height, but this can make some students in the class uncomfortable or self-conscious even if the data doesn't come from class members themselves. What other topics can you think of that should probably be avoided? Can you think of some good substitutes?

Exercise 2-3 The 2nd caffeine-and-heart-rate study that we discussed measured pulse in 2 ways (using an app and the traditional manual way), and we used those to highlight measurement variation. Is there still measurement variation if you only measure something in 1 way? Explain

Exercise 2-4 In what way should we plan for each type of variation as we design a study?

- ***(a)*** How do we plan for natural variation?
- ***(b)*** How do we plan for measurement variation?
- ***(c)*** How do we plan for induced variation?

Exercise 2-5 Is it possible for a study to <u>not</u> have one or more of the types of variation? For each type, describe/give an example of a kind of study that wouldn't have that type of variation and why, or explain why that's impossible/not interesting.

- ***(a)*** Natural variation
- ***(b)*** Measurement variation
- ***(c)*** Induced variation
- ***(d)*** Sampling variation

Exercise 2-6 A student tells you about their project: "I wanted to see how much handwashing cuts down on germs on my hands. I rubbed my hands on a slice of bread then sealed it in a plastic bag. Then I washed my hands, rubbed them on another slice of bread, and sealed that in a plastic bag. I put both on my shelf for a month. At the end, I counted 121 fuzzy spots on the no-wash slice then 93 on the hands-washed slice. I also got 2 of my friends to count: one got 105 then 98, the other got 109 then 101." What would you say to this student, as their teacher?

Exercise 2-7 Thinking back to Candy Question B (the number of red candies in 6 separate pulls of 10 candies each): would it be more statistical to not say how many red candies are in the big container by asking the question "If someone got 6 red candies in a sample of 10 candies, what can you say about the % red in the big jar?". Suppose that you are a teacher starting a lesson on sampling variability. What are the pros and cons of asking this new question, instead of the question as it was posed?

Exercise 2-8 One of the studies by Shaughnessy et al. (2003) asked:

The Dice Task. Consider rolling a normal six-sided die. Imagine you threw a die 60 times. Fill in the table below to show how many times each number might come up. Why do you think this?

Number on Die	How many times it might come up
1	
2	
3	
4	
5	
6	
Total	60

What do you think is the most common response that students give to this task? What would be the best response? Can you give a better phrasing of the prompt that might lead them to a better response?

Exercise 2-9 When we talked about Sampling Variation, we used an example about heart rates, and focused on the mean (a statistic) rather than on individual people's values. When we discussed the studies done by Shaughnessy et al. (2003), we talked about "how many red candies" out of 10. Is "how many red candies" an individual data value or a statistic? Explain.

Exercise 2-10 Thinking back to the Shaughnessy (2003) studies, let's investigate what we really should expect. What is a typical amount of variation when you draw 10 things from a big jar with 60% red? We will use an applet to simulate it.

1) Go to lock5stat.com > StatKey > Sampling Distributions > Proportion (http://www.lock5stat.com/StatKey/sampling_1_cat/sampling_1_cat.html)
2) Click on "Edit Proportion" and type in 0.60
3) Click on "Choose samples of size n=" and enter 10.
4) Press "generate 1 sample" and see how many "reds" get drawn, in the "count" box on the right-hand side of the screen, about mid-way down the screen. Notice that it also records the #reds as a dot in the large dotplot in the middle of the screen.
5) Press "generate 1 sample" a few more times, trying to get a feel for what numbers tend to occur.
6) Press the "Generate 1000 Samples" button and observe the resulting large dotplot, and its mean and Standard Deviation (called "standard error" in the plot).
7) Based on that, answer the overall questions:
   - You pull out a handful of 10 candies and count the number of reds. How many red candies do you expect? ____ Also, explain why.
   - Suppose six of your classmates did this experiment, each of them pulling out 10 candies. What do you think is likely to occur for the numbers of red candies that each classmate would pull out? __ __ __ __ __ __ Also, explain why.

Exercise 2-11 Why is household income often a better way to view poverty than personal income, among adults? Which topic(s) from this lesson is this related to?

Exercise 2-12 One of the Statistics Habits of Mind is to "Ensure the best measure of an attribute of interest." How is this related to studies in educational policy that use standardized test scores to measure outcomes?

Choose one of the next two exercises (2-13 or 2-14):

Exercise 2-13 The Common Core State Standards for Mathematics includes "Standards for Mathematical Practice," at http://bit.ly/CCSS_SMP [3].

How do they relate to the Statistics Habits of Mind in this lesson? For each Statistical Habit of Mind, list the related Standard(s) for Mathematical Practice, and describe why/give an example

Statistical Habit of Mind	Related Standard(s) for Mathematical Practice from CCSS, with explanation/reasoning/examples

[3] Or http://www.corestandards.org/Math/Practice/

Exercise 2-14 As we go through the Statistical Investigation Cycle, various Statistics Habits of Mind are helpful at different stages. Which ones are most helpful at which stages? Fill in this table:

Stage	Helpful Statistical Habits of Mind (often more than one)
Formulate Questions	
Collect/Consider the Data	
Analyze Data	
Interpret Results	

Mathematics of Doing, Understanding, Learning, and Educating for Secondary Schools

MODULE(S^2):

Statistics for Secondary Mathematics Teaching

Module 2: Statistical Inference

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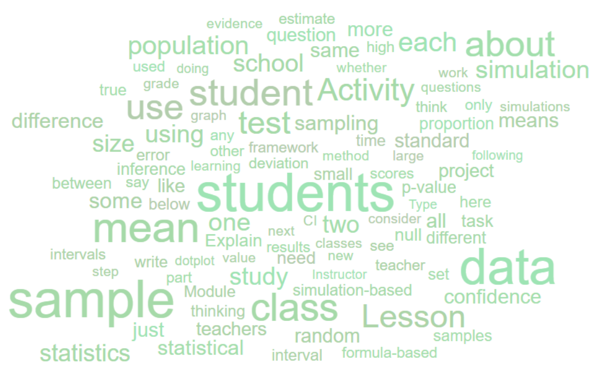

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For more information about the MODULE(S^2) Project and other MODULE(S^2) materials please visit www.modules2.com

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Contents

MODULE(S²) Overview **4**

Overview of Module 2: Statistical Inference **5**

Overview Table for Module 2 11

Lesson 1: Simulation-Based Informal Hypothesis Test: Comparing 2 Groups (means) (Length: ~90 minutes) **14**

Overview 14

In-Class Resources for Lesson 1 18

Unit Overview 18

Activity 1: Is There A Difference? 19

Activity 2: Do The Shuffle 22

Exercises 25

Lesson 2: Random Assignment Simulations with Technology, Paired Studies, and TN STAR study (Length: ~90 minutes) **28**

Overview 28

In-Class Resources for Lesson 2 31

Activity 3: Technology For Simulations 31

Activity 4: Study Design, and Effect of Class Size on Students 38

Exercises 44

Lesson 3: Formal Hypothesis Tests (Length: ~90 minutes) **46**

Overview 46

In-Class Resources for Lesson 3 49

Activity 5: HCCC framework 49

Activity 6: Additional considerations when doing statistical inference 55

Exercises 63

Lesson 4: Confidence Intervals (Length: ~90 minutes) **70**

Overview 70

In-Class Resources for Lesson 4 73

Activity 7: Candy! 73

Activity 8: Confidence Intervals 76

Exercises 82

Lesson 5: Teaching with Simulations (Length: ~90 minutes) 90

Overview 90

In-Class Resources for Lesson 5 93

Activity 9: Using simulations in the classroom 93

Activity 10: Student CHALLENGES when using simulations 105

Exercises 112

Lesson 6: One sample Hypothesis Test for a Mean (formula-based) (Length: ~90 minutes) . 119

Overview 119

In-Class Resources for Lesson 6 123

Activity 11: Central Limit Theorem 123

Activity 12: Teacher Perceptions 129

Exercises 138

Lesson 7: Formula-Based Hypothesis Test for a Difference of Means (Length: ~90 minutes)144

Overview 144

In-Class Resources for Lesson 7 147

Activity 13: Formula-Based Hypothesis Testing for the Difference in Means 147

Activity 14: Student Conceptions and Multiple Testing 152

Exercises 155

Lesson 8: Wrap-Up and Projects! (Length: ~75 minutes) 160

Overview 160

In-Class Resources for Lesson 8 164

Activity 15: Learning Progressions for Inference 164

Activity 16: Projects! 168

Exercises 176

MODULE(S²) Overview

If you are just using this Module and not Module 1 from the Statistics modules, please find Module 1 and read its general overview. If you cannot find it, please contact one of the authors.

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The Statistics Modules are:

1) Study Design and Exploratory Data Analysis (EDA)
2) Inference (confidence intervals and hypothesis tests)
3) Association (linear regression, two-way categorical tables, and chi-squared analysis)

Equity and Social Justice Topics and Conversations

Statistics classes are a natural place to look at data from the real world. We have chosen many of our data sets to focus on equity and social justice, often in contexts that are important to the US education system. This requires some student preparation, in order to have successful courageous conversations. We strongly recommend that you read Module 1 Lesson 3 to get a sense of the preparation needed for this. If you are in the habit of recording class sessions, you should strongly consider not recording these conversations, so students feel a bit more free to speak up.

Overview of Module 2: Statistical Inference

This module's Key Concept is *Inferential statistics enables us to infer, though with uncertainty, beyond the data we have to a broader set of individuals or circumstances.*

Teachers Doing a Big Project

A meaningful task for persons who are learning statistics is to design and carry out a statistical investigation cycle to investigate a topic of personal interest that they choose. We have included information about doing such a project in Module 2 Activity 16, but instructors may choose to start the project much earlier in the semester. See the project pacing thoughts in the instructor material at the start of Lesson 8. If you start the project process early, students might not yet know all the statistical analysis techniques they'll need eventually, but they will probably learn about them in time to do their formal proposals.

Below, we explain some of the decisions we have made in writing these materials.

A Simulation-Based Approach

The Common Core State Standards specify that margins of error will be developed using simulations rather than the use of formulas like SD/sqrt(n). There is a movement in the statistics-education community to use simulations as the initial basis for inference (hypothesis tests and confidence intervals) in "Stat 101" types of courses. To quote Case and Jacobbe (2018), "Developers of curricula that employ simulations as the primary means of teaching inference have published evaluations to suggest students in simulation-based courses compare favorably to students in theory-based courses (e.g., Garfield, delMas, & Zieffler, 2012; Tintle, Topliff, Vanderstoep, Holmes, & Swanson, 2012; Tintle, VanderStoep, Holmes, Quisenberry, & Swanson, 2011)" For more, read the Simulation-Based Inference (SBI) blog, https://www.causeweb.org/sbi/

Our materials start with a simulation-based approach in Lessons 1-3 (focused on hypothesis testing (HT)) and 4 (focused on confidence intervals (CI)), and then introduce formula-driven HT in Lessons 6-7 and wrap things up in Lesson 8. We skip formula-based CI entirely. You might need to re-assure any teachers in your class who might have had formula-based statistics that this simulation-based work is still important—it has some advantages when compared with the common formulas in Introductory Statistics. Not the least of these is that simulation-based methods are what the Common Core calls for! On the other hand, the stats community's initial optimism about simple simulation-based CIs (that they don't require distributional assumptions when sample sizes are small) has turned out to not be correct (Hesterberg 2014). Still, they are a more intuitive for students of Introductory Statistics than formula-based CIs, at least as an introduction. Care in helping teachers understand the value and relationship between simulation-based inference and formula-based inference is needed. Attention to differences between population distribution, sample distribution, and sampling distribution from the beginning can assist with this.

Our materials are written for an in-person class where the instructor can distribute index cards that teachers can then shuffle and move around. In an all-online class that isn't possible, so you should plan ahead: after doing Activity 1 together, have students work through the first part of Activity 2 on their own (up to the start of Question 2-c), including writing out values on index cards and doing the shuffling and computation of the mean from their own shuffle. This probably means it's good to end one class period with Activity 1 and then start the next class period with Activity 2 Question 2-c.

Nuanced Approach to "Statistical Significance"

A recent editorial by the executive director of the ASA and others (Wasserstein et al. 2019) argues that the phrase "statistical significance" is hopelessly misleading to people who have not had statistics (and many who have!) and the phrase should be abandoned:

> …it is time to stop using the term "statistically significant" entirely. Nor should variants such as "significantly different," "$p < 0.05$," and "nonsignificant" survive, whether expressed in words, by asterisks in a table, or in some other way.

Furthermore, the editorial discourages us from using a cutoff for p-values like 0.05 to decide between saying "reject the null hypothesis" vs "fail to reject the null hypothesis":

> …using bright-line rules for justifying scientific claims or conclusions can lead to erroneous beliefs and poor decision making.... A label of statistical significance adds nothing to what is already conveyed by the value of p; in fact, this dichotomization of p-values makes matters worse.

Some of the primary motivations are:

a) A statement like "reject the null hypothesis and conclude that the means of the two distributions are different" takes a situation with some uncertainty (perhaps H0 is true and we just got unlucky with the sample) and makes it sound like a certain conclusion.
b) The scientific literature is distorted by long-standing practices of only publishing studies that show "statistical significance"--the scientific community should be able to "see" studies that did not find evidence of effects, for many reasons.
c) The phrase "statistically significant" is too easily confused with practical significance.
d) Regarding 0.05 specifically: a p-value just below 0.05 isn't actually very strong evidence. The ASA's statement on p-values says "a p-value near 0.05 taken by itself offers only weak evidence against the null hypothesis". Fisher originally meant it to indicate that something was perhaps worth further thought, rather than being conclusive. At the USCOTS 2019 conference, Wasserstein said "It's more like a right-swipe on Tinder." It's

an expression of interest, not commitment. Christine Franklin says it's "sufficient evidence to question the plausibility" of H0, rather than making H0 "implausible".

e) When a binary decision must be made, it's often best if the consequences (such as costs) of false positives, true positives, false negatives, and true negatives should be considered.

Overall, we could say that the push is to move away from asking "is there a difference" and toward "how big is the difference".

We have decided in writing these materials to still teach the method (reject H_0 if p<0.05) that is taught in many Stat 101 courses (and is required in AP Statistics), since a vast amount of existing literature uses it, and due to inertia, other classes will continue teaching it for years. We are avoiding telling them to say that such a result is statistically significant; instead, we say that traditionally, people have called it "statistically significant" (using quotes to show that we, the authors, don't advocate using that phrase to describe a study or result). We do offer some advice to those using these materials about the current movement away from yes/no hypothesis test conclusions, and toward more thinking about effect sizes and confidence intervals.

Avoiding the term "Standard Error"

We have decided to mostly avoid the term "standard error", since it is often confusing for students. You may choose to define it for them if you wish. We do refer to it in one or two places, since it is shown in the randomization dotplot or bootstrap dotplot in Lock5stat.com/StatKey applets. Usually, we use the terms SD(xbar) or SD(xbar1-xbar2) instead. We also refer to SE in Module 3 Lesson 5 when discussing uncertainty in the slope of a trendline.

De-Emphasizing "Proportions" CI and HT methods

Stat 101-type courses typically include material on doing HT and CI for proportions as well as HT and CI for means. Indeed, they often use inference on proportions as a stepping-stone to inference on means, since we can avoid *t* distributions when doing proportions. We have decided to focus just on means in the in-class material, since it is possible to apply the same basic simulation methods to proportions, but using binary/indicator variables (using the numeric values 0 and 1). We leave this idea of using binary values in a test of means to homework problem, Exercise 8-2, in the 1-sample confidence-interval setting. Exercise 12-3 introduces a formula-based approach for a 1-sample test for proportion. The conditions for the methods to be valid (usually written as np>=10 and n(1-p)>=10) are perhaps a bit different than the conditions for methods involving means, but we are aiming first for overall ideas, leaving technical conditions as a lower priority. This is because the Common Core does not even require formal methods, and mentions inference only for means, not proportions.

Skipping Student's *t* distribution

We have had to make tough calls on prioritizing material. We have decided to only use the Normal distribution, and not Student's *t*, for the following reasons:

- The CCSS includes the Normal but not the *t*.
- While AP Statistics does include the *t*, these materials are not meant to be enough to prepare people to teach AP Statistics.
- The "Statistical Education of Teachers" document calls for pre-service teachers to take 3 statistics courses, and these materials are only meant to cover the first of those 3, plus some of the 2nd. They can see the *t* distribution in a later course, ideally.
- The computations to get a *t* score and a *z* score are the basically same, so establishing the concept of a *z* score leads into later learning to use the *t* distribution.
- Cobb (2007) leads us to think of the *t* distribution as an adjustment to an approximation anyway (a Ptolemaic "epicycle"), distracting from the central line of statistical reasoning appropriate for an intro-stats course.

You might still want to tell your students that there is something called a "*t*-test" that they might read about in academic articles, and that they can think of it as almost the same as what we are learning.

Our initial goal was to only use data sets with n>120 or so, so that the sample standard deviation would be a very good estimate of the population standard deviation, and we could easily justify using z instead of t. However, there are many interesting and important data sets that are not that large. We decided to use them even though the sample sizes were smaller, favoring student motivation and important contexts over technical precision.

With the recent efforts within the ASA to move away from letting everything depend on a "reject/fail to reject" decision (and avoiding the term "statistically significant"), perhaps the perceived importance of "precise" *p*-values and the *z* versus *t* distinction at the Intro Stat level will lessen as well. See Wasserstein, Schirm, Lazar (2019), https://www.tandfonline.com/doi/full/10.1080/00031305.2019.1583913

Emphasizing 1-tailed tests rather than 2-tailed

For the most part, we concentrate on 1-tailed alternative hypotheses rather than 2-tailed, even though the usual advice in an Introductory Statistics class is to default to two-tailed. We decided this partly because it makes the intuition easier, and partly because in the push by ASA leaders to move away from reject/fail-to-reject testing, it has been suggested that a null hypothesis value of the minimum interesting (or, clinically relevant) effect size makes more sense than a null hypothesis value of zero. Such a test would then naturally have a 1-tailed alternate hypothesis. A potential downside is that 2-tailed tests map more naturally to confidence intervals, but that is still in the reject/fail-to-reject mindset, and is discouraged by the push to avoid making artificially binary decisions based on a HT or CI.

Link Between z-Score and Relative Difference of Means

In Module 1 Lesson 8 Activity 16 we explored a statistic for comparing two groups that is basically "Cohen's d", (xbar1 – xbar2)/SD(group 1), based on the 7th-grade CCSS standard "CCSS.MATH.CONTENT.7.SP.B.3: Informally assess the degree of visual overlap of two numerical data distributions with similar variabilities, measuring the difference between the centers by expressing it as a multiple of a measure of variability." This can be seen as related to statistical inference in two somewhat opposite ways:

a) "d" is a version of the z-score for a 2-group difference-of-means hypothesis test but it does not include the sample sizes at all. In this sense it's a stepping stone to the test statistic for a 2-group difference-of-means hypothesis test.
b) "d" is a way of expressing the "effect size" of whatever is different between the groups (such as a medical treatment vs. a control group), in a way that is not prone to issues of statistical significance vs. practical significance because it does not include the sample size. In this sense it's worthy of use on its own, without being a stepping stone to hypothesis tests, especially considering the troubles with the traditional reject/fail-to-reject HT discussed above.

The concept does not appear in the CCSS at the high school level. We have worked it and its links to HT into this module in the Exercises for Lesson 3 Activity 5, and Lesson 7 Activity 13; you can search for 'Cohen' inside this instructor's version file to find the comments for those places.

Pre-requisites from Module 1

Module 2's lessons 6 and 7 require using the Standard Deviation and z-scores, which are covered near the end of Module 1. Lesson 6 also uses CODAP, so it would be handy if students had already seen it; Module 1 has an intro-to-CODAP assignment to be done outside of class that you could assign here if they haven't used CODAP.

The project assignment in Lesson 8 includes reflections on the Statistical Investigation Cycle and the Statistical Habits of Mind that are in the first lesson of Module 1.

Note on Technology. In Module 2, we mostly use StatKey (http://Lock5stat.com/statkey), a web site with a collection of simulation-based statistical tools (also called randomization-based). We are grateful to the Lock family for their permission to use it in this text, even though they have their own very good Intro-Stat textbook, "Unlocking the Power of Statistics," that goes with their website.

It would have been nice if we could use CODAP, as we did in Module 1, but it is not currently able to do the kind of simulations that we need to do. It does have a "sampler" feature but it can't do 2-group situations well, and even with 1 sample, it's hard to compute the upper and lower ends of a bootstrap confidence interval, for example. We do use CODAP for a few incidental things in Module 2, and use it for a 1-sample simulation in Lesson 6; students are also free to use it for anything they like.

Students might want to take different subsets of the Tennessee STAR data for their projects. Examples might include doing separate analyses by race, gender, or free-lunch status of the students, or by teacher characteristics. To do this subsetting by row, you can load a CSV file into Excel or Google Sheets, then use the Auto-Filter feature and perhaps Pivot Tables to aggregate, then save the resulting sheet back to a CSV file. You can email one of this book's authors, andrew.ross@emich.edu , for help. Lock5stat doesn't currently have a row-filtering feature. It's also possible to do row filtering in CODAP, by graphing the column(s) of interest then selecting some dots with the mouse, then hiding unselected rows, but you would still have to export it back to a CSV file.

Note on Exercises. The first number in each exercise number shows which Activity it most relates to, but some offer opportunities to mix knowledge from various activities.

Overview Table for Module 2

Lesson	Projected Length	Content	In-Class Activities	Homework Notes	Connections and Notes
1: Simulation-Based Informal Hypothesis Test: Comparing 2 Groups (means)	90 min	Simulation-based hypothesis test for comparing means of two groups	Activity 1: Is There a Difference? Activity 2: Do The Shuffle	Exercise 1-1 is pre-work for Lesson 2 Activity 4.	Activity 1 picks up the final question asked in Activity 8 of Module 1. We use by-hand/physical-manipulative methods to conduct the simulations in this lesson, leaving repetition-via-technology to Lesson 2. Also, we use only informal/everyday vocabulary, leaving formal vocabulary to Lesson 3.
2: Random Assignment Simulations with Technology, Paired Studies, and TN STAR study	90 min	Simulation-based hypothesis test for comparing means of two groups; Paired study design	Activity 3: Technology for Simulations Activity 4: Study Design, and Effect of Class Size on Students		Activity 3 uses the same task as Activity 2, but uses the technology tool StatKey to carry out the simulation process. Activity 4's look at paired study design connects to Module 1. The mini-framework for simulation steps in Activity 4 summarizes the work done with simulation-based hypothesis tests in Activities 2&3, and it will be incorporated into a broader framework for hypothesis tests in Activity 5.

3: Formal Hypothesis Tests	90 min.	Simulation-based hypothesis test for a mean; Type I and Type II errors; Practical vs. statistical significance; framework for the logic of simulation-based inference	Activity 5: HCCC framework Activity 6: Additional considerations when doing statistical inference		Activity 5 uses the same context (TN STAR study) as Activity 4.
4: Confidence Intervals	90 min.	Meaning of confidence intervals; Simulation-based confidence interval for a mean	Activity 7: Candy! Activity 8: Confidence Intervals	Exercise 7-1 is pre-work for Lesson 5 Activity 9.	Question 7-p in Activity 7 makes a connection to different sources of error and variability learned in Module 1. Activity 8 uses the class data collection in Lesson 3 about hours of sleep per night. Activity 8 connects to the HCCC framework learned in Activity 5. In Activity 9, a video recording of a middle school class working through the Candy task done in Activity 7 will be analyzed.
5: Teaching with Simulation	90 min.	Knowledge of students' learning experiences when doing simulation-based inference	Activity 9: Using Simulations in the Classroom Activity 10: Student Challenges when Using Simulations	Exercise 9-1 is pre-work for Lesson 6 Activity 12. Exercise 10-5 is a Written Simulation of Practice.	Activity 9 connects back to the Candy task done in Activity 9 and the Election task done in Exercise 7-1. Activity 10 connects back to the Case/Jacobbe framework for the logic of simulation-

					based inference that was introduced in Lesson 3.
6: One-sample Hypothesis Test for a Mean (formula-based)	90 min.	Central Limit Theorem; One sample hypothesis test for a mean; Teacher expectations of students and families related to stereotypes	Activity 11: Central Limit Theorem Activity 12: Teacher Perceptions	Exercise 12-6 is a Video Simulation of Practice.	This lesson connects to Lesson 3 where the simulation-based hypothesis test for a mean was taught. The context for Activity 11, household incomes in the U.S., was previously used in Activity 16 of Module 1.
7: Formula-Based Hypothesis Test for a Difference of Means	90 min.	Formula-based hypothesis test for the difference in means; Common student conceptions when conducting hypothesis tests; Issues with multiple testing	Activity 13: Formula-based Hypothesis Testing for the Difference in Means Activity 14: Student Conceptions and Multiple Testing		The portion of Activity 14 about Multiple Testing is optional.
8: Wrap-Up and Projects!	75 min.	Learning progression for inference in current curriculum standards; Common student interpretations of *p*-values	Activity 15: Learning Progressions for Inference Activity 16: Projects!	Many of the exercises in the exercise set synthesize topics in Module 2.	Activity 16 describes doing an extended statistical project. It does not follow the format of other activities in these materials. This project can be started earlier in the course, and should continue through the end of the course.

Responding to Student Thinking:
What Makes for a Good Teacher Response?

- Asking questions is often better than making statements.
- It can be helpful to give an example of the student's reasoning but taken to extremes (either numerical extremes or context extremes where the answer should be clear).
- Simply telling the student the right way to think is often not as helpful as one might hope—they need to see why their reasoning isn't correct.
- High quality responses to student work:
    - Move students toward the student learning objective;
    - Draw on and are consistent with the student thinking presented and research on students' mathematical development; and
- Leave space for student's thinking (i.e., the teacher should not do all of the thinking; the student needs to be prompted to think about the concept)

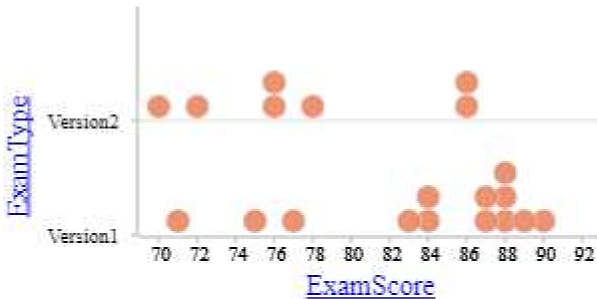
In-Class Resources for Lesson 1

UNIT OVERVIEW

This unit is about Statistical Inference: using statistics, trying to infer what is going on in the real world when all we have is a sample.

ACTIVITY 1: IS THERE A DIFFERENCE?

Suppose you are teaching a class and decide to give 2 versions of an exam to help prevent cheating. 13 students were given version 1, and 7 students were given version 2, with the versions being assigned randomly. Version 1 is the exam you usually give, and Version 2 is the same questions but in a different order. You're mostly concerned that changing the order might have made the exam harder than you are used to. The resulting exam scores are shown below. This same scenario was at the very end of Module 1, but now we consider it more deeply.



http://bit.ly/CODAPtwoexams00 [2]

Version 2 scores: 76 76 86 86 70 78 72

Version 1 scores: 71 77 88 88 75 87 84 87 83 88 84 90 89

Question 1-a Do you see much evidence of a difference in the exam scores for the two versions? Explain your thoughts.

Question 1-b Suppose that some of your fun-loving students were in the school's mock-trial team, and they decided to get some extra practice by "taking you to court" for doing two versions of the exam.

i. What would they be trying to prove, from a quantitative point of view?

ii. What would your argument be, from a quantitative point of view?

[2] Or https://codap.concord.org/releases/latest/static/dg/en/cert/index.html#shared=77201

Question 1-c If we wanted to show that two data sets had different distributions, could we do that by showing that their numerical summary measures are different? Or would that not be sufficient? Explain.

Question 1-d What is the best way to summarize the test scores of each test version? Explain, then discuss as a class.

Question 1-e Using your class's answer to the previous question, how much of a difference is there between the two groups of data shown above?

Question 1-f If you as the teacher could repeat this with new random assignment of your students to exam versions, and re-summarize the results, do you think it would still show the same amount of difference between the two versions? Explain.

Question 1-g Based on what we've seen so far, and your intuition, do you think there's a difference between the mean exam score of the two versions generally, not just in our sampled data? In other words, do you think Version 2 is inherently harder, resulting in the mean exam score of Version 2 being lower in general than Version 1? Explain.

Two Big Questions

Before we go any farther, consider two Big Questions that we will be working to answer in this Module. Write down your thoughts on these Big Questions:

Big Question 1: Let's suppose that we ran a clinical trial to compare the effectiveness of a new medicine at treating a condition vs. the established medicine for treating this condition. How can we decide whether one is better than the other, if graphing the data that we collected doesn't show an obvious difference?

Big Question 2: Let's suppose we studied the standardized test scores of large classes vs. small classes in a particular grade across a sample of some school districts in a state. We could easily compute the difference between the average score for students in small classes and the average score for students in large classes in the sample school districts, but how would we get a margin of error for it, to provide an estimate of the average difference in standardized test scores for all districts in the state?

ACTIVITY 2: DO THE SHUFFLE

Think back to the situation where you gave 2 versions of a test, from Activity 1, above.

Question 2-a If the two versions of the test really made no difference, what would the difference of the mean scores be, in the long run (if we could repeat it with many different groups of students at the same level)?

Question 2-b What kind(s) of value(s) would we expect for the difference of the means "in the short run" (for just one class, with 20 students), again if the two versions of the test really made no difference in the long run?

Here is a method that we can use to judge the evidence about whether the two groups (test versions) make any difference overall. We'll temporarily assume that the two versions of the test have no effect on scores, so it won't matter if we mix up which version each student got, then simulate what might happen. Each person in our class today should…

1) Start with 20 index cards since there are 20 data values, then write one data value on each index card.
2) Mix all those cards together in a random order/shuffle them.
3) Deal out 7 cards into one pile and 13 into another pile
4) Take the mean of the values in each pile separately, then subtract those two means. Use (mean of 13-card pile) minus (mean of 7-card pile).
5) Write that difference-of-means on a post-it (please write it big so it's easily visible) and contribute it to the class dot-plot.
6) Repeat from step 2, if time remains.

Question 2-c Record the class's dot-plot here, for your later reference:

```
 +----+----+----+----+----+----+----+----+----+----+
-10   -8   -6   -4   -2    0    2    4    6    8    10
```

Question 2-d Why do we mix the cards together into one pile initially (step 2)?

Question 2-e Why do we deal out 7 cards into one pile and 13 into another (step 3)?

Question 2-f What does each post-it note represent?

Question 2-g If the two exams were equally hard, does it seem plausible that we would see the difference in means as big as what we saw in our original data? Explain your thinking.

Question 2-h Why did we do the simulation?

Question 2-i What did the simulation tell you?

Question 2-j How does the simulation relate to the original context of the situation?

Question 2-k Does our simulation assume that whether a person is assigned to version 1 or version 2 of the test has no effect on anyone else's scores? Explain.

Question 2-l What could we do to improve on our index-card simulation?

Exercises

Exercise 1-1 Due before next class: In Lesson 2 Activity 4, answer Question 4-d (after the heading "Effect of Class Size on Students", then read the Tennessee STAR study description that follows it (all the way up to Question 4-e), then answer Question 4-e.

Exercise 1-2 (optional) The scenario we used in Activity 1 was about having two versions of an exam in a class. Consider whether the teacher should publicize that there is more than one version (perhaps by using different color paper for the versions):

- ***(a)*** What would the effects (if any) be on the class's behavior if you told them that there is more than one version?
- ***(b)*** What would the effects (if any) be on the statistical study of whether the versions were different?
- ***(c)*** Overall, is it wise to tell the students that there is more than one version?

Exercise 1-3 Suppose that a similar 2-exam-versions situation had produced data like this: Version 1, scores 56,58,53,48,55. Version 2, scores 88,81,84,92,86,91. Clearly there's a big difference—this question isn't about analyzing the numbers. The question is, can you conclude that the two different versions <u>caused</u> the difference? Explain.

Exercise 1-4 Again, for the similar situation in Exercise 1-3, can you conclude that the big difference between the two exam versions' scores will apply to a class in the same subject taught by your colleague in the next time block? Explain.

Exercise 1-5 (optional) Explain how the concept of "signal and noise" applies to the 2-exam-versions situation. What is the "signal"? What is the "noise"?

Exercise 2-1 We called our index-card-shuffling work a "simulation". How is it the same as, or different than, simulations in other contexts:

- ***(a)*** trainee airplane pilots using a "flight simulator", or trainee doctors and nurses using a plastic body that is a "patient simulator", and
- ***(b)*** an engineer running a simulation of breaking strength of a bridge using a computer, or a physics teacher using a computer to simulate the trajectory of a water balloon launched from a student's catapult?

Exercise 2-2 Suppose that once the two-version exam was graded, your students asked about how the two versions' scores compared to each other. You want to show them the scores, and the analysis/simulation. However, you don't have time to actually have them do the simulation. They haven't had any high-school-level statistics yet. Write down a paragraph or two of what you would say, possibly including graphs. Don't worry about giving "spoilers" for future lessons they might have—go ahead and spoil your students!

Exercise 2-3 After your explanation in the exercise above, one of your students said, "But different numbers of people took the two versions, so we can't do anything to compare them." How would you respond, as the teacher? Remember that asking questions which probe and push their thinking is often a better approach than simply making statements.

Exercise 2-4 Here are some Common Core State Standards from 7th grade:

7.SP.2: Use data from a random sample to draw inferences about a population with an unknown characteristic of interest. Generate multiple samples (or simulated samples) of the same size to gauge the variation in estimates or predictions.

7.SP.4: Use measures of center and measures of variability for numerical data from random samples to draw informal comparative inferences about two populations.

(a) Which one of them is more applicable to the two-exam-versions situation? Explain why.

(b) In what way(s) does even that standard that you chose not entirely apply to the two-exam-versions situation?

(c) Regardless of that mismatch in part (b), do the best you can to apply that standard to the two-exam-versions situation, by doing appropriate computations and writing a concluding sentence or two. Hint: this does not require a simulation.

(d) Does your analysis in part (c) align fairly well with the results of your class simulation, or are they somewhat different? Quite different? If they are different, speculate on why.

Exercise 2-5 Here are some Common Core State Standards for high school:

HSS.IC.B.4 : Use data from a sample survey to estimate a population mean or proportion; develop a margin of error through the use of simulation models for random sampling.

HSS.IC.B.5: Use data from a randomized experiment to compare two treatments; use simulations to decide if differences between parameters are significant.

(a) Which of these is more related to the two-exam-versions situation? Explain.

(b) How do these 2 standards relate to Big Question 1 and Big Question 2?

Exercise 2-6 Do the simulation results give you a more accurate or more precise estimate of the difference of the two version's means? Explain.

Exercise 2-7 Did the two-exams situation involve random assignment, random selection, both, or neither? Did our simulation involve random assignment, random selection, both, or neither?

Exercise 2-8 Suppose that you asked some students whether they had gotten a good night's sleep before for a big statewide standardized test, and what score they got. You're wondering, of course, if good sleep is associated with a higher mean score than not getting good sleep. You got the following data set:

Got good sleep: 840 860 960 980

Did not: 770 930 850

Design a simulation to see whether a good sleep made a difference in the scores. Do at least 25 repetitions (feel free to recruit friends and family to help do all the repetitive work of the simulation). Describe your simulation, and the results you got.

Mathematics of Doing, Understanding, Learning, and Educating for Secondary Schools

MODULE(S²):

Statistics for Secondary Mathematics Teaching

Module 3: Statistical Association

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Contents

MODULE(S²) Overview 5

Overview of Module 3: Statistical Association 5

Overview of Module 3 Part I: Quantitative Association 6

Lesson 1: Introduction to Quantitative Association (Length: ~90 minutes) 8

Overview 8

In-Class Resources for Lesson 1 14

Module Overview (Part I) 14

Activity 1: Introduction 15

Exercises 27

Lesson 2: Informal Fit Methods, and Correlation Coefficient (Length: ~90 minutes) 33

Overview 33

In-Class Resources for Lesson 2 35

Activity 2: Signal and Noise 35

Activity 3: Correlation Coefficient 37

Exercises 43

Lesson 3: Formal Trendlines and Interpreting Slope Without Causation (Length: ~90 minutes) 51

Overview 51

In-Class Resources for Lesson 3 55

Activity 4: Formal Line Of Best Fit, Residual Plots 55

Activity 5: Lurking Variables, Interpreting Slope, and Extrapolation 63

Exercises 70

Lesson 4: Coefficient of Determination, and Nonlinear Modeling (Length: ~90 minutes) 80

Overview 80

In-Class Resources for Lesson 4 84

Activity 6: $R2$, and Software Output (optional) 84

Activity 7: Model Selection 90

Exercises.. 100

Lesson 5: Transformations and Inference (optional) (Length: ~90 minutes)........................... 104

Overview 104

In-Class Resources for Lesson 5 106

Activity 8: Transformations (optional) 106

Activity 9: Inference (optional) 113

Exercises.. 123

Overview of Module 3 Part II: Categorical Association 126

Lesson 6: Introduction to Categorical Association (Length: ~90 minutes)............................ 129

Overview 129

In-Class Resources for Lesson 6 135

Unit Overview (Part II)... 135

Activity 10: Introduction to Categorical Association 136

Activity 11: Joint and Marginal Relative Frequencies, and Tables with No Association 144

Exercises.. 151

Lesson 7: Interpreting Two-Way Tables and Graphs (Length: ~90 minutes)......................... 156

Overview 156

In-Class Resources for Lesson 7 158

Activity 12: Analyzing Student Thinking 158

Activity 13: Other Graphs for Two-Way Tables 166

Exercises.. 171

Lesson 8: Association, Causation, and Evidence (optional) (Length: ~90 minutes)................ 176

Overview 176

In-Class Resources for Lesson 8 180

Activity 14: Association vs Causation (optional).. 180

Activity 15: $\chi 2$ Analysis Thoughts (optional) 185

Exercises.. 190

Lesson 9: Simulation-Based Hypothesis Testing and Curriculum Differences (optional) (Length: ~90 minutes) **193**

Overview 193

In-Class Resources for Lesson 9 196

Activity 16: Simulation for Inference (optional) 196

Activity 17: Curriculum Standards: CCSS-M through AP Statistics (optional) 204

Exercises 207

MODULE(S^2) Overview

If you are just using this Module and not Module 1 from the Statistics modules, please find Module 1 and read its general overview. If you cannot find it, please contact one of the authors.

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The Statistics Modules are:

1) Study Design and Exploratory Data Analysis (EDA)
2) Inference (confidence intervals and hypothesis tests)
3) Association (linear regression, two-way categorical tables, and chi-squared analysis)

Equity and Social Justice Topics and Conversations

Statistics classes are a natural place to look at data from the real world. We have chosen many of our data sets to focus on equity and social justice, often in contexts that are important to the US education system. This requires some student preparation, in order to have successful courageous conversations. We strongly recommend that you read Module 1 Lesson 3 to get a sense of the preparation needed for this. If you are in the habit of recording class sessions, you should strongly consider not recording these conversations, so students feel a bit more free to speak up.

Overview of Module 3: Statistical Association

This module's Key Concept is *Association means that information about one variable changes our idea about what happened with the other variable, but does not establish a causation relationship.* Developing a deep understanding of this key concept is a goal of each lesson. Notice that the key concept has two parts: what association means and detecting it (e.g., through making and interpreting a scatterplot and a model for the data (like a regression line)), and the inability to make causation conclusions based just on association ("correlation is not causation").

Both Part I and Part II have an Activity regarding association-is-not-causation. If you feel that the class understood it well enough in Part I, it is reasonable to skip the similar activity in Part II.

Unlike Modules 1 and 2, we see some of the lessons in Module 3 as optional, so we have marked them as such. In particular, the lessons regarding statistical inference (hypothesis tests or confidence intervals) for trendlines and for categorical association are optional.

Overview of Module 3 Part I: Quantitative Association

This is the start of Part I of Module 3, which is focused on association of quantitative variables, using scatterplots and trendlines. Part I is not a pre-requisite for Part II.

Note that for modeling data with a linear form, we start with the informal line of best fit in Lesson 1 before moving to the formal line of best fit in Lesson 3. This is done to match the progression in the CCSS-M standards. Your teachers are likely familiar with the formal line of best fit (least-squares regression line), but may need help unpacking their understanding of that and taking a step 'back' to think about what a best fit line really is. Their understanding of lines from mathematics may cause some cognitive dissonance with the use of linear models in statistics.

Note on Technology. Module 3 mostly uses CODAP, but when we do inference (confidence intervals and hypothesis tests), we use Lock5stat.com/StatKey ; both are free online tools. Teachers in your class will often need a laptop and internet access in class and on the homework.

Note on Exercises. The first digit in each exercise number shows which Activity it most relates to, but some offer opportunities to mix knowledge from various activities.

Lesson	Projected Length	Content	In-Class Activities	Homework Notes	Connections and Notes
1: Introduction to Quantitative Association	90 min	Meaning of statistical association; Scatterplots; Informal line of best fit	Module overview (Part I) Activity 1: Introduction	Exercise 1-1 is pre-work for Lesson 2.	The meaning of association stays consistent from Part I to Part II of this module. Activity 1 builds on the work done in Modules 1 and 2 regarding responding to student thinking.
2: Informal Fit Methods, and Correlation Coefficient	90 min	Conceptualizing linear regression as the search for a signal in noisy data; Correlation coefficient	Activity 2: Signal and Noise Activity 3: Correlation Coefficient	Exercise 2-1 is a Video Simulation of Practice.	Activity 2 connects to Module 1 Activity 16 where the conception of statistics about signal and noise was introduced. Activity 3 references z-scores, which were

					introduced in Module 1 Activity 16.
3: Formal Trendlines, and Interpreting Slope without Causation	90 min	Least-Squares Regression Line; Residual plots; Lurking variables; Interpolation and extrapolation	Activity 4: Formal Line of Best Fit, Residual Plots Activity 5: Lurking Variables, Interpreting Slope, and Extrapolation		Activity 4's introduction to the formal line of best fit (the least-squares regression line) connects to the discussion of informal line of best fit in Activity 1, Exercise 1-1, and Exercise 2-1.
4: Coefficient of Determination, and Nonlinear Modeling (ACTIVITY 6 OPTIONAL)	90 min	R^2 statistic; Reading software output for regression; Selection of a model for data that follows a non-linear form	Activity 6 (OPTIONAL): R^2 and Software Output Activity 7: Model Selection	Exercise 6-1 is pre-work for Lesson 6 Activity 10.	Activity 8 will explain the mechanics behind the non-linear models studied in Activity 7.
5: Transformations and Inference (OPTIONAL)	90 min	Fitting exponential, power, and logarithmic functions through linear transformations; Inference procedures for the slope of the regression line; Prediction Interval	Activity 8 (OPTIONAL): Transformations Activity 9 (OPTIONAL): Inference	Exercise 8-1 is the same as exercise 6-1. It is pre-work for Lesson 6 Activity 10.	Activity 8 explains the mechanics behind the non-linear models studied in Activity 7. Activity 9 connects to Module 2's focus on statistical inference and use of StatKey for simulation-based inference procedures, and signal+noise.

Responding to Student Thinking:

What Makes for a Good Teacher Response?

- Asking questions is often better than making statements.
- It can be helpful to give an example of the student's reasoning but taken to extremes (either numerical extremes or context extremes where the answer should be clear).
- Simply telling the student the right way to think is often not as helpful as one might hope—they need to see why their reasoning isn't correct.
- High quality responses to student work:
    - Move students toward the student learning objective;
    - Draw on and are consistent with the student thinking presented and research on students' mathematical development; and
    - Leave space for student's thinking (i.e., the teacher should not do all of the thinking; the student needs to be prompted to think about the concept).

In-Class Resources for Lesson 1

MODULE OVERVIEW (PART I)

This module is about "statistical association". For now, we can say that:

Two variables are **associated** if knowing one of them gives you information about how the other one behaves.

For example, people's height is associated with their foot length, so if you know someone's height you have meaningful information that informs your estimation of their foot length. Later, we will discuss the distinction between variables being associated and one variable causing changes in another one. For now, keep in mind that associations can happen because of outside variables that influence the variables we are studying, so we cannot make causative claims in many cases.

Part I of this module is about **numeric** variables, also sometimes called **quantitative** variables.:

Quantitative variables are variables with numeric values that we can do useful arithmetic on.

These are often the result of counting or measuring things. Common examples include anything in years, days, seconds, meters, kilograms, liters, degrees Celsius, or dollars. Percents are also included. What is not included in Part 1 are categorical variables regarding things like racial categories, political party, states (Oregon, Florida, etc.), and major (Math, History, etc.)—we will discuss how to deal with variables like those in Part 2.

ACTIVITY 1: INTRODUCTION

We will start this module with an activity you can do with students. You might have seen bridge-building competitions where students spend weeks gluing together complicated truss structures out of wood, and each team's bridge is tested with weights until it breaks. To save time, we will use a much simpler bridge: simple strands of uncooked spaghetti with no glue. Students measure how much weight the bridge can hold by adding coins (quarters) to a cup suspended under the bridge. The "strength" column shows how many quarters it took to break the bridge. This video shows part of a spaghetti bridge investigation so you can get an idea of what this entails: https://www.youtube.com/watch?v=jaW6mEV-8Sg ; they used marbles for weights but we will imagine using coins, US quarters in particular. You might design the study and collect some data in class, or your class might just use this data set, which is also available in CODAP at http://bit.ly/codapbridge00 [1].

Trial#	Type	Bridge Span(cm)	#sticks of spaghetti	strength
1	Wi$eValue Spaghetti	10	1	3
2	Wi$eValue Spaghetti	10	2	24
3	Wi$eValue Spaghetti	10	3	62
4	Wi$eValue Spaghetti	10	4	47
5	Wi$eValue Spaghetti	10	5	72
6	Wi$eValue Spaghetti	10	7	105
7	Wi$eValue Spaghetti	10	7	93
8	Wi$eValue Spaghetti	10	8	102
9	Wi$eValue Spaghetti	10	9	97
10	Wi$eValue Spaghetti	10	10	97
11	Wi$eValue Spaghetti	10	11	122
12	Wi$eValue Spaghetti	10	11	136

Question 1-a Put yourself in the frame of mind of an 8th grade student. If you wanted to predict the strength of a bridge with 13 sticks of spaghetti based on the data above or similar data, how would you do it? What about at 6 sticks of spaghetti, where no data was collected?

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[1] or https://codap.concord.org/releases/latest/static/dg/en/cert/index.html#shared=36380

Question 1-b Suppose one of your students is using CODAP's movable line feature to place the informal line of best fit. The equation of the informal line of best fit they find is $strength = 10.7(sticks) + 10$. How could you judge if the line was reasonable for the data?

We will continue using this data in later activities. Next, we will look at what happened in a classroom when a teacher did a lesson that also involved using a data set to make predictions.

Introducing Students to the Topic of Statistical Association

You will be watching a series of videos that depict what happened when Ms. Tuck, a secondary mathematics teacher, began an activity called "Previous Travelers". The videos are based on documentation of a real class session; Ms. Tuck is a pseudonym.

The videos are stored in a YouTube playlist here: http://bit.ly/AssociationPL [2]. Watch Video 1. Solve the presented "Previous Travelers" task yourself, determining the best way to use the data to make an accurate prediction regarding how many pounds of beans a family of 20 will use. The data from the "Previous Travelers" activity is presented below for your reference.

Number of people	Pounds of beans used
5	61
8	95
6	56
7	75
11	125
10	135
5	80
7	100
10	103
6	75
8	100
7	105
9	125
12	150
10	125

Your solution:

[2] Or https://www.youtube.com/playlist?list=PLLT-D-NcgUv6I3kQJuRplDjo1B4ZnrfLv

Question 1-c Describe at least 3 different ways you anticipate that students will solve the task and their resulting predictions.

Question 1-d The following table presents an adaption of ***Considerations for design and implementation of statistical tasks*** (Tran & Lee, 2015) that can be used to consider the components of a statistical task. Complete the rightmost column with respect to the "Previous Travelers" task.

Guidelines for Statistical Tasks

Component of a Statistics Task	**Questions to Consider**	**Responses based on "Previous Travelers" task**
1. Learning Goal	What learning goals does the task aim for students to accomplish? Does the task focus on answering questions that are statistical, versus mathematical? *Module 1 Activity 2 discusses the difference.* e.g., If the task asks students to use computations or graphs, are these used in support of statistically analyzing the data to inform making a decision OR are they only used to simply practice an algorithm or create a graph?	
2. Data	Does the task call for the use of data (either to collect or use already collected data to answer)? Does the data appear to come from a real source?	
3. Context	Is context a salient part when solving the problem? Is the context likely to be of interest to the students engaging in the task?	
4. Investigation Cycle	Does the task address only one phase of a statistical investigation, some phases, or all phases of the cycle? Consider the appropriate phases below as applicable to the intent of the task:	

Components 5-8 will address each phase of the statistical investigation cycle.		
5. Pose a Question	Is the question already posed (by teachers, or curriculum developers) or do students have opportunities to pose statistical questions based on their interest? What type of variability does the task attend to?	
6. Collect or Consider Data	Does the task offer opportunities for students to plan to collect data: sampling, sample size, attribute, and measurement? Do students conduct the data collection? OR If the task uses pre-collected data, is a context provided so that students are aware of the measurement issues and how data were collected?	
7. Analyze Data	Does the task offer opportunities for students to decide on the types of graphical representation and/or numerical statistics to use when analyzing data? Does the task afford students to use alternative representations to shed light on the trends of data?	
8. Interpret Results	Does the task ask students to incorporate context when making claims/inferences about the data? Does the task expect students' claims to account for uncertainty?	

Question 1-e Watch Video 2. As the teacher, how would you respond to this student's suggestion to encourage statistical thinking? Be specific when describing this and future responses to students, writing exactly what you would say and/or draw in your response.

Question 1-f Watch Video 3. Ms. Tuck chose to point out there are multiple families with 10 members in them. How effective was that response? Explain. Base your assessment on the student's response to the teacher.

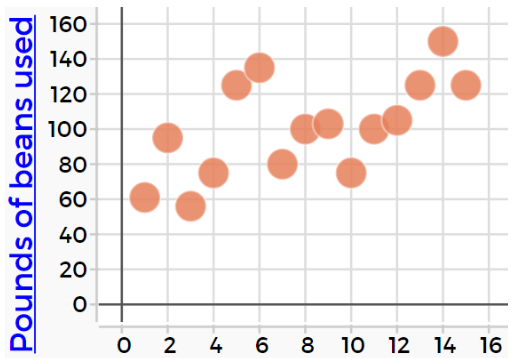
Question 1-g How would you respond to the student named Barry?

Question 1-h Watch Video 4. Answer the prompt presented at the end of the video: If you were the teacher, what would you do in this situation to make the transition from the students' scale-up approach to the line of best fit approach?

Question 1-i Watch Video 5. How does Ms. Tuck's transition compare to yours as described in the previous question? How effective do you think Ms. Tuck's transition is?

Question 1-j Watch Video 6, which brings out struggles that students are having with aspects of the posed task. Describe each of these 'stumbling blocks', potential sources or reasons for them, and responses you would give to these students to help them overcome these blocks and move forward in their learning.

While it didn't happen to any of Ms. Tuck's students, another common stumbling block for students involves deciding on axes for the plot. It's fairly common that students would not choose appropriate variables to put on the axes. For example, a common initial plot made by students is case-based with case identifier (e.g., name or family number) on one axis and value of one of the attributes (for example, pounds of beans used) on the other axis. Below is an example of this type of graph for a portion of the data from the Previous Travelers task. At first glance this type of graph may appear to be a scatterplot, but it's really a univariate, case-based graph.



Question 1-k Watch Video 7. The student who draws the red line to show her line of best fit starts to operationalize her meaning of the line of best as something that "follows the trend", stating that the line should be "where most dots are at". Based on the student's drawing of the line, what do you think she means by this?

Question 1-l One of the students is struggling with the idea that there can be more than one line of best fit. What would you say to this student?

Question 1-m At the end of the video a student states "The best fit line. It has to start at zero, right?". Why do you think the student thinks the line has to start at zero? Is this a valid point? How would you respond to this student?

Question 1-n Does Ms. Tuck's implementation of Problem 2 maintain the cognitive demand of the task? If so, how does the teacher do so? If not, what does the teacher do to diminish the level of cognitive demand? Make sure to consider the questions posed by Ms. Tuck.

Question 1-o How would you continue the lesson from the point that Video 7 ends?

The two data sets we have seen so far are moderately similar to each other. Next, we will look at a variety of data sets that are all suitable for use in this module, even though they can look very different.

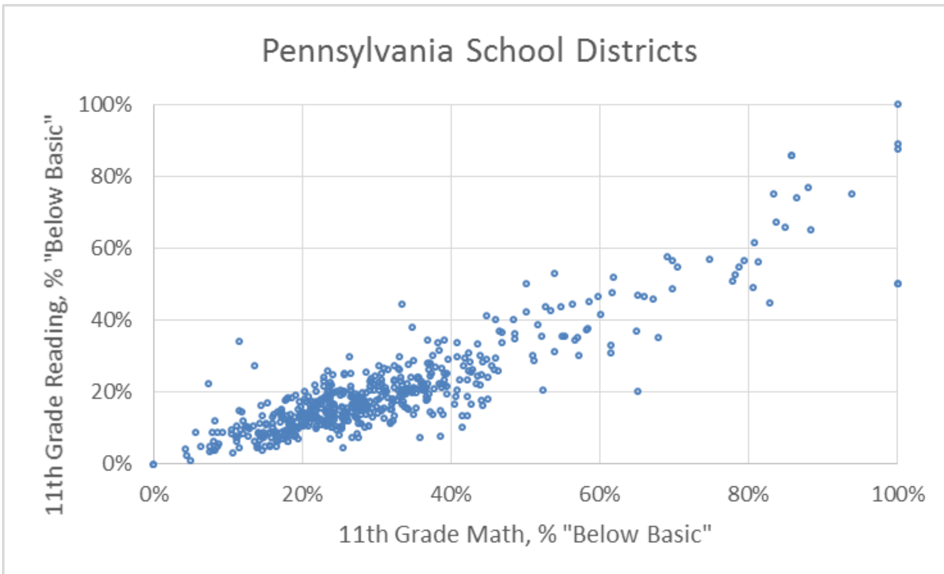
A Variety of Scatterplots

This portion of the module is about measuring the association between two quantitative variables and making predictions about one variable using the other variable. We often use scatterplots to visualize these data sets. When analyzing scatterplots, consider these features:

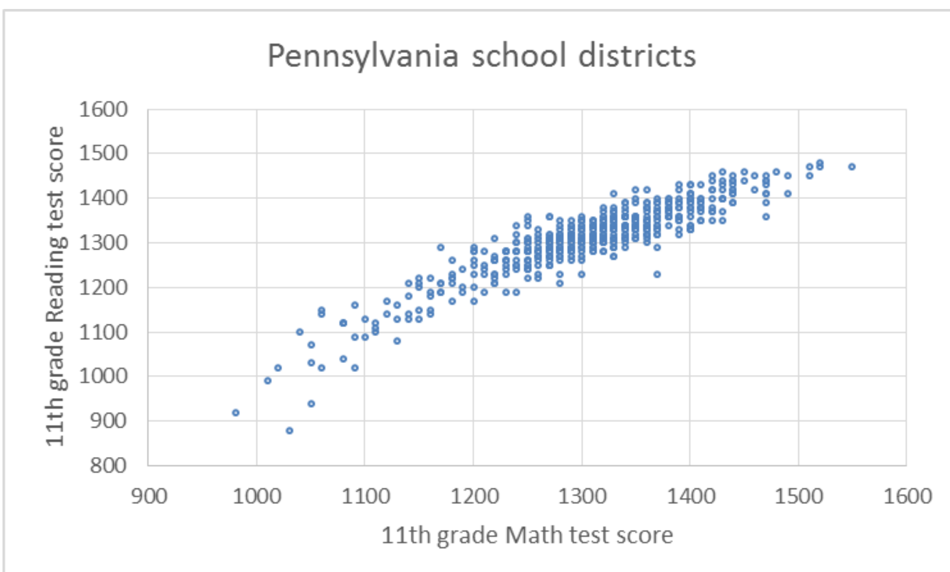
- Form: What function type (e.g., linear, quadratic, exponential) do the data follow?
- Strength: How well do the data follow that form? The closer the points in the plot follow that form, the stronger the association.
- Direction: *For data with a linear form*, the data has a positive association if the line of best fit has a positive slope and a negative association if the line of best fit has a negative slope.
- Unusual features: Are there clusters of points or unusual points that don't fit the trend of the other points in the dataset?

Studies with different designs can produce scatterplots that look different, but that doesn't mean we can't use the methods of this module on them. If students only ever see one or two types, they might not realize that fitted lines can be helpful for all of these types. Consider these scatterplots:

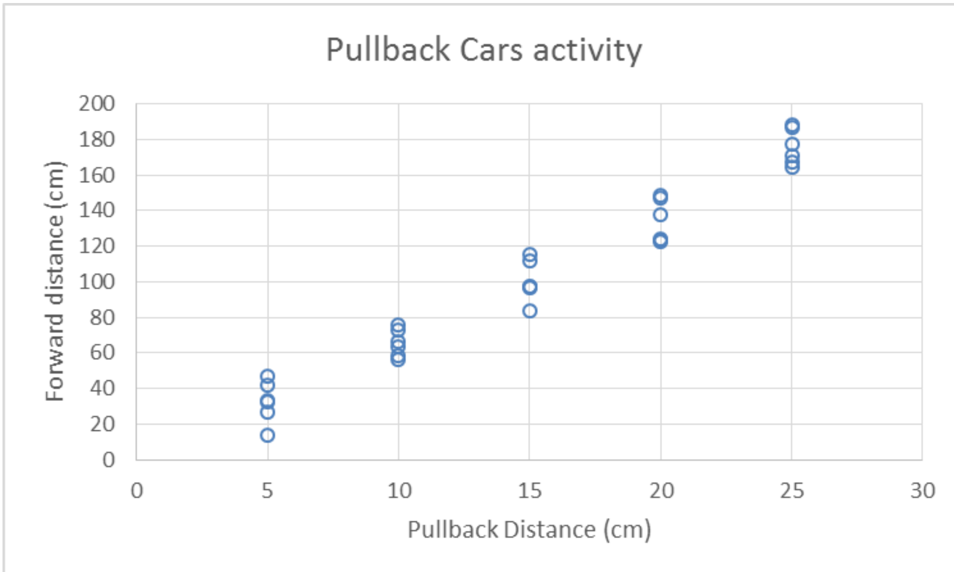
1) In 2005, the state of Pennsylvania released a spreadsheet of data about each school district, including the results of standardized tests. While there are many equity-related problems with judging students, teachers, and schools via standardized tests, we will take the results at face value for now. Here is a graph showing the percent of students in each district whose 11th grade math score was judged "Below Basic", and the same for 11th grade reading scores:



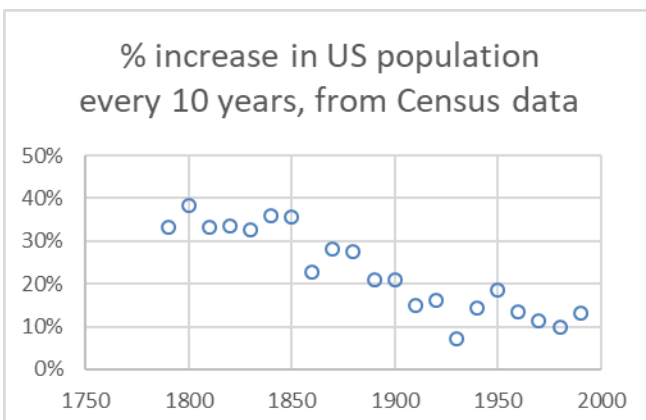
2) Similarly, we can look at the test scores themselves for 11th grade math and reading:



3) A fun in-class activity is to use toy cars that wind up as you pull them backwards, then roll themselves forwards once you let go. Here is a graph that you might get from an activity like that:



4) If we plot the percent increase in the US population from one census to another (every 10 years), we get this graph:



Question 1-p Analyze the first scatterplot for its form, strength, direction, and unusual features. Interpret your analysis in the context of the data.

Question 1-q Name some differences between these graphs. Think about things like: do they represent observational or experimental studies? How are their x values arranged or chosen?

Question 1-r It has been said that "the definition of insanity is doing the same thing over and over and expecting different results." How does this relate to these graphs and a statistical way of thinking?

Question 1-s Sometimes students want to apply the "vertical line test" (to test if a relationship is a function) to scatterplots like these. How would you reply to students who make that suggestion?

Question 1-t For which of these graphs would "making good predictions" be your top priority? For which of these graphs would "understanding what is going on" be your top priority?

References

Tran, D., & Lee, H. (2015). Considerations for design and implementation of statistics tasks. In *Teaching statistics through data investigations MOOC-Ed,* Friday Institute for Education Innovation. http://fi-courses.s3.amazonaws.com/tsdi/unit_3/CDIST.pdf

https://www.propublica.org/article/minority-neighborhoods-higher-car-insurance-premiums-white-areas-same-risk

Exercises

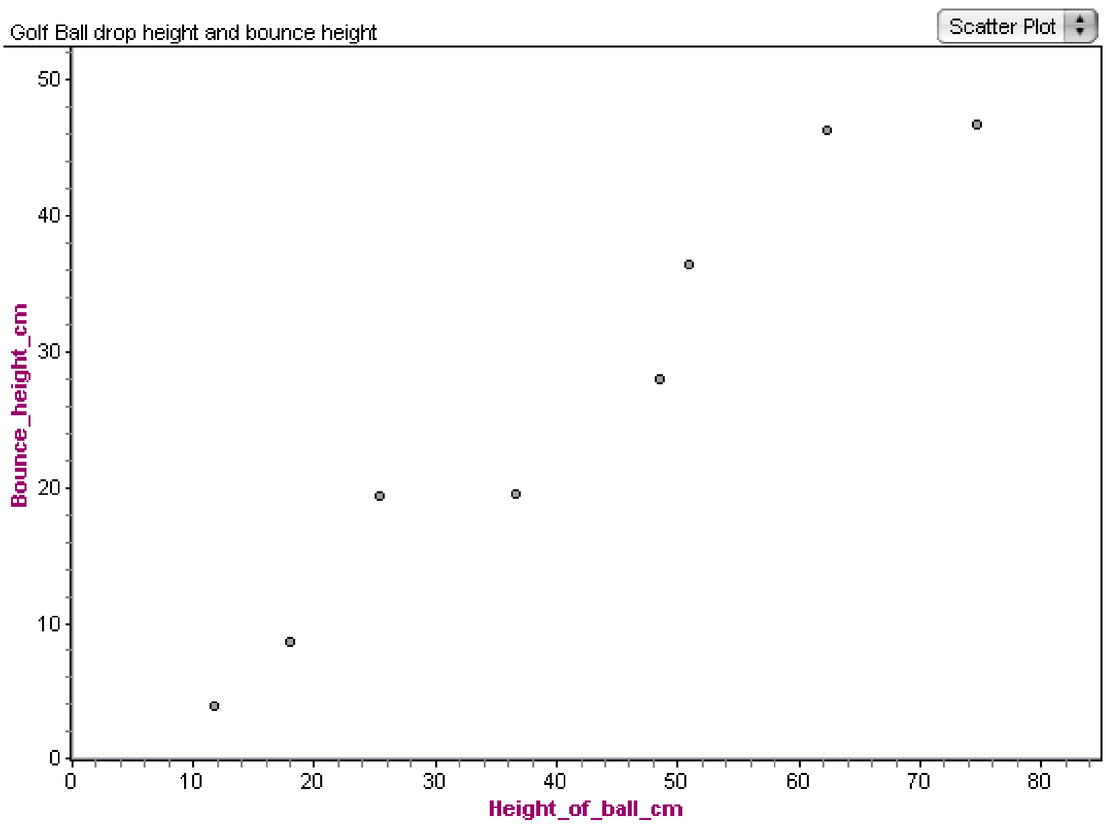
Exercise 1-1 Due before next class:

Considering Student Approaches to Placing the Informal Line of Best Fit

In eighth grade, students begin to learn about association of quantitative variables and place a line of best fit informally (i.e., by eye, without technology) for data that displays a linear association. The Common Core State Standard that addresses this is below:

> CCSS.Math.Content.8.SPA.2: Know that straight lines are widely used to model relationships between two quantitative variables. For scatter plots that suggest a linear association, informally fit a straight line, and informally assess the model fit by judging the closeness of the data points to the line.

1) Complete this task: Consider the following scatterplot which shows data about how high a golf ball bounces when it is dropped from different heights. Since this data shows a linear association, informally fit a straight line to it using a piece of uncooked spaghetti.



In settings where actual 8th graders worked on this task, students came up with their own approaches for placing the informal line of best fit. For example, numerous students responded to the task by asking if they could break the spaghetti into small pieces in order to 'connect the dots' in the graph.

Next, you will watch and analyze some videos that depict other approaches taken by students to place the informal line of best fit for this task.

2) Go to the YouTube playlist here: http://bit.ly/AssociationPL [3].
3) Watch Video 8, then complete the tables on the next 2 pages to summarize and analyze students' approaches to placing the line.
4) Pick one of the students and describe in detail your response to the student as their teacher. Be specific when describing this and future responses to students, writing exactly what you would say and/or draw in your response.

5) Watch Video 9. Summarize why some of the students (1/3 of the class) prefer Angelo's line and what this implies about their developing conception of the line of best fit.

6) Summarize why other students (2/3 of the class) prefer Barbara's line and what this implies about their developing conception of the line of best fit.

7) What would you do next in the class session, picking up at the end of Video 9? Consider how to provide experiences for students that would further develop their conception of the line of best fit and have them understand that lines of best fit are assessed based on the closeness of the data points to the line.

[3] Or https://www.youtube.com/playlist?list=PLLT-D-NcgUv6I3kQJuRplDjo1B4ZnrfLv

Student name	Maggie	Ashara
Approach for placing the line		
Potential reasons/sources for approach		
Is the approach generalizable (i.e., would the approach work for other data sets with a linear trend)? If not, draw at least one counter example (a scatterplot that will produce a poor line of best fit using the student's approach and the line).		

Student name	Dee	Randall
Approach for placing the line		
Potential reasons/sources for approach		
Is the approach generalizable (i.e., would the approach work for other data sets with a linear trend)? If not, draw at least one counter example (a scatterplot that will produce a poor line of best fit using the student's approach and the line).		

Exercise 1-2 Read this study description, then answer items (a, b, c, d, e):

Car Insurance and Racial Issues

Do people of color face more issues when getting car insurance than white people do, in the U.S.? Decades ago, race-based redlining outlined regions on various city maps all across the country where residents were to be denied mortgages and various other services because neighborhoods were considered risky. While intentional redlining is illegal now, are there still price differences? A recent study by ProPublica and Consumer Reports gathered a dataset of car insurance prices for a hypothetical person in a wide variety of neighborhoods (grouped at the zip-code level) in Illinois, Missouri, Texas, and California. We will focus on Illinois, and exclude the city of Chicago because it has insurance rules that are different than the rest of the state. The study's authors got a rate quote based on a hypothetical 30-year-old single female teacher with a good driving record and excellent credit[4], for each of the listed zip codes, from a variety of companies. While the report's data set is downloadable, we could not get permission to use it, so we made a simulated version that behaves similarly. You can access it at this link: http://bit.ly/codapcarinsurancerace01 [5]

The column "wnh_pct" stands for White-Not-Hispanic(-or-Latino) Percent, which is the racial measurement that the study used. The column "combined premium" indicates the price that the company quoted for car liability insurance (which includes only bodily injury and property damage for other people that the insured driver causes). All of the data is from the year 2014. CODAP tip: You can make a scatterplot by clicking on "Graph", then dragging column headings from the table to the x or y axes of the graph. If you drag a column heading from the table into the middle of the graph, it will color the points based on that variable's values.

(a) Is there any evidence of a relationship between car insurance prices and racial composition of a zip code? Investigate the data, and explain what you did and what you found.

(b) If you did find a relationship between wnh_pct and prices, is it evidence of systemic inequity? Explain.

[4] Why race is not specified: Insurance companies want to know the age, gender, marital status, occupation, and driving record of their customers, to estimate risk/average payout before quoting a price. They are not allowed to ask about race, so the race of this hypothetical person did not need to be specified. The issue of using credit scores in setting insurance prices is more complicated—some states prohibit it since credit scores are correlated with income, which is also correlated with race.

[5] Or https://codap.concord.org/app/static/dg/en/cert/index.html#shared=https%3A%2F%2Fcfm-shared.concord.org%2FUawUu8atSM7s5TuTF7eS%2Ffile.json

(c) Historically, car insurance companies have explained higher rates in some zip codes by pointing out that some places have more "congestion" than others, and so drivers who live in those places tend to cost more in claims each year. What is new about this study is that the study's authors got data on the average payout per policy per year in the various zip codes, so the study could adjust for "congestion" in that way. In the data set we've provided, the column "C.A.P." stands for "Congestion-Adjusted Price", and it tries to say what the prices might be if all neighborhoods had the same congestion (we will discuss later how we modeled this). Is there any evidence of a relationship between these adjusted prices and the racial composition of a zip code? Investigate the data, and explain what you did and what you found.

(d) If you did find a relationship between wnh_pct and congestion-adjusted prices, is it evidence of systemic inequity? Explain.

(e) Turn back to the T-chart in Module 1 Lesson 3 (Question 6-c). Considering your findings from the data you just investigated, update your chart (add, cross out, or annotate items.)

Choose one of the following three exercises to complete (1-3, 1-4, or 1-5):

Exercise 1-3 Bridge-building competitions are often done in middle or high school to see who can build the strongest bridge. How do you imagine them occurring? Reflecting on what you learned in the spaghetti bridge investigation, can you think of ways that middle school or high school bridge-building competitions could be improved?

Exercise 1-4 What are the good and bad things about the design of the bridge-strength study that gave the data shown in Activity 1 (and at http://bit.ly/codapbridge00)? How might you fix the bad things?

Exercise 1-5 With the variety of scatterplots we have earlier in this activity, how would you respond to a student who said "you can tell if the study was observational or experimental by whether there are repeated x values or not" ?