5.5.9 $\vec B = \vec r \times \left\{ \nabla \times \left[\vec r \times (\vec A \times \vec r) \right] \right\}$

$\vec A$ konst , $\vec r$ ortsvektorn

$\vec B = \vec r \times (\nabla \times \vec C)$

$$C_i = \varepsilon_{ijk} r_j (\vec A \times \vec r)_k$$
$$= \varepsilon_{ijk} r_j \varepsilon_{klm} A_l r_m \qquad \left[\varepsilon_{ijk} \varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{lj} \right]$$
$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{lj}) r_j A_l r_m$$
$$= A_i r_j r_j - r_i A_j r_j$$
$$= \left(\vec A r^2 - \vec r (\vec r \cdot \vec A) \right)_i$$

$$B_i = \left(\vec r \times (\nabla \times \vec C) \right)_i$$
$$= \varepsilon_{ijk} r_j \varepsilon_{klm} \partial_l C_m$$
$$= \varepsilon_{ijk} r_j \varepsilon_{klm} \partial_l (A_m r^2 - r_m A_n r_n)$$
$$= \left\{ \begin{array}{l} \partial_l A_m = 0 \\ \partial_l r_m r_n = r_m \delta_{ln} + r_n \delta_{ml} \Rightarrow \partial_l r^2 = 2 r_l \end{array} \right\}$$
$$= \varepsilon_{ijk} r_j \varepsilon_{klm} \Big(2 A_m r_l - \underbrace{A_n r_m \delta_{ln}}_{*} - \underbrace{A_n r_n \delta_{lm}}_{\#} \Big)$$

$\#$: $\varepsilon_{klm} A_n r_n \delta_{lm} = \underbrace{\varepsilon_{kll}}_{=0} A_n r_n$

$*$: $\varepsilon_{klm} A_n r_m \delta_{ln} = \varepsilon_{klm} A_l r_m \underset{l \leftrightarrow m}{=} \varepsilon_{kml} A_m r_l = - \varepsilon_{klm} A_m r_l$

$$\Rightarrow B_i = \varepsilon_{ijk} r_j \varepsilon_{klm} 3 A_m r_l$$
$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{lj}) r_j 3 A_m r_l$$
$$= 3 r_i r_j A_j - 3 A_n r_j r_j$$

$$\Rightarrow \boxed{\vec B = 3 \vec r (\vec r \cdot \vec A) - 3 \vec A r^2}$$

12.6.5

$$\operatorname{Det} M = \underbrace{\varepsilon^{rst} M^1_r M^2_s M^3_t}_{A} = \underbrace{\tfrac{1}{6}\varepsilon^{rst} \varepsilon_{lmn} M^l_r M^m_s M^n_t}_{B}$$

(A)

$$\operatorname{Det} M = \begin{array}{c|c|c|c|} & 1 & 2 & 3 \\ \hline M^1_r & 0 & 0 & 0 \\ \hline M^2_s & & & \\ \hline M^3_t & & & \\ \hline \end{array}$$

rst	123	132	231	213	312	321
ε	1	−1	1	−1	1	−1

$\Rightarrow$ A ok!

(B) visa $A = B$

byt plats på 1 & 2 i A

$$\varepsilon^{rst} M^2_r M^1_s M^3_t = \varepsilon^{rst} M^1_s M^2_r M^3_t$$

$$r \leftrightarrow s \rightarrow = \varepsilon^{srt} M^1_r M^2_s M^3_t$$

$$= -\varepsilon^{rst} M^1_r M^2_s M^3_t = -A$$

Notera: funkar på alla parbyten & skiljer bara i tecken.
Detta kan identifieras med lmn i ε_{lmn}

$$\varepsilon^{rst} M^l_r M^m_s M^n_t = \varepsilon^{lmn} A$$

men $\varepsilon^{lmn}\varepsilon_{lmn} = 3! = 6$

$$\Leftrightarrow \quad \varepsilon^{rst}\varepsilon_{lmn} M^l_r M^m_s M^n_t = \varepsilon^{lmn}\varepsilon_{lmn} A = 6A$$

$\Rightarrow$ $A = B$

12.6.5 forts

$$\det(M) = \frac{1}{D!}\,\varepsilon^{a_1\ldots a_D}\varepsilon_{b_1\ldots b_D} M_{a_1}^{\;b_1}\ldots M_{a_D}^{\;b_D}$$

Visa $\operatorname{Det} M = \frac{1}{3}\operatorname{Tr} M^3 - \frac{1}{2}\operatorname{Tr} M \operatorname{Tr} M^2 + \frac{1}{6}(\operatorname{Tr} M)^3$

Utnyttja eq (5.6) $\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

skriv som

$$\delta^{tl}\varepsilon_{rst}\varepsilon_{lmn} = \delta_{rm}\delta_{sn} - \delta_{rn}\delta_{sm}$$

$$\delta^{tm}\varepsilon_{rst}\varepsilon_{lmn} = \delta^{tm}\varepsilon_{rst}\varepsilon_{mnl} = \delta_{rn}\delta_{sl} - \delta_{rl}\delta_{sn}$$

$$\delta^{tn}\varepsilon_{rst}\varepsilon_{lmn} = \delta^{tn}\varepsilon_{rst}\varepsilon_{nlm} = \delta_{rl}\delta_{sm} - \delta_{rm}\delta_{sl}$$

$$\Rightarrow \quad \varepsilon_{rst}\varepsilon_{lmn} = \det \begin{array}{cc} \left|\begin{matrix} \delta_{tl} & \delta_{tm} & \delta_{tn} \\ \delta_{rl} & \delta_{rm} & \delta_{rn} \\ \delta_{sl} & \delta_{sm} & \delta_{sn} \end{matrix}\right| & \begin{matrix} t \\ r \\ s \end{matrix} \\ \begin{matrix} l & \quad m & \quad n \end{matrix} & \end{array}$$

$$\Rightarrow \det M = \frac{1}{6}\varepsilon^{rst}\varepsilon_{lmn} M_r^{\;l} M_s^{\;m} M_t^{\;n}$$

$$= \frac{1}{6}\Big(\delta_l^{\;t}\big(\delta_m^{\;r}\delta_n^{\;s} - \delta_n^{\;r}\delta_m^{\;s}\big) + (l\to m\to n) + (l\to n\to m)\Big) \times$$

$$\times\, M_r^{\;l} M_s^{\;m} M_t^{\;n}$$

övre index är kvar
nedre permuterar
$r\to s\to t$
& $r\to t\to s$

$$\to \;= \frac{1}{6}\Big(M_r^{\;t}\big(M_s^{\;r}M_t^{\;s} - M_s^{\;s}M_t^{\;r}\big) + M_s^{\;t}\big(M_t^{\;r}M_r^{\;s} - M_t^{\;s}M_r^{\;r}\big)$$

$$+\, M_t^{\;t}\big(M_r^{\;r}M_s^{\;s} - M_r^{\;s}M_s^{\;r}\big)\Big)$$

$$= \frac{1}{6}\Big(\operatorname{Tr} M^3 - \operatorname{Tr} M \operatorname{Tr} M^2 + \operatorname{Tr} M^3 - \operatorname{Tr} M^2 \operatorname{Tr} M$$

$$+\, (\operatorname{Tr} M)^3 - \operatorname{Tr} M \operatorname{Tr} M^2\Big)$$

$$= \frac{1}{6}(\operatorname{Tr} M)^3 - \frac{1}{2}\operatorname{Tr} M \operatorname{Tr} M^2 + \frac{1}{3}\operatorname{Tr} M^3$$