3.6.9 $\vec{B}$ givet i sfäriska koord

$$\vec{B}(r,\theta,\varphi) = \frac{B_0 a}{r\sin\theta}\left(\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta} + \hat{\varphi}\right)$$

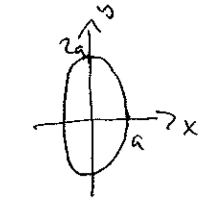
Bestäm kurvintegralen av $\vec{B}$ längs kurvan C

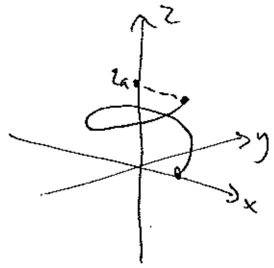
med parameterframställningen

$C: \vec{r} = \left(a\cos\alpha,\ 2a\sin\alpha,\ \frac{a\alpha}{\pi}\right)$ från $(a,0,0)$ till $(a,0,2a)$

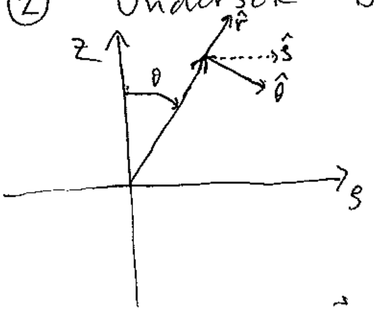
① Undersök kurvan

$x = a\cos\alpha$
$y = 2a\sin\alpha$ $\Rightarrow$ ellips!





② Undersök $\vec{B}$



$\Rightarrow$ $\rho = r\sin\theta$
$\hat{\rho} = \sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}$

så $\vec{B} = \frac{B_0 a}{\rho}\left(\hat{\rho} + \hat{\varphi}\right)$ $\Rightarrow$ Cylinder!

③ Parametrisera? C - ellips B - cylinder, gå via xyz?

Vill lösa $\int_C \vec{B}\cdot d\vec{r} = \int_0^{2\pi} \vec{B}(\vec{r}(\alpha))\cdot\frac{d\vec{r}}{d\alpha}\,d\alpha$

Behöver
$\vec{B}(\vec{r}(\alpha))$ & $\frac{d\vec{r}}{d\alpha}$

$$\frac{d\vec{r}}{d\alpha} = \left(-a\sin\alpha,\ 2a\cos\alpha,\ \frac{a}{\pi}\right)$$

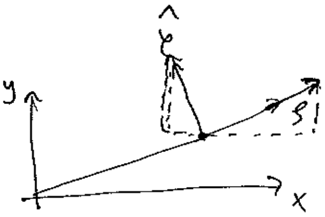
för $\vec{B}(\vec{r}(\alpha))$ behövs

$$\rho = \sqrt{x^2+y^2} \underset{\vec{r}}{\overset{\uparrow}{=}} a\sqrt{\cos^2\alpha + 4\sin^2\alpha} = a\sqrt{1+3\sin^2\alpha}$$

$$\hat{\rho} = (x\hat{x} + y\hat{y})/\rho$$

$$\hat{\varphi} = (-y\hat{x} + x\hat{y})/\rho$$

från



(tips: testa $\varphi = 0,\ \pi/2$)

$$\hat{\rho} + \hat{\varphi} = \frac{x\hat{x} + y\hat{y} - y\hat{x} + x\hat{y}}{\rho} = \cancel{a\hat{x}(\cos\alpha\,\hat{x} + 2a\sin\alpha\,\hat{y} - 2a}$$

$$= \frac{\cancel{a}\,\hat{x}(\cos\alpha - 2\sin\alpha) + \cancel{a}\,\hat{y}(\cos\alpha + 2\sin\alpha)}{\cancel{a}\sqrt{1+3\sin^2\alpha}}$$

$$\oint \vec{B}(\vec{r}(\alpha))\cdot\frac{d\vec{r}}{d\alpha}\,d\alpha = B_0 a\int_0^{2\pi}\frac{-\sin\alpha(\cos\alpha - 2\sin\alpha) + 2\cos\alpha(2\sin\alpha + \cos\alpha)}{1+3\sin^2\alpha}\,d\alpha$$

$$= B_0 a\int_0^{2\pi}\frac{\overset{\text{udda}}{3\cos\alpha\sin\alpha} + 2}{\underset{\text{jämn}}{1+3\sin^2\alpha}}\,d\alpha$$

(udda: $f(-x) = -f(x)$)

$$= 2B_0 a\underbrace{\int_0^{2\pi}\frac{1}{1+3\sin^2\alpha}\,d\alpha}_{=\pi}$$

$$= 2\pi B_0 a$$

Alternativ lösning i cylindriska koordinater.
$\vec{B}$ i cylindriska koordinater:

$$\vec{B}(\vec{r}) = \frac{B_0 a}{\rho}\left(\hat{r} + \hat{\varphi}\right), \tag{1}$$

kurvan i cylindriska koordinater:

$$\vec{r}(\alpha) = \rho(\alpha)\hat{\rho}(\alpha) + \alpha\frac{a}{\pi}\hat{z}, \tag{2}$$

där

$$\rho(\alpha) = a\sqrt{1 + 3\sin^2\alpha}, \tag{3}$$

$$\hat{\rho}(\alpha) = \begin{pmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{pmatrix} \qquad \Rightarrow \frac{\partial\hat{\rho}}{\partial\alpha} = \begin{pmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{pmatrix} = \hat{\varphi}(\alpha) \tag{4}$$

Beräkna derivatan av $\vec{r}(\alpha)$:

$$\begin{aligned} \frac{d\vec{r}}{d\alpha} &= \frac{\partial\rho}{\partial\alpha}\hat{\rho}(\alpha) + \rho(\alpha)\underbrace{\frac{\partial\hat{\rho}(\alpha)}{\partial\alpha}}_{=\hat{\varphi}} + \frac{a}{\pi}\hat{z} \\ &= \frac{a}{2}\frac{3\sin\alpha\cos\alpha}{\sqrt{1+3\sin^2\alpha}}\hat{\rho}(\alpha) + \rho(\alpha)\hat{\varphi} + \underbrace{\frac{a}{\pi}\hat{z}}_{\text{spelar ingen roll i integralen}} \end{aligned} \tag{5}$$

Vägintegralen:

$$\begin{aligned} I &= \int_{\mathcal{C}} \vec{B}(\vec{r}) \cdot d\vec{r} = \int_0^{2\pi} \vec{B}(\vec{r}(\alpha)) \cdot \frac{d\vec{r}}{d\alpha} d\alpha \\ &= B_0 a \int_0^{2\pi} \frac{1}{\rho(\alpha)}\left(\hat{\rho} + \hat{\varphi}\right) \cdot \left(\frac{\partial\rho(\alpha)}{\partial\alpha}\hat{\rho} + \rho(\alpha)\hat{\varphi}\right) d\alpha \\ &= B_0 a \int_0^{2\pi} \underbrace{\frac{\partial\rho(\alpha)/\partial\alpha}{\rho(\alpha)}}_{=\frac{f'(x)}{f(x)} \Rightarrow \ln|f(x)|} + \underbrace{\frac{\rho(\alpha)}{\rho(\alpha)}}_{=1} d\alpha = B_0 a \left(\ln|\rho(\alpha)| + \alpha\right)\Big|_0^{2\pi} \\ &= B_0 a \left(1 - 1 + 2\pi - 0\right) = 2\pi B_0 a. \end{aligned} \tag{6}$$

4.5.23 $\vec{F}$ ges av $\vec{F} = -\nabla\phi$

där $\phi(x,y,z) = 6xyz + 2xy$

Beräkna integralen $\oint_S \vec{F} \times d\vec{S} = -\int_V \nabla \times F \, dV = 0$

$S: x^2 + y^2 + (3z-1)^2 = 1$

① ytan? sluten

② fältet? inga singulariteter => Stokes/Gauss ok!

③ kan vi använda någon av integralsatserna?

$\partial V \to V \Rightarrow$ Gauss $\int_S \vec{F} \cdot d\vec{S} = \int_V \nabla \cdot \vec{F} \, dV$

inte exakt samma!

Välj ett smart $\vec{F}$ för att införa kryssprodukt

testa $\vec{F} = \vec{a} \times \vec{v}$, $\vec{a}$ konst, $v = v(x,y,z)$

Gauss: $\int_{\partial V} \underbrace{(\vec{a} \times \vec{v}) \cdot dS}_{} = \int_V \underbrace{\nabla \cdot (\vec{a} \times \vec{v})}_{} \, dV$

$= dS \cdot (\vec{a} \times \vec{v})$ $\quad = (\underbrace{\nabla \times \vec{a}}_{=0}) \cdot \vec{v} - \vec{a} \cdot (\nabla \times \vec{v})$

$= \vec{a} \cdot (\vec{v} \times dS)$ $\quad = -\vec{a} \cdot (\nabla \times \vec{v})$

$$\Leftrightarrow \int_{\partial V} \vec{a} \cdot (\vec{v} \times d\vec{S}) = -\int_V \vec{a} \cdot (\nabla \times \vec{v}) \, dV$$

$$\Leftrightarrow \vec{a} \cdot \int_{\partial V} \vec{v} \times d\vec{S} = -\vec{a} \cdot \int_V \nabla \times \vec{v} \, dV$$

$$\Leftrightarrow \int_{\partial V} \vec{v} \times d\vec{S} = -\int_V \nabla \times \vec{v} \, dV$$

$$\Rightarrow \oint_S \vec{F} \times d\vec{S} = -\int_V \nabla \times \vec{F} \, dV = +\int_V \underbrace{\nabla \times (\nabla\phi)}_{=0} \, dV = 0$$