Exercice 1. *(Variation totale)*
Soit $b = (b_1, \dots, b_n)^T$ un vecteur de $\mathbb{R}^n$. On se pose le problème suivant :

$$\min_{x \in \mathbb{R}^n} \frac{1}{2}\|x - b\|^2 + \eta \sum_{i=1}^{n-1} |x_{i+1} - x_i| \,. \tag{1}$$

1. Expliquer brièvement l'effet que peut avoir le deuxième terme (dit de régularisation). Autrement dit, intuitivement, en quoi la solution x^* de ce problème va-t-elle différer de b ?
2. Montrer que le problème (1) peut être réécrit

$$\min_{x \in \mathbb{R}^n} f(x) + g(Mx)$$

   où $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ et $g : \mathbb{R}^m \cup \{+\infty\}$ sont des fonctions convexes et M est une matrice dont on donnera les dimensions.
3. Calculer les opérateurs proximaux de f et de g.
4. Pour $n = 5$, expliciter M et $M^T M$.
5. Écrire les itérations de l'ADMM de pas $\sigma > 0$ pour la résolution de (1).
6. Montrer que l'algorithme se réduit à une succession de seuillages doux et de résolution de systèmes linéaires.

Exercice 2. *(Distributed optimization)*
A database is distributed on a computer network composed of N parallel workers. Each worker i has a private cost function $f_i : \mathcal{X} \to \mathbb{R}$ where $\mathcal{X}$ is a Euclidean space. The aim is to find a minimizer of the function

$$f(x) = \sum_{i=1}^{N} f_i(x) \,.$$

We define the function $F(x_1, \dots, x_N) = \sum_{i=1}^{N} f_i(x_i)$ on $\mathcal{X}^N \to \mathbb{R}$. One can therefore reformulate the problem as

$$\min F(x_1, \dots, x_N) \;\; \text{s.t.} \quad x_1 = \dots = x_N \,. \tag{2}$$

1. State that problem (2) is equivalent to the minimization of $F(x) + \iota_{C_N}(x)$ on $x \in \mathcal{X}^N$ where C_N is the indicator function of a linear space C_N which you will specify.
2. Write the iterations of ADMM for that problem, making clear the communications between workers that are needed at each step of the algorithm.
3. Explicit the algorithm in the case where

$$f_i(x) = \frac{1}{2}\|A_i x - b\|^2 \,.$$

We now assume that the workers are connected through a graph structure. Let $G = (V, E)$ be a graph with $V = \{1, \ldots, N\}$ and E is a set of edges such that $\{i, j\} \in E$ if and only if the workers i and j can communicate.

4. Under what condition on the graph have we
$$\iota_{C_N}(x) = \sum_{\{i,j\} \in E} \iota_{C_2}(x_i, x_j)\,?$$
5. For any $e = \{i, j\}$ in E $(i < j)$, we define the matrix $M_e : \mathcal{X}^N \to \mathcal{X}^2$ such that $M_e x = (x_i, x_j)^T$. We define the matrix $M : \mathcal{X}^N \to \mathcal{X}^{2|E|}$ such that $Mx = (M_e x)_{e \in E}$. Show that $\iota_{C_N}(x) = g(Mx)$ where g is a function that will be specified.
6. Write and simplify the iterations of ADMM, making clear the communications between workers that are needed at each step of the algorithm.