Homework #8
Algorithms I
600.463
Spring 2017

Due on: Thursday, April 27th, 11:59pm
Late submissions: will NOT be accepted
Format: Please start each problem on a new page.
Where to submit: On Gradescope, under HW8.
Please type your answers; handwritten assignments will not be accepted.
To get full credit, your answers must be explained clearly,
with enough details and rigorous proofs.

April 28, 2017

1 Problem 1 (20 points)

A bipartite graph $G = (V, E)$, where $V = L \cup R$, is *d-regular* if every vertex $v \in V$ has degree exactly d.

(a) Show that for every *d-regular* bipartite graph, $|L| = |R|$.

(b) Model the maximum *d-regular* bipartite matching as a max-flow problem. Show that the max-flow value from s to t in the formulation is $|L|$.

(c) Prove that every *d-regular* bipartite graph has a matching of cardinality $|L|$.

Problem 1 Answer

(a) Show that for every *d-regular* bipartite graph, $|L| = |R|$.

Solution: The proof can be simply described in the following manner.T he definition of d-regular just means that for each vertex in the L, there will be d edges connecting to a vertex in R. For every edge in the graph, connect the node in R with a node in L. Then, the total number of edges is $|R|$*d and $|L|$*d, and therefore by closure $|L| = |R|$.

(b) Model the maximum *d-regular* bipartite matching as a max-flow problem. Show that the max-flow value from s to t in the formulation is $|L|$.

Solution: In CLRS, chapter 26, particularly in the the figure 26.3, a bipartite matching problem is reduced to a network flow and this is the exact idea that we need to use to model the d-regular bipartite matching as a max-flow problem. In the figure, every edge in the graph has unit capacity, and so we can derive a function where there is a unit flow out of every edge from s and every edge into t. As all edges will have unit capacity, we can assume further that all edges have flow 1, from s in L and all vertices in R to t. Since every edge from s is accounted for and used in the flow, then this is a maximum flow and therefore the max-flow value from s to t is indeed $|L|$.

(c) Prove that every *d-regular* bipartite graph has a matching of cardinality $|L|$.

Solution: In CLRS Chapter 26, we get a theorem specifically Theorem 26.11 which states the Integral Theorem. If the capacity function c takes only integral values, then the maximum flow f produced by the Ford-Fulkerson method has the properpty that $|f|$ is integer-valued. For all vertices u and v the value $f(u, v)$ of the flow is an integer. And so since every edge in the max-flow values in the formulation showed above has an integral capacity, then there must be an integral flow with the value $|L|$ as described in the theorem. And so every d-regular bipartite graph has a matching of cardinality L.

2 Problem 2 (20 points)

In the *maximum k-cut* problem, we are given an undirected graph $G = (V, E)$, and non-negative weights $w_{ij} \geq 0$ for all $(i, j) \in E$. The goal is to partition the vertex set V into k parts $V_1, \ldots, V_k$ so as to maximize the weight of all edges whose endpoints are in different parts (i.e., $\max_{(i,j)\in E: i\in V_a, j\in V_b, a\neq b} w_{ij}$).
Give a randomized $\frac{k-1}{k}$-approximation algorithm for the MAX k-CUT problem.
Hint: please review Chapter 5.1 of the book—"The Design of Approximation Algorithms" (Williamson and Shmoys 2010), and try to solve this problem using similar ideas.

Problem 2 Answer

Here is an algorithm that partitions the vertex set V into k parts and maximizes the weight of all edges whose end-points are in different parts.

function RANDOM-APPROXIMATION
    for all $V_1, \ldots, V_k$ **do**
        set to $null$
    end for
    for all $i \leftarrow$ 1 to n **do**
        uniform selection x from $[0, 1]$
        if x ∈ [j/k, (j + 1)/k]
            V$_{j+1} \leftarrow V_{j+1} \cup i$
    end for
end function

The algorithm is supposed to produce as many partitions as there are k. In layments term, the goal is to have a partitioning of [0, 1) into the propoer sub-ranges where there is also one subset per subrange. Let's note the following: (i.e., $\max_{(i,j)\in E: i\in V_a, j\in V_b, a\neq b} w_{ij}$).

The entire function for assignment of the variables to the randomized uniform selection of x will run in polynomial time, where the random variable is either 1 or 0, or in other words, when there is an edge connecting vertices in the different sets the variable will be set to 1 and 0 in all other cases. So, an edge can either connect vertices in the same subset or connect vertices in different subsets. It will be 1 if for some $a, b, i \in V_a$ and $j \in V_b$ and a! =b.Here is an approximation analysis proof (where W is the random variable) that proves mathematically what the end goal is that I have described above with much more precision and detail. This link was found accompanied with a Google search result of Williamson and Shmoys 2010 Chapter 5.1 (courses.csail.mit.edu/6.891-s00/pss2.ps). Ultimately, we can find that the summation of $(i, j) \in E$ of w_{i*j} multiplied by $(k-1)/k$ and thus simplified one step further that $\frac{k-1}{k}$-approximation algorithm works for this MAX k-CUT problem