Simple Linear Regression

Models

- Representation of some phenomenon
- Mathematical model is a mathematical expression of some phenomenon
- Often describe relationships between variables
- Types
  - Deterministic models
  - Probabilistic models

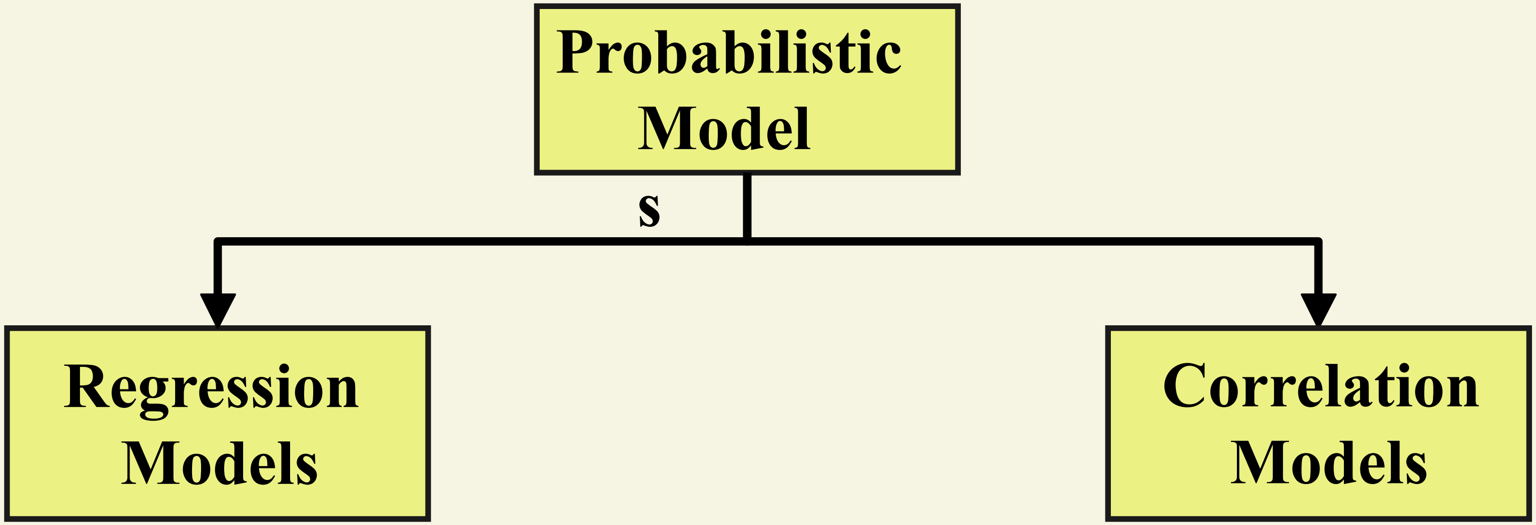
Deterministic Models

- Hypothesize exact relationships
- Suitable when prediction error is negligible
- Example: force is exactly mass times acceleration
  - $F = m \cdot a$

Probabilistic Models

- Hypothesize two components
  - Deterministic
  - Random error
- Example: sales volume (*y*) is 10 times advertising spending (*x*) + random error
  - $y = 10x + \varepsilon$
  - Random error may be due to factors other than advertising

Types of Probabilistic Models



Probabilistic Model

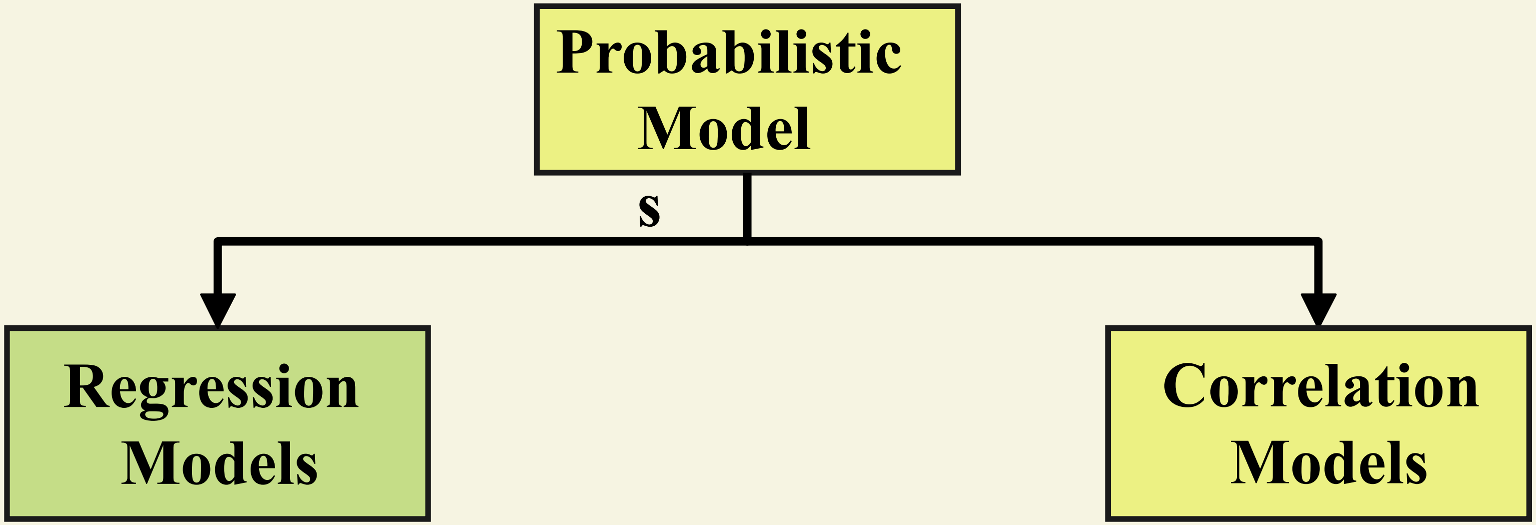
s

Regression Models

Correlation Models

Regression Models

Types of Probabilistic Models



Regression Models

- Answers ‘What is the relationship between the variables?’
- Equation used
  - One numerical dependent (response) variable
    - What is to be predicted
  - One or more numerical or categorical independent (explanatory) variables
- Used mainly for prediction and estimation

Regression Modeling Steps

1. Hypothesize deterministic component
2. Estimate unknown model parameters
3. Specify probability distribution of random error term
   - Estimate standard deviation of error
4. Evaluate model
5. Use model for prediction and estimation

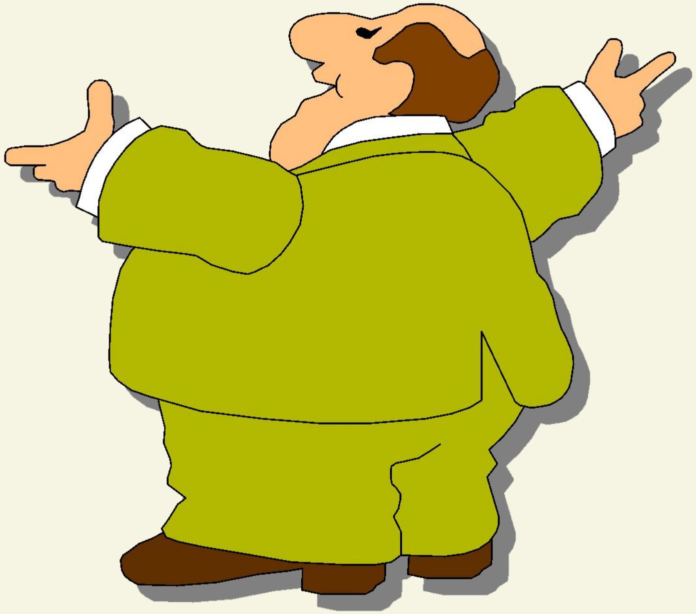
Model Specification

Regression Modeling Steps

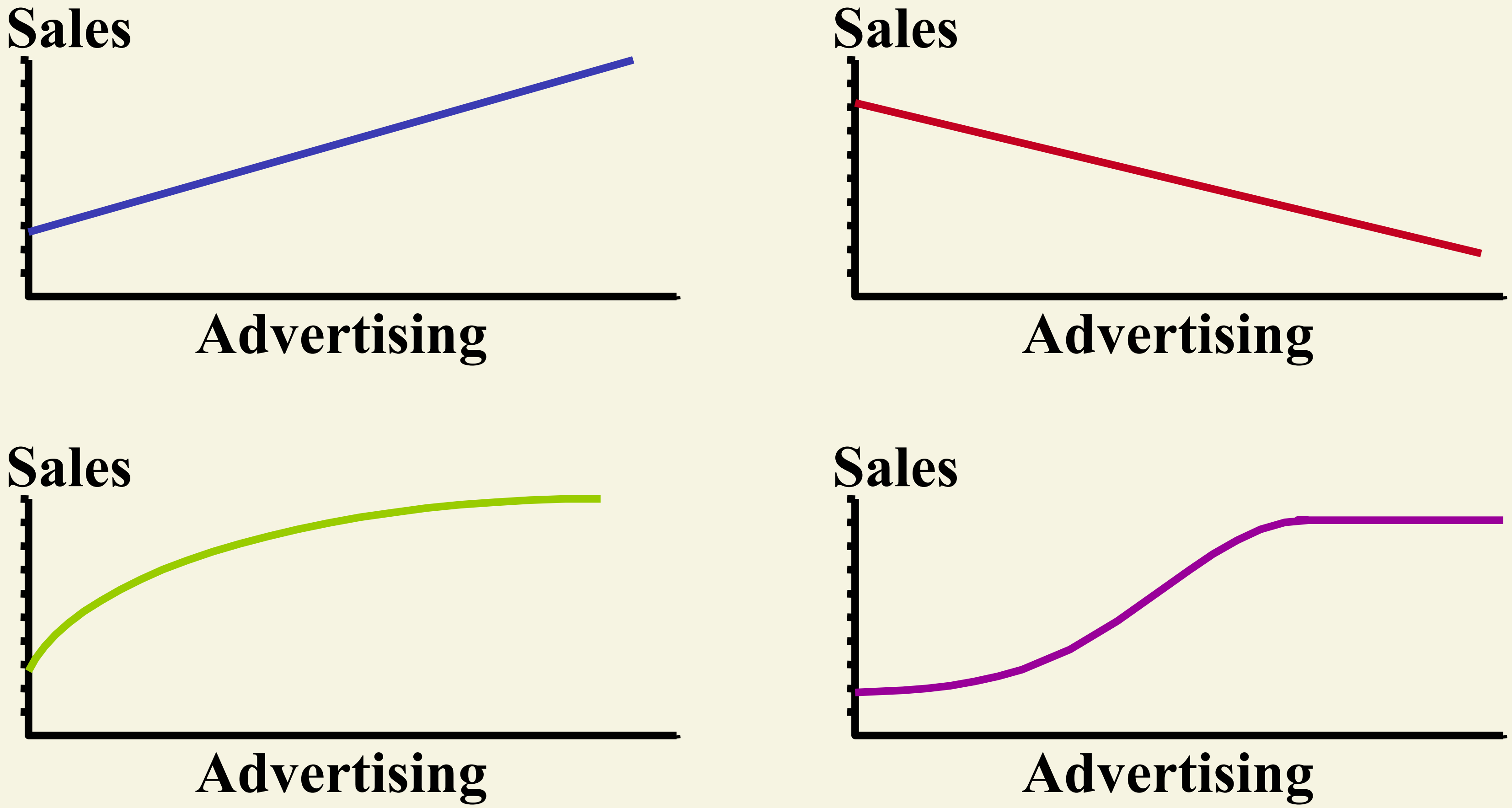
1. **Hypothesize deterministic component**
2. Estimate unknown model parameters
3. Specify probability distribution of random error term
   - Estimate standard deviation of error
4. Evaluate model
5. Use model for prediction and estimation

Model Specification Is Based on Theory

- Theory of field (e.g., Sociology)
- Mathematical theory
- Previous research
- ‘Common sense’



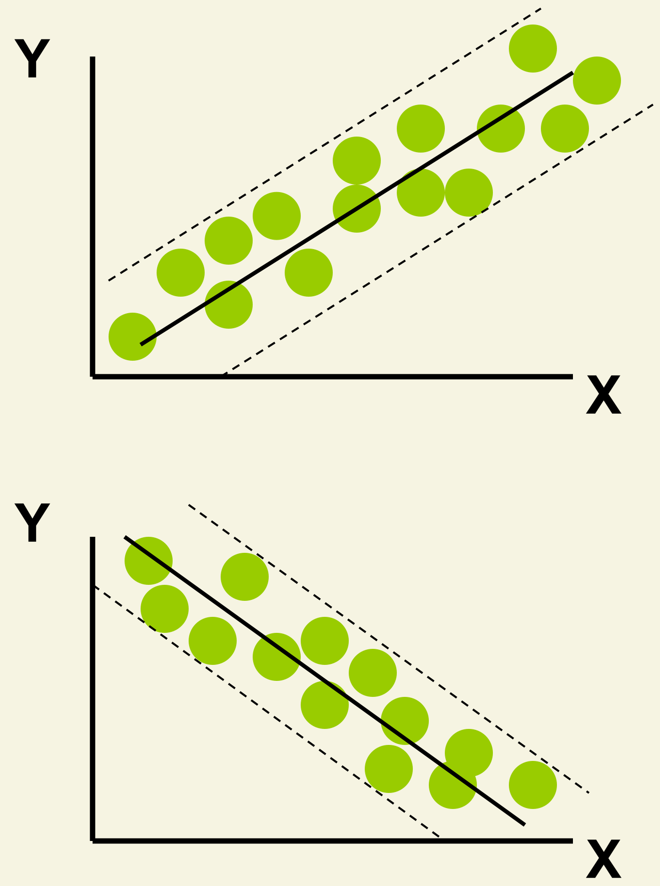
Thinking Challenge: Which Is More Logical?



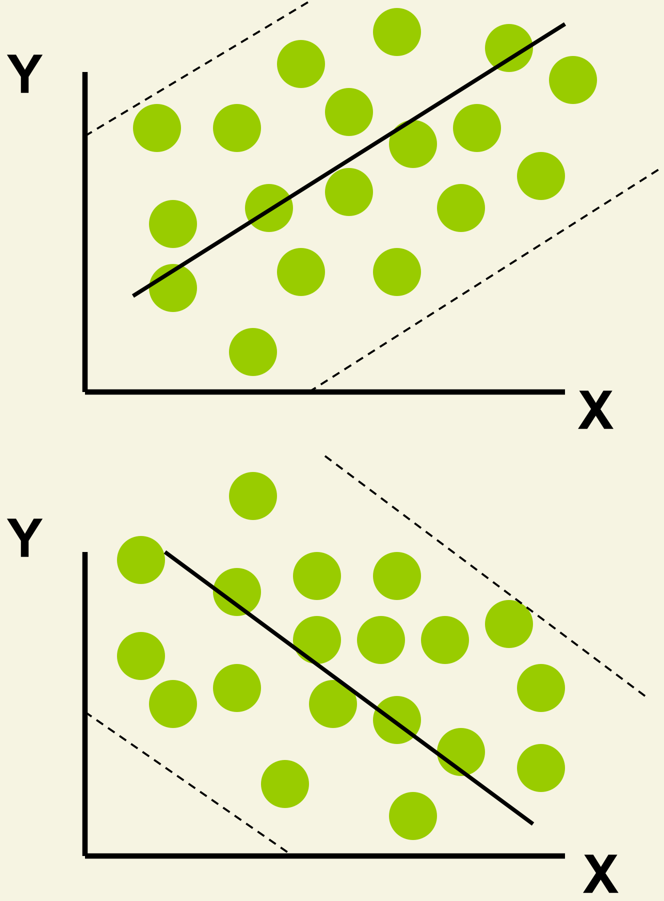
Types of Relationships

(continued)

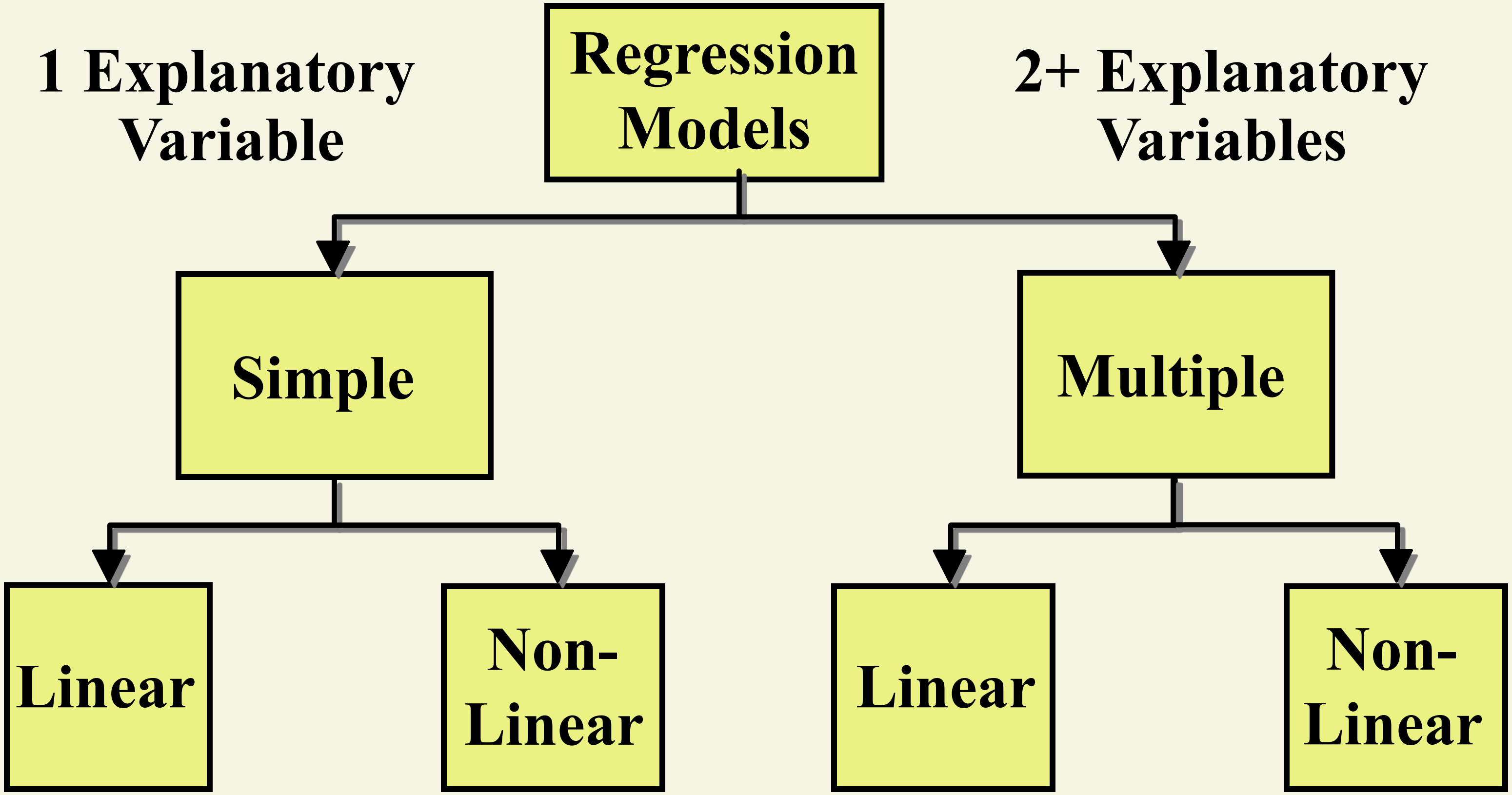
Strong relationships



Weak relationships



Types of Regression Models



1 Explanatory Variable

Regression Models

2+ Explanatory Variables

Simple

Multiple

Linear

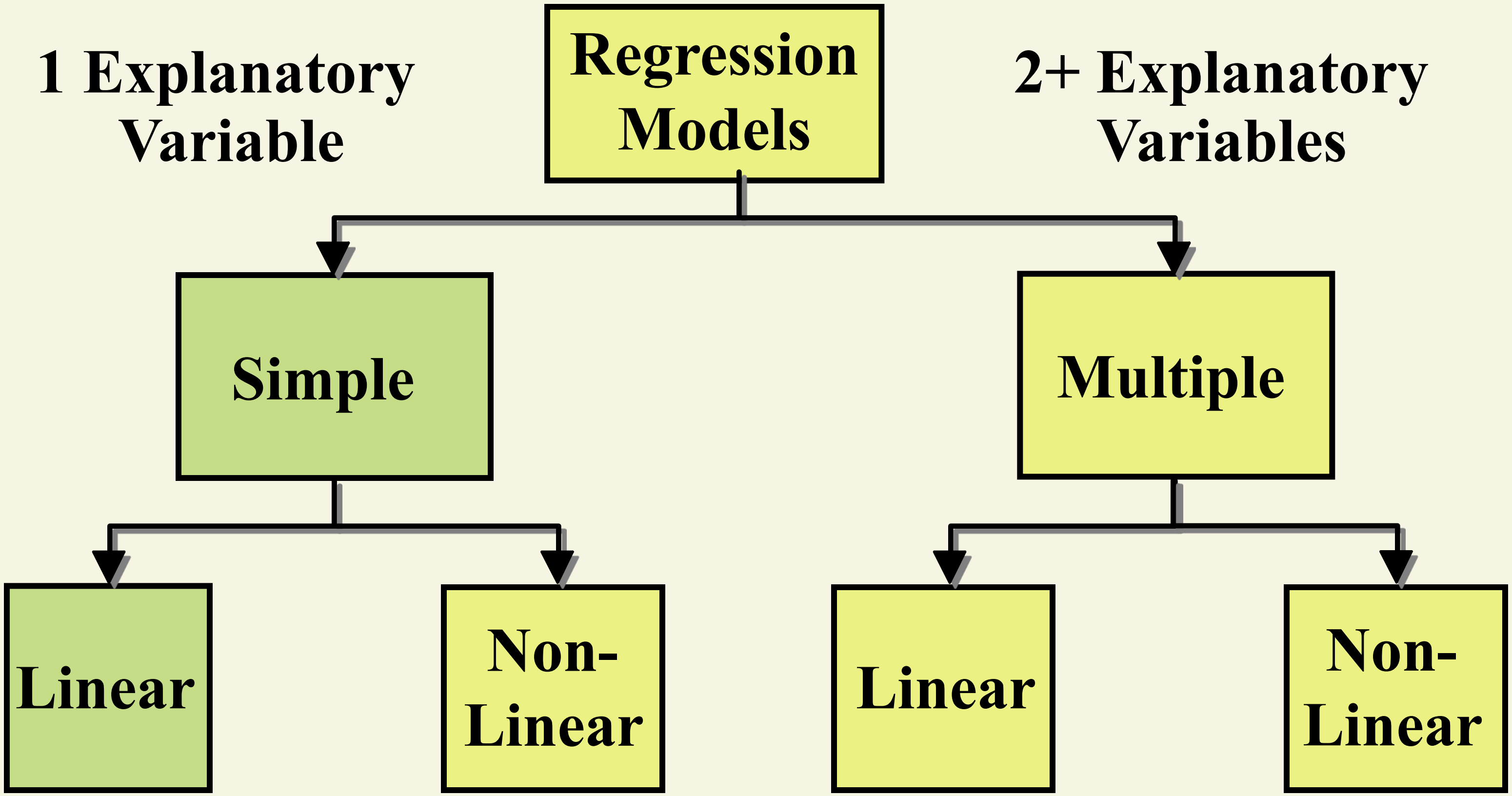
Non-Linear

Linear

Non-Linear

Linear Regression Model

Types of Regression Models



Regression Models

1 Explanatory Variable

2+ Explanatory Variables

Simple

Multiple

Linear

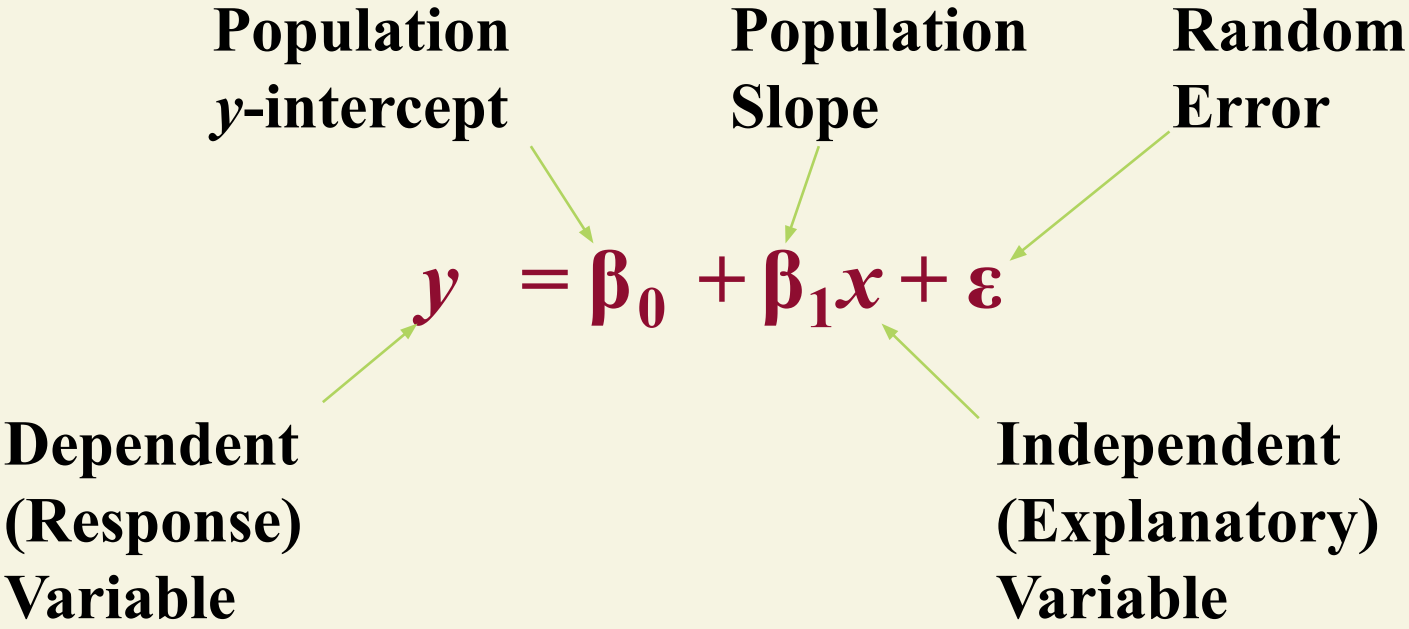
Non-Linear

Linear

Non-Linear

Linear Regression Model

Relationship between variables is a linear function



Population *y*-intercept

Population Slope

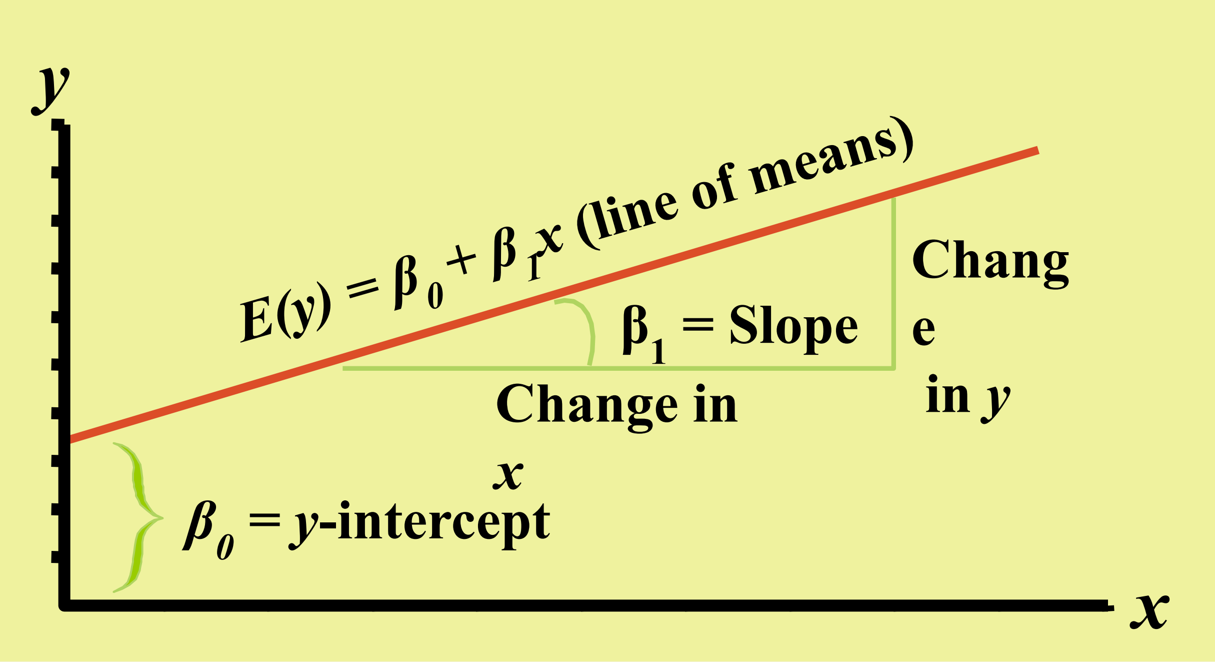
Random Error

$$y = \beta_0 + \beta_1 x + \varepsilon$$

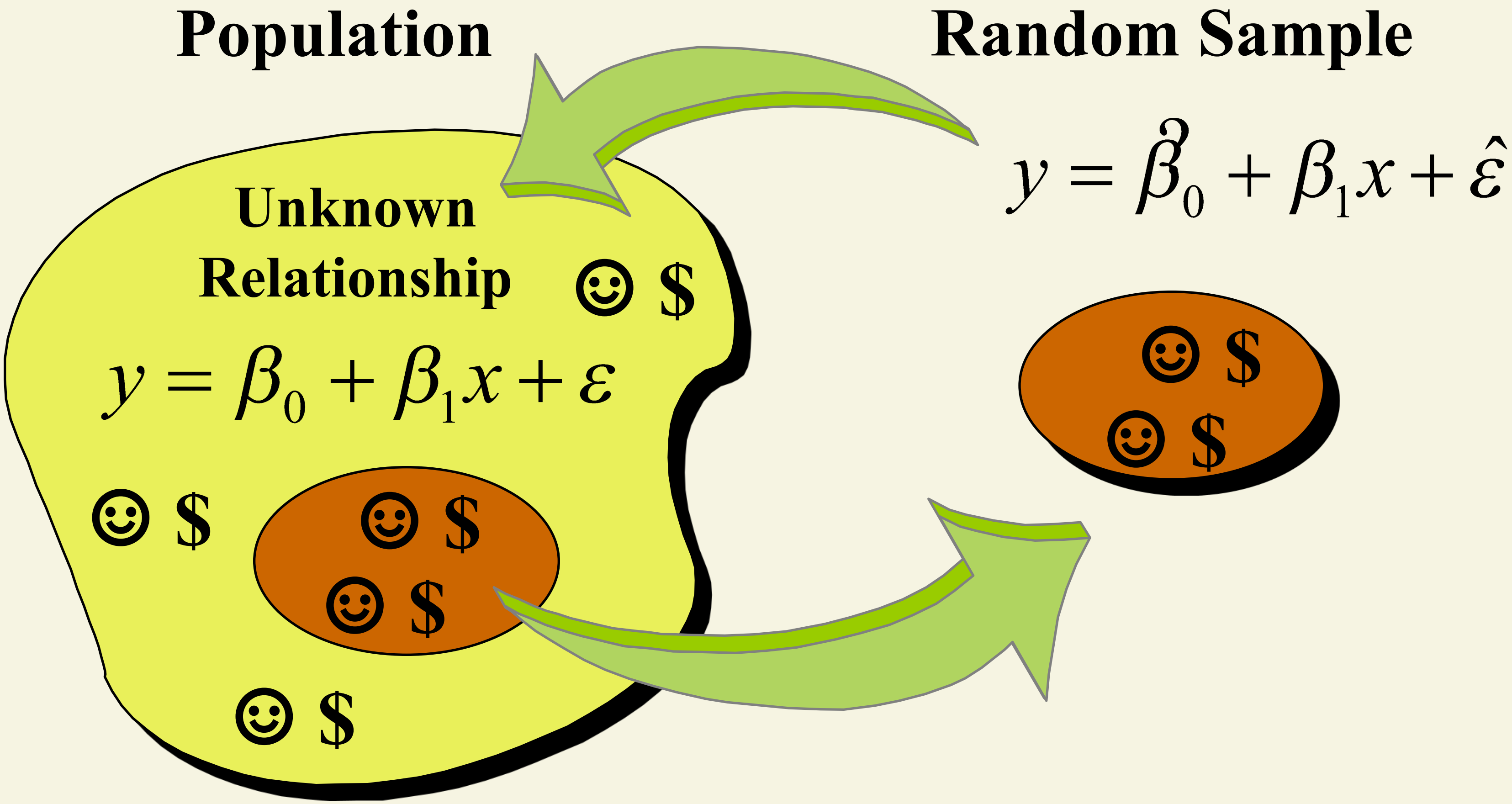
Dependent (Response) Variable

Independent (Explanatory) Variable

Line of Means



Population & Sample Regression Models



Population

Unknown Relationship

$y = \beta_0 + \beta_1 x + \varepsilon$

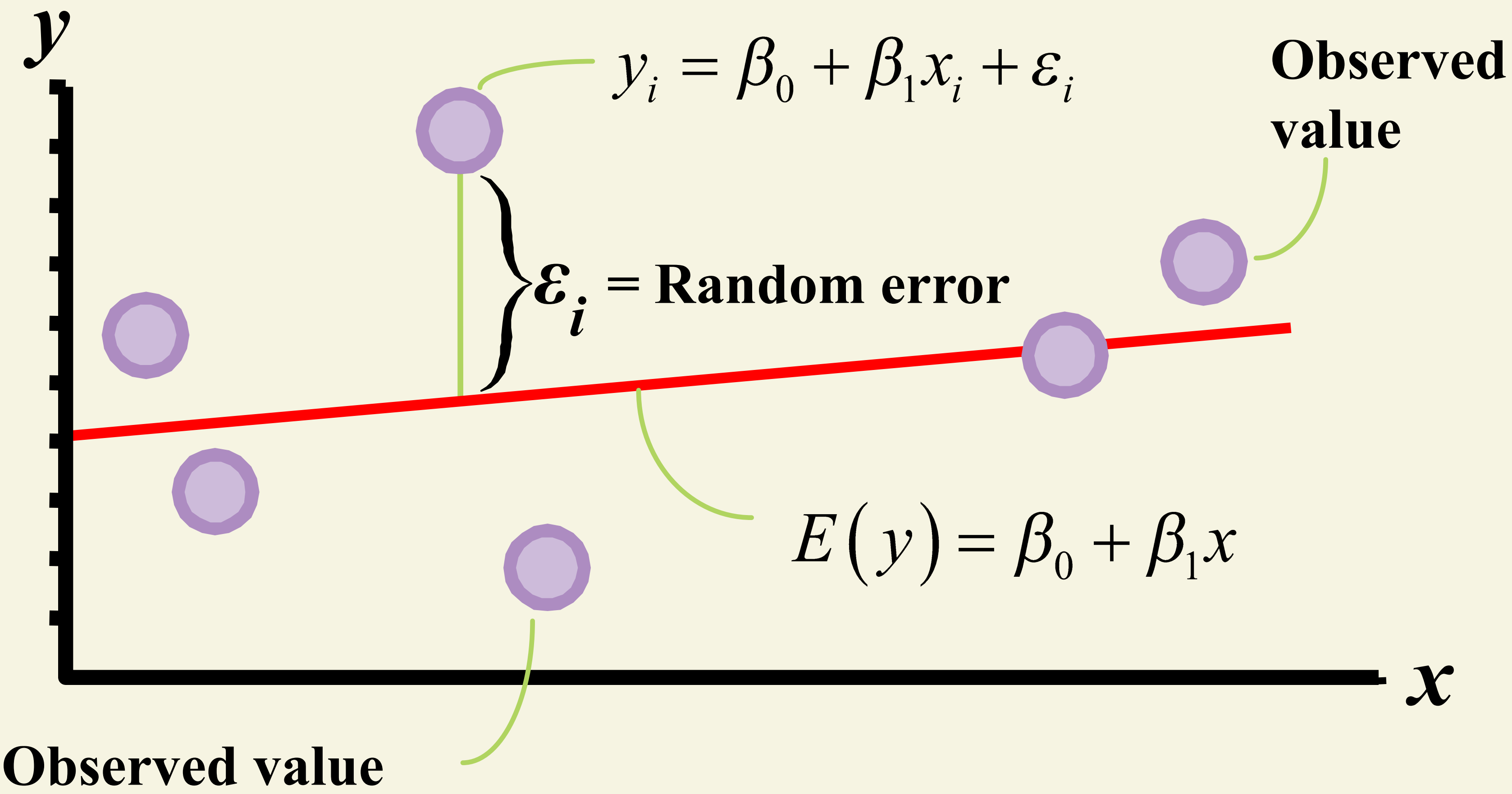
☺ $ ☺ $ ☺ $ ☺ $ ☺ $

Random Sample

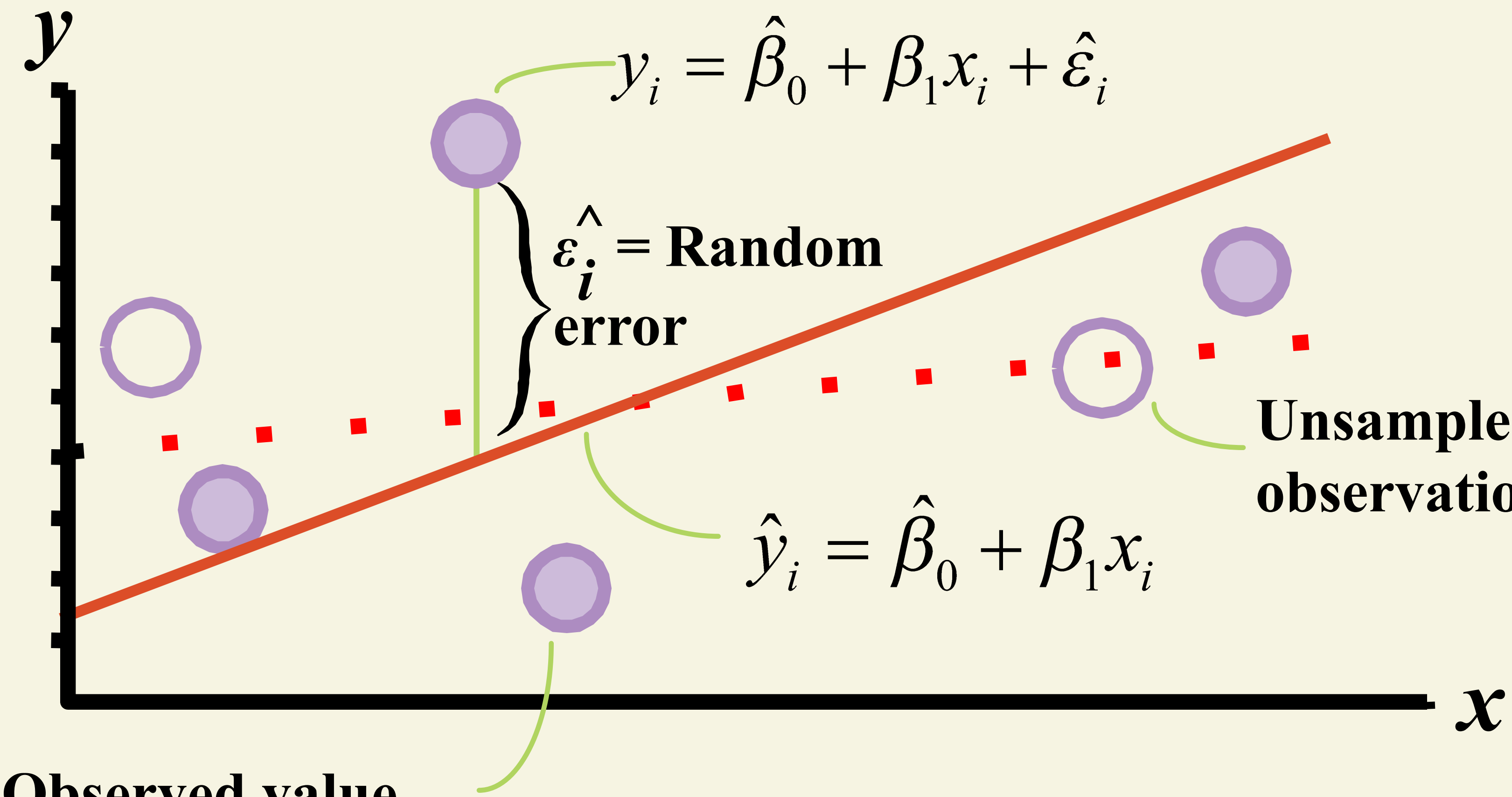
$y = \hat{\beta}_0 + \beta_1 x + \hat{\varepsilon}$

☺ $ ☺ $

Population Linear Regression Model


y
$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$
Observed value
ε_i = Random error
$E(y) = \beta_0 + \beta_1 x$
x
Observed value

Sample Linear Regression Model



$y_i = \hat{\beta}_0 + \beta_1 x_i + \hat{\varepsilon}_i$

$\hat{\varepsilon}_i$ = Random error

$\hat{y}_i = \hat{\beta}_0 + \beta_1 x_i$

Unsampled observation

Observed value

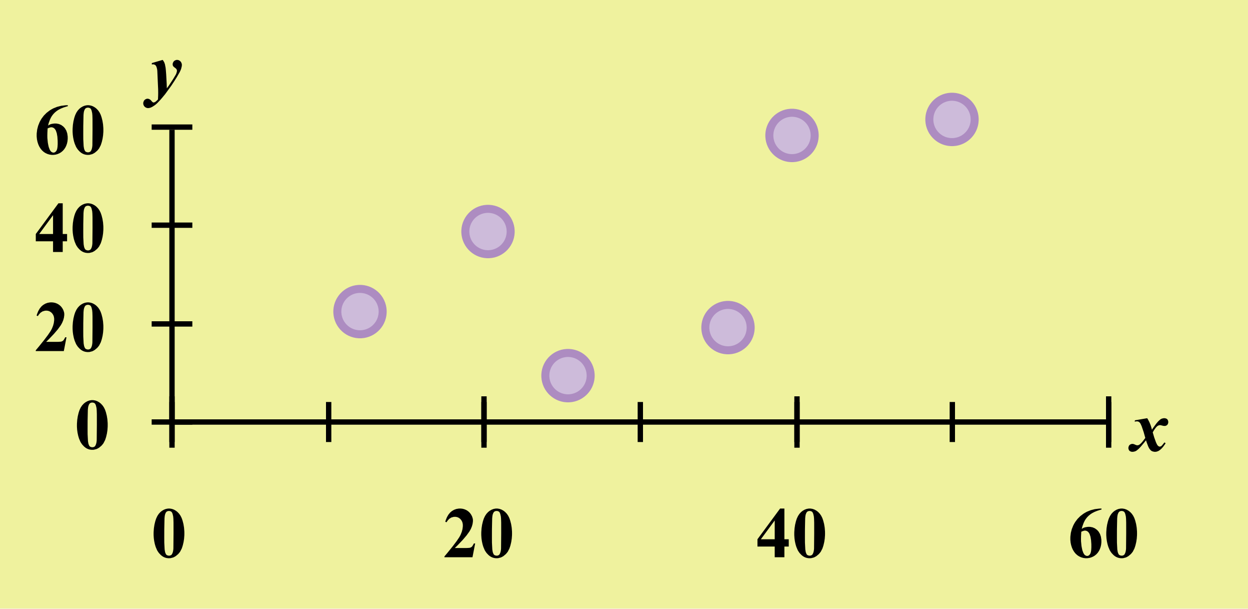
Estimating Parameters: Least Squares Method

Regression Modeling Steps

1. Hypothesize deterministic component
2. **Estimate unknown model parameters**
3. Specify probability distribution of random error term
    - Estimate standard deviation of error
4. Evaluate model
5. Use model for prediction and estimation

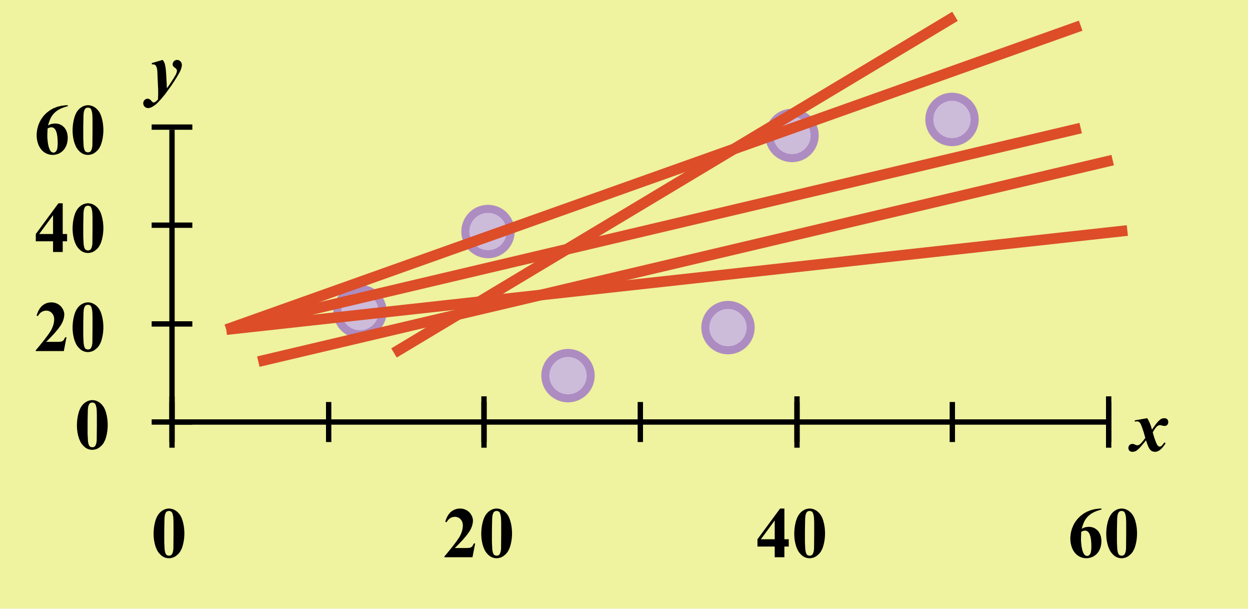
Scattergram

1. Plot of all (x_i, y_i) pairs
2. Suggests how well model will fit



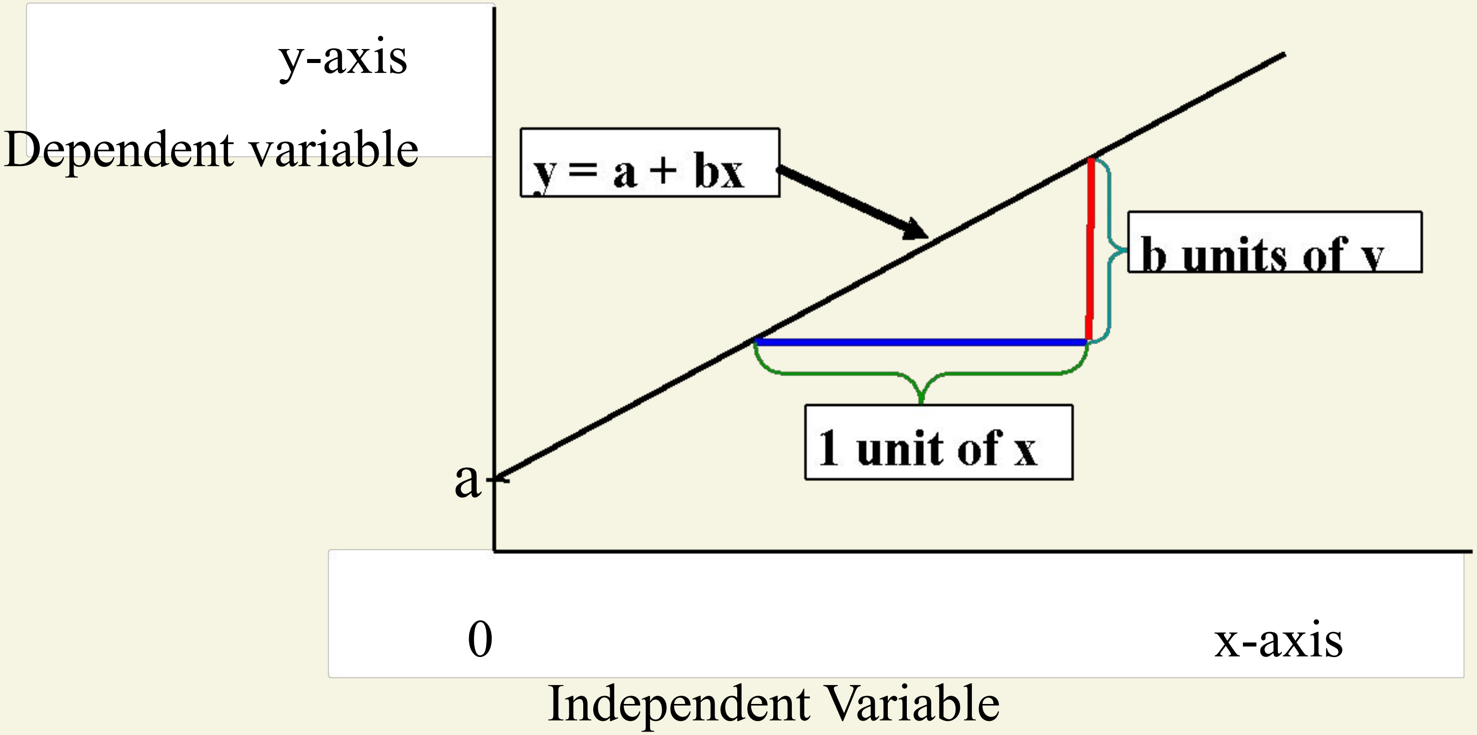
Thinking Challenge

- How would you draw a line through the points?
- How do you determine which line ‘fits best’?



Simple Linear Regression Concepts

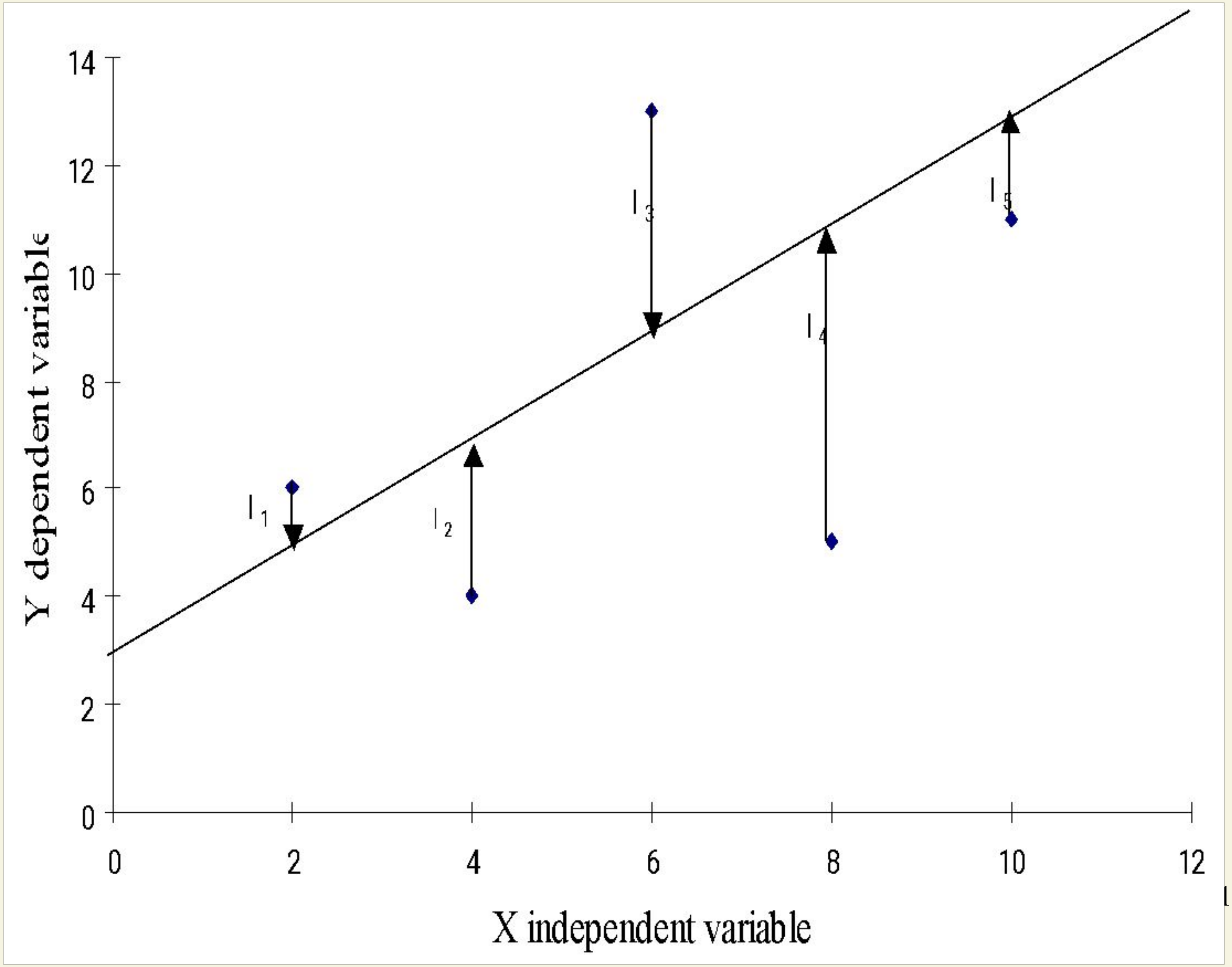
In general, simple linear regression finds the best straight line for describing the relationship between two variables. In its simplest form, which is what we consider here, it does not do a very good job of assessing how well the line describes the data, but nevertheless provides useful information.

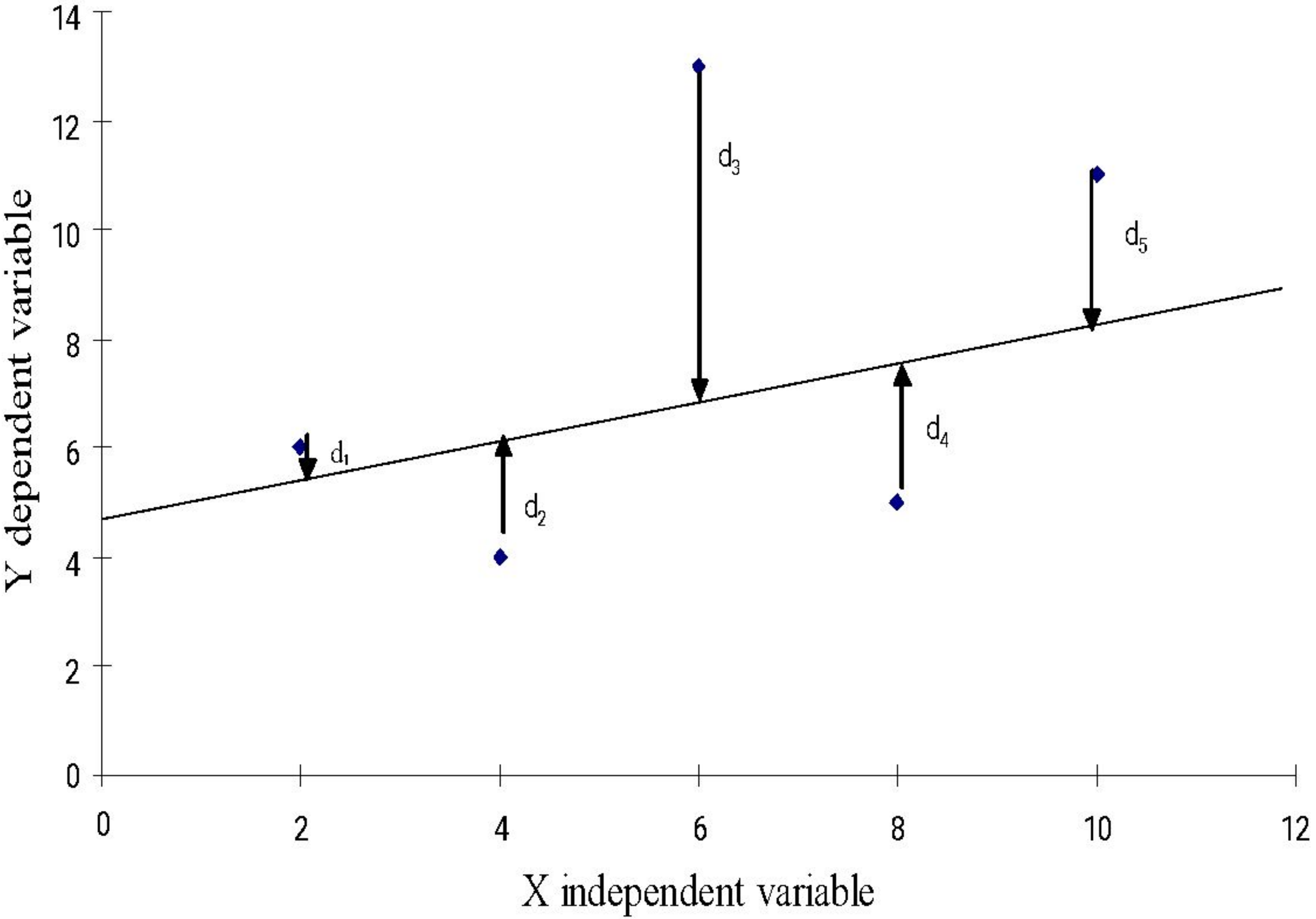


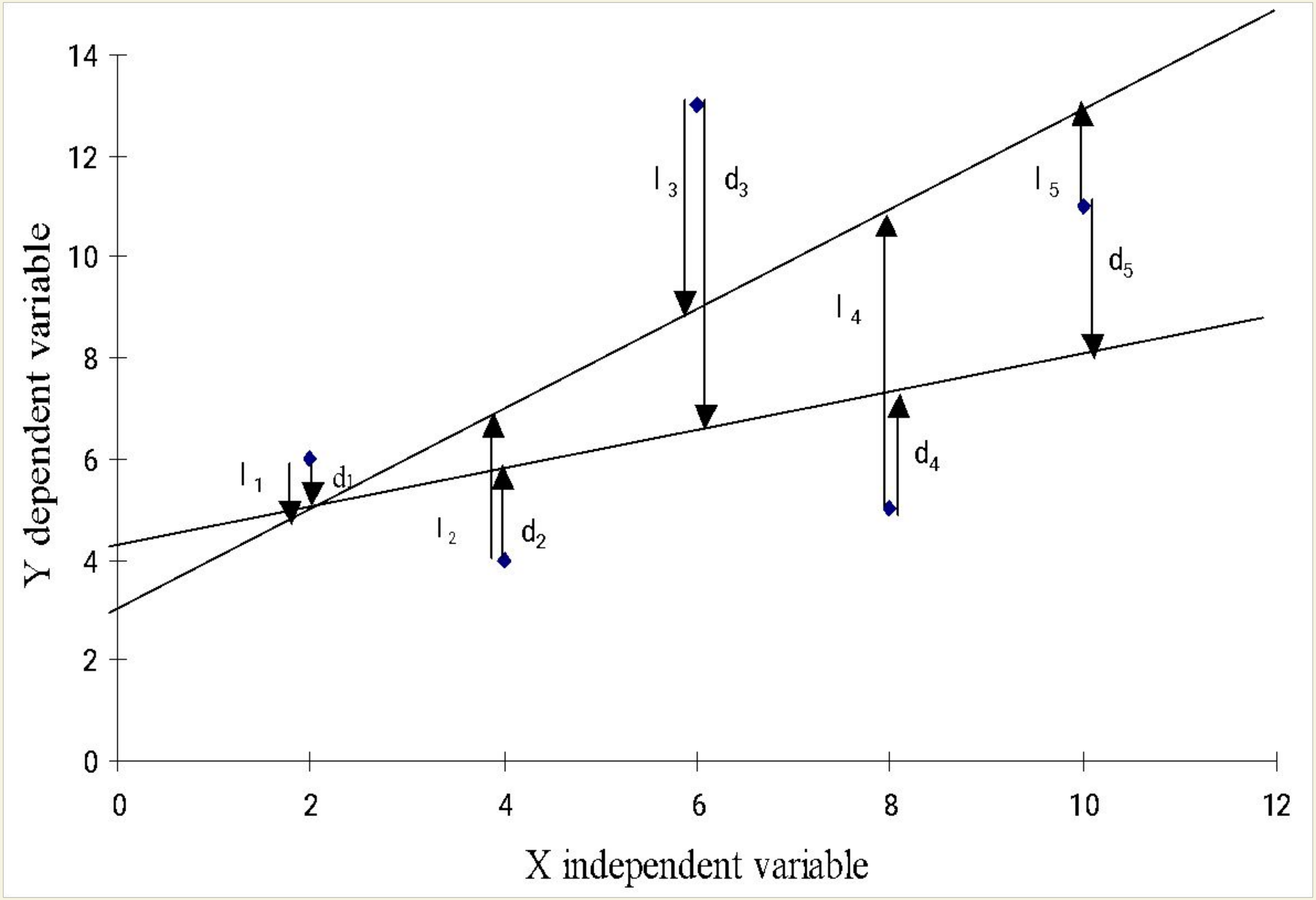
a = Intercept, that is, the point where the line crosses the y-axis, which is the value of y at **x** = 0.

b = Slope of the regression line, that is, the number of units of increase (positive slope) or decrease (negative slope) in **y**

The Regression Line




14
12
10
8
6
4
2
0
Y dependent variable
0
2
4
6
8
10
12
X independent variable
d_1
d_2
d_3
d_4
d_5


14
12
10
8
6
4
2
0
0
2
4
6
8
10
12
Y dependent variable
X independent variable
l_1
d_1
l_2
d_2
l_3
d_3
l_4
d_4
l_5
d_5

The context for simple linear regression is that we have a random sample of persons from a set of well-defined populations, each defined by a specific value for x-variable. We have measurements of another variable, the y-variable so that we have two variables for each person. For simple linear regression, we focus on a straight line that depicts the relationship between these two variables. The best straight line is the one for which the sum of the squared vertical distances of each point from the line is the least. This "least squares" line has ***slope***

$$b = \frac{\sum xy - \sum x \sum y / n}{\sum x^2 - (\sum x)^2 / n} = \frac{SS(xy)}{SS(x)},$$

and ***intercept*** $a = \overline{y} - b\overline{x}.$

For this situation, the sample line

$$y = a + bx$$

is an estimate of the population line

$$Y = \alpha + \beta x,$$

and a and b are estimates of α and β respectively. For a specific value of x, such as x = 10, the value for y calculated from the regression equation is

$$\hat{y} = a + b(x = 10),$$

which is called the regression estimate of Y at the val

Least Squares

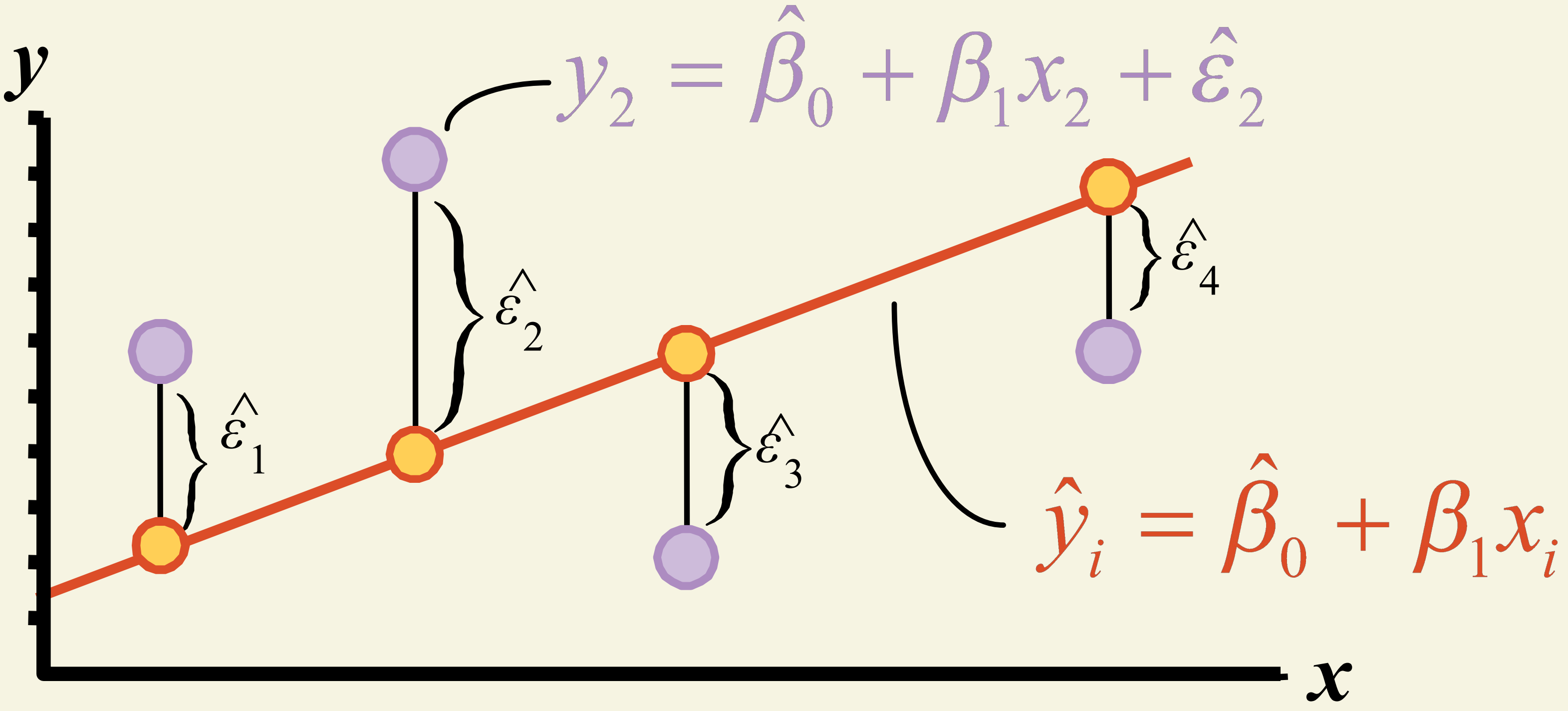
- ‘Best fit’ means difference between actual *y* values and predicted *y* values are a minimum
  - *But* positive differences off-set negative

$$\sum_{i=1}^{n}\left(y_i - \hat{y}_i\right)^2 = \sum_{i=1}^{n}\hat{\varepsilon}_i^2$$

- Least Squares minimizes the Sum of the Squared Differences (SSE)

Least Squares Graphically

LS minimizes $\sum_{i=1}^{n} \hat{\varepsilon}_i^2 = \varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2$



y
$y_2 = \hat{\beta}_0 + \beta_1 x_2 + \hat{\varepsilon}_2$
$\hat{\varepsilon}_1$
$\hat{\varepsilon}_2$
$\hat{\varepsilon}_3$
$\hat{\varepsilon}_4$
$\hat{y}_i = \hat{\beta}_0 + \beta_1 x_i$
x

Coefficient Equations

Prediction Equation $\hat{y} = \hat{\beta}_0 + \beta_1 x$

Slope $$\hat{\beta}_1 = \frac{SS_{xy}}{SS_{xx}} = \frac{\sum_{i=1}^{n} x_i y_i - \frac{\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n}}{\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}}$$

***y*-intercept** $\hat{\beta}_0 = \overline{y} - \beta_1 \overline{x}$

Computation Table

x_i	y_i	x_i^2	y_i^2	x_iy_i
x_1	y_1	x_1^2	y_1^2	x_1y_1
x_2	y_2	x_2^2	y_2^2	x_2y_2
:	:	:	:	:
x_n	y_n	x_n^2	y_n^2	x_ny_n
Σx_i	Σy_i	Σx_i^2	Σy_i^2	Σx_iy_i

Interpretation of Coefficients

1. Slope ($\hat{\beta}_1$)
   - Estimated y changes by $\hat{\beta}_1$ for each 1unit increase in x
     - If $\hat{\beta}_1 = 2$, then Sales (y) is expected to increase by 2 for each 1 unit increase in Advertising (x)

2. Y-Intercept ($\hat{\beta}_0$)
   - Average value of y when $x = 0$
     - If $\hat{\beta}_0 = 4$, then Average Sales (y) is expected to be 4 when Advertising (x) is 0

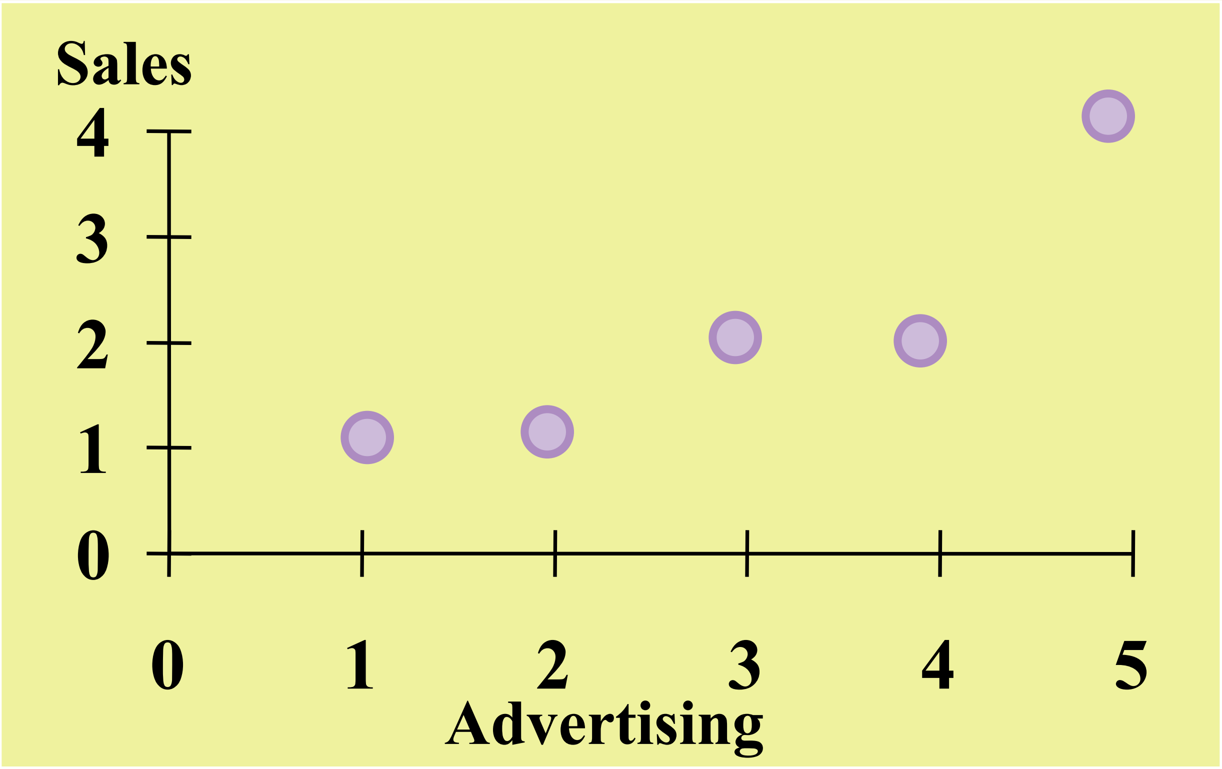
Least Squares Example

You're a marketing analyst for Hasbro Toys. You gather the following data:

Ad $	Sales (Units)
1	1
2	1
3	2
4	2
5	4

Find the **least squares line** relating sales and advertising.

Scattergram
Sales vs. Advertising



Parameter Estimation Solution Table

x_i	y_i	x_i^2	y_i^2	x_iy_i
1	1	1	1	1
2	1	4	1	2
3	2	9	4	6
4	2	16	4	8
5	4	25	16	20
15	10	55	26	37

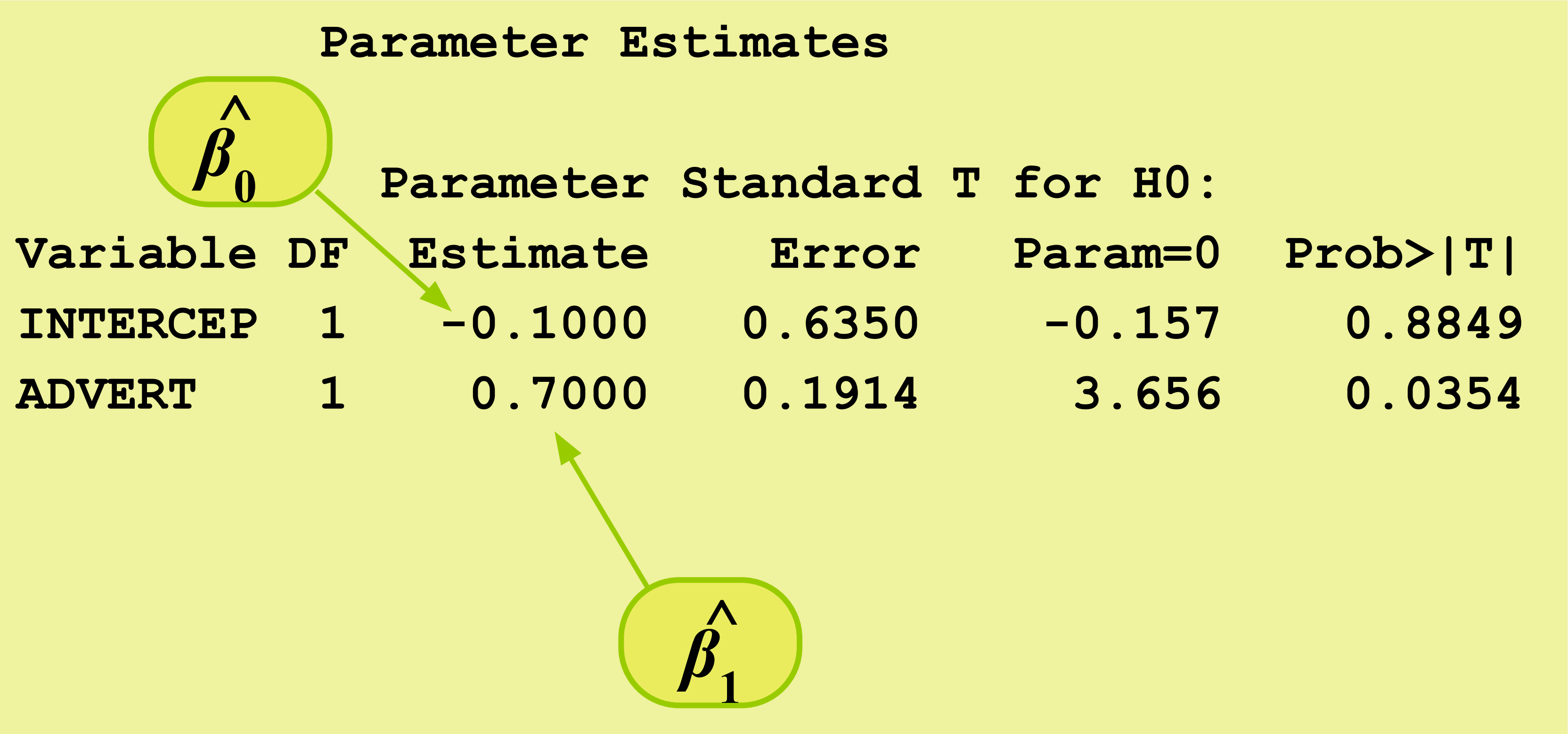
Parameter Estimation Solution

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} x_i y_i - \frac{\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n}}{\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}} = \frac{37 - \frac{(15)(10)}{5}}{55 - \frac{(15)^2}{5}} = .70$$

$$\hat{\beta}_0 = \overline{y} - \beta_1 \overline{x} = 2 - (.70)(3) = -.10$$

$$\hat{y} = -.1 + .7x$$

Parameter Estimation Computer Output



```
          Parameter Estimates

            Parameter Standard T for H0:
Variable DF  Estimate    Error    Param=0   Prob>|T|
INTERCEP  1   -0.1000   0.6350     -0.157     0.8849
ADVERT    1    0.7000   0.1914      3.656     0.0354
```

$\hat{\beta}_0$

$\hat{\beta}_1$

$$\hat{y} = -.1 + .7x$$

Simple Regression Example

The following data are diastolic blood pressure (DBP) measurements taken at different times after an intervention for n = 5 persons. For each person, the data available include the time of the measurement and the DBP level. Of interest is the relationship between these two variables.

	Time		DPB		
Patient	x	x^2	y	y^2	xy
1	0	0	72	5,184	0
2	5	25	66	4,356	330
3	10	100	70	4,900	700
4	15	225	64	4,096	960
5	20	400	66	4,356	1,320
Sum	50	750	338	22,892	3,310
Mean	10		67.6		
n	5		5		

For the blood pressure data,

$$\bar{x} = 50/5 = 10,$$

$$\bar{y} = 338/5 = 67.6,$$

the slope is

$$b = \frac{\sum xy - \sum x \sum y / n}{\sum x^2 - (\sum x)^2 / n} = \frac{SS(xy)}{SS(x)},$$

$$b = \frac{3,310 - (50)(338)/5}{750 - (50)^2/5} = -0.28$$

and the intercept is

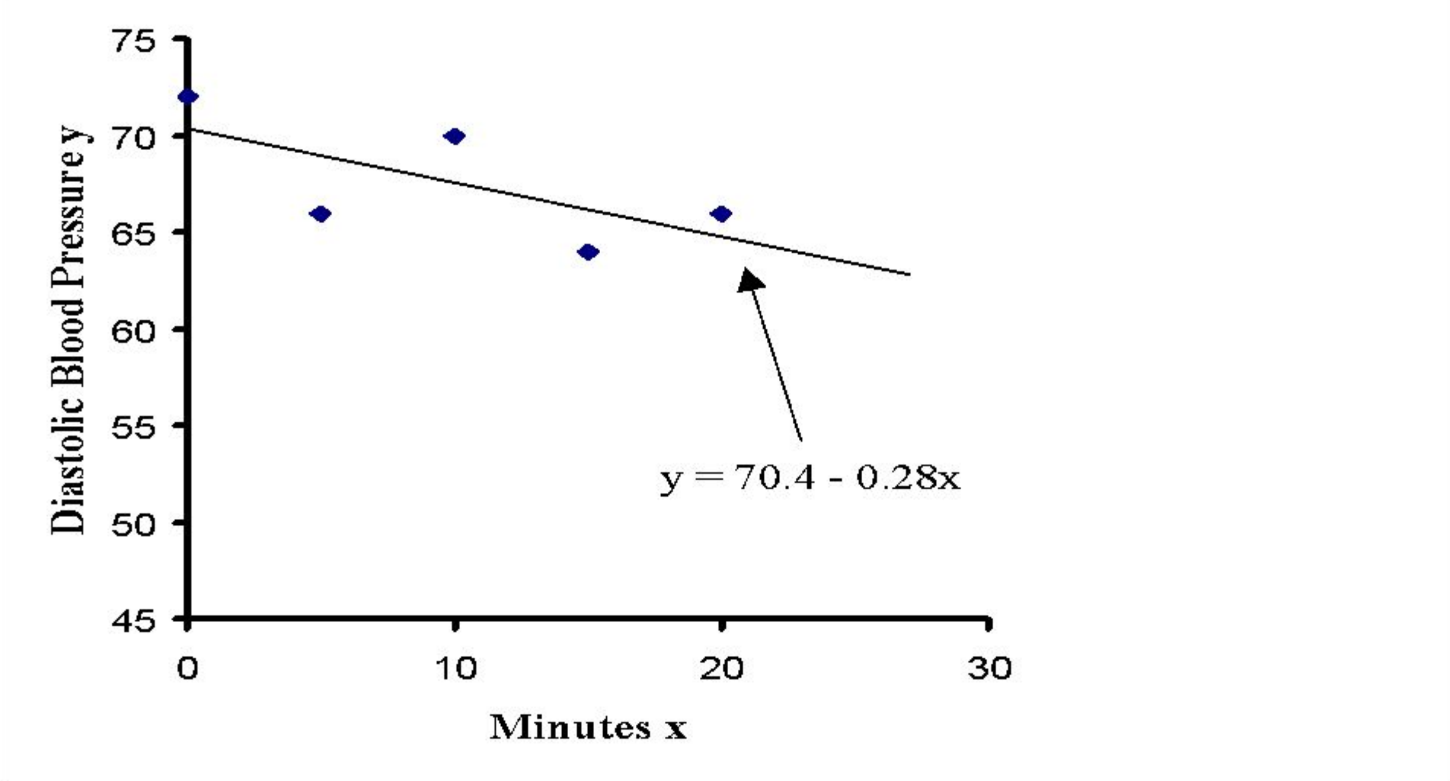
$$a = \bar{y} - b\bar{x},$$

$$a = 67.6 - (-0.28)10 = 70.4$$

The best line is

$$y = a + bx = 70.4 - 0.28x$$

Patient	Time x	DBP y
1	0	72
2	5	66
3	10	70
4	15	64
5	20	66

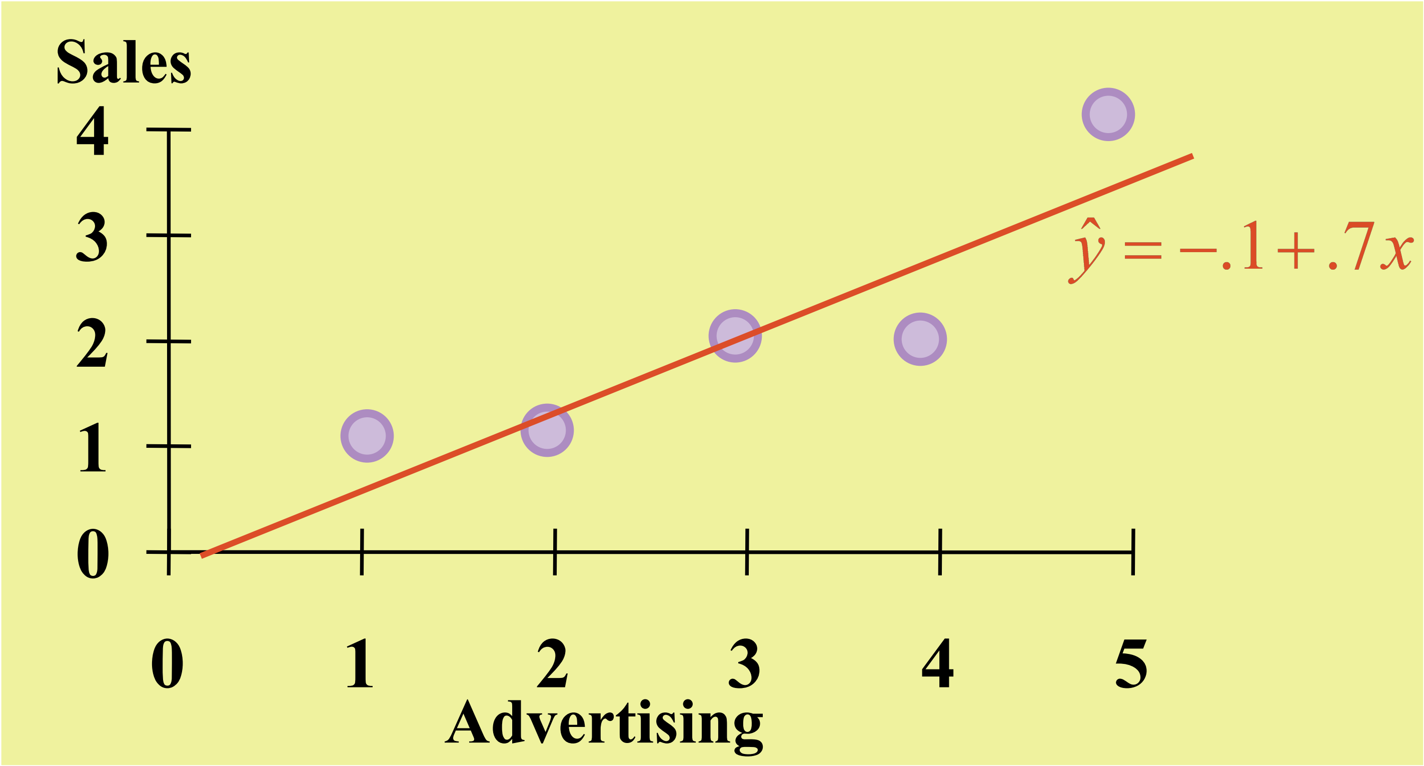


Coefficient Interpretation Solution

1. Slope ($\hat{\beta}_1$)
   - Sales Volume (y) is expected to increase by .7 units for each $1 increase in Advertising (x)

2. Y-Intercept ($\hat{\beta}_0$)
   - Average value of Sales Volume (y) is -.10 units when Advertising (x) is 0
     - Difficult to explain to marketing manager
     - Expect some sales without advertising

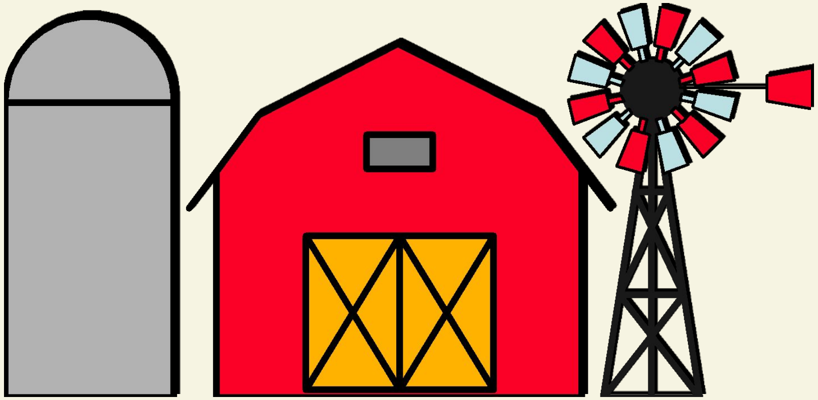
Regression Line Fitted to the Data



Least Squares Thinking Challenge

You're an economist for the county cooperative. You gather the following data:

Fertilizer (lb.)	Yield (lb.)
4	3.0
6	5.5
10	6.5
12	9.0



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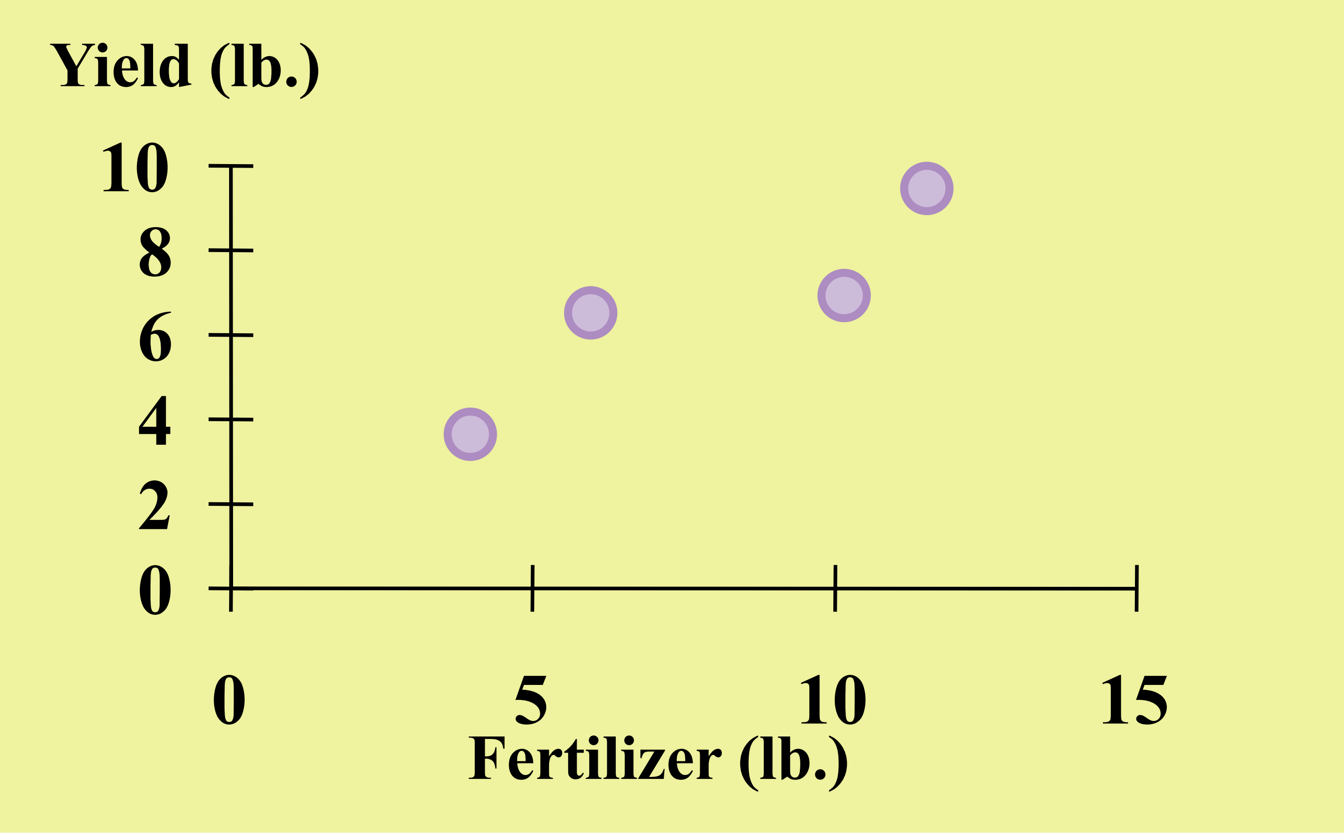
Find the **least squares line** relating crop yield and fertilizer.

Scattergram
Crop Yield vs. Fertilizer*



Parameter Estimation Solution Table*

x_i	y_i	x_i^2	y_i^2	$x_i y_i$
4	3.0	16	9.00	12
6	5.5	36	30.25	33
10	6.5	100	42.25	65
12	9.0	144	81.00	108
32	24.0	296	162.50	218

Parameter Estimation Solution*

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n} x_i y_i - \frac{\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n}}{\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}} = \frac{218 - \frac{(32)(24)}{4}}{296 - \frac{(32)^2}{4}} = .65$$

$$\hat{\beta}_0 = \overline{y} - \beta_1 \overline{x} = 6 - (.65)(8) = .80$$

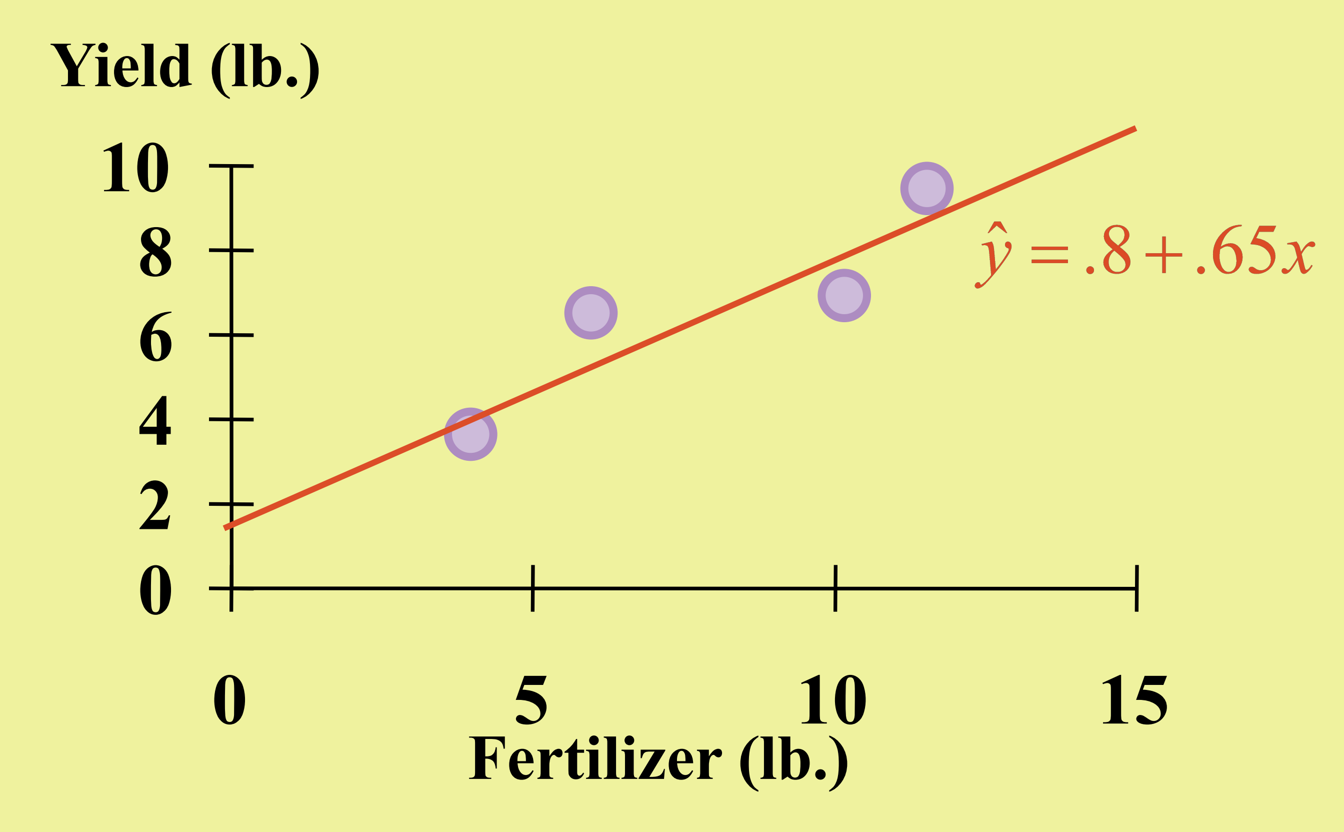
$$\hat{y} = .8 + .65x$$

Coefficient Interpretation Solution*

1. Slope ($\hat{\beta}_1$)
   - Crop Yield (*y*) is expected to increase by .65 lb. for each 1 lb. increase in Fertilizer (*x*)

2. Y-Intercept ($\hat{\beta}_0$)
   - Average Crop Yield (*y*) is expected to be 0.8 lb. when no Fertilizer (*x*) is used

Regression Line Fitted to the Data*



Probability Distribution of Random Error

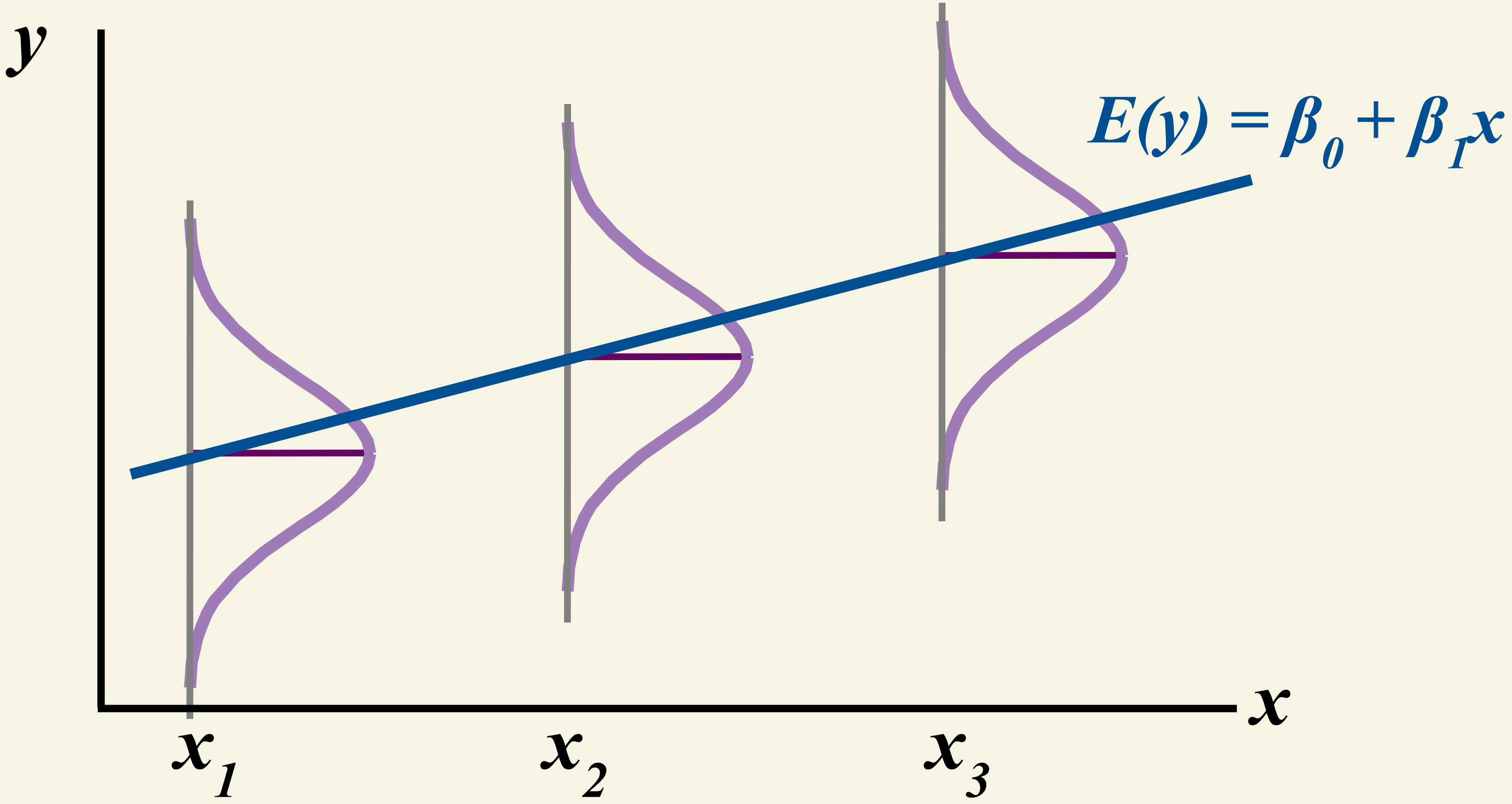
Regression Modeling Steps

1. Hypothesize deterministic component
2. Estimate unknown model parameters
3. **Specify probability distribution of random error term**
   - Estimate standard deviation of error
4. Evaluate model
5. Use model for prediction and estimation

Linear Regression Assumptions

1. Mean of probability distribution of error, ε, is 0
2. Probability distribution of error has constant variance
3. Probability distribution of error, ε, is normal
4. Errors are independent

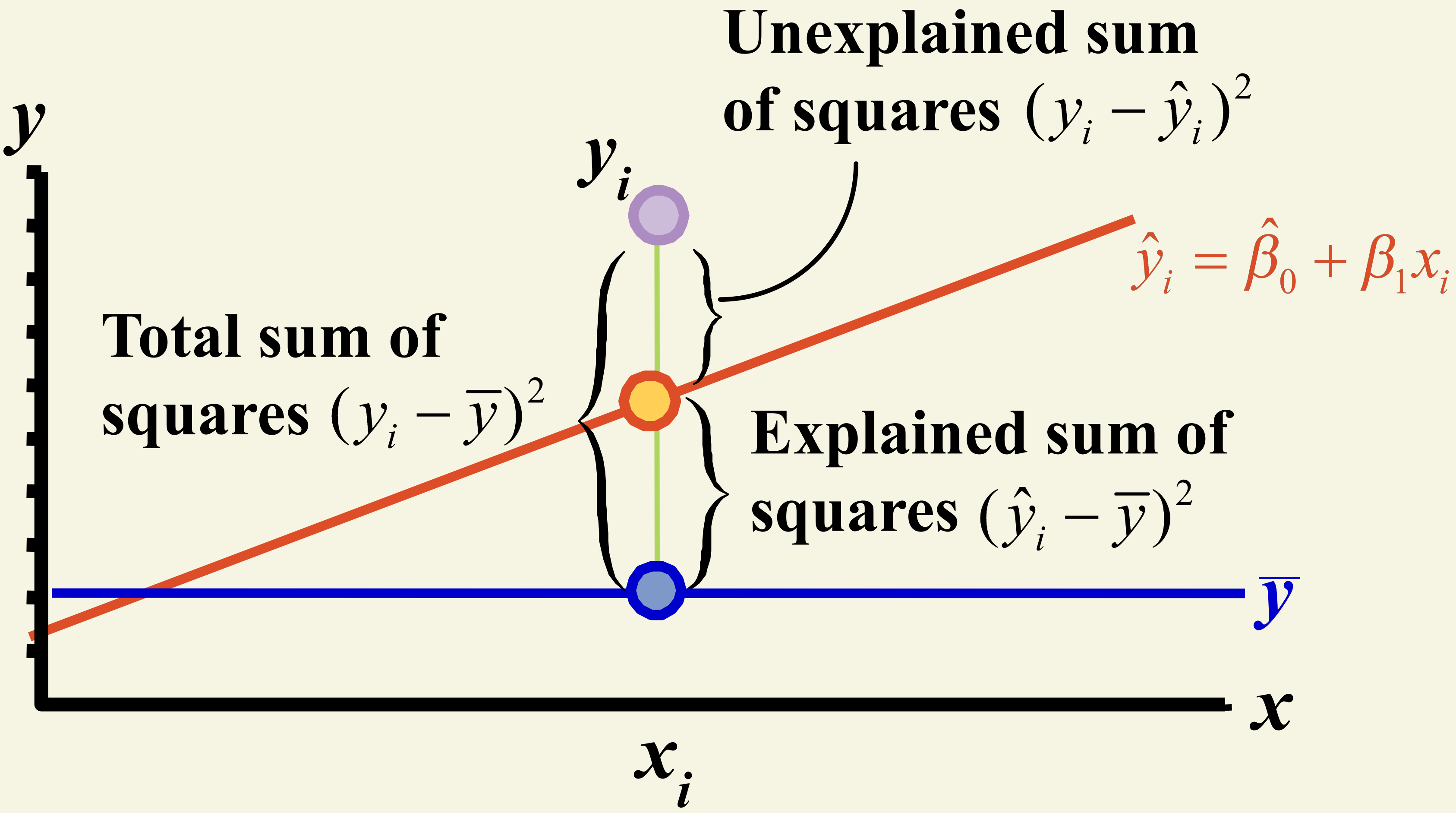
Error Probability Distribution



Random Error Variation

- Variation of actual y from predicted y, $\hat{y}$
- Measured by standard error of regression model
  - Sample standard deviation of $\hat{\varepsilon}$: s
- Affects several factors
  - Parameter significance
  - Prediction accuracy

Variation Measures



Unexplained sum of squares $(y_i - \hat{y}_i)^2$

Total sum of squares $(y_i - \overline{y})^2$

Explained sum of squares $(\hat{y}_i - \overline{y})^2$

$\hat{y}_i = \hat{\beta}_0 + \beta_1 x_i$

Estimation of σ^2

$$s^2 = \frac{SSE}{n-2} \quad where \quad SSE = \sum \left(y_i - \hat{y}_i \right)^2$$

$$s = \sqrt{s^2} = \sqrt{\frac{SSE}{n-2}}$$

Calculating SSE, s^2, s Example

You're a marketing analyst for Hasbro Toys. You gather the following data:

Ad $	Sales (Units)
1	1
2	1
3	2
4	2
5	4

Find **SSE**, $\mathbf{s^2}$, and **s**.

Calculating SSE Solution

x_i	y_i	$\hat{y} = -.1 + .7x$	$y - \hat{y}$	$(y - \hat{y})^2$
1	1	.6	.4	.16
2	1	1.3	-.3	.09
3	2	2	0	0
4	2	2.7	-.7	.49
5	4	3.4	.6	.36
				SSE=1.1

Calculating s^2 and s Solution

$$s^2 = \frac{SSE}{n-2} = \frac{1.1}{5-2} = .36667$$

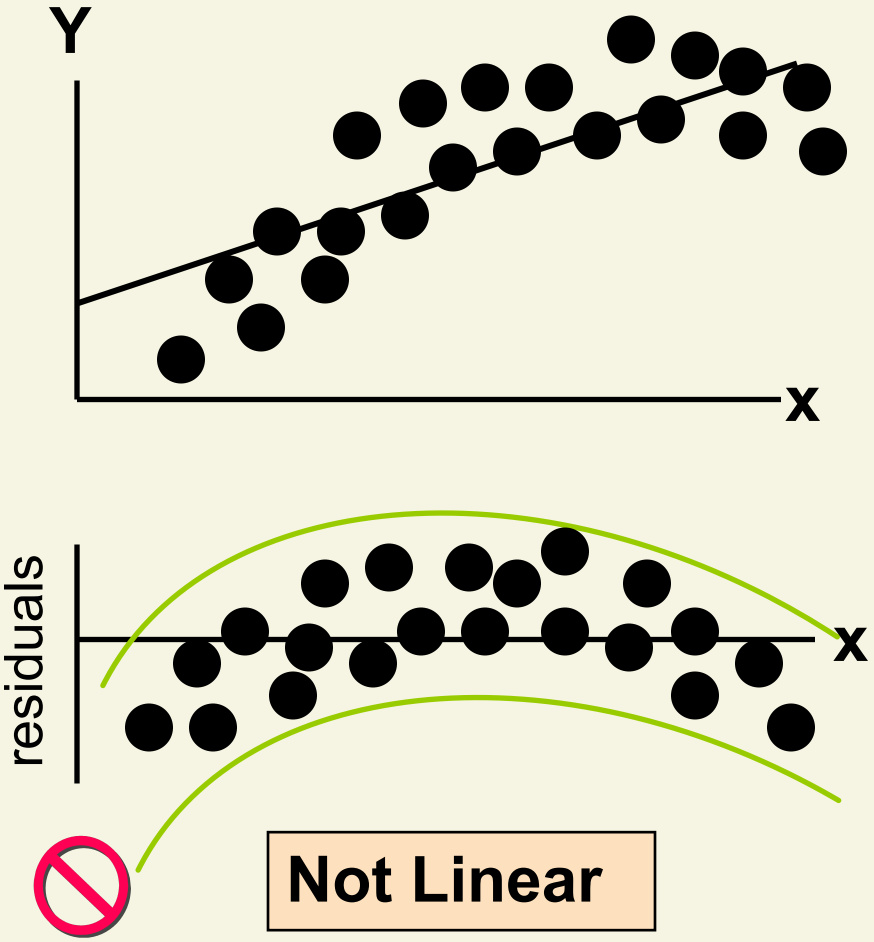
$$s = \sqrt{.36667} = .6055$$

Residual Analysis

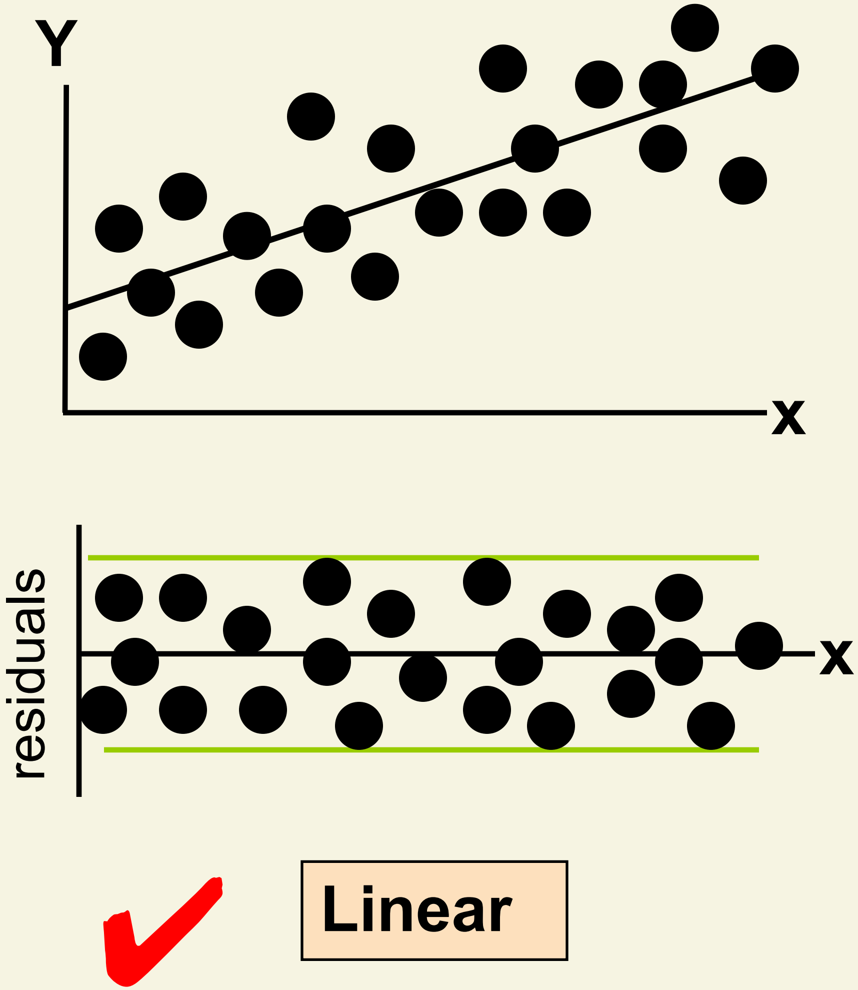
$$\boxed{e_i = Y_i - \hat{Y}_i}$$

- The residual for observation i, e_i, is the difference between its observed and predicted value
- Check the assumptions of regression by examining the residuals
  - Examine for linearity assumption
  - Evaluate independence assumption
  - Evaluate normal distribution assumption
  - Examine for constant variance for all levels of X (homoscedasticity)

Residual Analysis for Linearity



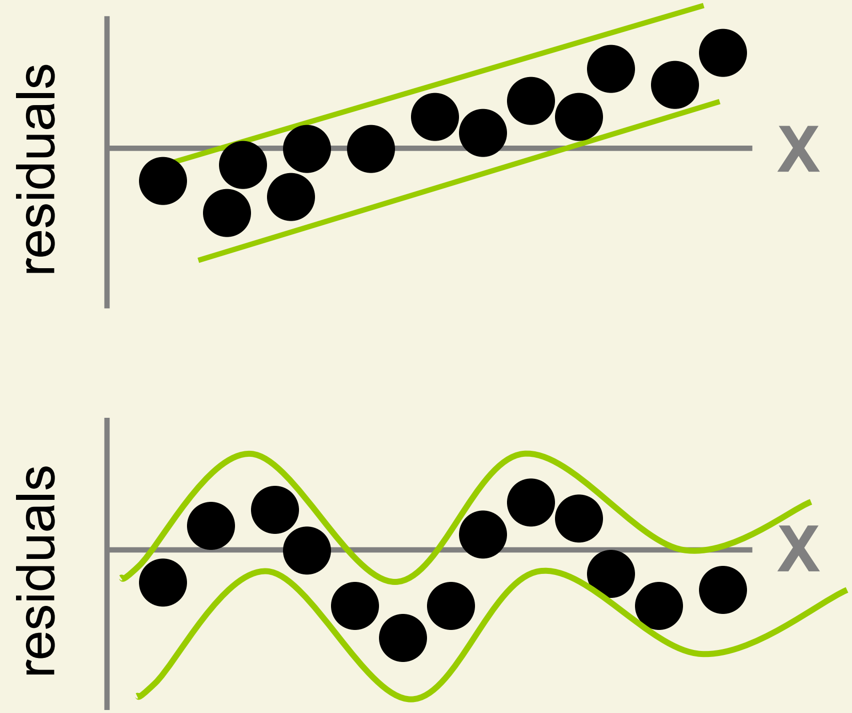
Not Linear



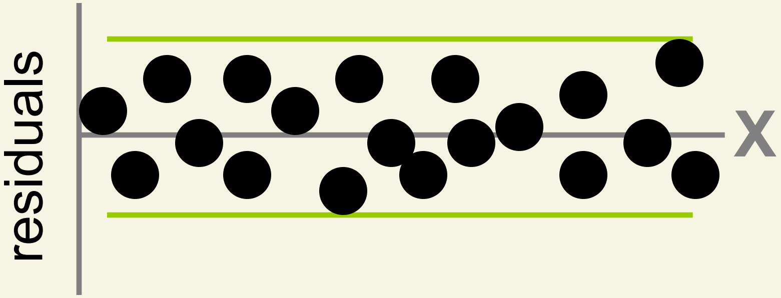
✓ Linear

Residual Analysis for Independence

🚫 Not Independent



✓ Independent

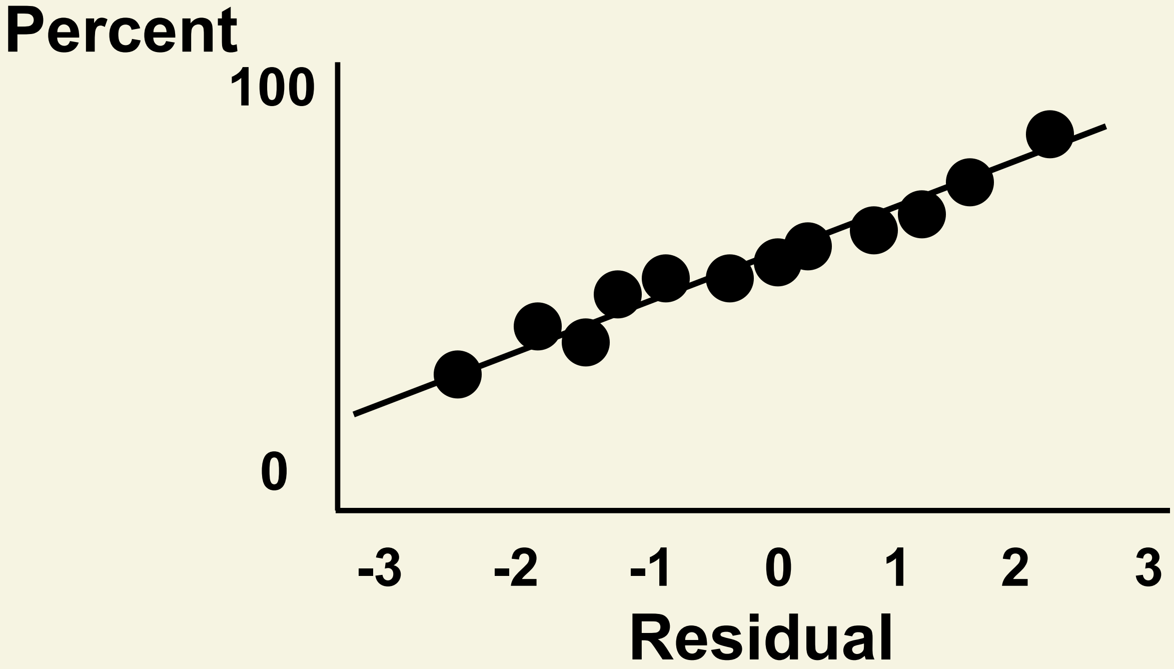


Checking for Normality

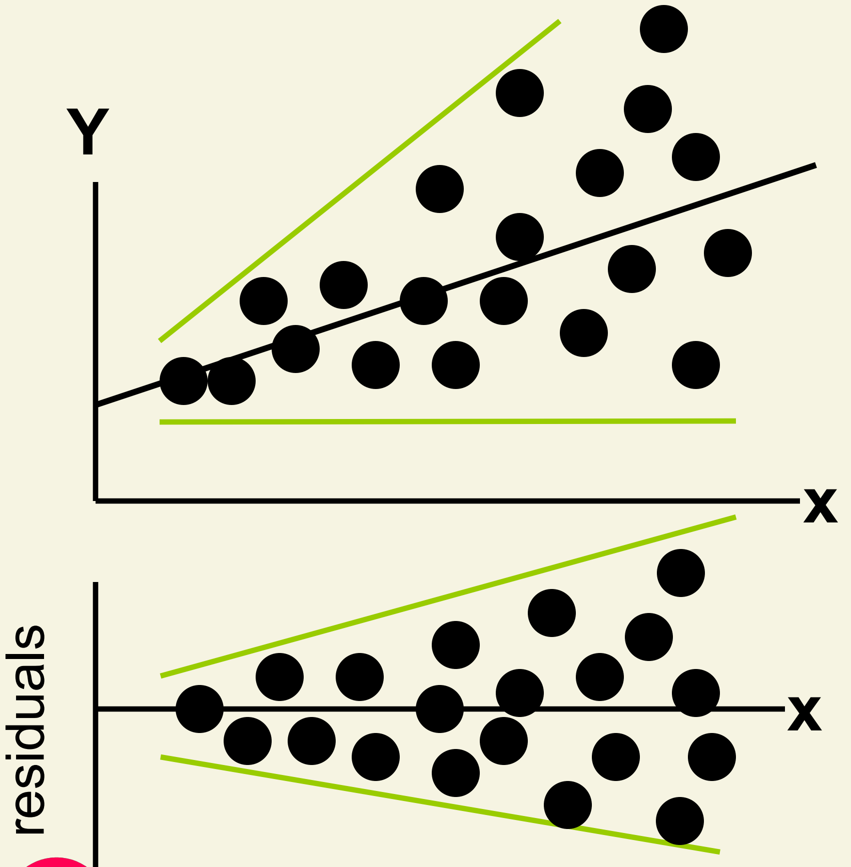
- Examine the Stem-and-Leaf Display of the Residuals
- Examine the Boxplot of the Residuals
- Examine the Histogram of the Residuals
- Construct a Normal Probability Plot of the Residuals

Residual Analysis for Normality

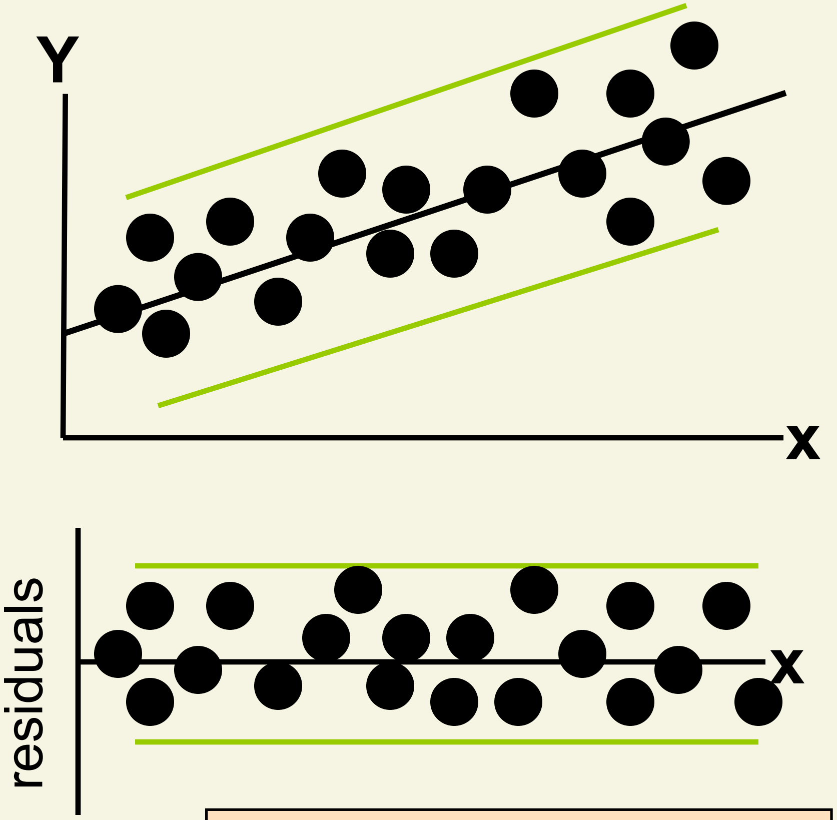
When using a normal probability plot, normal errors will approximately display in a straight line



Residual Analysis for Equal Variance



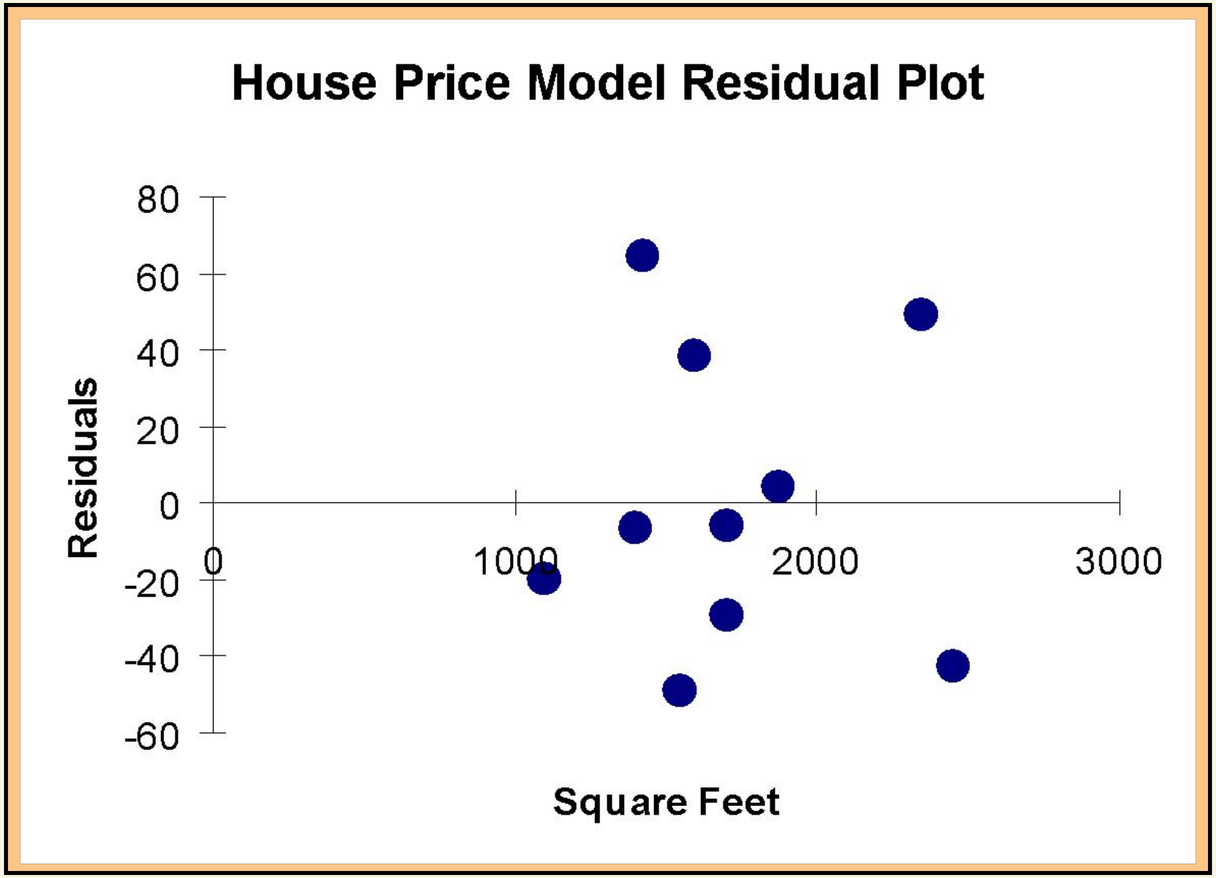
🚫 Non-constant variance



✓ Constant variance

Simple Linear Regression Example: Excel Residual Output

RESIDUAL OUTPUT		
	Predicted House Price	*Residuals*
1	251.92316	-6.923162
2	273.87671	38.12329
3	284.85348	-5.853484
4	304.06284	3.937162
5	218.99284	-19.99284
6	268.38832	-49.38832
7	356.20251	48.79749
8	367.17929	-43.17929
9	254.6674	64.33264
10	284.85348	-29.85348



Does not appear to violate any regression assumptions

Evaluating the Model

Testing for Significance

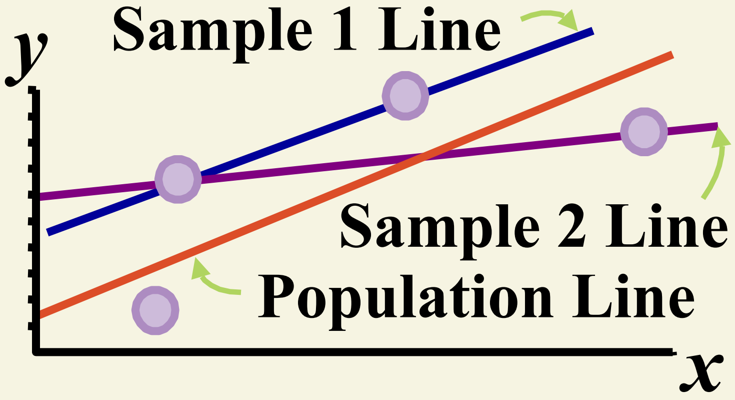
Regression Modeling Steps

1. Hypothesize deterministic component
2. Estimate unknown model parameters
3. Specify probability distribution of random error term
   - Estimate standard deviation of error
4. **Evaluate model**
5. Use model for prediction and estimation

Test of Slope Coefficient

- Shows if there is a linear relationship between *x* and *y*
- Involves population slope β_1
- Hypotheses
  - H_0: $\beta_1 = 0$ (No Linear Relationship)
  - H_a: $\beta_1 \neq 0$ (Linear Relationship)
- Theoretical basis is sampling distribution of slope

Sampling Distribution of Sample Slopes



All Possible
<u>Sample Slopes</u>

Sample 1: 2.5

Sample 2: 1.6

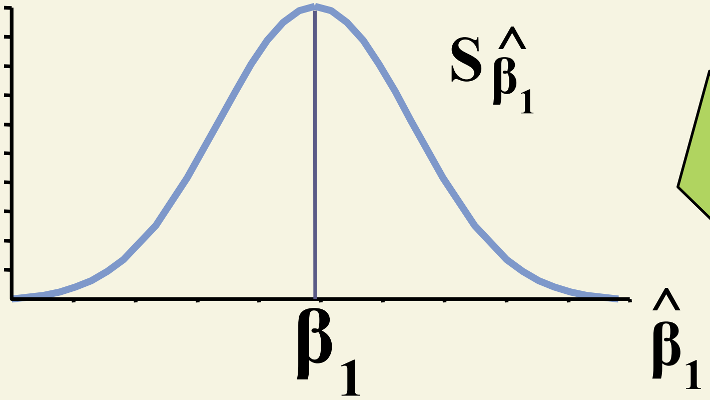
Sample 3: 1.8

Sample 4: 2.1

:   :

Very large number of sample slopes

Sampling Distribution



Slope Coefficient Test Statistic

$$t = \frac{\hat{\beta}_1}{S_{\hat{\beta}_1}} = \frac{\beta_1}{s \Big/ \sqrt{SS_{xx}}} \qquad df = n - 2$$

where

$$SS_{xx} = \sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i \right)^2}{n}$$

Test of Slope Coefficient Example

You’re a marketing analyst for Hasbro Toys.

You find $\hat{\beta}_0 = -.1$, $\hat{\beta}_1 = .7$ and $s = .6055$.

Ad $	Sales (Units)
1	1
2	1
3	2
4	2
5	4

Is the relationship **significant** at the **.05** level of significance?

Solution Table

x	y_i	x^2	y_i^2	$x_i y$
i1	1	i1	1	i 1
2	1	4	1	2
3	2	9	4	6
4	2	16	4	8
5	4	25	16	20
15	10	55	26	37

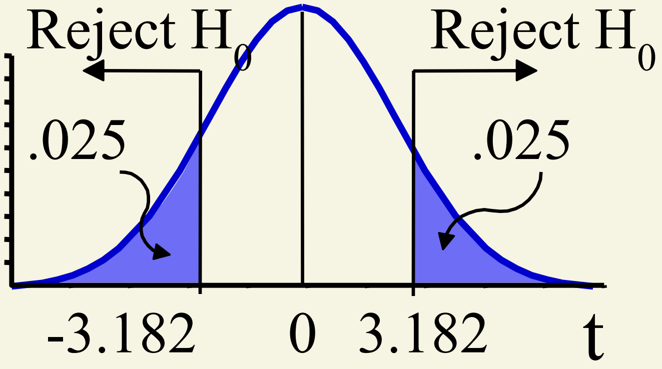
Test Statistic Solution

$$S_{\hat{\beta}_1} = \frac{S}{\sqrt{SS_{xx}}} = \frac{.6055}{\sqrt{55 - \frac{(15)^2}{5}}} = .1914$$

$$t = \frac{\hat{\beta}_1}{S_{\hat{\beta}_1}} = \frac{.70}{.1914} = 3.657$$

Test of Slope Coefficient Solution

- H_0: $\beta_1 = 0$
- H_a: $\beta_1 \neq 0$
- $\alpha = .05$
- df = 5 - 2 = 3
- Critical Value(s):



Test of Slope Coefficient Solution

Test Statistic:

$$t = \frac{\hat{\beta}_1}{S_{\hat{\beta}_1}} = \frac{.70}{.1914} = 3.657$$

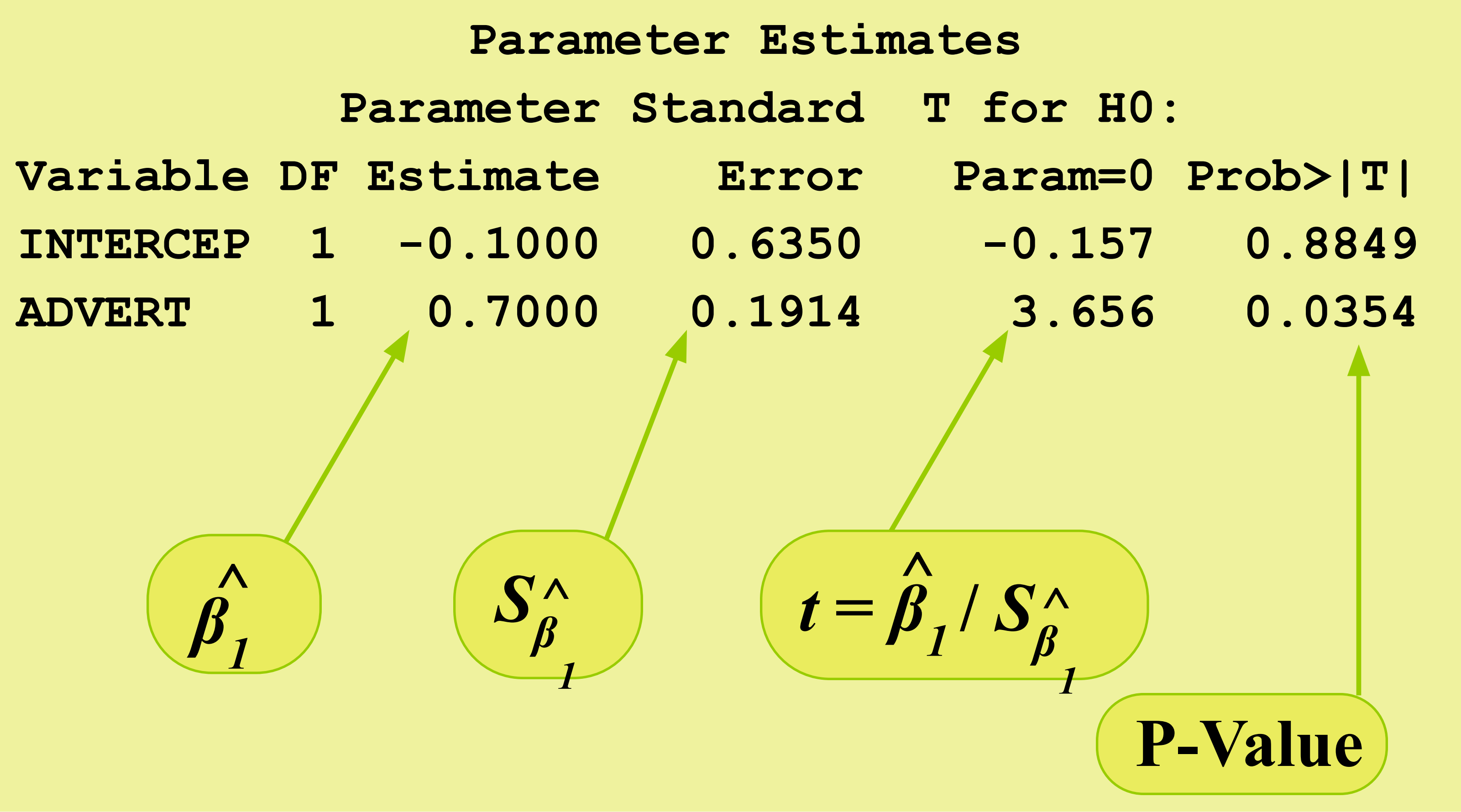
Decision:

Reject at α = .05

Conclusion:

There is evidence of a relationship

Test of Slope Coefficient Computer Output



```
                  Parameter Estimates
              Parameter Standard  T for H0:
Variable DF Estimate     Error    Param=0 Prob>|T|
INTERCEP  1  -0.1000    0.6350     -0.157   0.8849
ADVERT    1   0.7000    0.1914      3.656   0.0354
```

$\hat{\beta}_1$

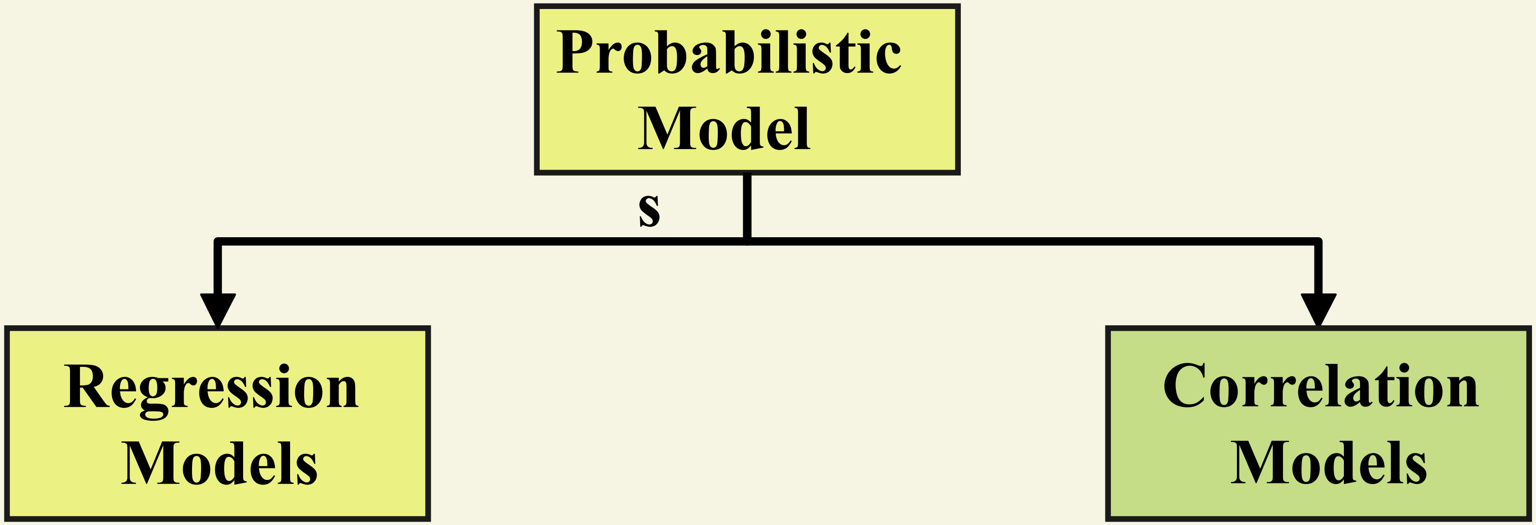
$S_{\hat{\beta}_1}$

$t = \hat{\beta}_1 / S_{\hat{\beta}_1}$

P-Value

Correlation Models

Types of Probabilistic Models



Probabilistic Model

s

Regression Models

Correlation Models

Correlation Models

- Answers ‘How strong is the **linear** relationship between two variables?’
- Coefficient of correlation
  - Sample correlation coefficient denoted *r*
  - Values range from –1 to +1
  - Measures degree of association
  - Does not indicate cause–effect relationship

Coefficient of Correlation

$$r = \frac{SS_{xy}}{\sqrt{SS_{xx}SS_{yy}}}$$

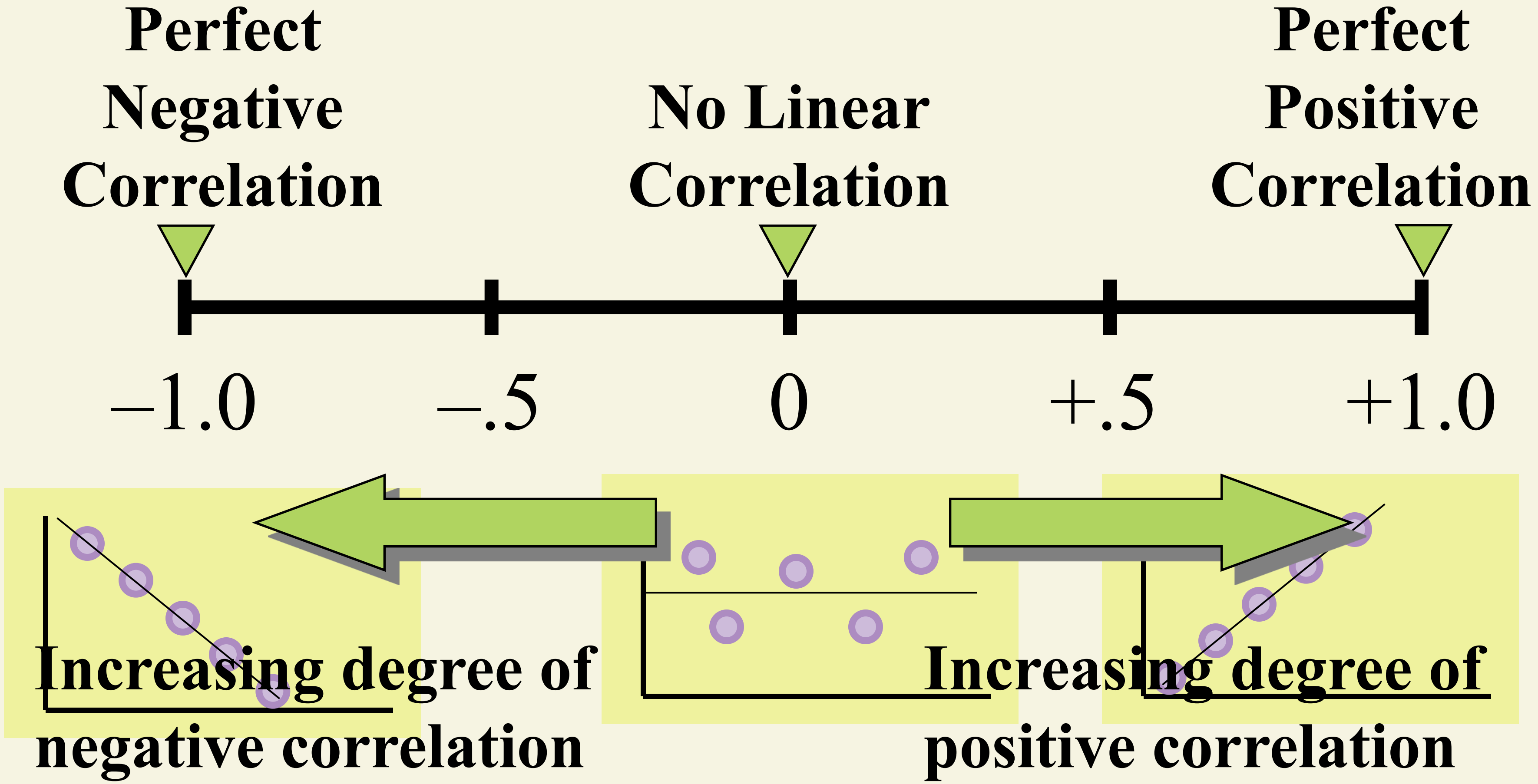
where

$$SS_{xx} = \sum x^2 - \frac{\left(\sum x\right)^2}{n}$$

$$SS_{yy} = \sum y^2 - \frac{\left(\sum y\right)^2}{n}$$

$$SS_{xy} = \sum xy - \frac{\left(\sum x\right)\left(\sum y\right)}{n}$$

Coefficient of Correlation Values



Perfect Negative Correlation

No Linear Correlation

Perfect Positive Correlation

–1.0 –.5 0 +.5 +1.0

Increasing degree of negative correlation

Increasing degree of positive correlation

Coefficient of Correlation Example

You're a marketing analyst for Hasbro Toys.

Ad $	Sales (Units)
1	1
2	1
3	2
4	2
5	4

Calculate the **coefficient of correlation**.

Solution Table

x	y_i	x^2	y_i^2	$x_i y$
i1	1	i1	1	i 1
2	1	4	1	2
3	2	9	4	6
4	2	16	4	8
5	4	25	16	20
15	10	55	26	37

Coefficient of Correlation Solution

$$SS_{xx} = \sum x^2 - \frac{\left(\sum x\right)^2}{n} = 55 - \frac{(15)^2}{5} = 10$$

$$SS_{yy} = \sum y^2 - \frac{\left(\sum y\right)^2}{n} = 26 - \frac{(10)^2}{5} = 6$$

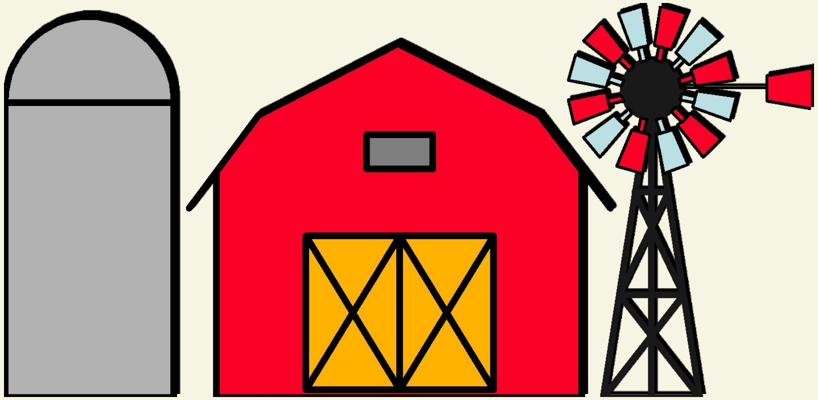
$$SS_{xy} = \sum xy - \frac{\left(\sum x\right)\left(\sum y\right)}{n} = 37 - \frac{(15)(10)}{5} = 7$$

$$r = \frac{SS_{xy}}{\sqrt{SS_{xx} SS_{yy}}} = \frac{7}{\sqrt{10 \cdot 6}} = .904$$

Coefficient of Correlation Thinking Challenge

You're an economist for the county cooperative.
You gather the following data:

Fertilizer (lb.)	Yield (lb.)
4	3.0
6	5.5
10	6.5
12	9.0



Find the **coefficient of correlation**.

Solution Table*

x_i	y_i	x_i^2	y_i^2	$x_i y_i$
4	3.0	16	9.00	12
6	5.5	36	30.25	33
10	6.5	100	42.25	65
12	9.0	144	81.00	108
32	24.0	296	162.50	218

Coefficient of Correlation Solution*

$$SS_{xx} = \sum x^2 - \frac{\left(\sum x\right)^2}{n} = 296 - \frac{(32)^2}{4} = 40$$

$$SS_{yy} = \sum y^2 - \frac{\left(\sum y\right)^2}{n} = 162.5 - \frac{(24)^2}{4} = 18.5$$

$$SS_{xy} = \sum xy - \frac{\left(\sum x\right)\left(\sum y\right)}{n} = 218 - \frac{(32)(24)}{4} = 26$$

$$r = \frac{SS_{xy}}{\sqrt{SS_{xx} SS_{yy}}} = \frac{26}{\sqrt{40 \cdot 18.5}} = .956$$

Coefficient of Determination

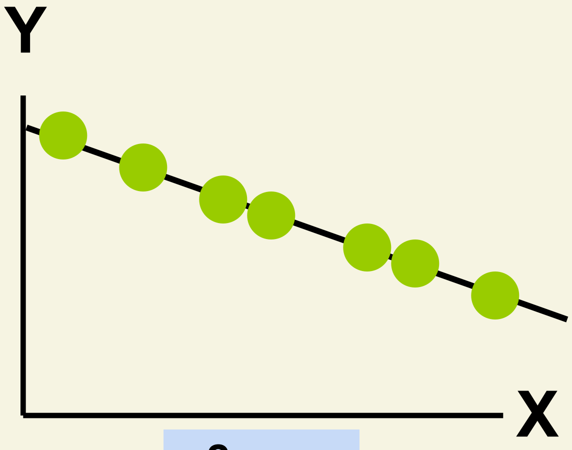
Proportion of variation ‘explained’ by relationship between *x* and *y*

$$r^2 = \frac{\text{Explained Variation}}{\text{Total Variation}} = \frac{SS_{yy} - SSE}{SS_{yy}}$$

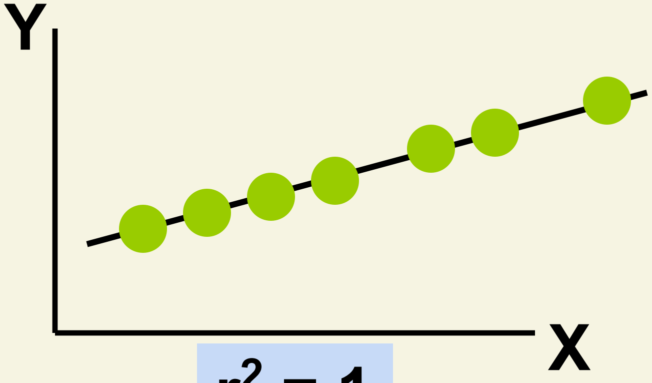
$$r^2 = (\text{coefficient of correlation})^2$$

$0 \le r^2 \le 1$

Examples of Approximate r^2 Values



$r^2 = 1$



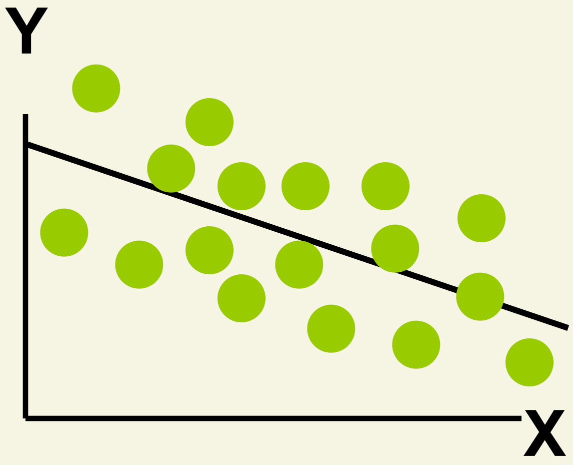
$r^2 = 1$

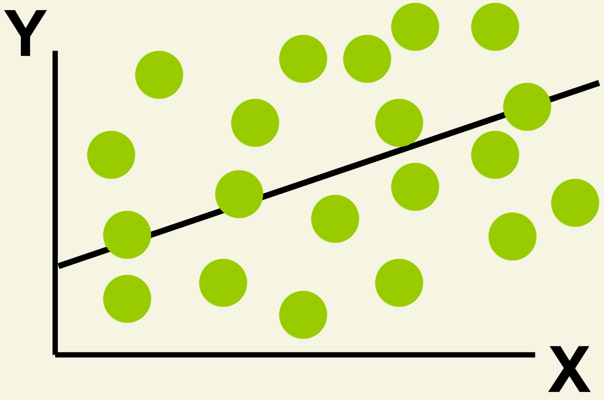
$r^2 = 1$

Perfect linear relationship between X and Y:

100% of the variation in Y is explained by variation in X

Examples of Approximate r^2 Values





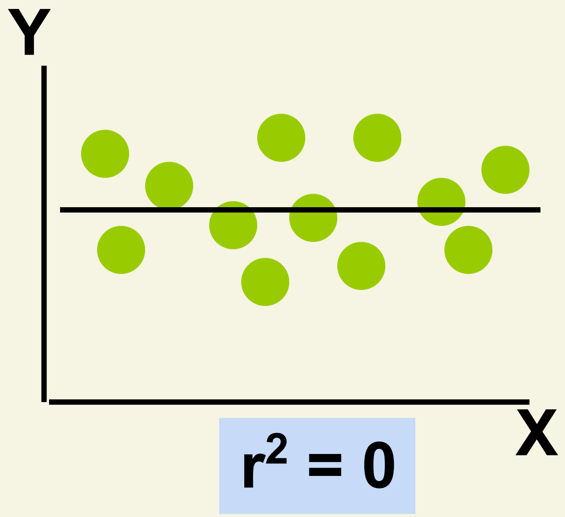
$0 < r^2 < 1$

Weaker linear relationships between X and Y:

Some but not all of the variation in Y is explained by variation in X

Examples of Approximate r^2 Values

$r^2 = 0$



No linear relationship between X and Y:

The value of Y does not depend on X. (None of the variation in Y is explained by variation in X)

Coefficient of Determination Example

You're a marketing analyst for Hasbro Toys.

You know $r = .904$.

Ad $	Sales (Units)
1	1
2	1
3	2
4	2
5	4

Calculate and interpret the **coefficient of determination**.

Coefficient of Determination Solution

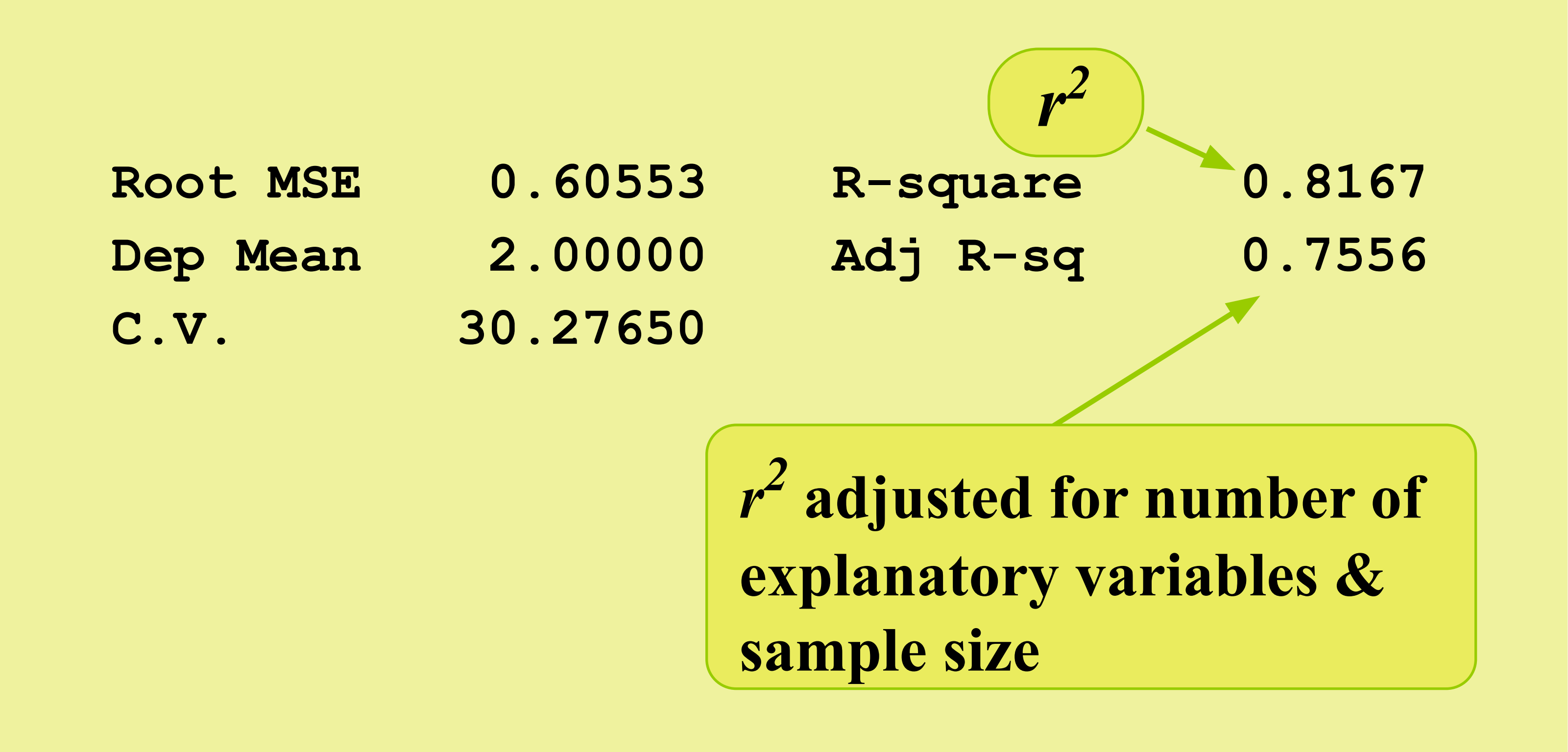
$$r^2 = (\text{coefficient of correlation})^2$$

$$r^2 = (.904)^2$$

$$r^2 = .817$$

Interpretation: About 81.7% of the sample variation in Sales (y) can be explained by using Ad \$ (x) to predict Sales (y) in the linear model.

r^2 Computer Output



```
Root MSE     0.60553    R-square      0.8167
Dep Mean     2.00000    Adj R-sq      0.7556
C.V.        30.27650
```

r^2

r^2 adjusted for number of explanatory variables & sample size

Using the Model for Prediction & Estimation

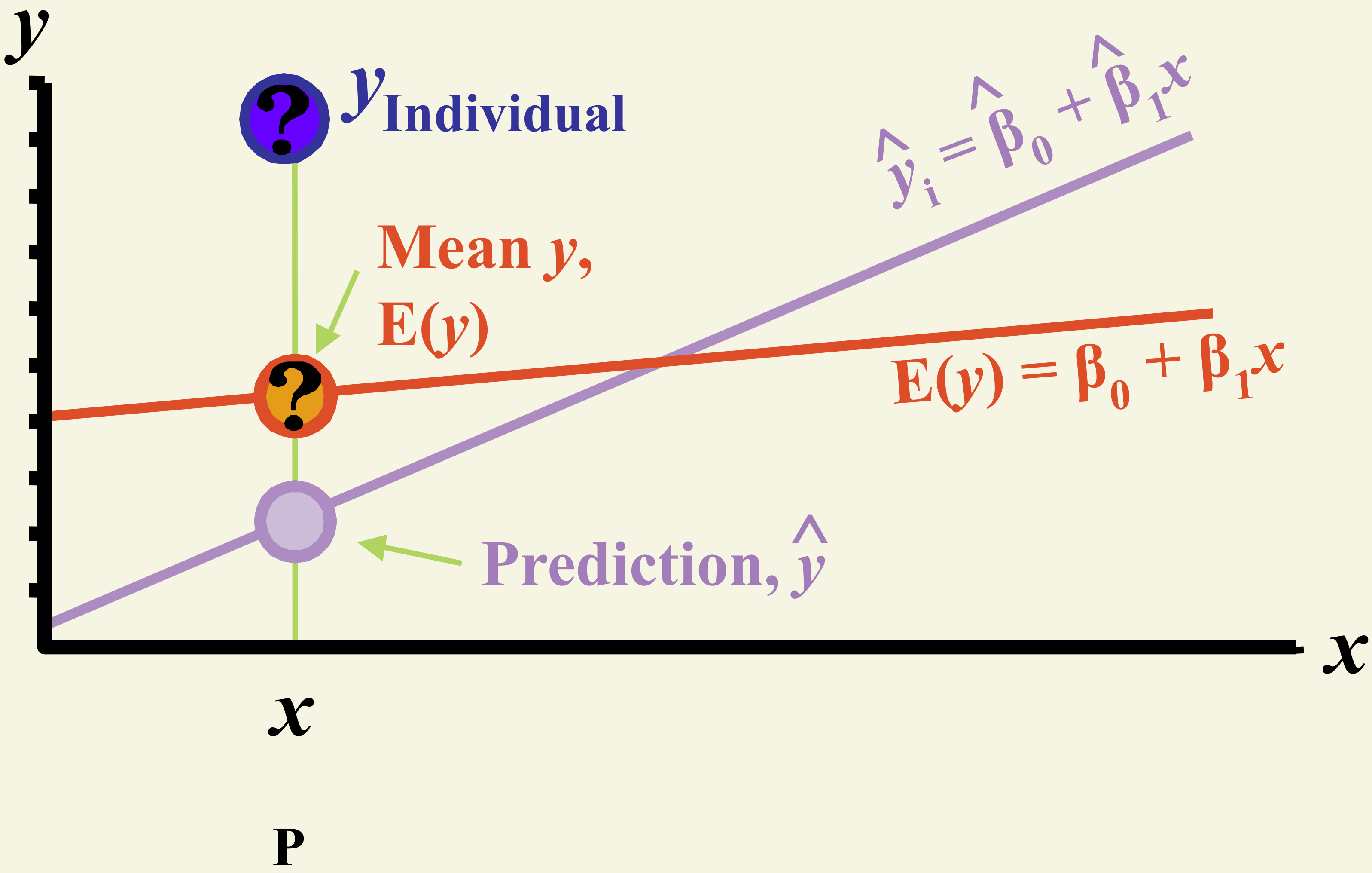
Regression Modeling Steps

1. Hypothesize deterministic component
2. Estimate unknown model parameters
3. Specify probability distribution of random error term
    - Estimate standard deviation of error
4. Evaluate model
5. **Use model for prediction and estimation**

Prediction With Regression Models

- Types of predictions
  - Point estimates
  - Interval estimates
- What is predicted
  - Population mean response $E(y)$ for given x
    - Point on population regression line
  - Individual response (y_i) for given x

What Is Predicted



Confidence Interval Estimate for Mean Value of *y* at *x* = *x*$_p$

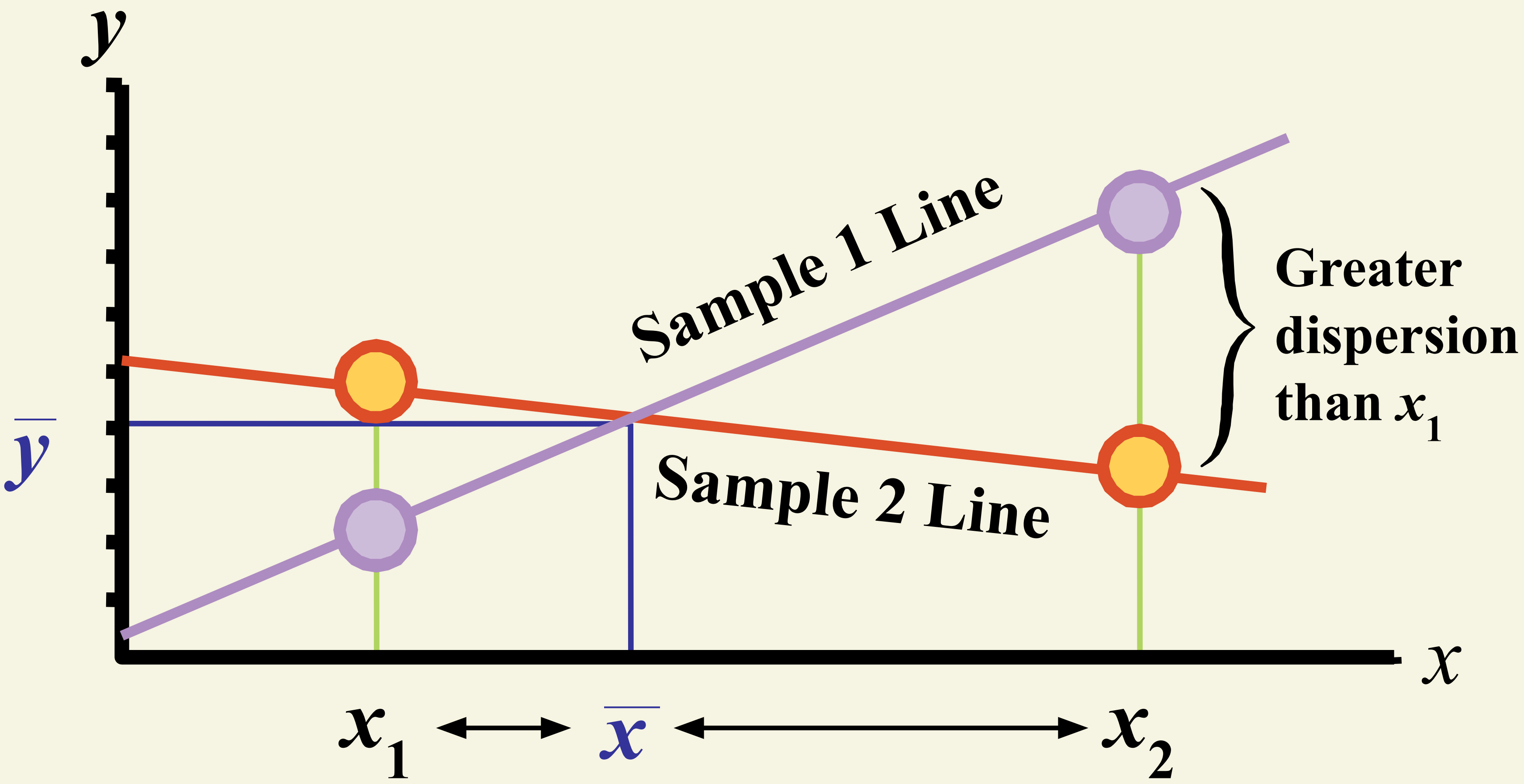
$$\hat{y} \pm t_{\alpha/2} S \sqrt{\frac{1}{n} + \frac{\left(x_p - \bar{x}\right)^2}{SS_{xx}}}$$

$$\text{df} = n - 2$$

Factors Affecting Interval Width

1. Level of confidence (1 – α)
   - Width increases as confidence increases
2. Data dispersion (*s*)
   - Width increases as variation increases
3. Sample size
   - Width decreases as sample size increases
4. Distance of x_p from mean $\bar{x}$
   - Width increases as distance increases

Why Distance from Mean?



Confidence Interval Estimate Example

You're a marketing analyst for Hasbro Toys.

You find $\hat{\beta}_0 = -.1$, $\hat{\beta}_1 = .7$ and $s = .6055$.

Ad $	Sales (Units)
1	1
2	1
3	2
4	2
5	4

Find a **95%** confidence interval for the **mean** sales when advertising is **$4**.

Solution Table

x_i	y_i	x_i^2	y_i^2	x_iy_i
1	1	1	1	1
2	1	4	1	2
3	2	9	4	6
4	2	16	4	8
5	4	25	16	20
15	**10**	**55**	**26**	**37**

Confidence Interval Estimate Solution

$$\hat{y} \pm t_{\alpha/2} s \sqrt{\frac{1}{n} + \frac{\left(x_p - \bar{x}\right)^2}{SS_{xx}}}$$

***x* to be predicted**

$$\hat{y} = -.1 + (.7)(4) = 2.7$$

$$2.7 \pm (3.182)(.6055)\sqrt{\frac{1}{5} + \frac{(4-3)^2}{10}}$$

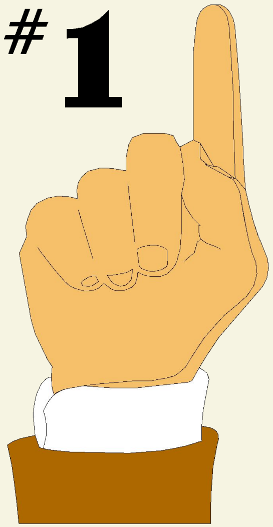
$$1.645 \leq E(Y) \leq 3.755$$

Prediction Interval of Individual Value of *y* at *x* = x$_p$

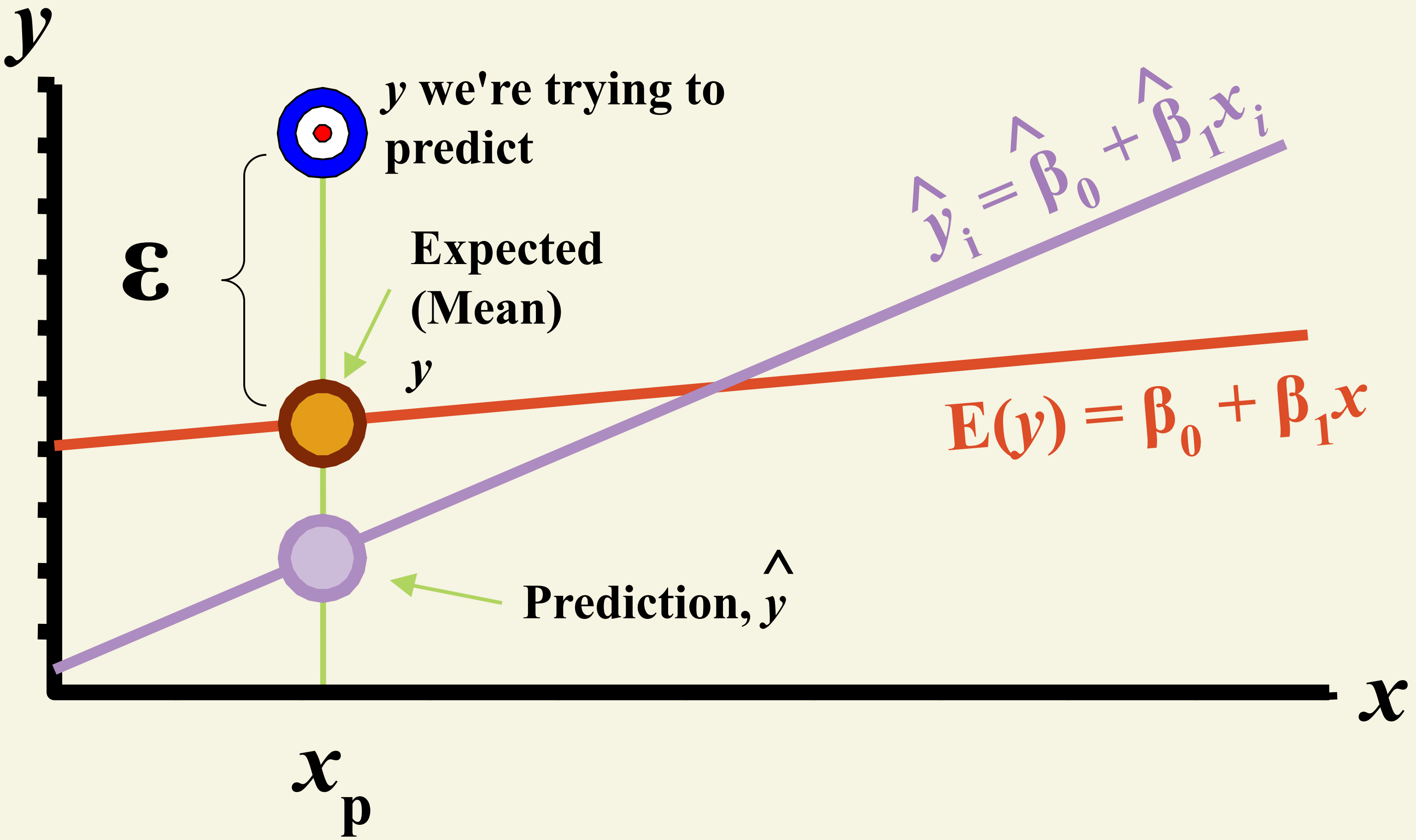
$$\hat{y} \pm t_{\alpha/2} S \sqrt{1 + \frac{1}{n} + \frac{\left(x_p - \bar{x}\right)^2}{SS_{xx}}}$$

Note!

$$\text{df} = n - 2$$



Why the Extra ‘S’?


y
y we're trying to predict
ε
Expected (Mean) y
$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$
$E(y) = \beta_0 + \beta_1 x$
Prediction, $\hat{y}$
x
x_p

Prediction Interval Example

You're a marketing analyst for Hasbro Toys.

You find $\hat{\beta}_0 = -.1$, $\hat{\beta}_1 = .7$ and $s = .6055$.

Ad $	Sales (Units)
1	1
2	1
3	2
4	2
5	4

Predict the sales when advertising is **$4**. Use a **95% prediction** interval.

Solution Table

x_i	y_i	x_i^2	y_i^2	$x_i y_i$
1	1	1	1	1
2	1	4	1	2
3	2	9	4	6
4	2	16	4	8
5	4	25	16	20
15	**10**	**55**	**26**	**37**

Prediction Interval Solution

$$\hat{y} \pm t_{\alpha/2} s \sqrt{1 + \frac{1}{n} + \frac{\left(x_p - \bar{x}\right)^2}{SS_{xx}}}$$

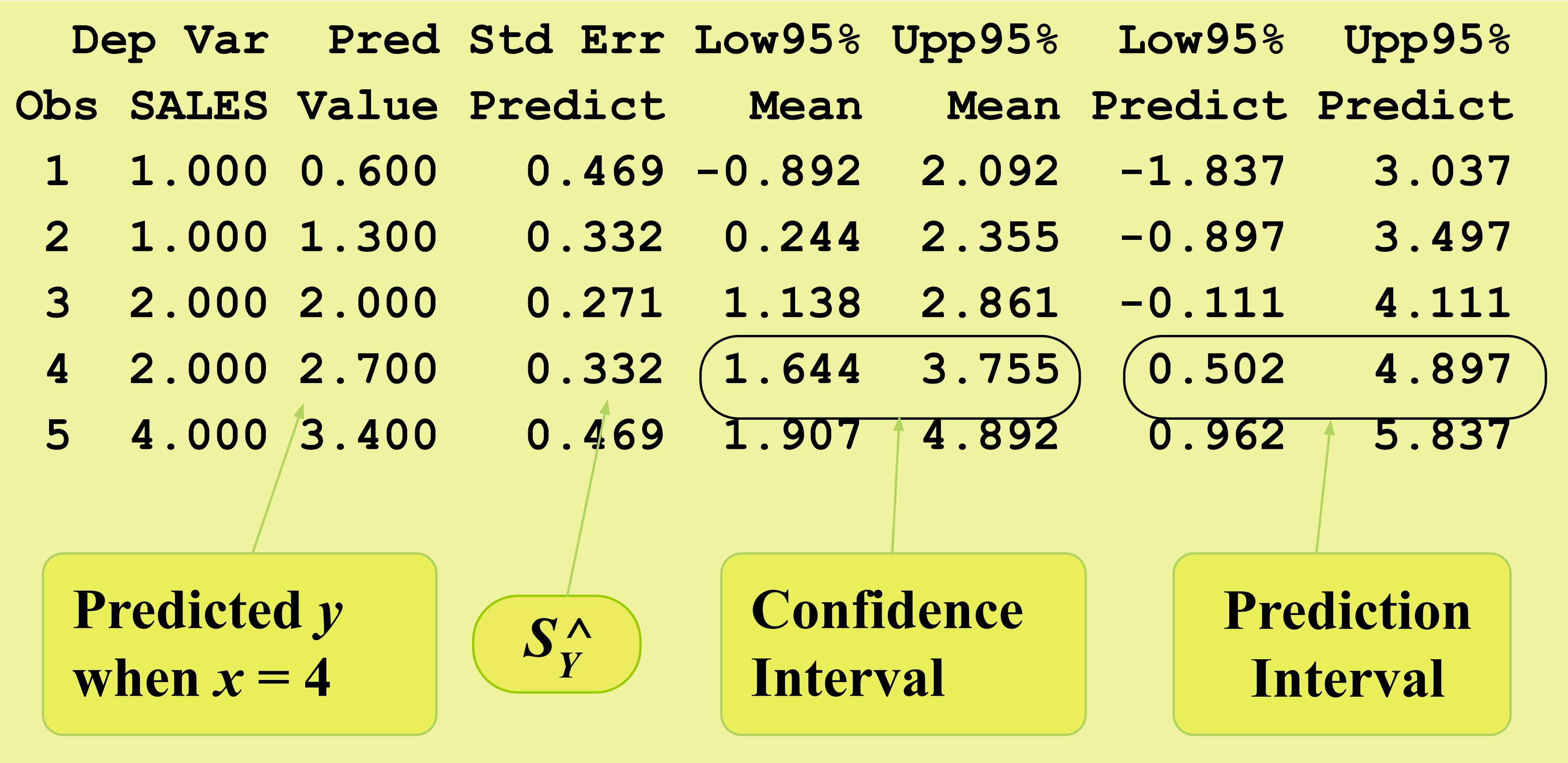
x **to be predicted**

$$\hat{y} = -.1 + (.7)(4) = 2.7$$

$$2.7 \pm (3.182)(.6055)\sqrt{1 + \frac{1}{5} + \frac{(4-3)^2}{10}}$$

$$.503 \leq y_4 \leq 4.897$$

Interval Estimate Computer Output



```
  Dep Var   Pred Std Err Low95% Upp95%  Low95%  Upp95%
Obs SALES Value Predict    Mean   Mean Predict Predict
 1  1.000 0.600   0.469 -0.892  2.092  -1.837   3.037
 2  1.000 1.300   0.332  0.244  2.355  -0.897   3.497
 3  2.000 2.000   0.271  1.138  2.861  -0.111   4.111
 4  2.000 2.700   0.332  1.644  3.755   0.502   4.897
 5  4.000 3.400   0.469  1.907  4.892   0.962   5.837
```

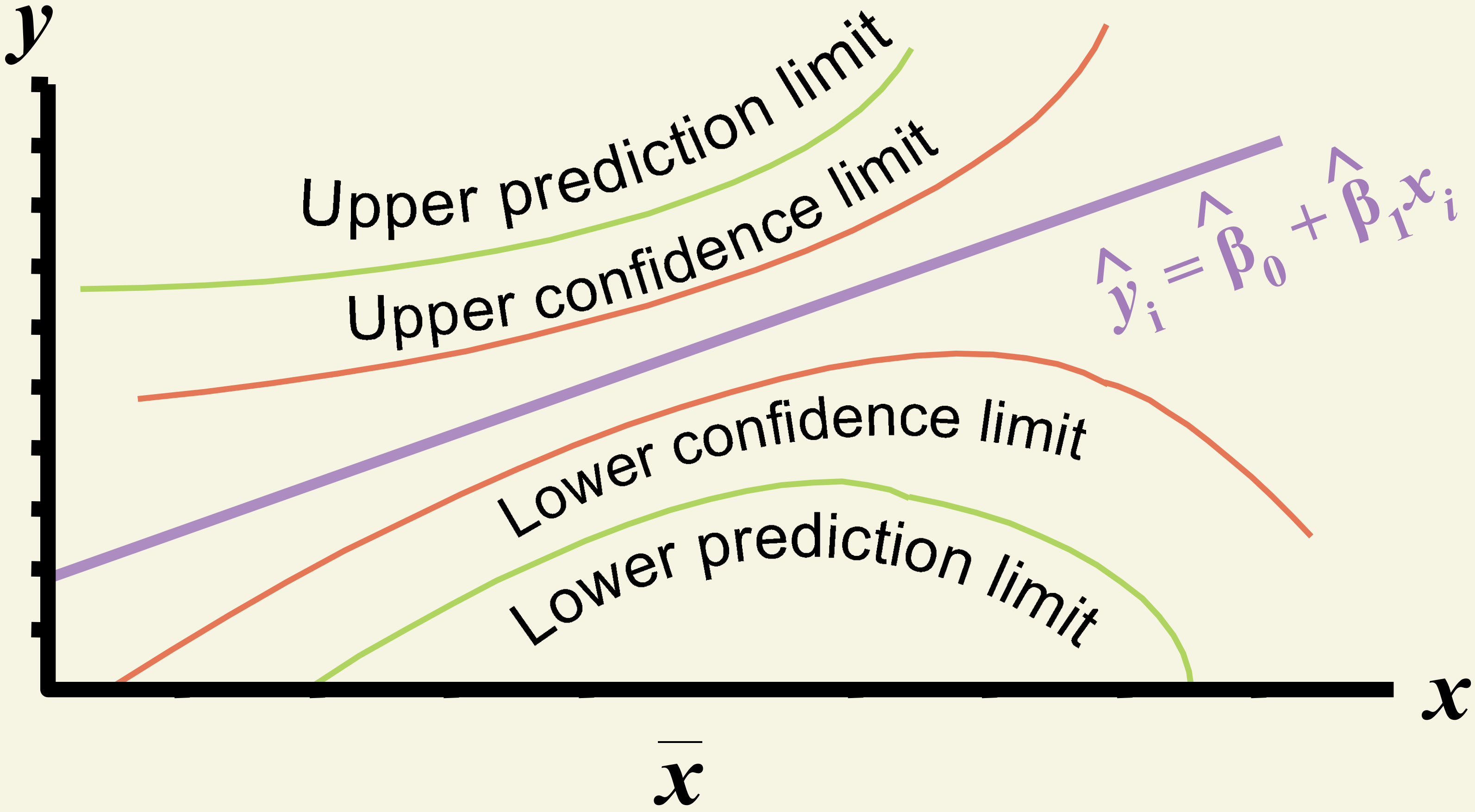
Predicted *y* when *x* = 4

$S_{\hat{Y}}$

Confidence Interval

Prediction Interval

Confidence Intervals v. Prediction Intervals



Conclusion

1. Described the Linear Regression Model
2. Stated the Regression Modeling Steps
3. Explained Least Squares
4. Computed Regression Coefficients
5. Explained Correlation
6. Predicted Response Variable