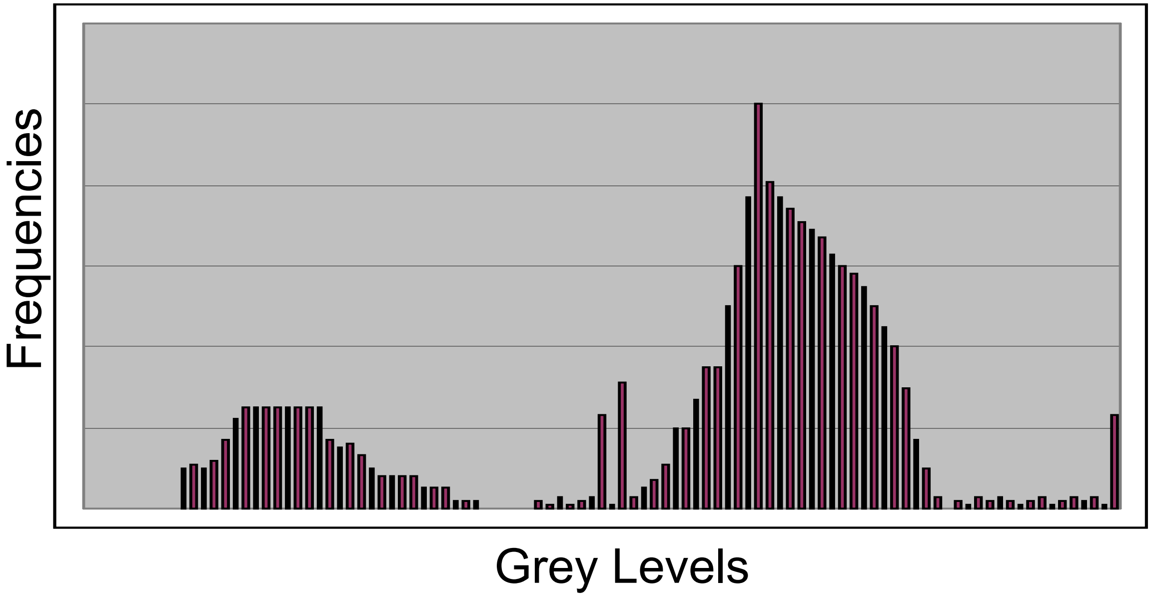
Intensity Transformations

Histogram Processing

Image Histograms

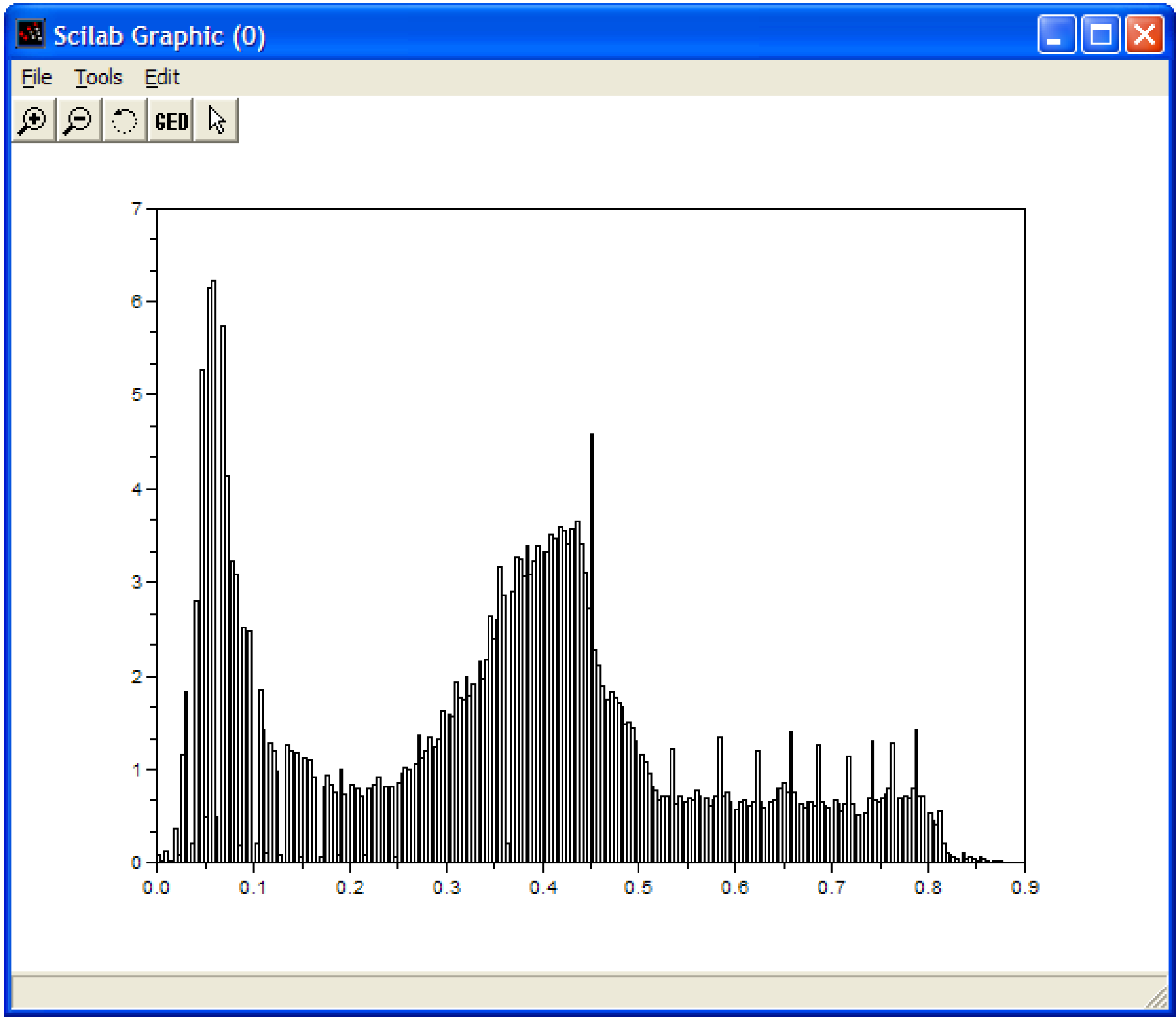
The histogram of an image shows us the distribution of grey levels in the image

Massively useful in image processing, especially in segmentation



Histogram Examples

Histogram Examples (cont…)



Histogram Processing

Histogram $h(r_k) = n_k$

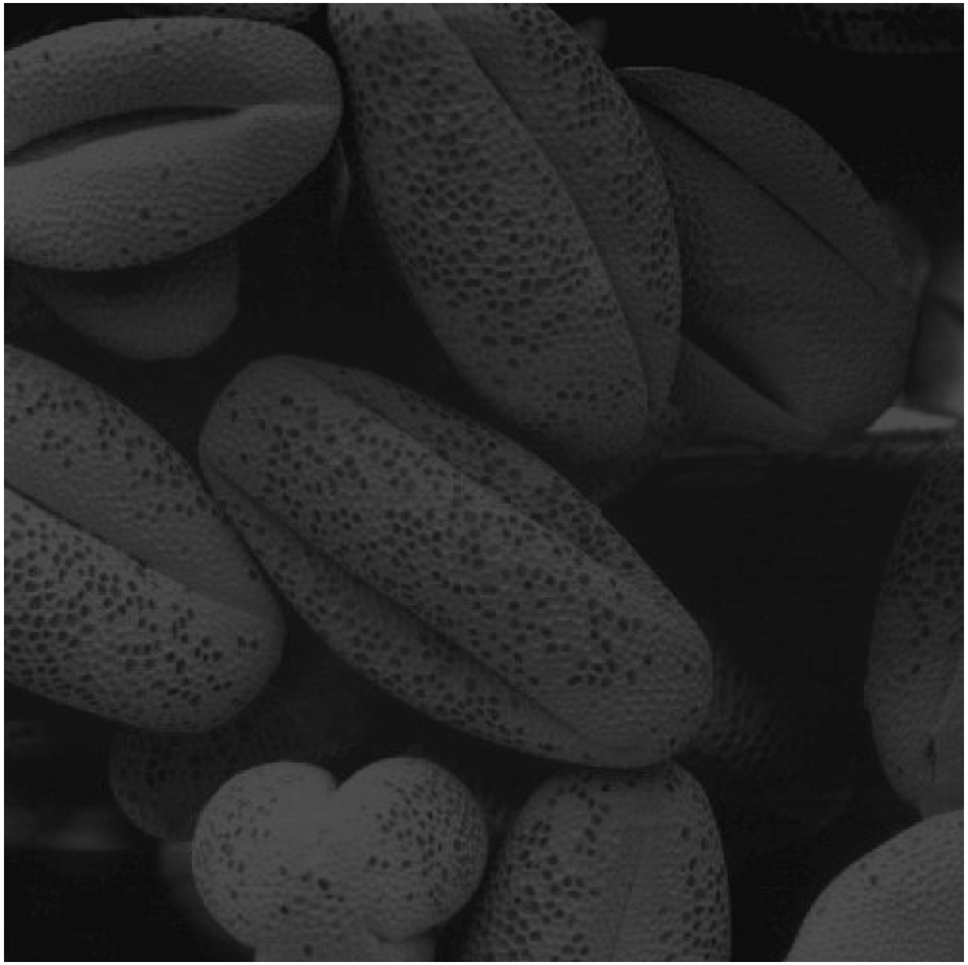
r_k is the k^{th} intensity value

n_k is the number of pixels in the image with intensity r_k

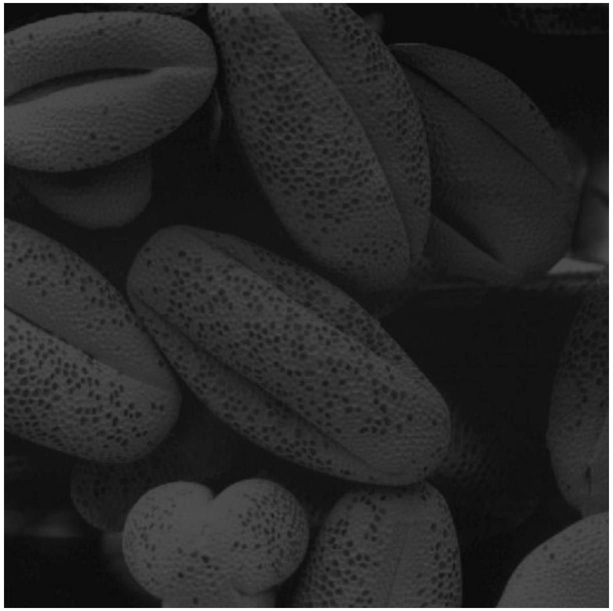
Normalized histogram $p(r_k) = \dfrac{n_k}{MN}$

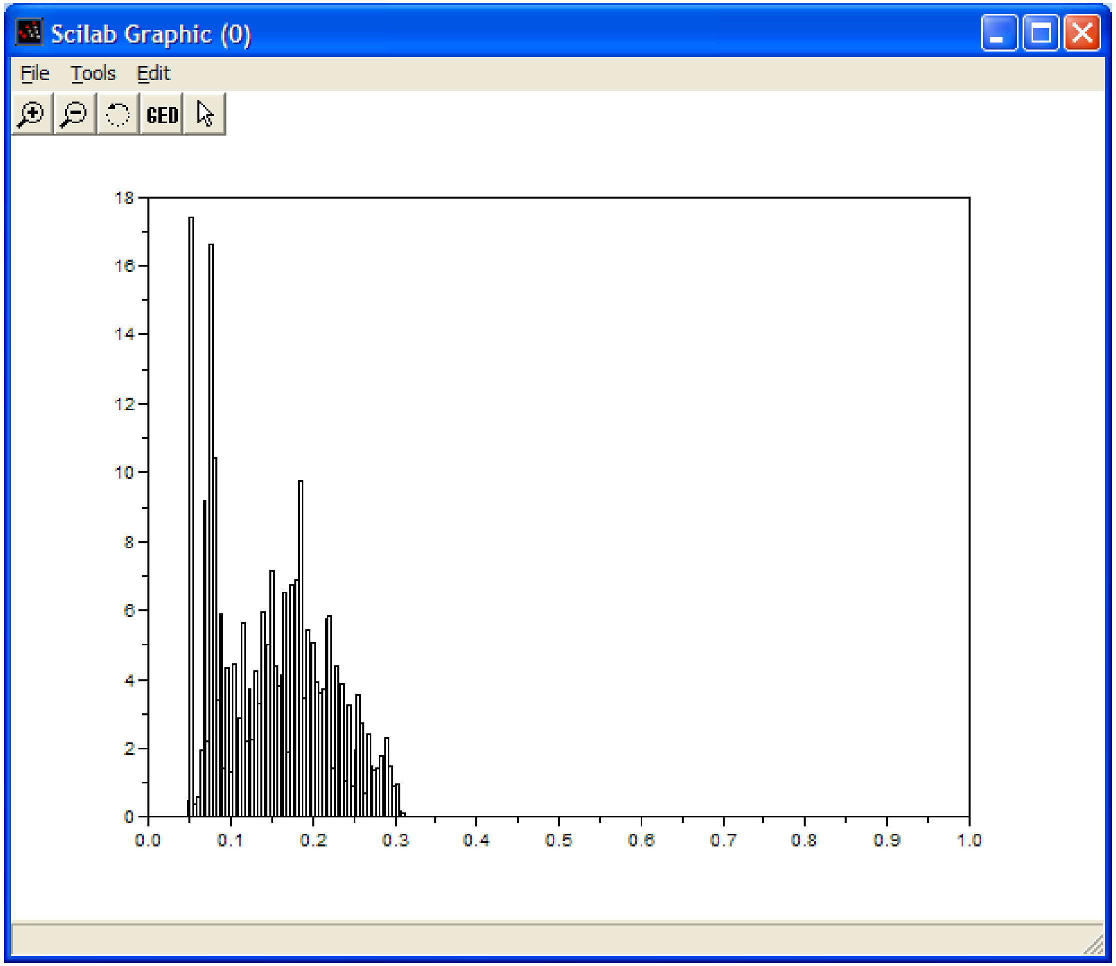
n_k: the number of pixels in the image of size $\mathrm{M} \times \mathrm{N}$ with intensity r_k

Histogram Examples (cont…)



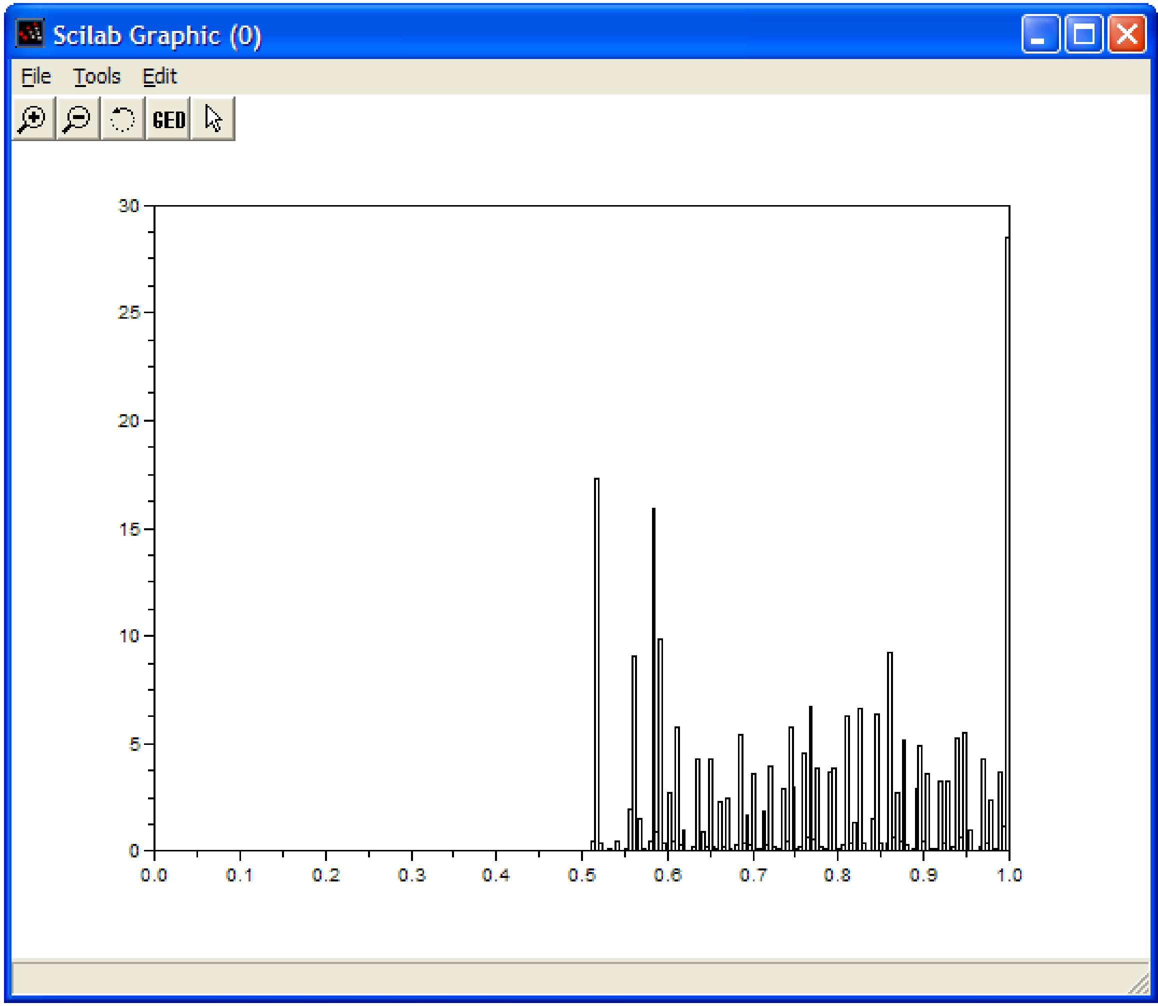
Histogram Examples (cont…)



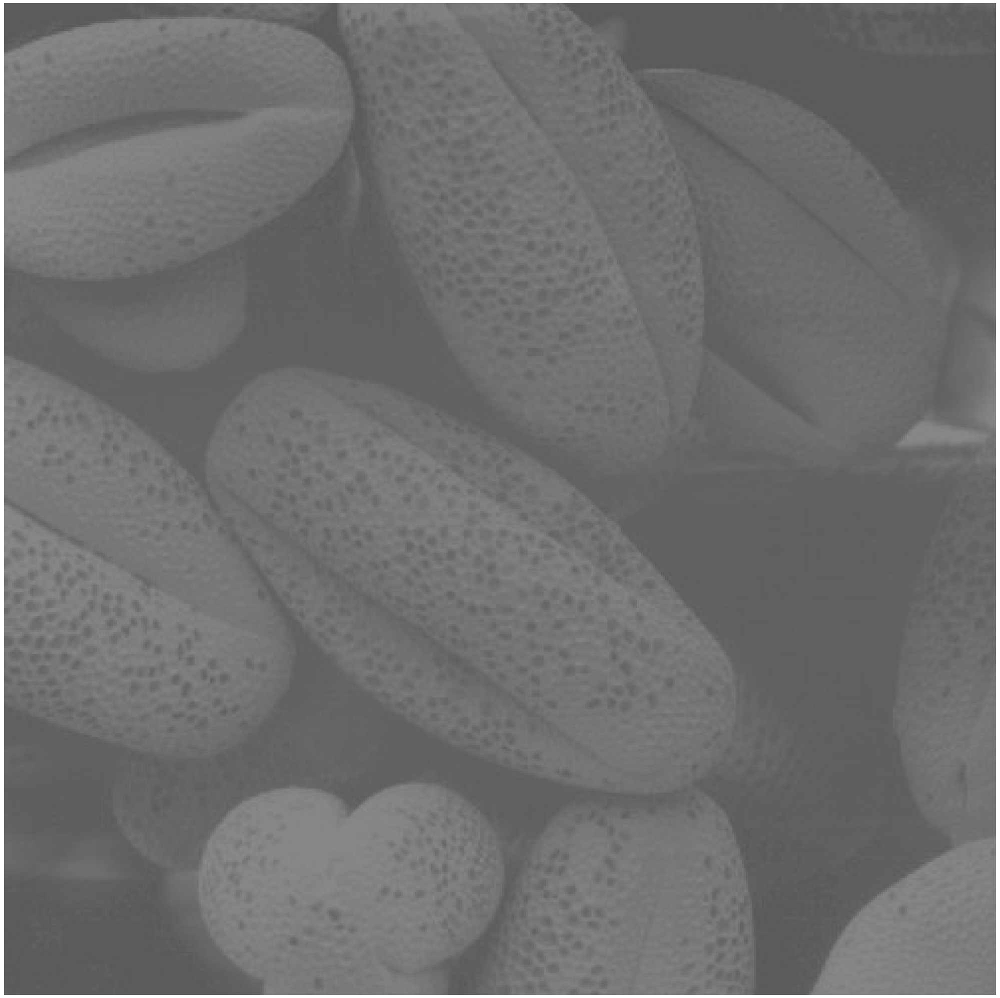


Histogram Examples (cont…)

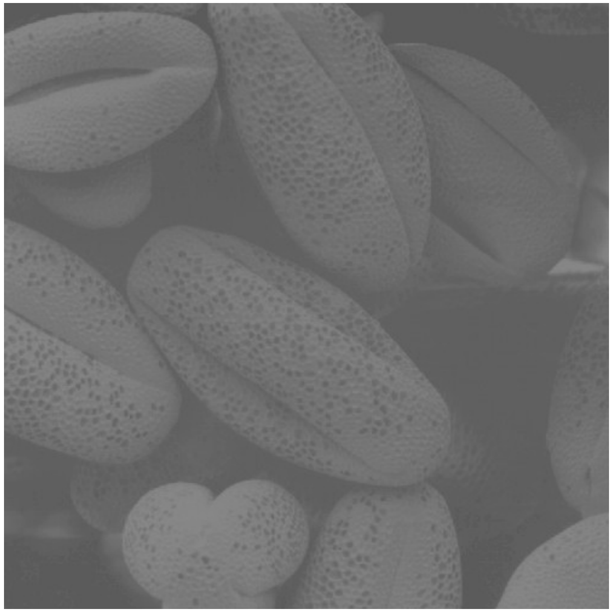
Histogram Examples (cont…)

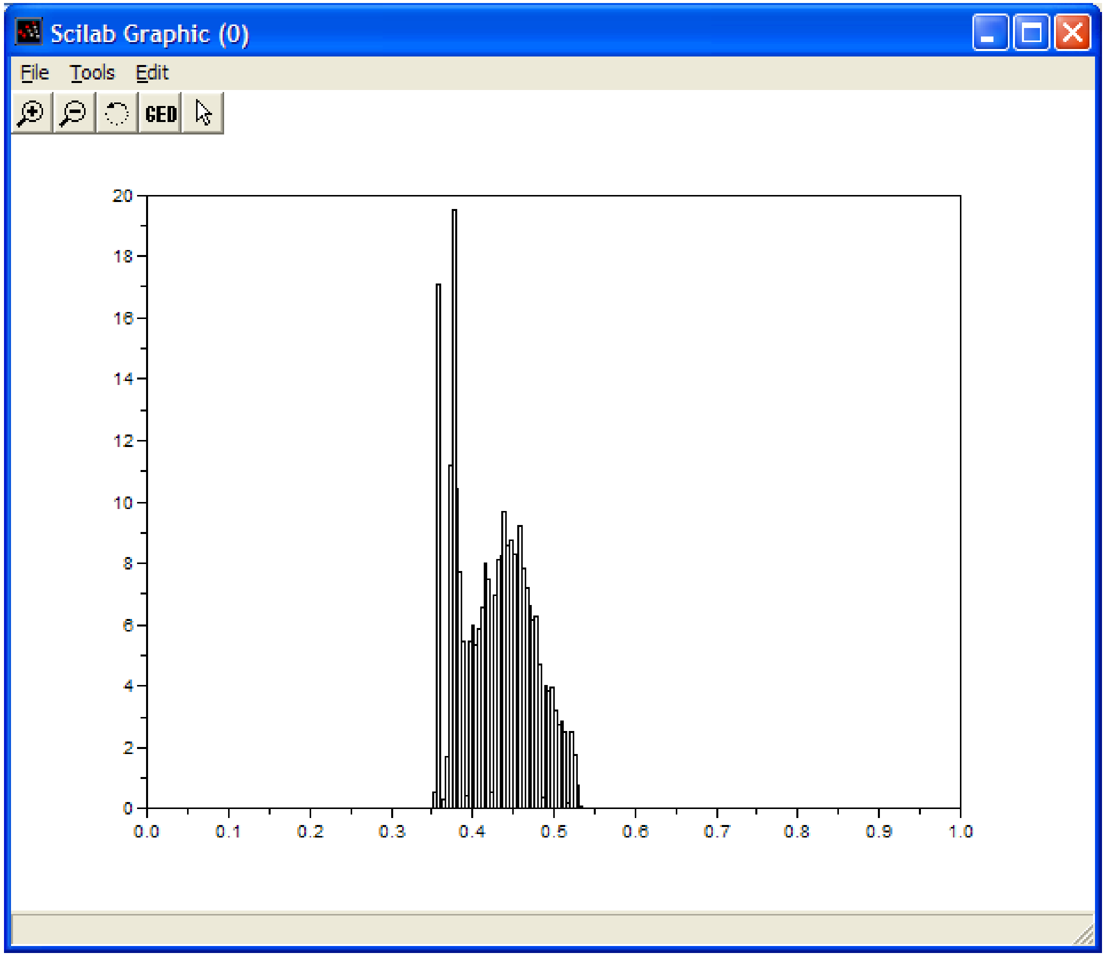


Histogram Examples (cont…)



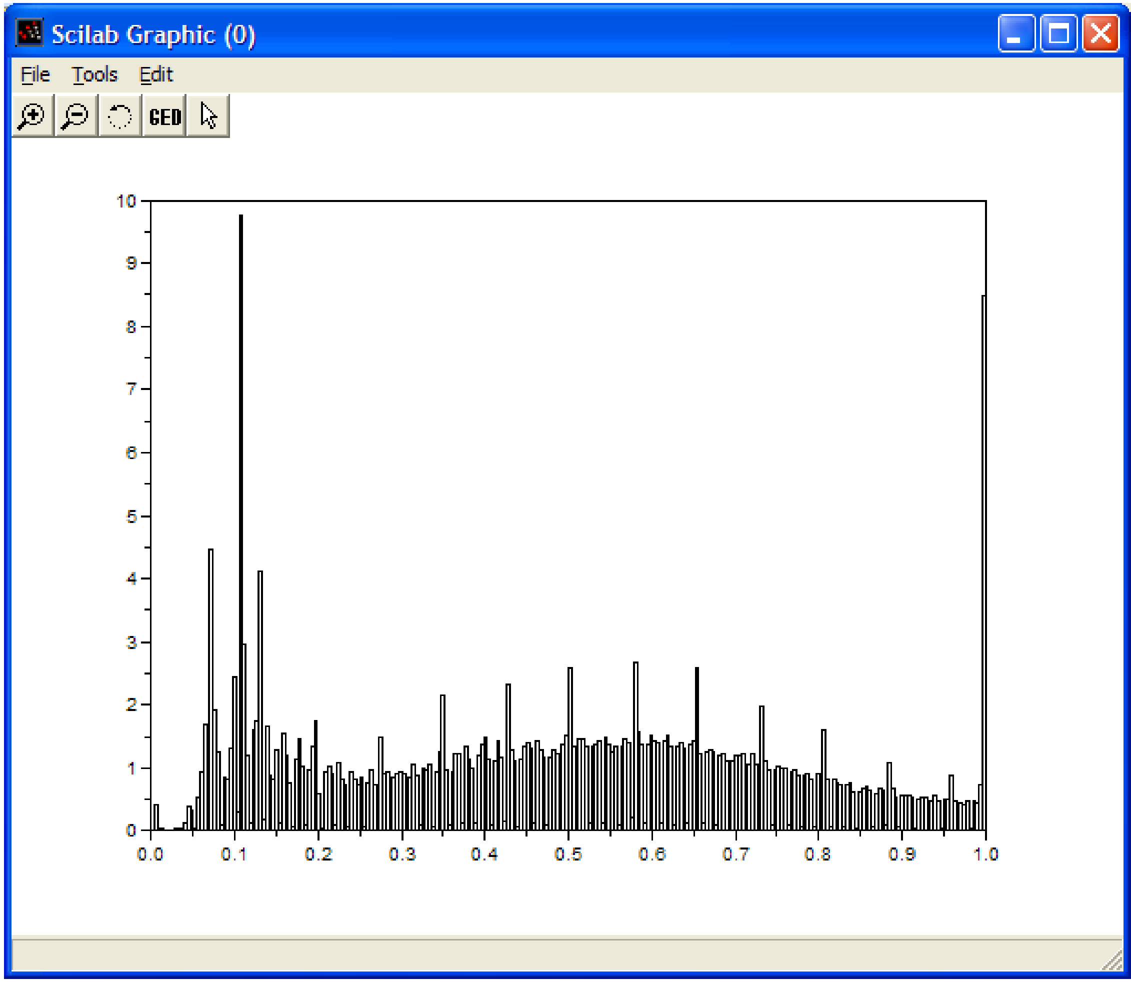
Histogram Examples (cont…)





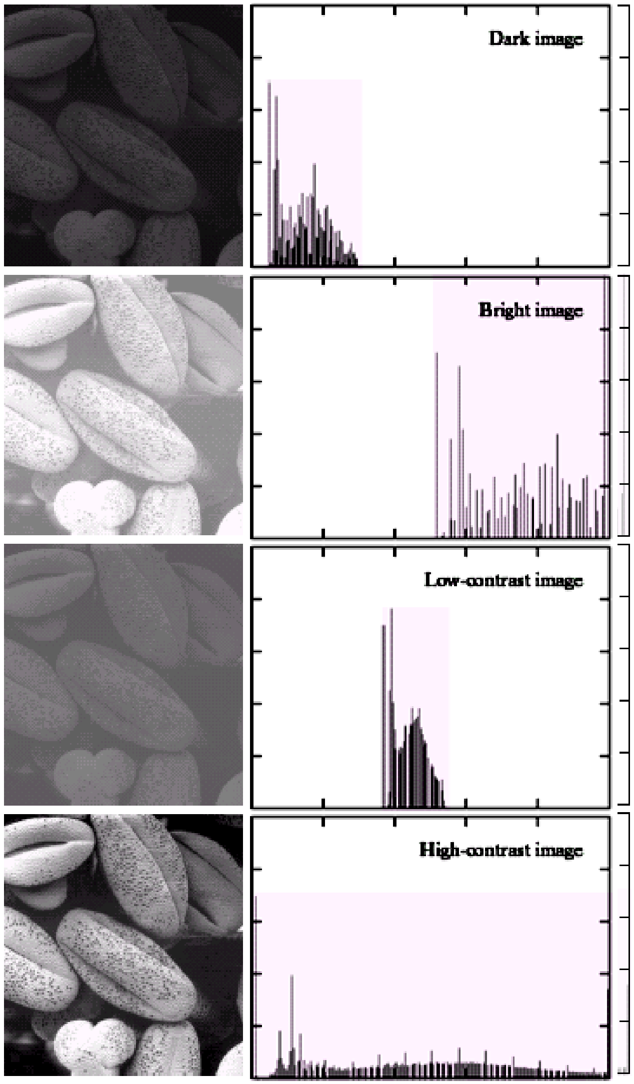
Histogram Examples (cont…)

Histogram Examples (cont…)



Histogram Examples (cont…)

- A selection of images and their histograms
- Notice the relationships between the images and their histograms
- Note that the high contrast image has the most evenly spaced histogram



Contrast Stretching

- We can fix images that have poor contrast by applying a pretty simple contrast specification
- The interesting part is how do we decide on this transformation function?

Histogram Equalization

The intensity levels in an image may be viewed as random variables in the interval [0, L-1].

Let $p_r(r)$ and $p_s(s)$ denote the probability density function (PDF) of random variables r and s.

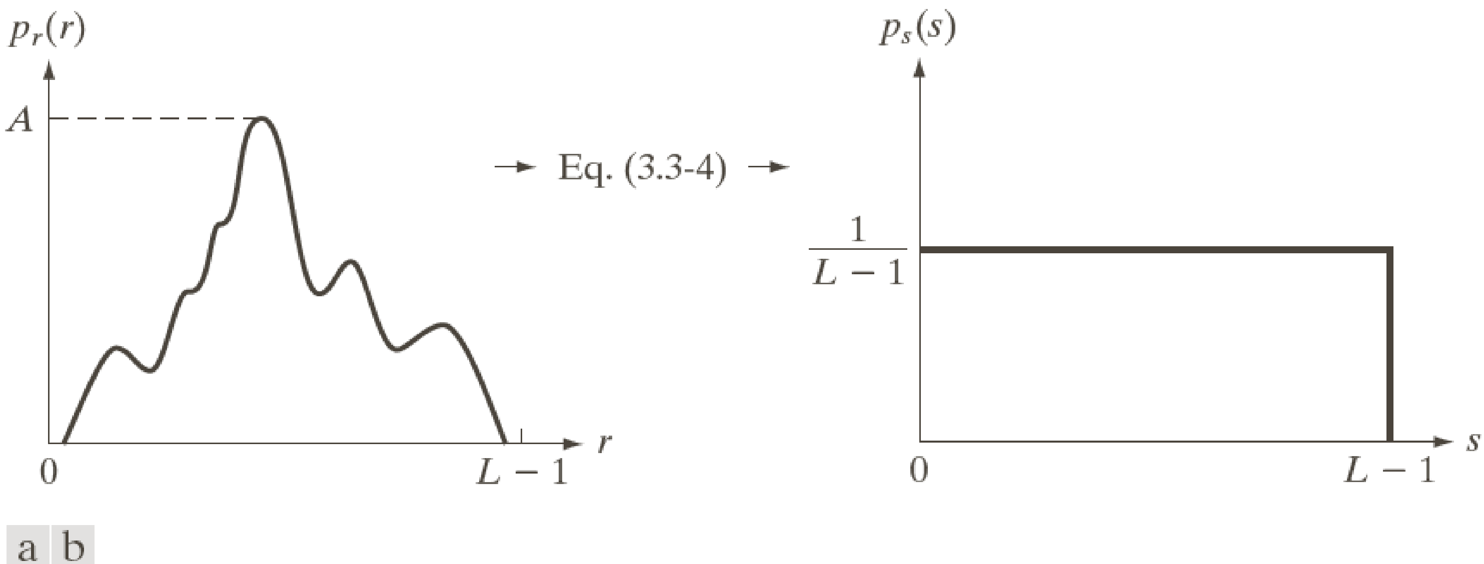


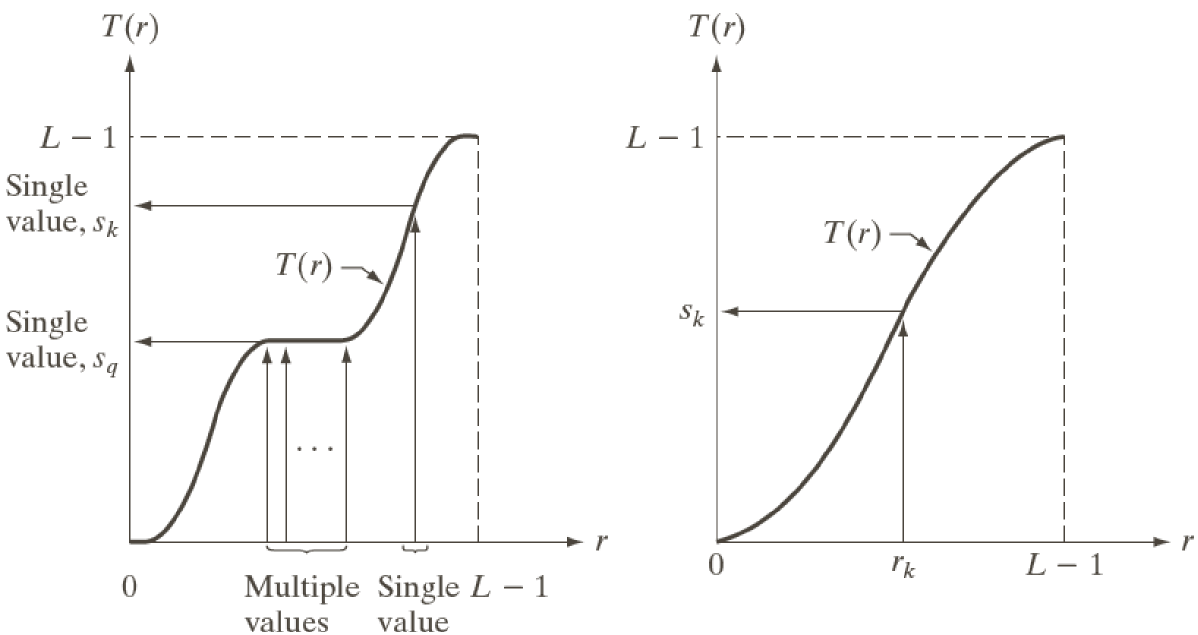
FIGURE 3.18 (a) An arbitrary PDF. (b) Result of applying the transformation in Eq. (3.3-4) to all intensity levels, r. The resulting intensities, s, have a uniform PDF, independently of the form of the PDF of the r's.

Histogram Equalization

$$s = T(r) \qquad 0 \le r \le L-1$$

a. T(r) is a strictly monotonically increasing function in the interval $0 \le r \le L\text{-}1$;

b. $0 \le T(r) \le L\text{-}1$ for $0 \le r \le L\text{-}1$.



a b

FIGURE 3.17 (a) Monotonically increasing function, showing how multiple values can map to a single value. (b) Strictly monotonically increasing function. This is a one-to-one mapping, both ways.

Histogram Equalization

$$s = T(r) \qquad 0 \le r \le L-1$$

$a.$ T(r) is a strictly monotonically increasing function in the interval $0 \le r \le L\text{-}1$;

$b.$ $0 \le T(r) \le L\text{-}1$ for $0 \le r \le L\text{-}1$.

$T(r)$ is continuous and differentiable.

$$p_s(s)ds = p_r(r)dr$$

Histogram Equalisation (cont...)

- We can view intensities r and s as random variables and their histograms as probability density functions (pdf) $p_r(r)$ and $p_s(s)$.
- Fundamental result from probability theory:
  - If $p_r(r)$ and $T(r)$ are known and $T(r)$ is continuous and differentiable, then

$$p_s(s) = p_r(r)\frac{1}{\left|\frac{ds}{dr}\right|} = p_r(r)\left|\frac{dr}{ds}\right|$$

Histogram Equalisation (cont...)

- The pdf of the output is determined by the pdf of the input and the transformation.
- This means that we can determine the histogram of the output image.
- A transformation of particular importance in image processing is the cumulative distribution function (CDF) of a random variable:

$$s = T(r) = (L-1)\int_{0}^{r} p_r(w)\,dw$$

Histogram Equalisation (cont...)

- It satisfies the first condition as the area under the curve increases as r increases.
- It satisfies the second condition as for $r=L\text{-}1$ we have $s=L\text{-}1$.
- To find $p_s(s)$ we have to compute

$$\frac{ds}{dr} = \frac{dT(r)}{dr} = (L-1)\frac{d}{dr}\int_0^r p_r(w)\,dw = (L-1)\,p_r(r)$$

Histogram Equalisation (cont...)

Substituting this result:

$$\frac{ds}{dr} = (L-1)\,p_r(r)$$

to

$$p_s(s) = p_r(r)\left|\frac{dr}{ds}\right|$$

Uniform pdf

yields

$$p_s(s) = p_r(r)\left|\frac{1}{(L-1)\,p_r(r)}\right| = \frac{1}{L-1}, \quad 0 \le s \le L-1$$

Histogram Equalisation (cont...)

The formula for histogram equalisation in the discrete case is given

$$s_k = T(r_k) = (L-1)\sum_{j=0}^{k} p_r(r_j) = \frac{(L-1)}{MN}\sum_{j=0}^{k} n_j$$

where

- r_k: input intensity
- s_k: processed intensity
- n_j: the frequency of intensity j
- MN: the number of image pixels.

Histogram Equalisation (cont...) Example

A 3-bit 64x64 image has the following intensities:

r_k	n_k	$p_r(r_k) = n_k/MN$
$r_0 = 0$	790	0.19
$r_1 = 1$	1023	0.25
$r_2 = 2$	850	0.21
$r_3 = 3$	656	0.16
$r_4 = 4$	329	0.08
$r_5 = 5$	245	0.06
$r_6 = 6$	122	0.03
$r_7 = 7$	81	0.02

$$s_k = T(r_k) = (L-1)\sum_{j=0}^{k} p_r(r_j)$$

Applying histogram equalization:

$$s_0 = T(r_0) = 7\sum_{j=0}^{0} p_r(r_j) = 7p_r(r_0) = 1.33$$

$$s_1 = T(r_1) = 7\sum_{j=0}^{1} p_r(r_j) = 7p_r(r_0) + 7p_r(r_1) = 3.08$$

Histogram Equalisation (cont...) Example

Rounding to the nearest integer:

$$s_0 = 1.33 \rightarrow 1 \quad s_1 = 3.08 \rightarrow 3 \quad s_2 = 4.55 \rightarrow 5 \quad s_3 = 5.67 \rightarrow 6$$

$$s_4 = 6.23 \rightarrow 6 \quad s_5 = 6.65 \rightarrow 7 \quad s_6 = 6.86 \rightarrow 7 \quad s_7 = 7.00 \rightarrow 7$$

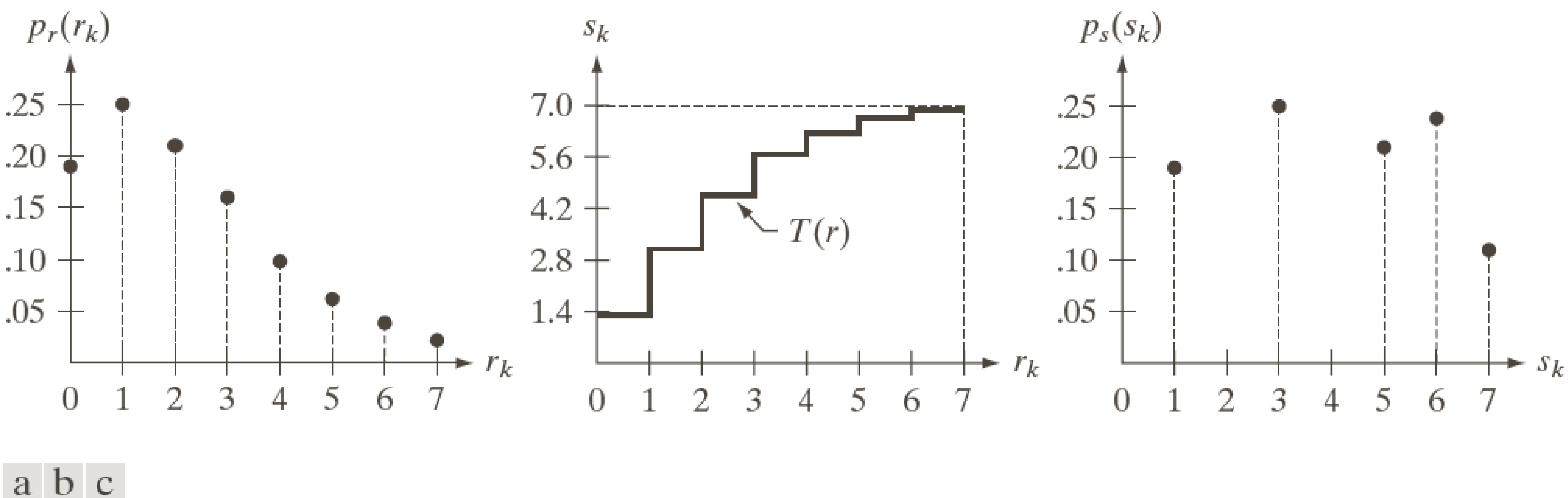


FIGURE 3.19 Illustration of histogram equalization of a 3-bit (8 intensity levels) image. (a) Original histogram. (b) Transformation function. (c) Equalized histogram.

Histogram Equalization (cont…) Example

Notice that due to discretization, the resulting histogram will rarely be perfectly flat. However, it will be extended.

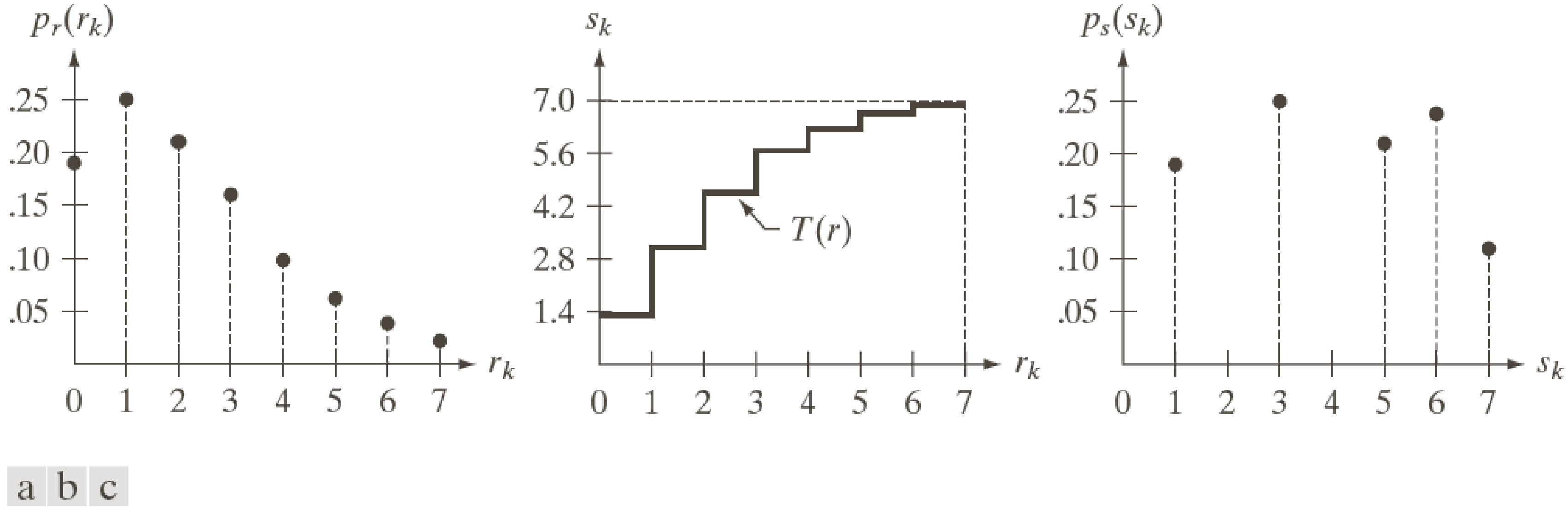
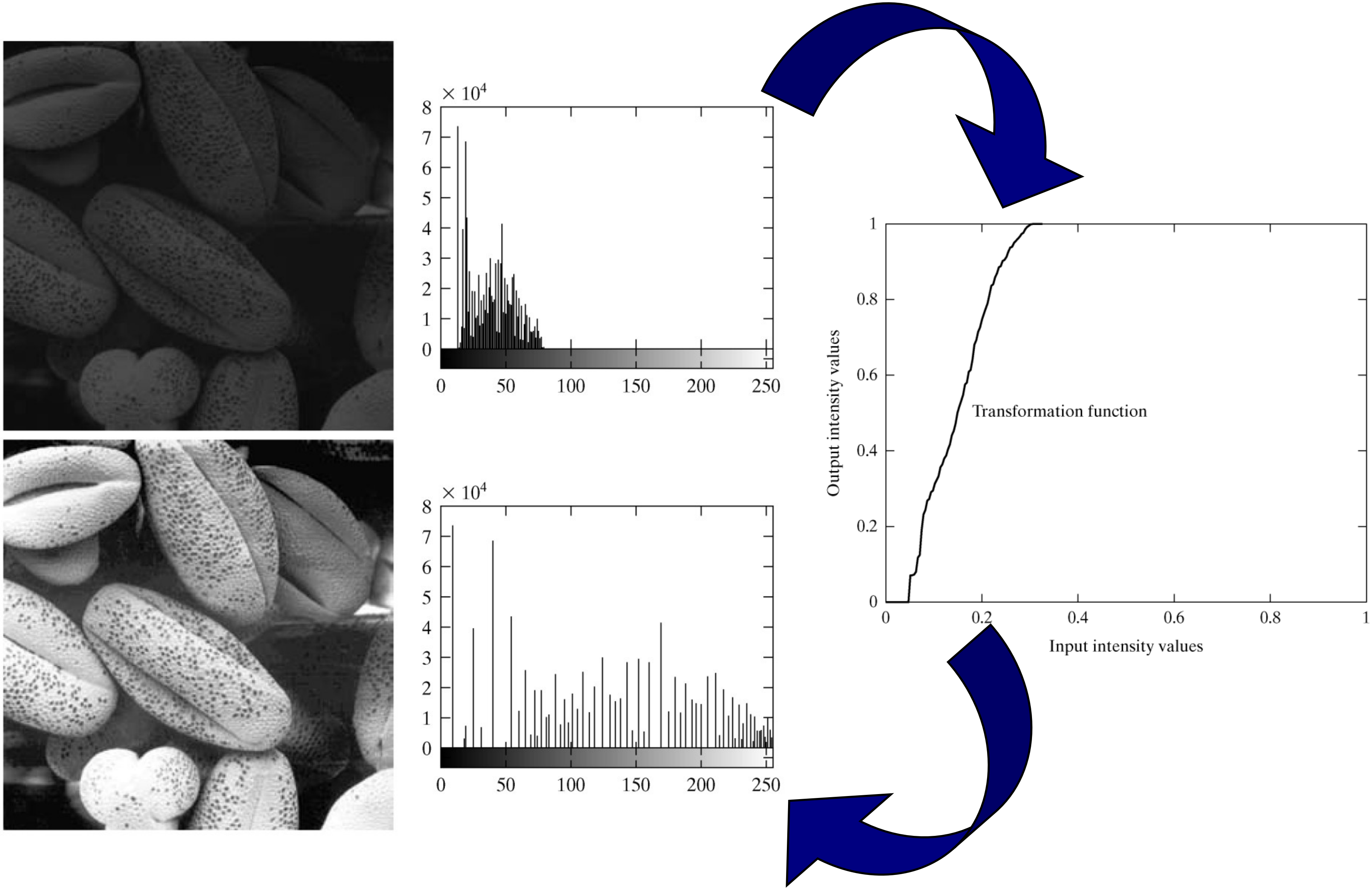
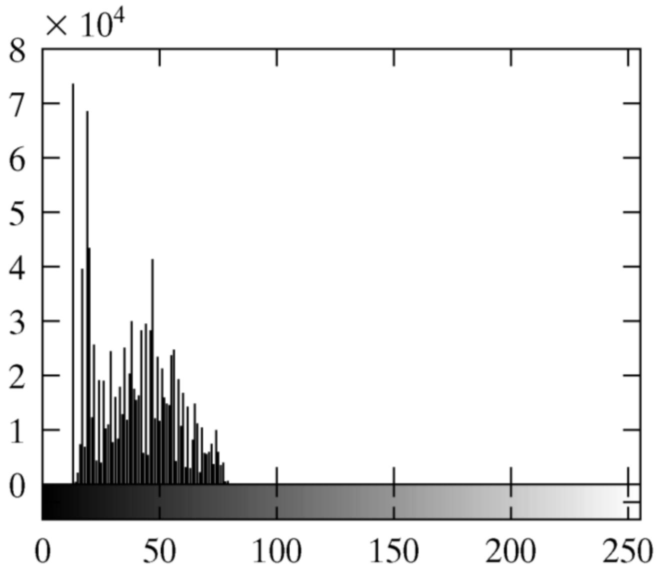


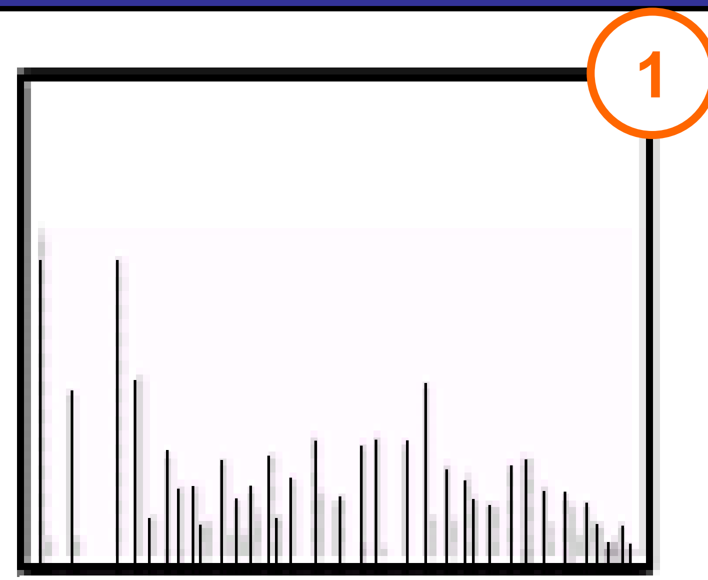
FIGURE 3.19 Illustration of histogram equalization of a 3-bit (8 intensity levels) image. (a) Original histogram. (b) Transformation function. (c) Equalized histogram.

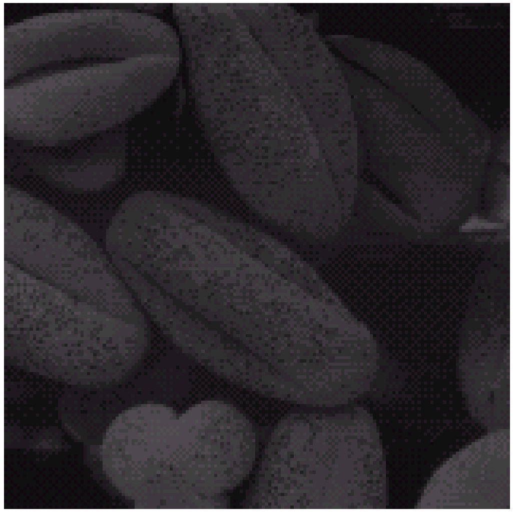
Equalisation Transformation Function



Equalisation Examples

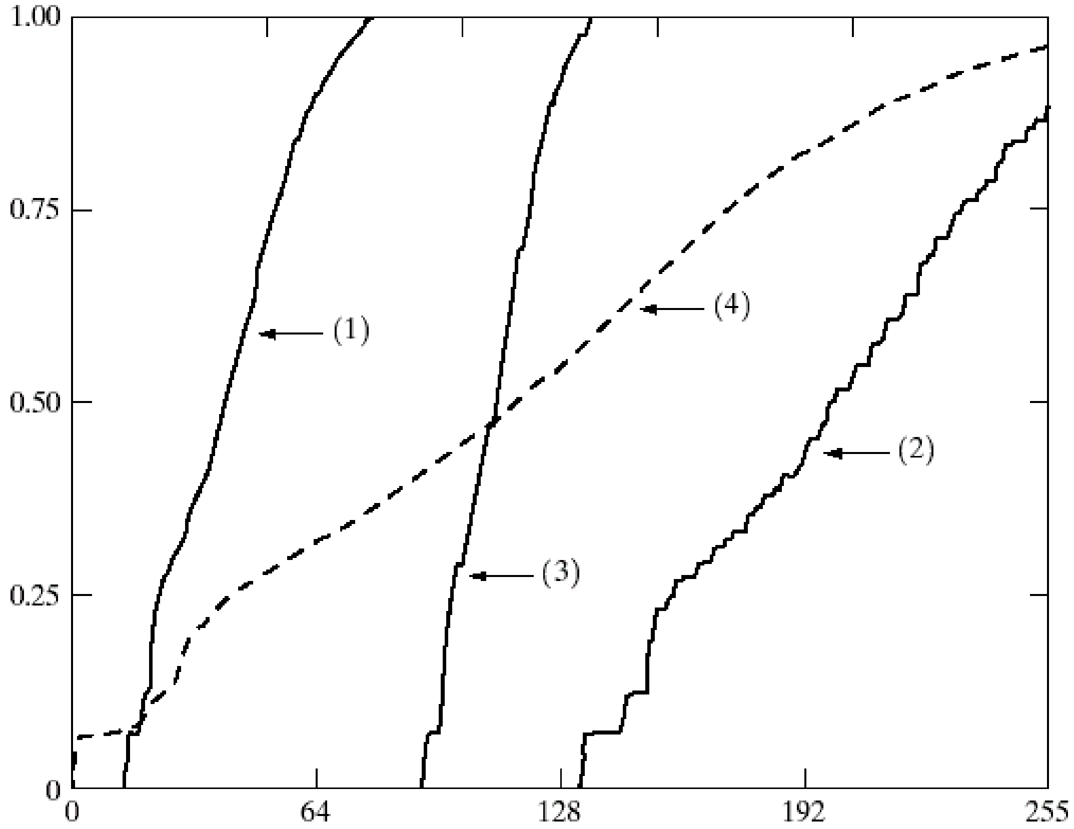




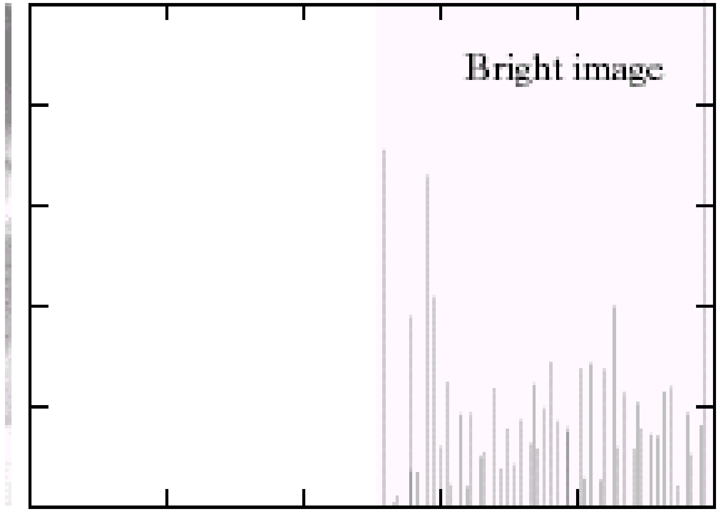


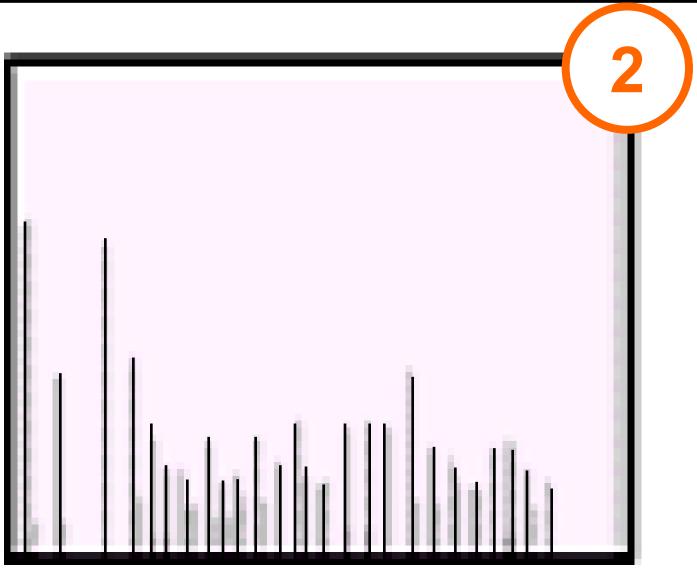
Equalisation Transformation Functions

The functions used to equalise the images in the previous example



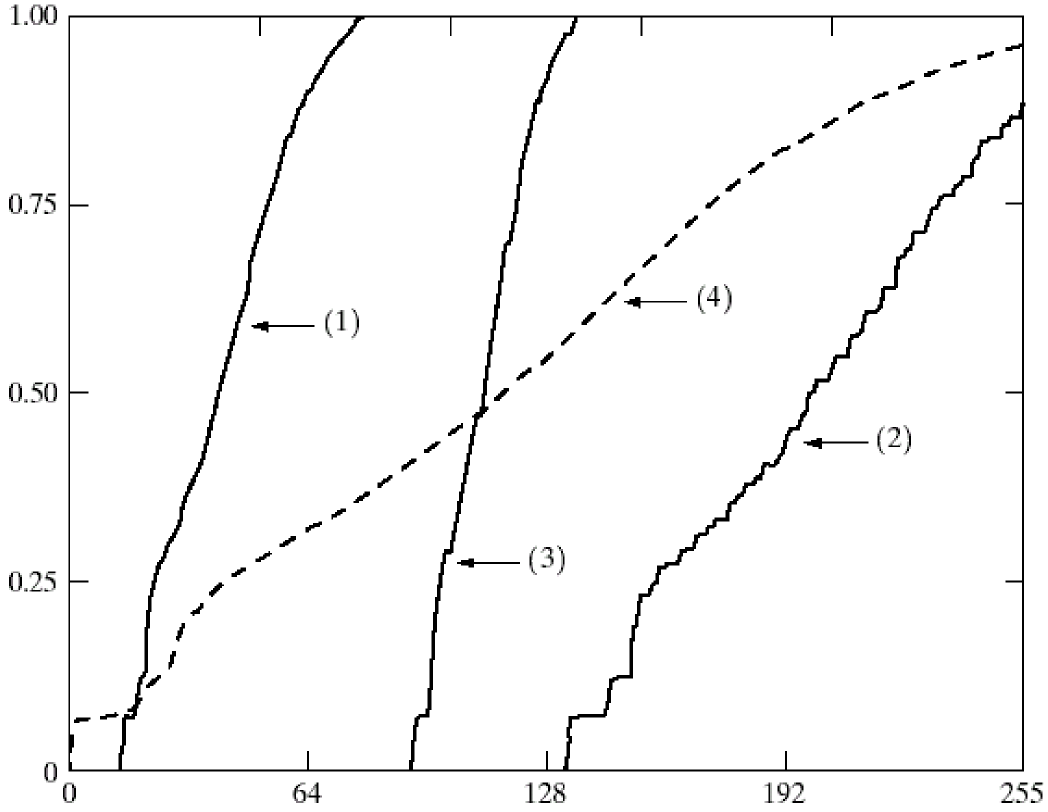
Equalisation Examples



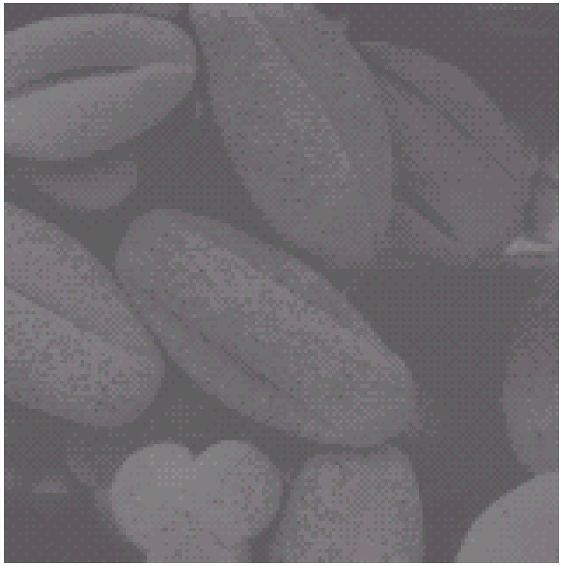


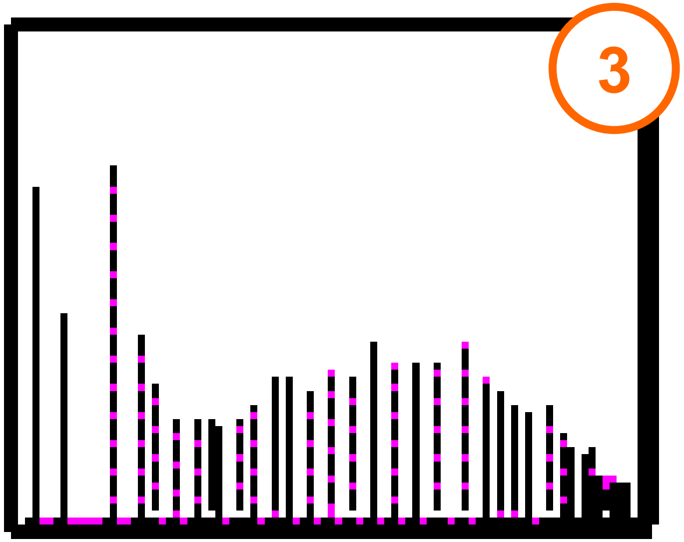
Equalisation Transformation Functions

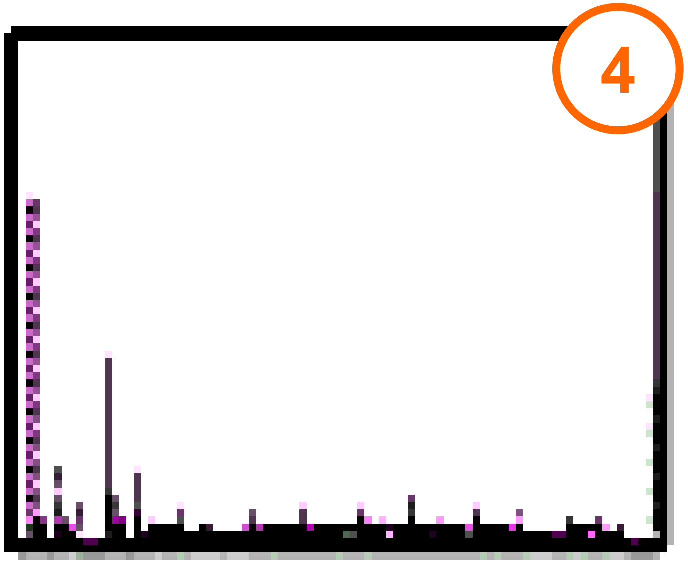
The functions used to equalise the images in the previous example



Equalisation Examples (cont…)

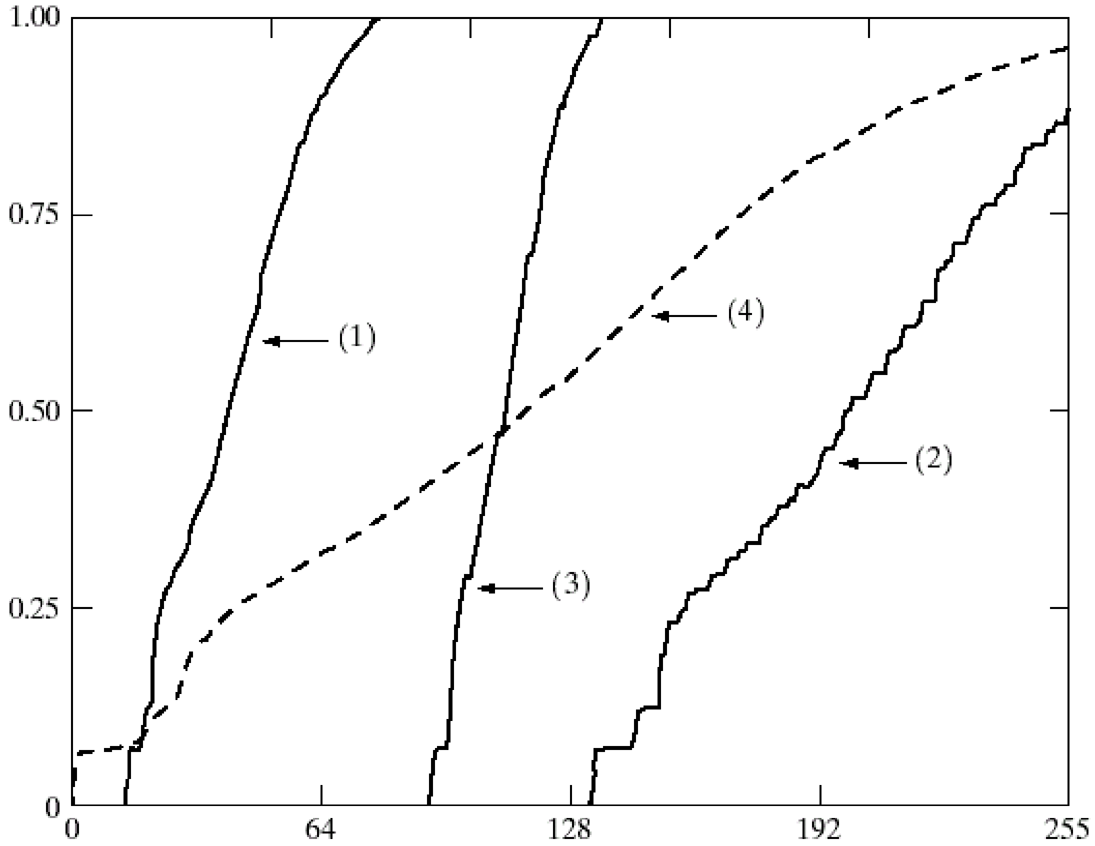






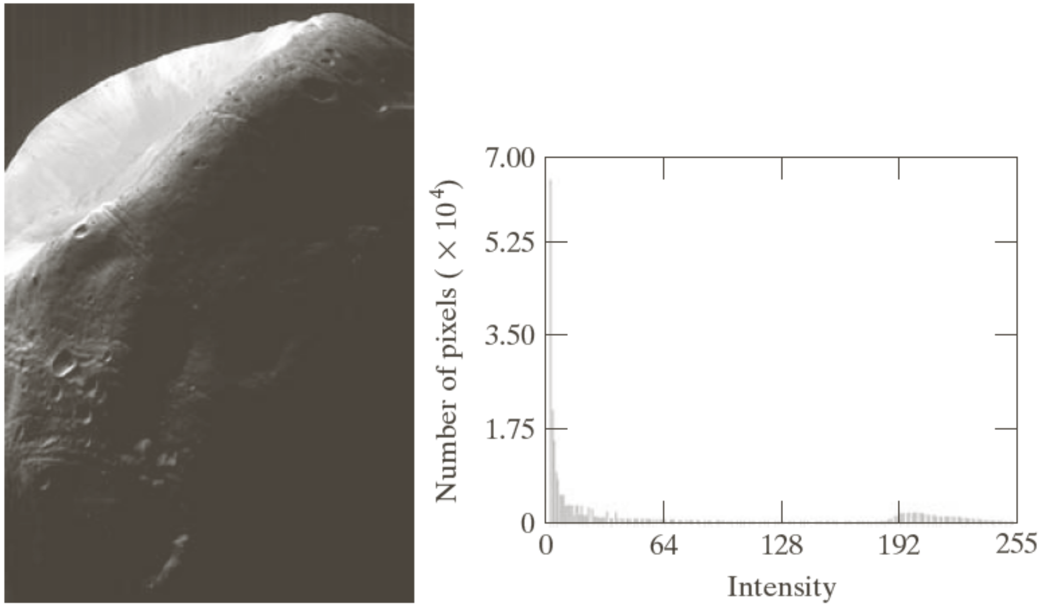
Equalisation Transformation Functions

The functions used to equalise the images in the previous examples



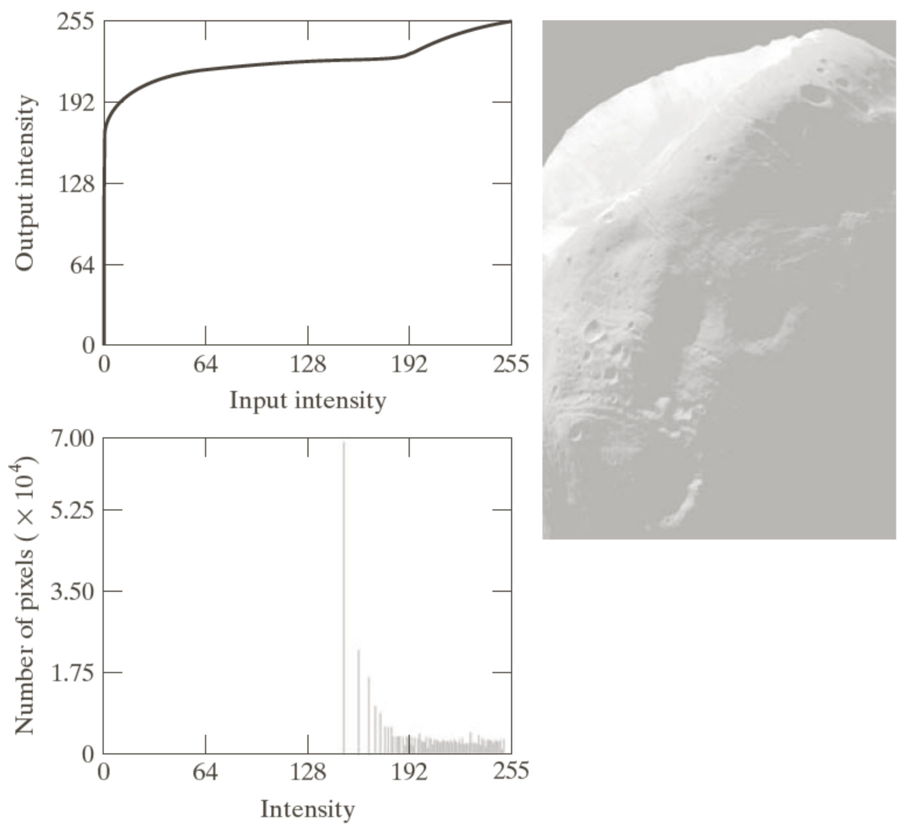
Histogram Specification

- Histogram equalization does not always provide the desirable results.



- Image of Phobos (Mars moon) and its histogram.
- Many values near zero in the initial histogram

Histogram Specification (cont...)



Histogram equalization

Histogram specification (cont.)

- In these cases, it is more useful to specify the final histogram.
- Problem statement:
  - Given $p_r(r)$ from the image and the target histogram $p_z(z)$, estimate the transformation $z=T(r)$.
- The solution exploits histogram equalization.

Histogram specification (cont…)

- Equalize the initial histogram of the image:

$$s = T(r) = (L-1)\int_0^r p_r(w)\,dw$$

$$G(z) = T(r)$$

- Equalize the target histogram:

$$s = G(z) = (L-1)\int_0^r p_z(w)\,dw$$

- Obtain the inverse transform: $z = G^{-1}(s) = G^{-1}(T(r))$

In practice, for every value of r in the image:

- get its equalized transformation $s=T(r)$.
- perform the inverse mapping $z=G^{-1}(s)$, where $s=G(z)$ is the equalized target histogram.

Histogram specification (cont…)

The discrete case:

•Equalize the initial histogram of the image:

$$s_k = T(r_k) = (L-1)\sum_{j=0}^{k} p_r(r_j) = \frac{(L-1)}{MN}\sum_{j=0}^{k} n_j$$

$$G(z) = T(r)$$

•Equalize the target histogram:

$$s_k = G(z_q) = (L-1)\sum_{i=0}^{q} p_z(r_i)$$

•Obtain the inverse transform: $z_q = G^{-1}(s_k) = G^{-1}(T(r_k))$

Histogram Specification (cont...) Example

Consider again the 3-bit 64x64 image:

r_k	n_k	$p_r(r_k) = n_k/MN$
$r_0 = 0$	790	0.19
$r_1 = 1$	1023	0.25
$r_2 = 2$	850	0.21
$r_3 = 3$	656	0.16
$r_4 = 4$	329	0.08
$r_5 = 5$	245	0.06
$r_6 = 6$	122	0.03
$r_7 = 7$	81	0.02

It is desired to transform this histogram to:

$$p_z(z_0) = 0.00 \quad p_z(z_1) = 0.00 \quad p_z(z_2) = 0.00 \quad p_z(z_3) = 0.15$$
$$p_z(z_4) = 0.20 \quad p_z(z_5) = 0.30 \quad p_z(z_6) = 0.20 \quad p_z(z_7) = 0.15$$

with $z_0 = 0, z_1 = 1, z_2 = 2, z_3 = 3, z_4 = 4, z_5 = 5, z_6 = 6, z_7 = 7.$

Histogram Specification (cont...) Example

The first step is to equalize the input (as before):

$$s_0 = 1,\ s_1 = 3,\ s_2 = 5,\ s_3 = 6,\ s_4 = 6,\ s_5 = 7,\ s_6 = 7,\ s_7 = 7$$

The next step is to equalize the output:

$$\begin{array}{llll} G(z_0) = 0 & G(z_1) = 0 & G(z_2) = 0 & G(z_3) = 1 \\ G(z_4) = 2 & G(z_5) = 5 & G(z_6) = 6 & G(z_7) = 7 \end{array}$$

Notice that $G(z)$ is not strictly monotonic. We must resolve this ambiguity by choosing, e.g. the smallest value for the inverse mapping.

Histogram Specification (cont...) Example

Perform inverse mapping: find the smallest value of z_q that is closest to s_k:

$s_k = T(r_i)$	$G(z_q)$
$s_0 = 1$	$G(z_0) = 0$
$s_1 = 3$	$G(z_1) = 0$
$s_2 = 5$	$G(z_2) = 0$
$s_3 = 6$	$G(z_3) = 1$
$s_4 = 6$	$G(z_4) = 2$
$s_5 = 7$	$G(z_5) = 5$
$s_6 = 7$	$G(z_6) = 6$
$s_7 = 7$	$G(z_7) = 7$

$$s_k \rightarrow z_q$$

$$1 \rightarrow 3$$

$$3 \rightarrow 4$$

$$5 \rightarrow 5$$

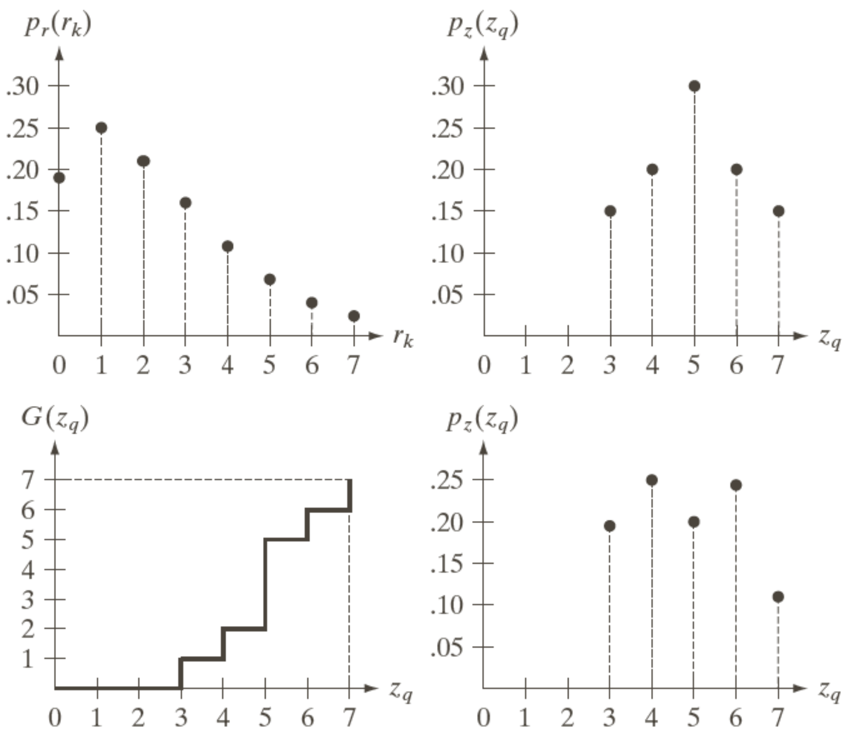
$$6 \rightarrow 6$$

$$7 \rightarrow 7$$

e.g. every pixel with value $s_0=1$ in the histogram-equalized image would have a value of 3 (z_3) in the histogram-specified image.

Histogram Specification (cont...) Example

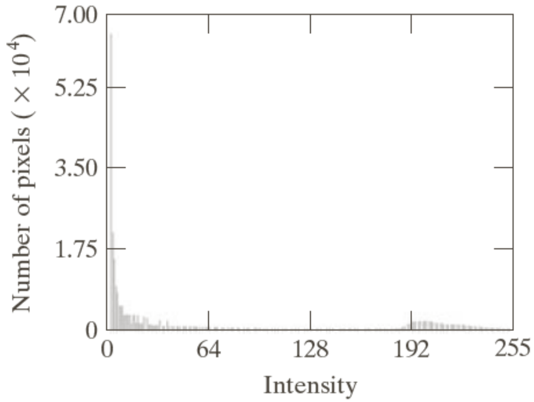
Notice that due to discretization, the resulting histogram will rarely be exactly the same as the desired histogram.

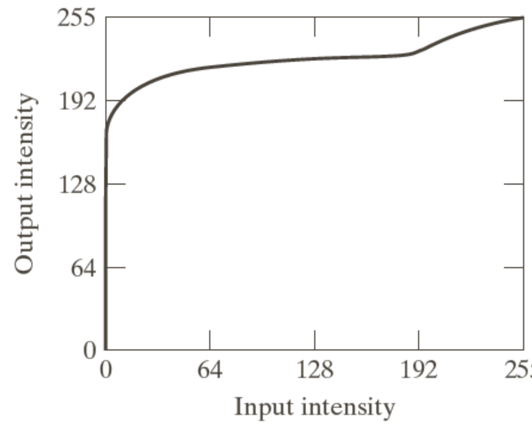


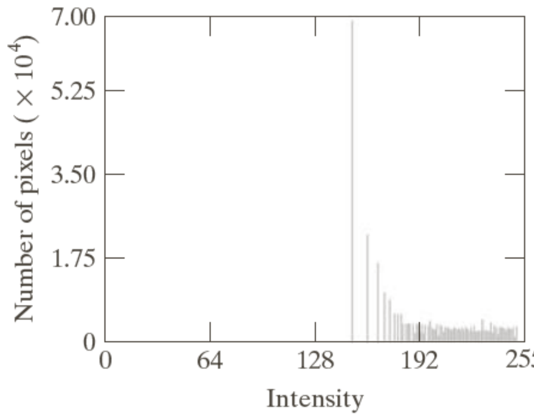
a	b
c	d

FIGURE 3.22
(a) Histogram of a 3-bit image. (b) Specified histogram. (c) Transformation function obtained from the specified histogram. (d) Result of performing histogram specification. Compare (b) and (d).

Histogram Specification (cont...)



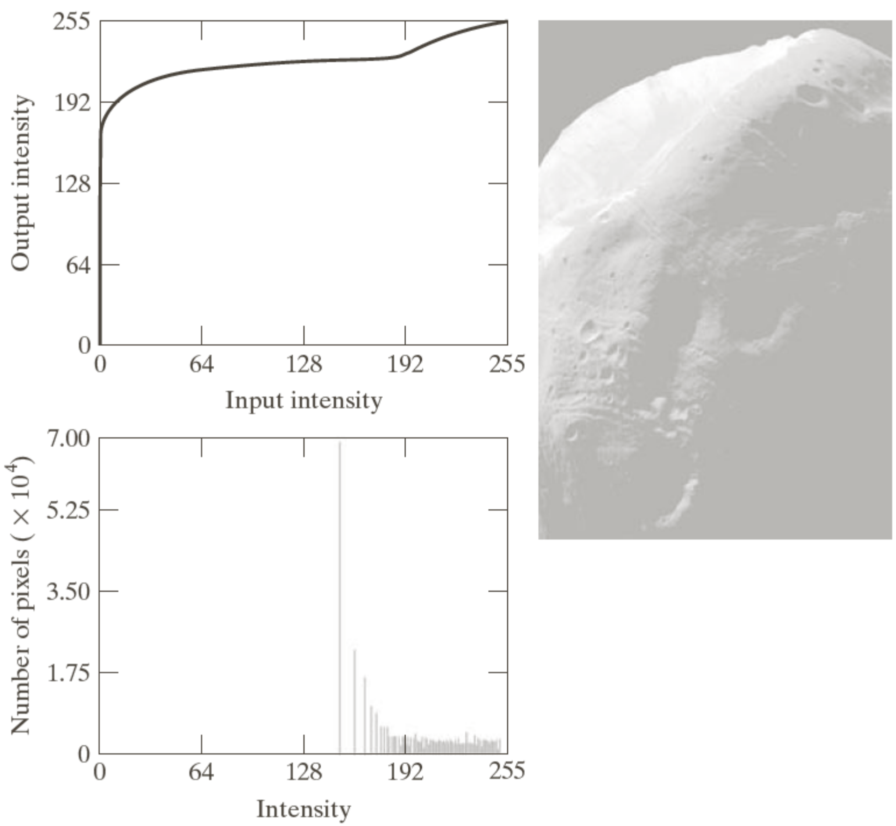




Original image

Histogram equalization

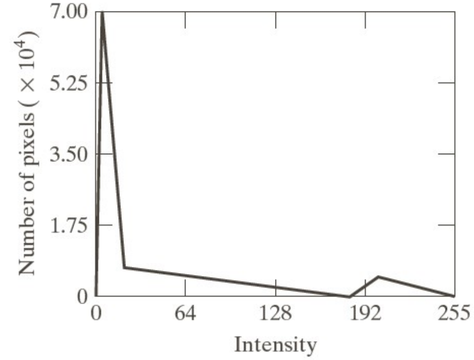
Histogram Specification (cont...)

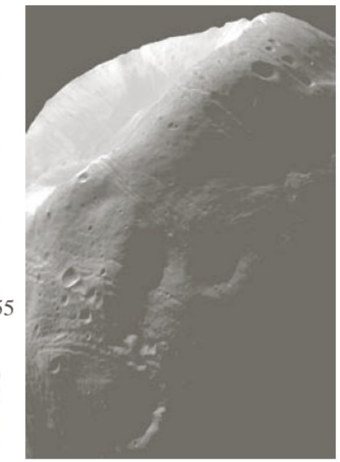


Histogram equalization

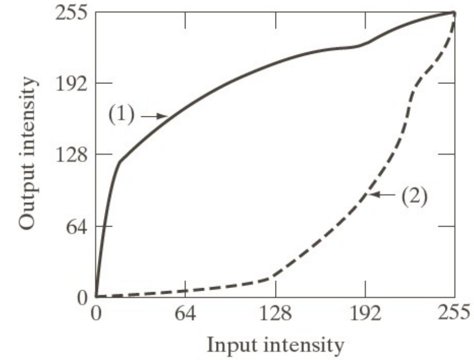
Histogram Specification (cont...)

Specified histogram

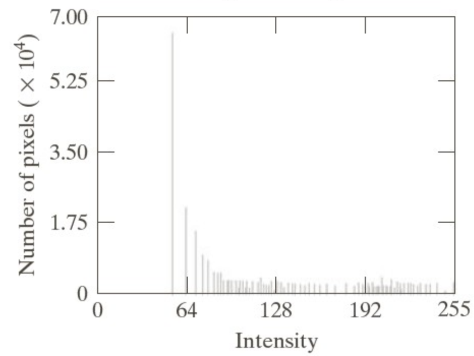




Transformation function and its inverse



Resulting histogram



Local Histogram Processing

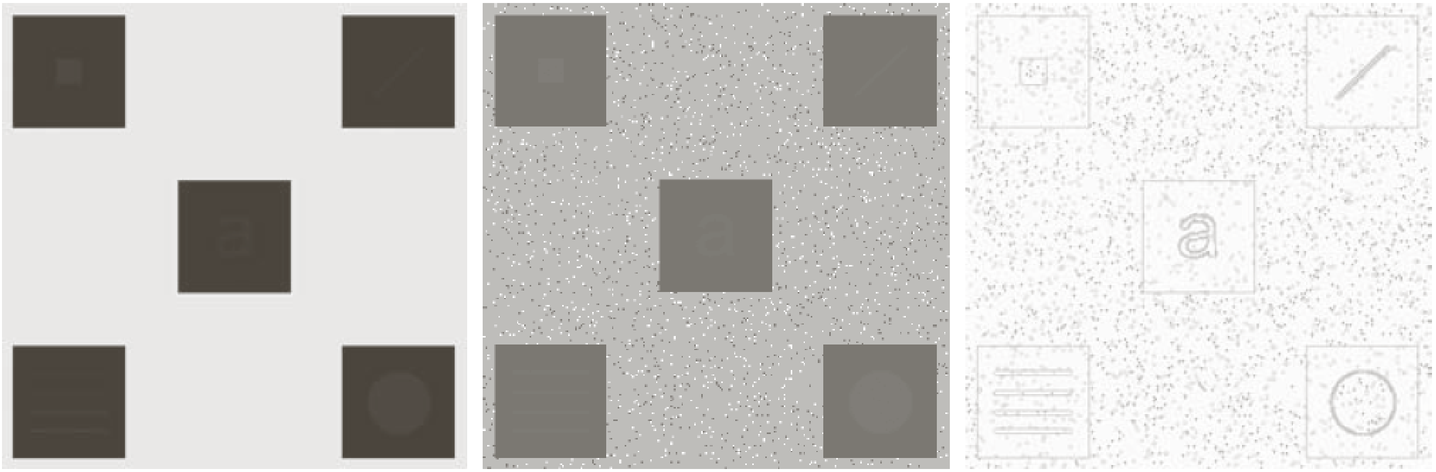


FIGURE 3.26 (a) Original image. (b) Result of global histogram equalization. (c) Result of local histogram equalization applied to (a), using a neighborhood of size 3 × 3.

- Image in (a) is slightly noisy but the noise is imperceptible.
- HE enhances the noise in smooth regions (b).
- Local HE reveals structures having values close to the values of the squares and small sizes to influence HE (c).

Summary

We have looked at:

- Different kinds of image enhancement
- Histograms
- Histogram equalisation
- Histogram specification

Next time we will start to look at spatial filtering and neighbourhood operations