COS 324 - S2020 Assignment 1

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Problem 1

Apply the Gram-Schmidt process to the rows of the following matrix U:

$$U = \begin{bmatrix} 3 & 4 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Part 1.A

Calculate $||U_{1,*}||$ and $||U_{2,*}||$.

$$||U_{1,*}|| = \sqrt{(3^2 + 4^2 + 0^2)} = 5$$

$$||U_{2,*}|| = \sqrt{0^2 + 1^2 + 1^2)} = \sqrt{2}$$

Part 1.B

Find the projection of $U_{2,*}$ onto $U_{1,*}$, which is $proj_{U_{1,*}}(U_{2,*})$.

$$proj_{U_{1,*}}(U_{2,*}) = \frac{U_{2,*} \cdot U_{1,*}}{||U_{1,*}||^2} U_{1,*}$$

$$proj_{U_{1,*}}(U_{2,*}) = \frac{(0*3) + (1*4) + (1*0)}{25} \begin{bmatrix} 3 & 4 & 0 \end{bmatrix}$$

$$proj_{U_{1,*}}(U_{2,*}) = \begin{bmatrix} \frac{12}{25} & \frac{16}{25} & 0 \end{bmatrix}$$

Part 1.C

Find $\tilde{U}_{2,*}$, given that $\tilde{U}_{2,*} = U_{2,*} - proj_{U_{1,*}}(U_{2,*})$

$$\tilde{U}_{2,*} = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} - \begin{bmatrix} \frac{12}{25} & \frac{16}{25} & 0 \end{bmatrix}$$

$$\tilde{U}_{2,*} = \begin{bmatrix} \frac{-12}{25} & \frac{9}{25} & 1 \end{bmatrix}$$

Part 1.D

What is the relation between $U_{1,*}$ and $\tilde{U}_{2,*}$? What is the relation between the $\frac{U_{1,*}}{||U_{1,*}||}$ and $\frac{\tilde{U}_{2,*}}{||\tilde{U}_{2,*}||}$?

To this I should say that $U_{1,*}$ and $\tilde{U}_{2,*}$ are orthogonal to each other since their dot products are equal to zero. In the same vein, $\frac{\tilde{U}_{2,*}}{||\tilde{U}_{2,*}||}$ are orthonormal to each other, so orthogonal and unitized.

Problem 2

Find the critical points of the following function:

$$f(x) = log_e(1 + e^{a-x}) + log_e(1 + e^{x-a})$$

Part 2.A

What is $f'(x)$?

$$\begin{aligned} f'(x) &= \frac{1}{1+e^{a-x}} * -e^{a-x} + \frac{1}{1+e^{x-a}} * e^{x-a} \\ &= \frac{-e^a}{e^a+e^x} + \frac{e^x}{e^a+e^x} \\ &= \frac{-e^a+e^x}{e^a+e^x} \end{aligned}$$

Part 2.B

At what point or points does $f'(x) = 0$, or does not exist?

At one critical point, when $x = a$.

$$\begin{aligned} \frac{-e^a+e^x}{e^a+e^x} &= 0 \\ -e^a + e^x &= 0 \\ e^x &= e^a \end{aligned}$$

Part 2.C

What is the point x^* such that $f'(x)$ is minimized?

This is the case when $x^* = x = a$. Applying the second derivative, it is found that the critical point is a minimum.

$$f''(x) = \frac{2e^{a+x}}{(e^a+e^x)^2}$$

$$f''(a) = \frac{2e^{2a}}{4e^{2a}} = \frac{1}{2} > 0$$