COS234:
Introduction To Machine Learning

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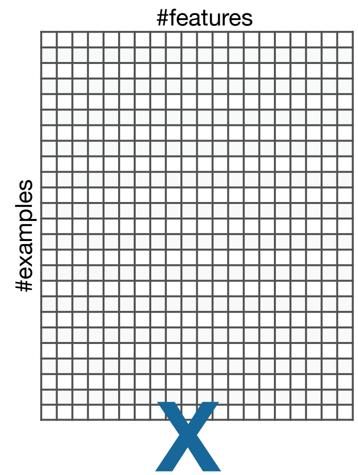
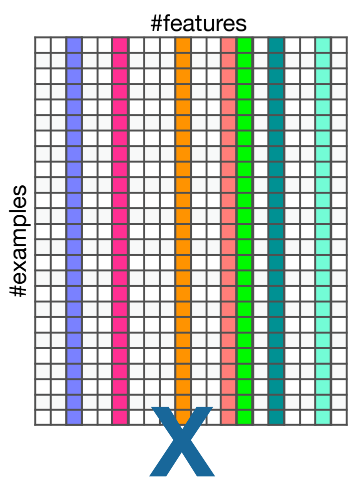
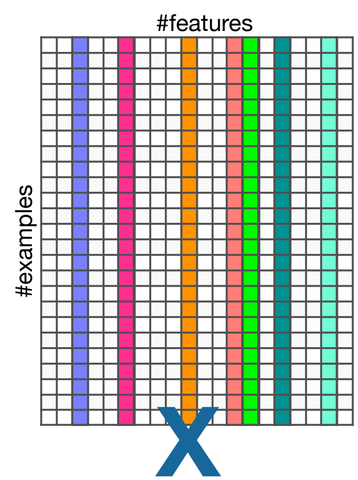
PRINCETON UNIVERSITY

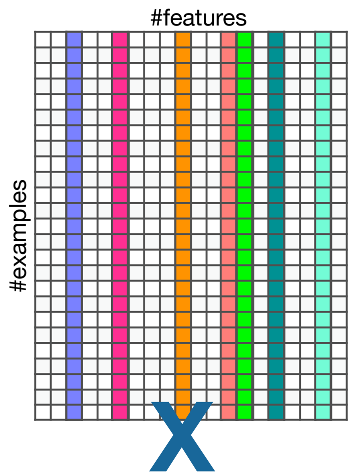
Topic: Gradient-Based Learning - Part I



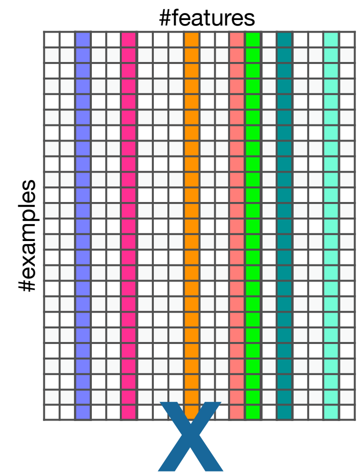




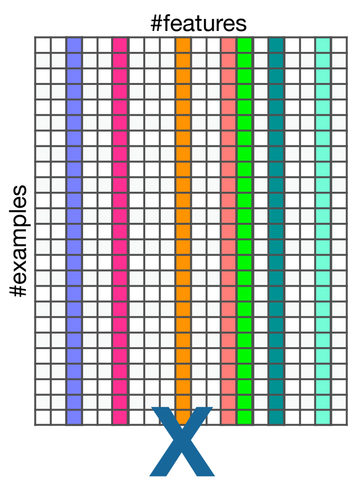
- Sequential: a column at a time



- Sequential: a column at a time
- Oblivious to “similar” examples



- Sequential: a column at a time
- Oblivious to “similar” examples
- Difficult to parallelize



- Sequential: a column at a time
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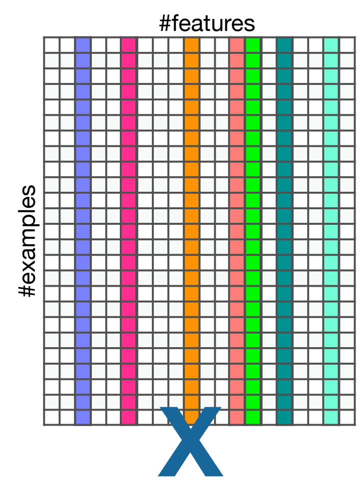
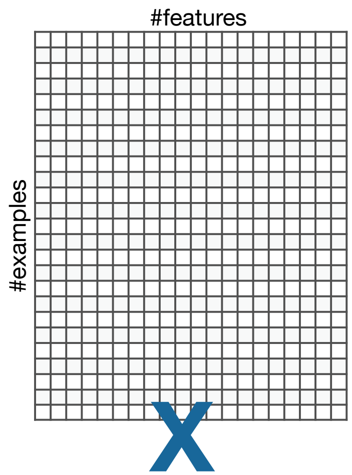
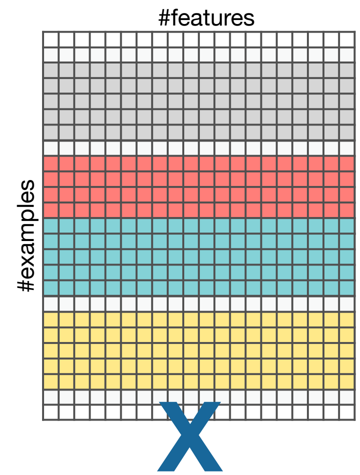
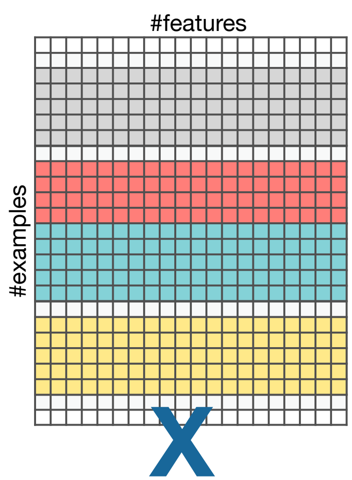
- Requires dedicated update per loss



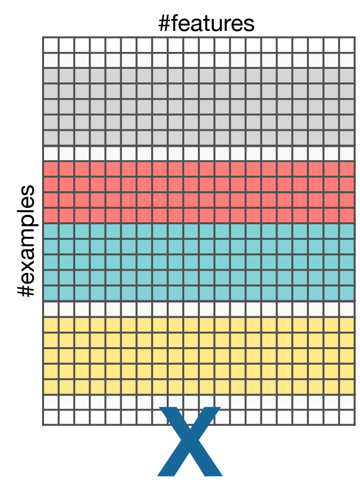
- Sequential: a column at a time
- Oblivious to “similar” examples
- Difficult to parallelize
- Requires dedicated update per loss
- Fails to work in non-linear settings



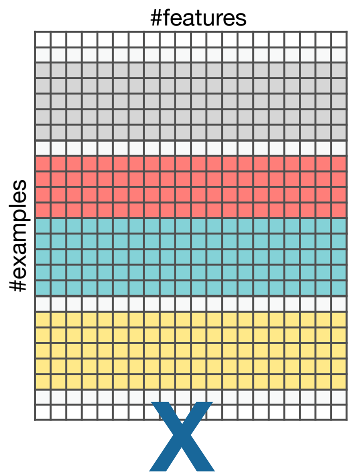




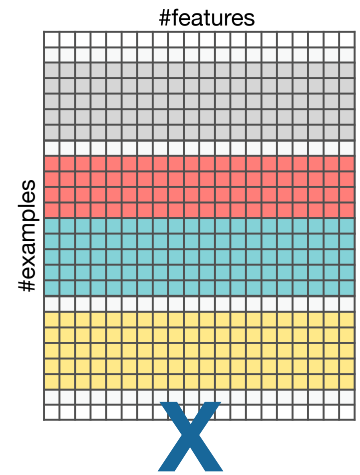
- Pick a small subset of rows of X to use



- Pick a small subset of rows of X to use
- Subset not necessarily consecutive rows



- Pick a small subset of rows of X to use
- Subset not necessarily consecutive rows
- Subset used to update all weights



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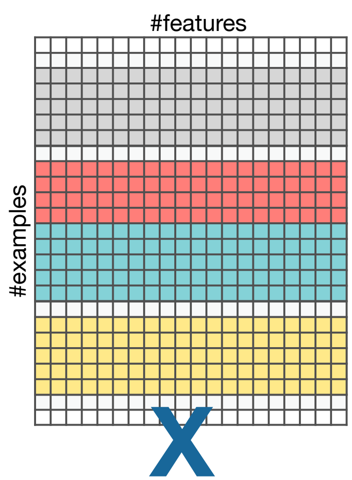
- Subset used to update all weights
- Update "local" to subset selected



- Pick a small subset of rows of X to use
- Subset not necessarily consecutive rows
- Subset used to update all weights
- Update "local" to subset selected
- Albeit local, effect is "global"

- a^* minimum point of f:

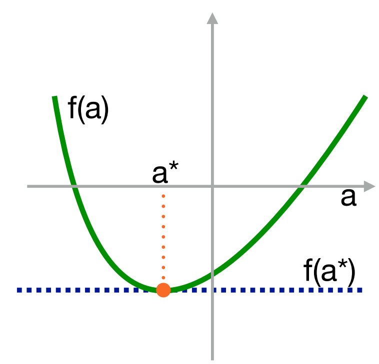
  $f(a^\star) \leq f(b)$ for all $b \neq a^\star$

- Derivate of f(a) at a^* is zero

  $$\left.\frac{df}{da}\right|_{a=a^\star} \equiv f'(a^\star) = 0$$

- Often no closed form solution for:

  $$\frac{df}{da} = 0$$



we can still make use of f' …

- $a^\star$ minimum point of f:

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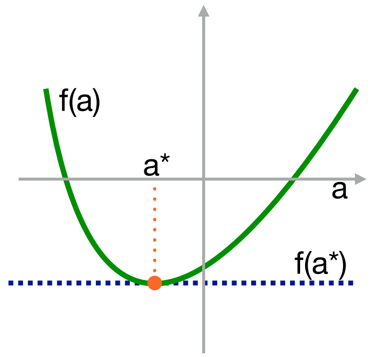
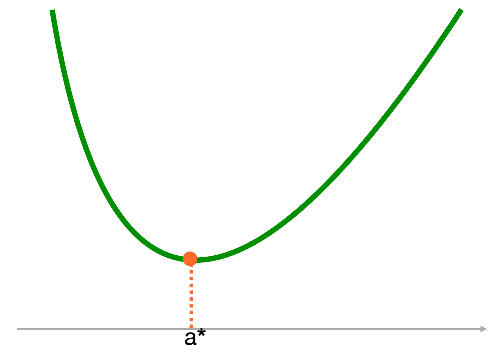


(Strict) Convexity



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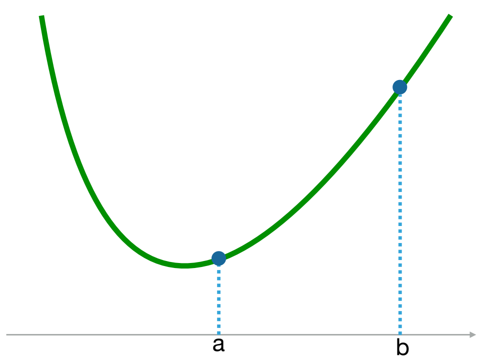

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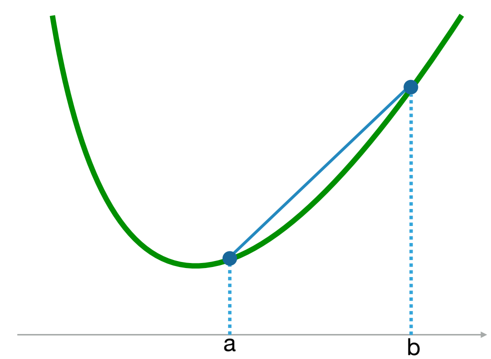

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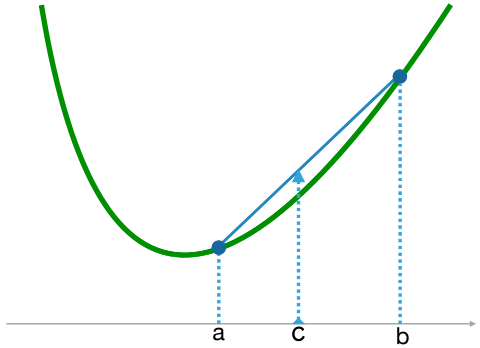
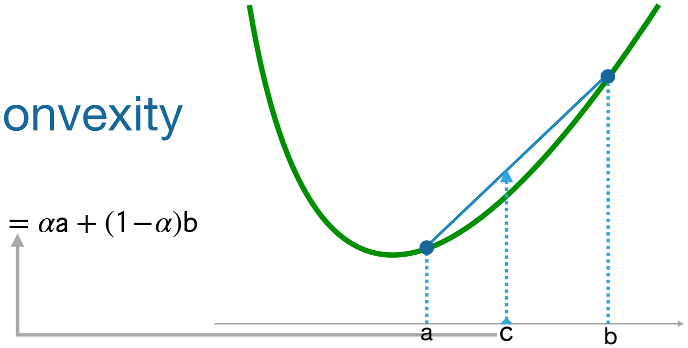


(Strict) Convexity

$$c = \alpha a + (1 - \alpha) b$$



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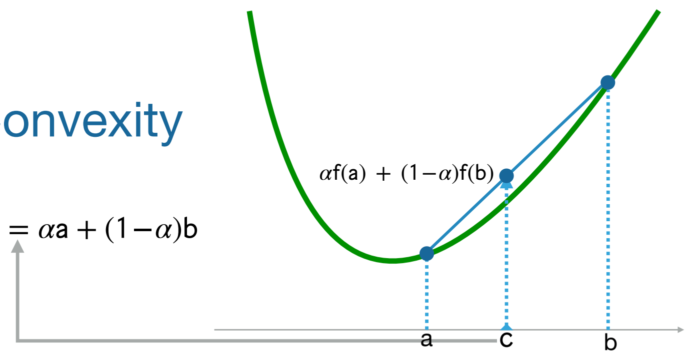

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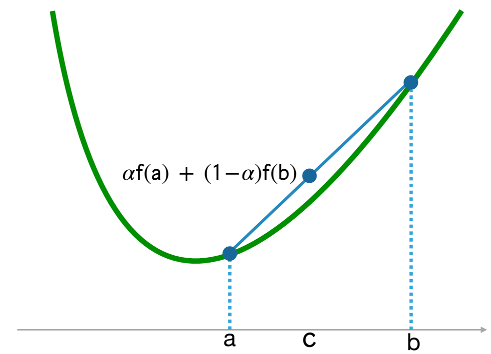

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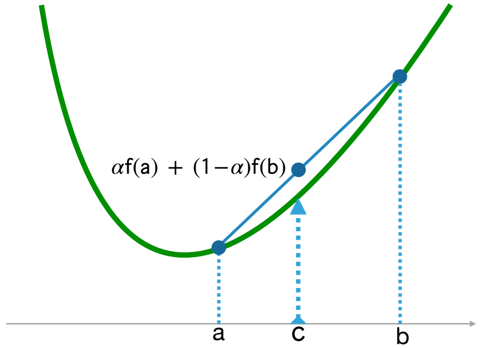

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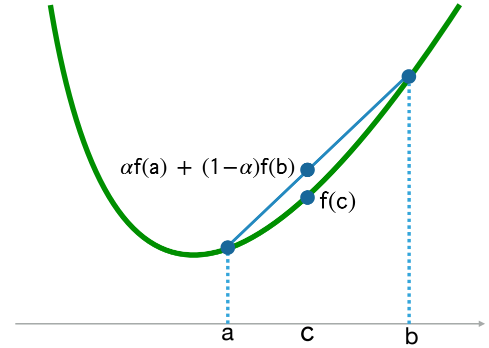
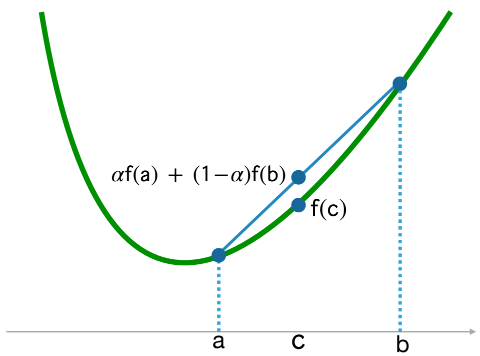


$$\alpha \in (0,1): \ \mathrm{f}\big(\alpha \mathrm{a} + (1-\alpha)\mathrm{b}\big) < \alpha \mathrm{f(a)} + (1-\alpha)\mathrm{f(b)}$$

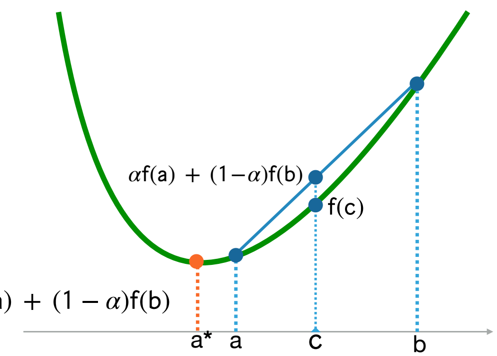
(Strict) Convexity

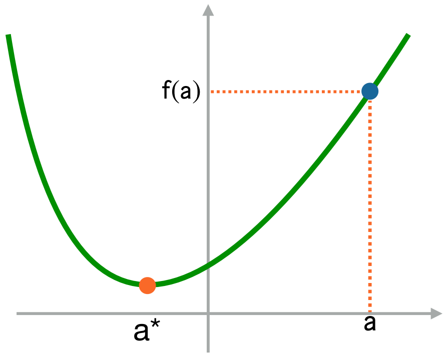


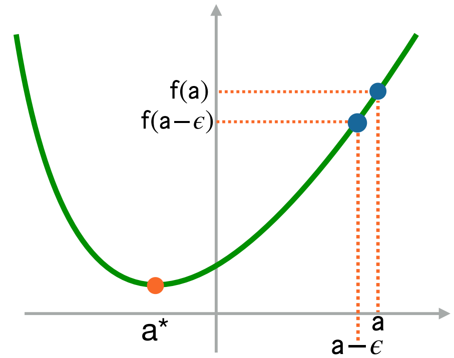
(Strict) Convexity

- a* is a unique minimum
- derivative is < 0 left to a*
- derivative is > 0 right to a*
- derivate of f(a) at a* is 0
- second derivative f''(a)>0

$$\alpha \in (0,1): \ \mathrm{f}\big(\alpha \mathrm{a} + (1-\alpha)\mathrm{b}\big) < \alpha \mathrm{f(a)} + (1-\alpha)\mathrm{f(b)}$$

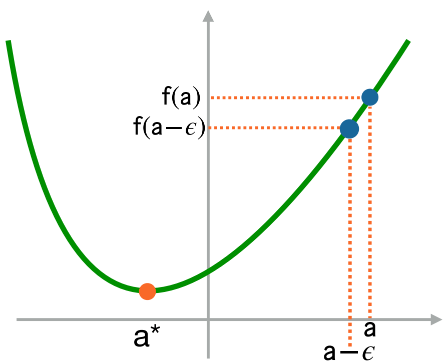






- if a to the right (a > a*) of a* then

$$f(a) > f(a-\epsilon)$$



- if a to the right (a > a*) of a* then

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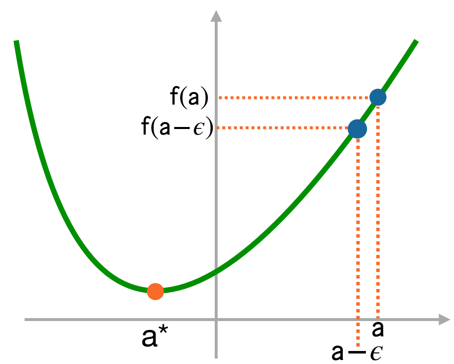
- Therefore

$$\frac{f(a) - f(a-\epsilon)}{\epsilon} > 0$$



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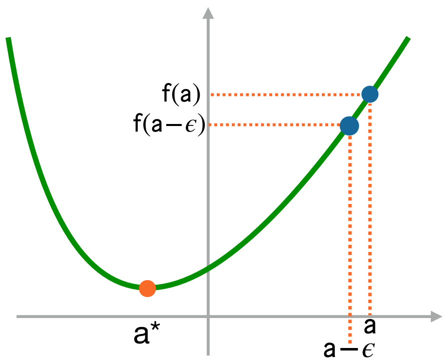
- Taking $\epsilon \to 0$ we get $f'(a) > 0$



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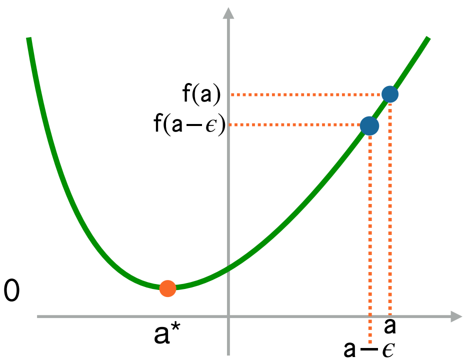

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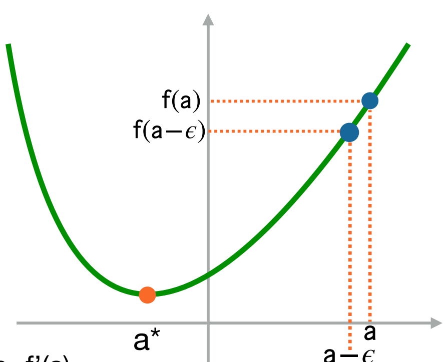
- Get closer to a* by going in direction -f'(a)



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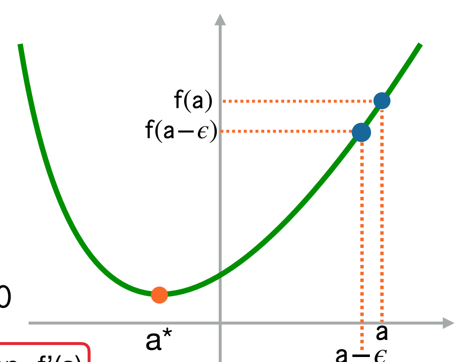

- if a to the right (a > a*) of a* then
$$f(a) > f(a-\epsilon)$$
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$$\frac{f(a) - f(a-\epsilon)}{\epsilon} > 0$$
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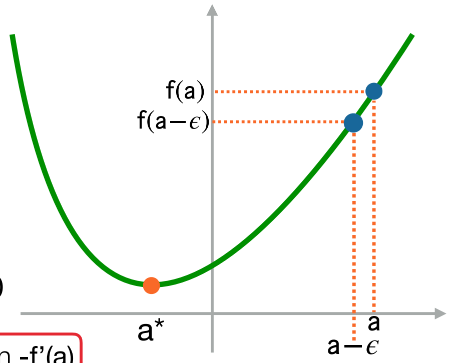
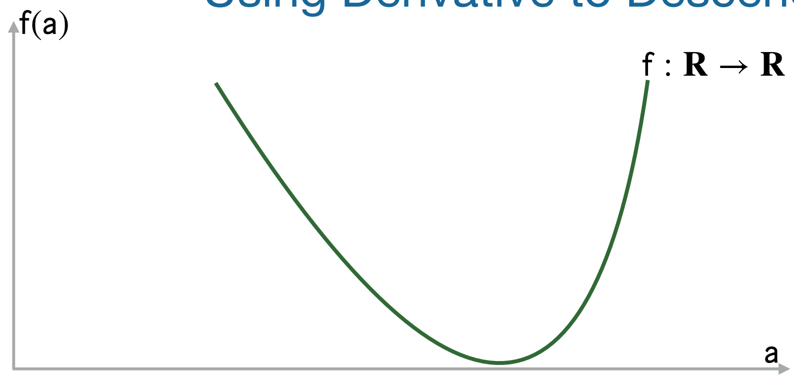


Using Derivative to Descend



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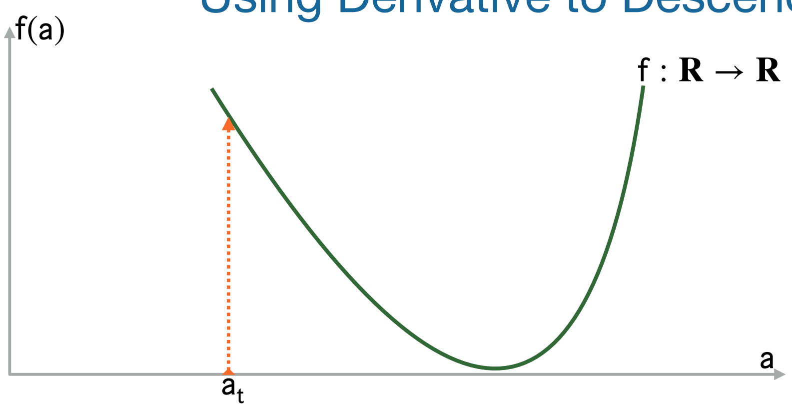

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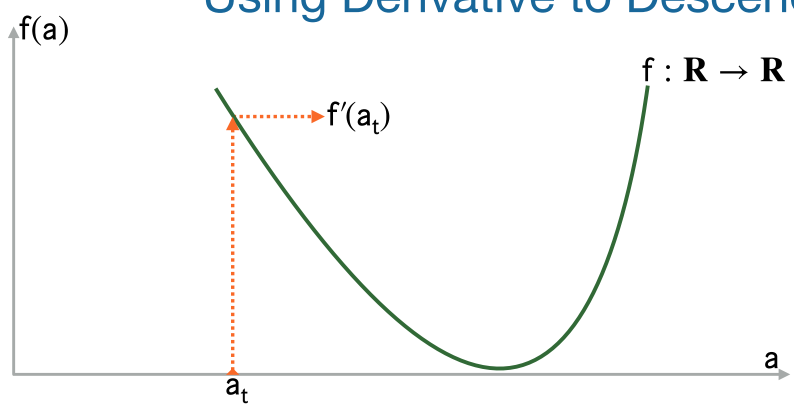

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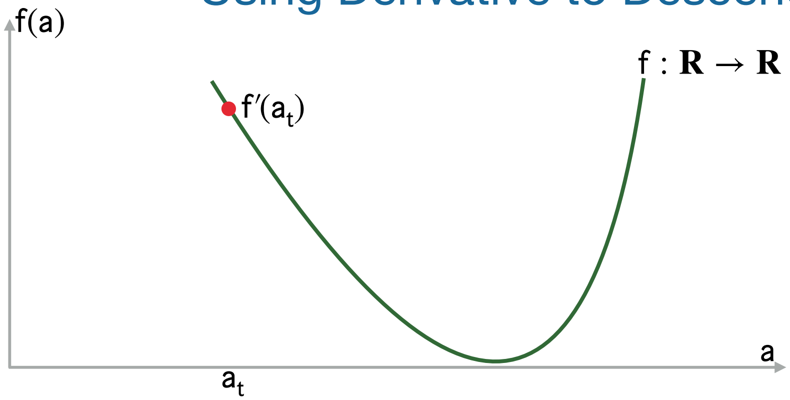

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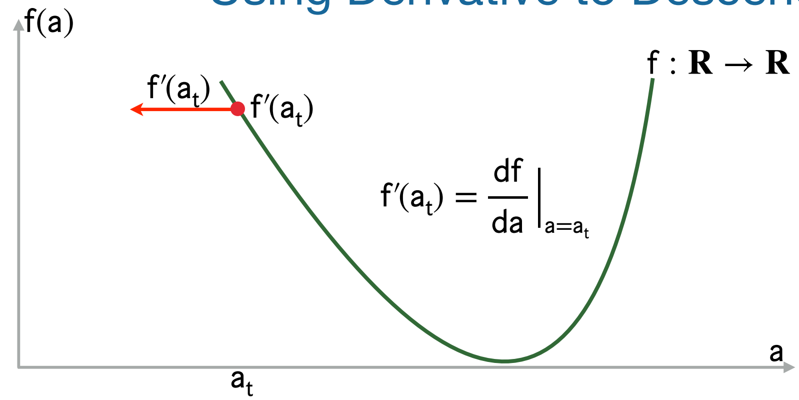

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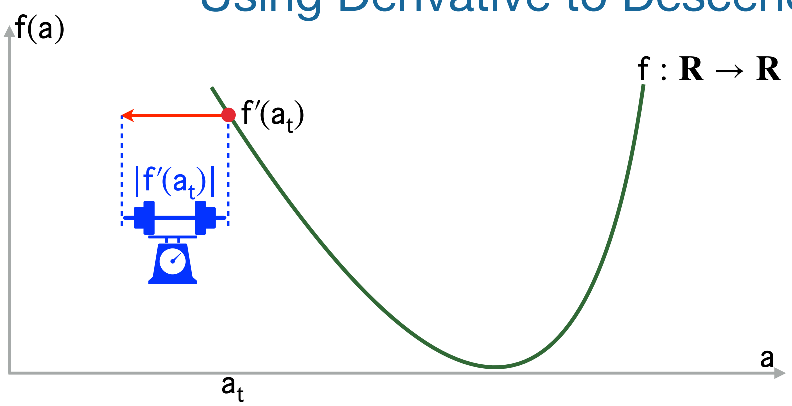

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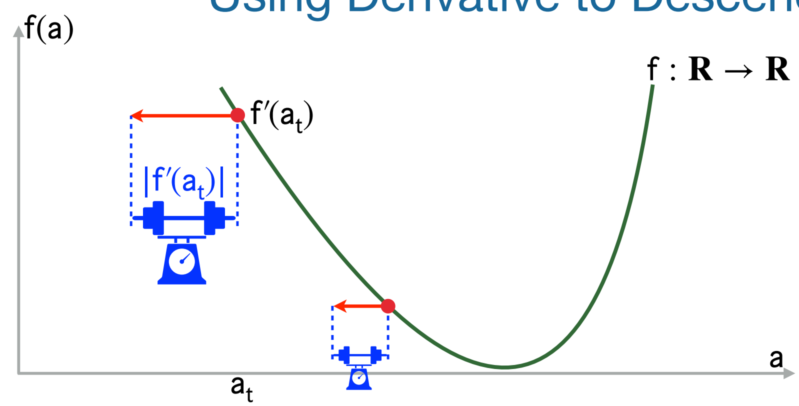

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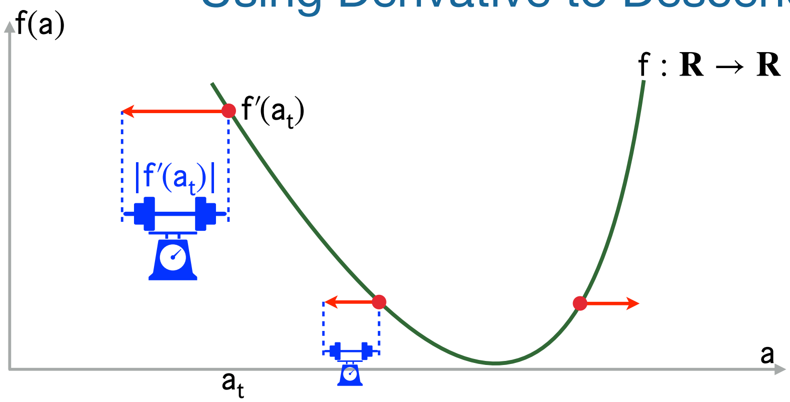
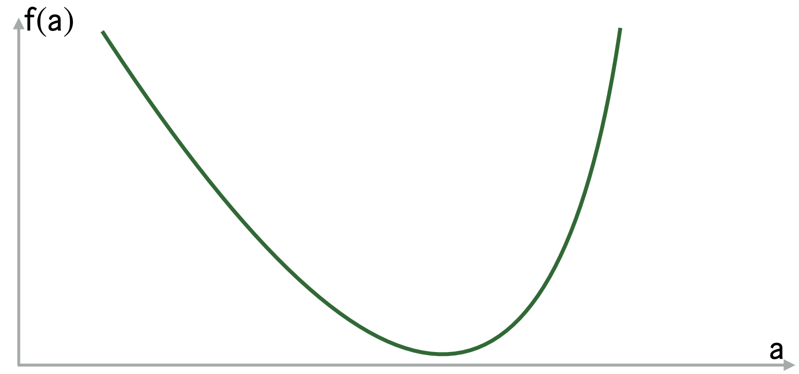
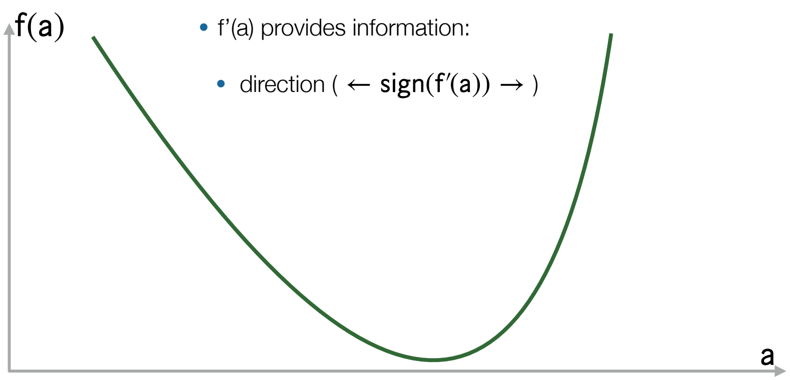


Iterative Derivative Procedure



Iterative Derivative Procedure

- f′(a) provides information:
  - direction (← sign(f′(a)) →)



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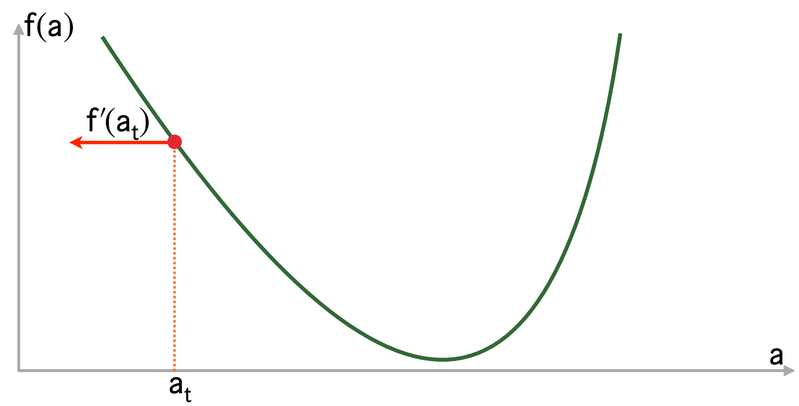

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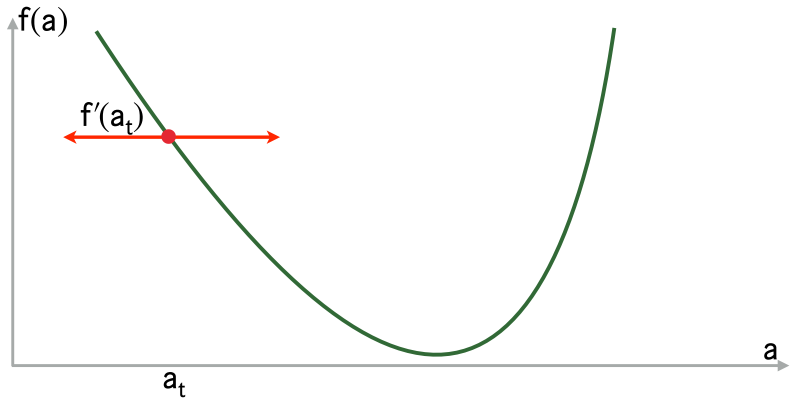

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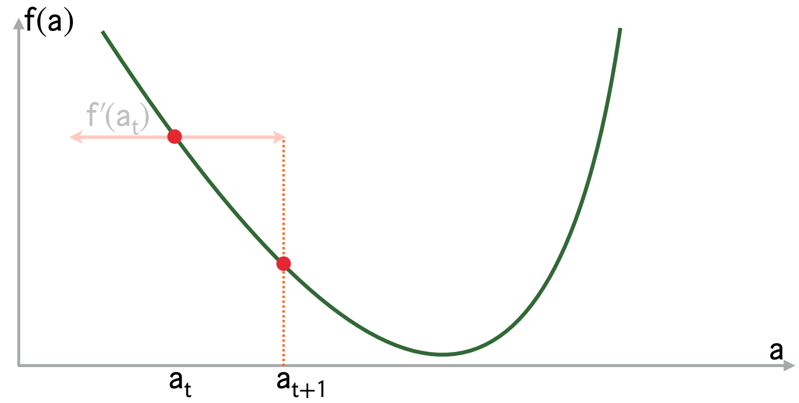

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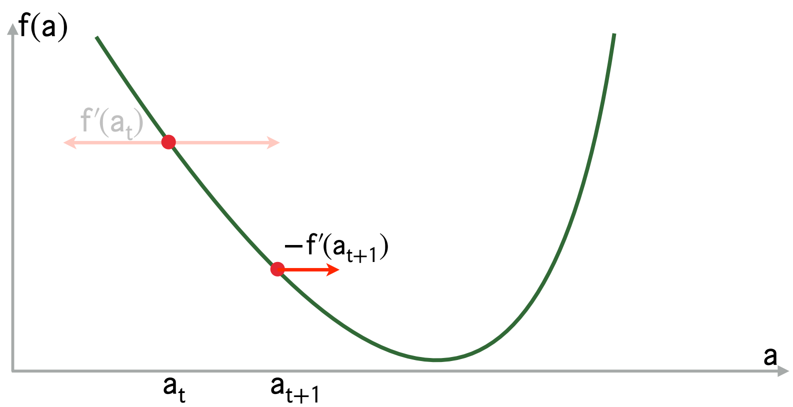

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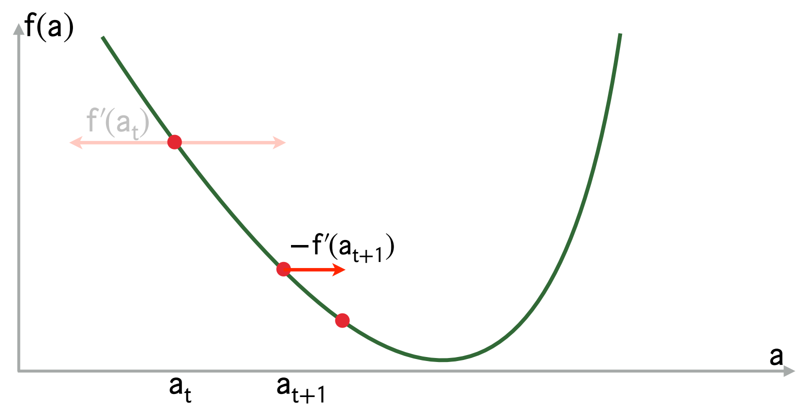

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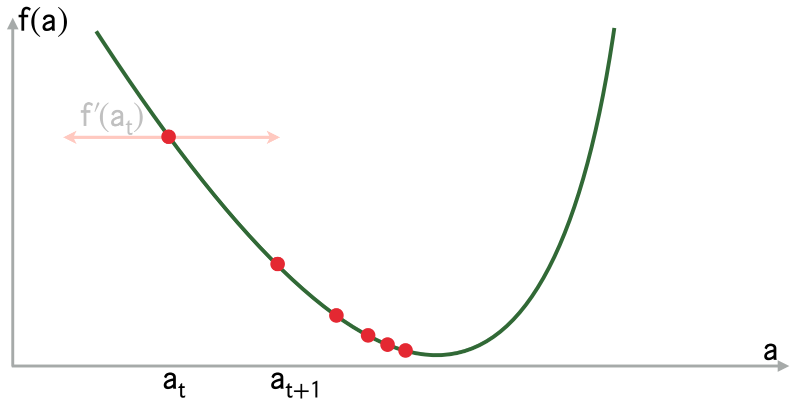

Iterative Derivative Procedure

- Input: function f
- Goal: find $\hat{a}$ such that $|f(\hat{a}) - f(a)| \leq \epsilon$
- Choose initial value a_1
- Loop:
  - $a_{t+1} \leftarrow a_t - \eta f'(a_t)$
  - Until …

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⭐learning rate
⭐step size

Learning Rate

- Crucial in many learning problems
- Fixed learning-rate can be used in certain circumstances
- Self-tuning procedure of learning-rate exist, notably AdaGrad
- In many applications:
  - Linear decrease $\eta_t = \dfrac{\eta_0}{b + st}$ where $\eta_0 \in [0.1, 1]$,
  - Sub-linear decreas $\eta_t \sim \dfrac{\eta_0}{\sqrt{t}}$

Example

- Find minimum of

$$f(x) = \log\left(1 + e^{x-a-\delta}\right) + \log\left(1 + e^{b-a-\delta}\right)$$

- Derivative is

$$f(x) = \frac{1}{1 + e^{a+\delta-x}} - \frac{1}{1 + e^{x+\delta-b}}$$

- a=1 b=2 δ=4.5
- Run with different initializations and learning rates $\left(\eta \in \{0.01, 4, 40\}\right)$

Implementation of f(x)

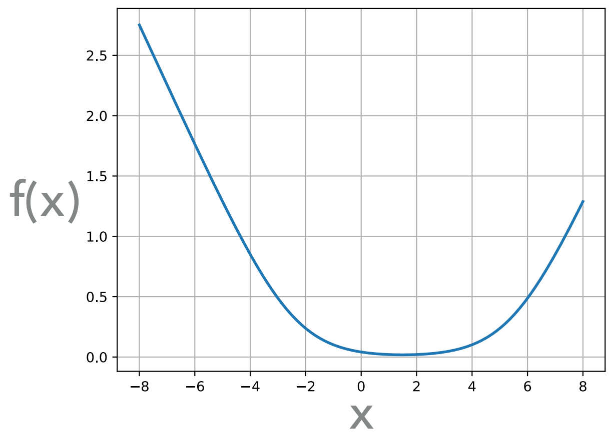
```
import numpy as np

def smooth_l1(a, b, delta):
    def smooth_l1_close(x):
        lloss = np.log(1 + np.exp(x - a - delta))
        rloss = np.log(1 + np.exp(b - x - delta))
        return 0.5 * (rloss + lloss)
    return smooth_l1_close
```

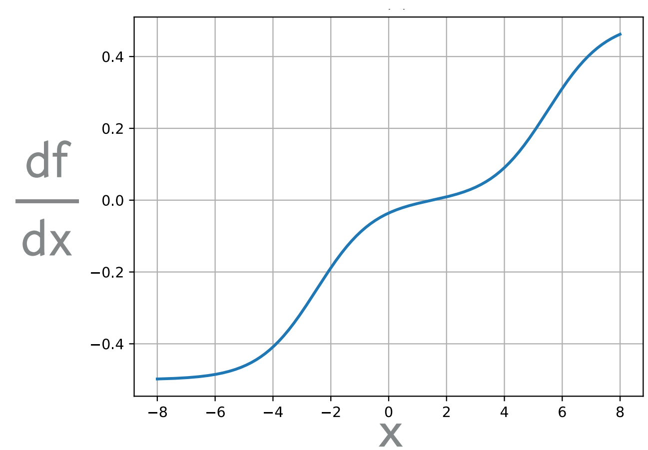

Implementation of f'(x)

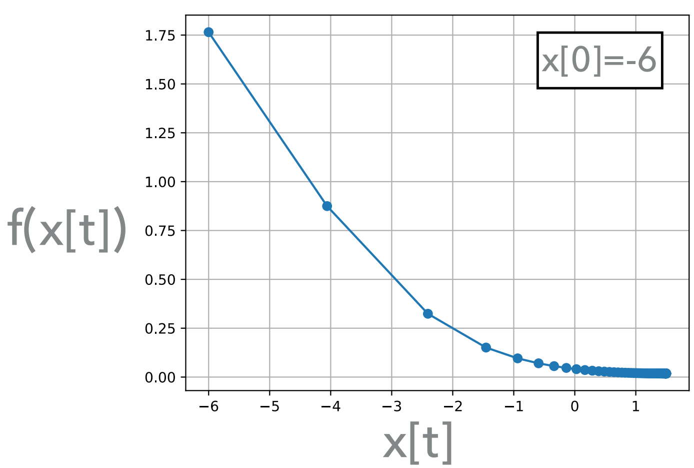
```
def smooth_l1_deriv(a, b, delta):
    def smooth_l1_deriv_close(x):
        rderiv = 1. / (1 + np.exp(a + delta - x))
        lderiv = 1. / (1 + np.exp(x + delta - b))
        return 0.5 * (rderiv - lderiv)
    return smooth_l1_deriv_close
```

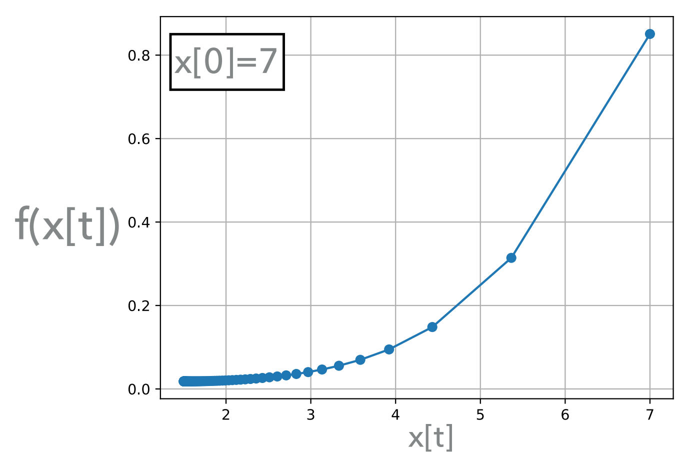

Derivative Descent

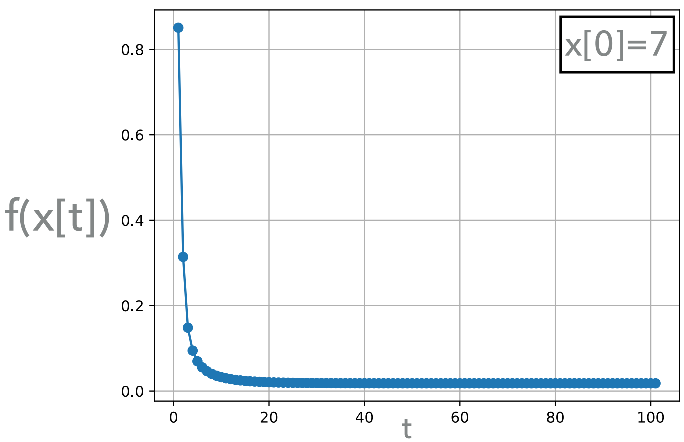
```
def derivative_descent(x0, deriv_func, eta):
    T = len(eta) + 1
    x = np.zeros(T)
    x[0] = x0
    for i in range(1, T):
        x[i] = x[i-1] - eta[i-1] deriv_func(x[i-1])
    return x
```

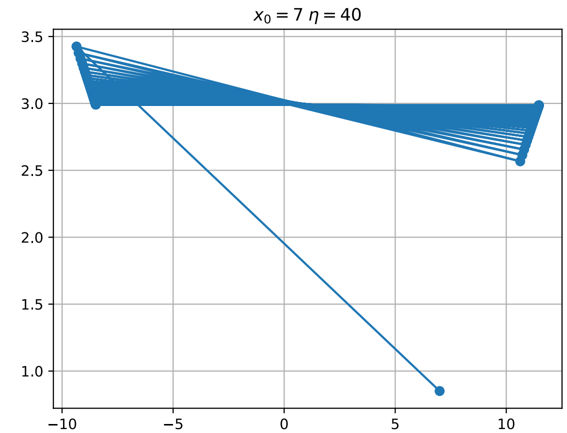

f(x)
x

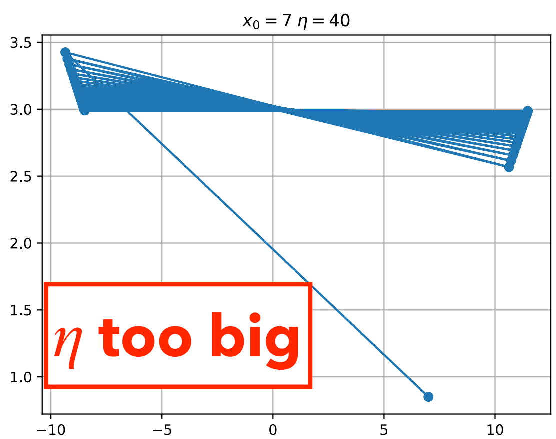

df
dx
x

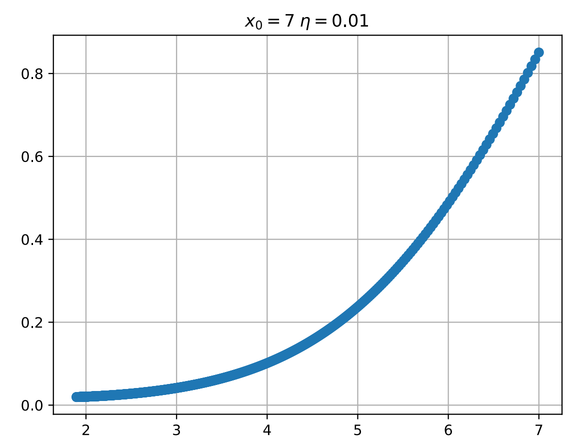

x[0]=-6
f(x[t])
x[t]

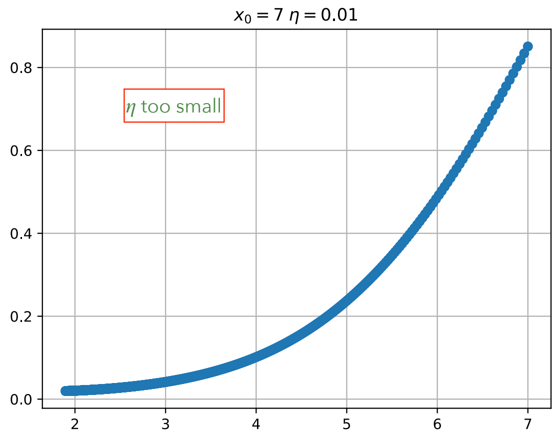

x[0]=7
f(x[t])
x[t]


x[0]=7
f(x[t])
t

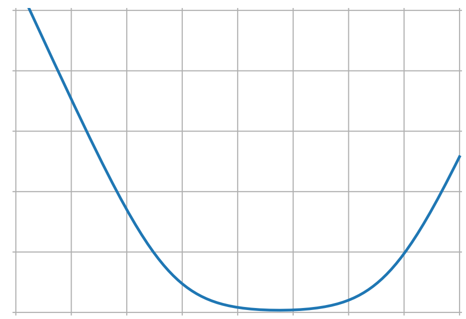

$x_0 = 7$ $\eta = 40$


$x_0 = 7$ $\eta = 40$
η too big


$x_0 = 7$ $\eta = 0.01$



function f is called β-smooth: $f(x) \leq f(x_0) + f'(x_0)(x - x_0) + \frac{b}{2}(x - x_0)^2$



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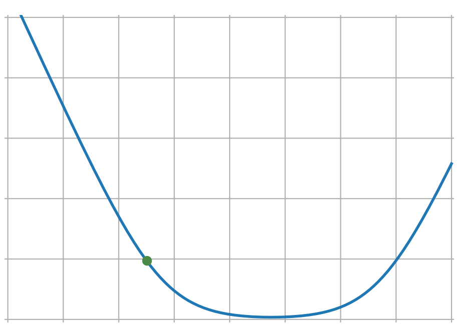

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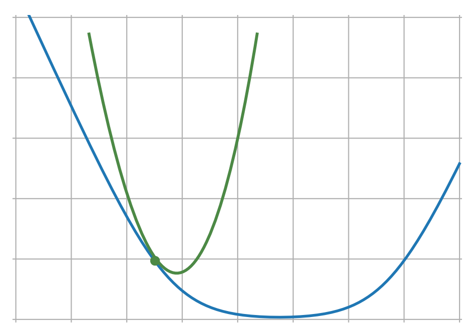

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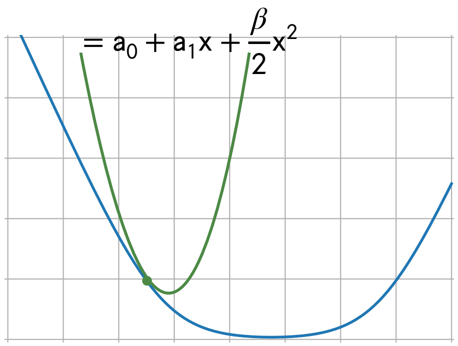

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$$\text{f is } \beta\text{-smooth} \quad \Leftrightarrow \quad f''(x) \le \beta$$

DD for Smooth Functions

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- Set $\forall t: \; \eta_t = \frac{1}{\beta}$ which gives

$$x_{t+1} = x_t - \frac{1}{\beta} f'(x_t) = x_t - \frac{1}{\beta} g_t$$

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- From Smoothness:

$$f(x_{t+1}) \le f(x_t) + g_t (x_{t+1} - x_t) + \frac{\beta}{2}(x_{t+1} - x_t)^2$$

$$f(x_{t+1}) \le f(x_t) \ - \ \frac{1}{\beta} g_t^2 \ + \ \frac{\beta}{2} \frac{g_t^2}{\beta^2} = f(x_t) \ - \ \frac{1}{2\beta} g_t^2$$

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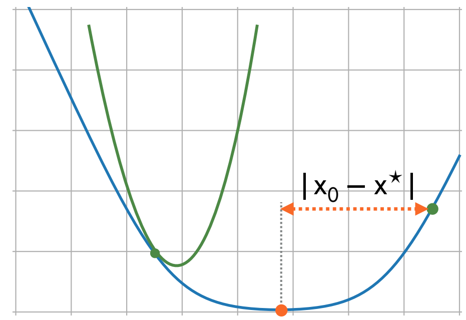
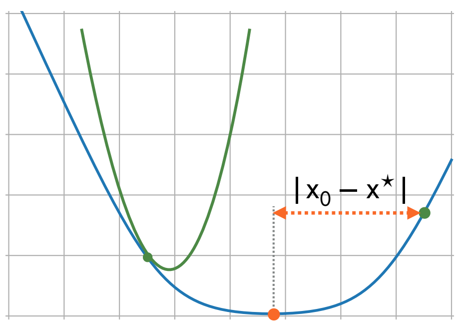
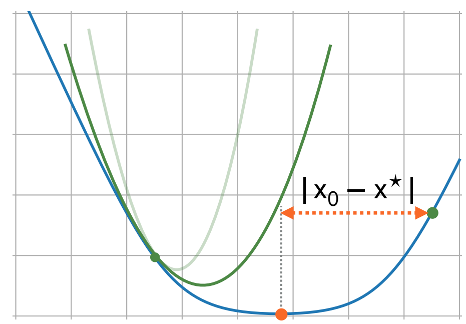
This gives $f(x_{t+1}) - f(x^\star) \leq \dfrac{2\beta |x_0 - x^\star|}{t}$





The closer to the optimum we start the faster we converge



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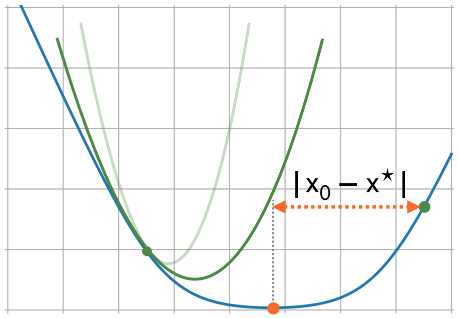

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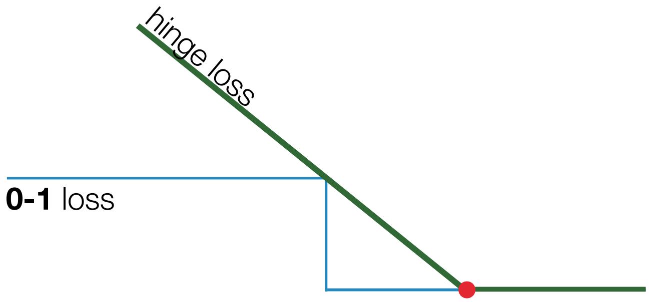
Non-smooth Functions



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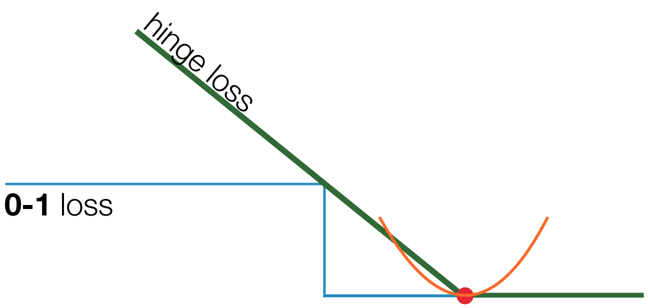

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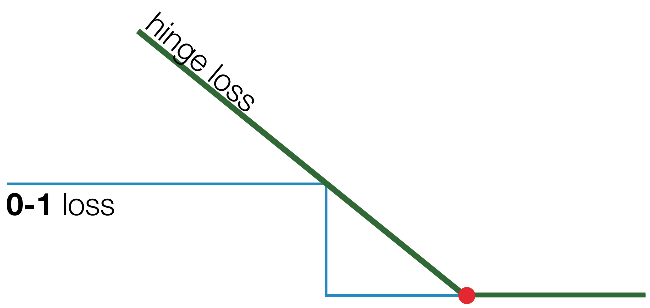

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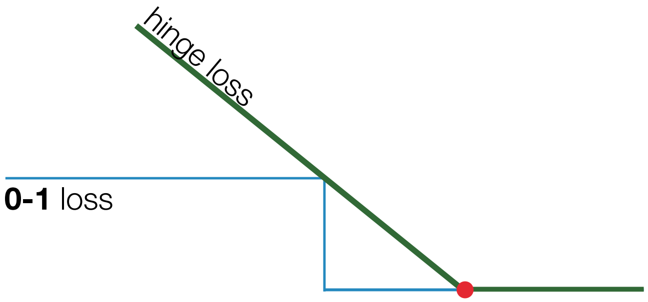
When loss function is not smooth:



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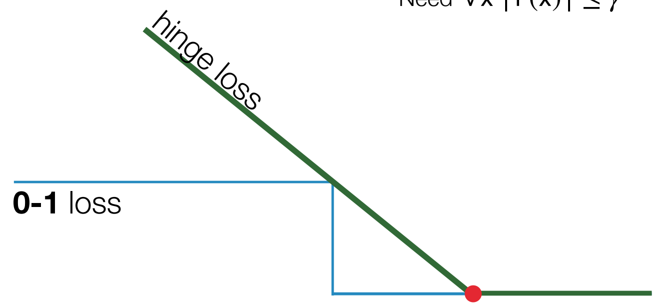

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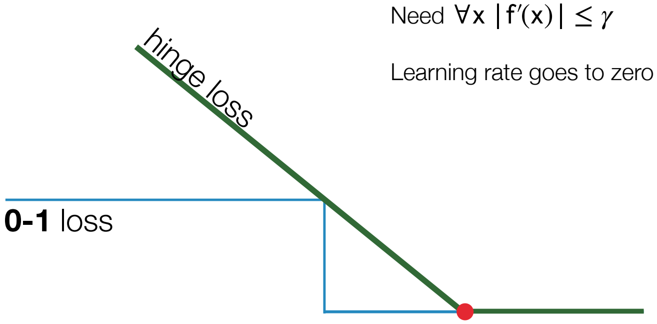

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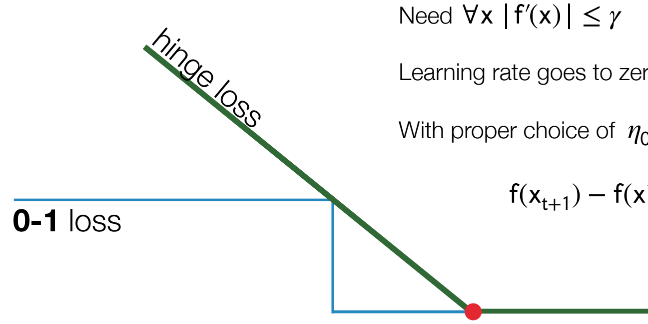

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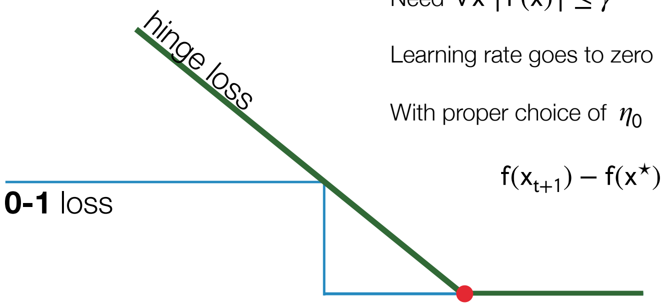

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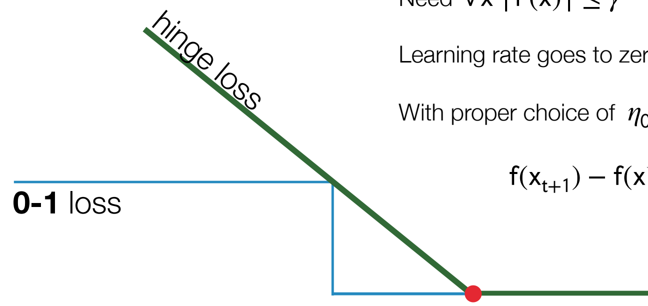

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Next…

From one variable to a vector of variables

Gradients and their properties

Convexity of multivariate functions

Smoothness of multivariate functions

Gradient Descent

Stochastic Gradient Descent (SGD)

SGD for generalized linear models

SGD for non-linear models