# Sets, Relations, Functions

Q1. Let $R_1, R_2, R_3$ be relations on a set $A$. Prove or disprove the following statements.

(a) $(R_1.R_2)^{-1} = R_2^{-1}.R_1^{-1}$.

(b) If $R_1$ and $R_2$ are both symmetric then so is $R_1.R_2$. What about the converse? Consider the same statement for other properties of a relation such as reflexive, transitive, antisymmetric, being a function etc.

(c) $(R_1 \cup R_2).R_3 = (R_1.R_3) \cup (R_2.R_3)$. What if union is replaced by intersection?

(d) Suppose $R_1.R_1^{-1} = I$. Prove that if $A$ is a finite set, then $R_1$ must be a bijection from $A$ to $A$. Give an example to show this is not true for arbitrary sets $A$.

Q2 Prove that the set of all finite subsets of natural numbers is countable. Similarly, the set of all finite sequences of natural numbers is countable.

Q3 Let $A_0, A_1, \ldots,$ be a sequence of sets such that for all $i$,

$$\bigcap_{j=0}^{i} A_j \neq \emptyset.$$

Does this imply $\exists x \forall i \; x \in A_i$? If so, prove it, else give examples of sets for which it is false. Suppose $n_0, n_1, \ldots$ is a sequence of numbers such that for all $i$, $gcd(n_0, n_1, \ldots, n_i) > 1$. Prove that there exists a prime $p$ such that $p | n_i$ for all $i$.

Q4. The transitive closure of a relation $R$ defined on a set $A$, denoted $R^+$, is defined as

$$R^+ = \bigcup_{i=1}^{\infty} R^i$$

where $R^i$ is defined inductively as $R^1 = R$ and $R^{i+1} = R^i.R$. Prove that $R^+$ is transitive. Prove that if $R'$ is any transitive relation that contains $R$, then $R^+ \subseteq R'$. $R^+$ is thus the smallest transitive relation that contains $R$.

Q5. Let $A$ be the set of bit strings of length $n$. Define a relation $R$ on $A$ by $s_1 R s_2$ iff $s_2$ can be obtained by shifting $s_1$ circularly by some number of places. Show that $R$ is an equivalence relation. What is the size of the smallest and largest equivalence class of $R$? How many different equivalence classes are there?