Assignment 1 Statistical inference

Wanchuang Zhu

23/02/2021

Assignments

Q1. Suppose $X \sim B(N,p)$, where $B(N,p)$ is the Binomial distribution whose Probability Mass Function is $P(X=k) = C_N^k p^k (1-p)^{N-k}$. The observations $(x_1, \cdots, x_n)$ are i.i.d samples of X. Given the prior distribution of p as a Beta distribution with parameters as α_0, β_0. The density function of the Beta distribution is $f(x \mid \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$. Derive the posterior distribution of p.

Ans 1.

Given that $X \sim B(N,p)$, the likelihood of i.i.d. samples $(x_i, \ldots, x_n)$ given the parameter p is written as,

$$f(x_1, \ldots, x_n \mid p) = \prod_{i=1}^{n} \binom{N}{x_i} p^{x_i} (1-p)^{N-x_i}$$

Given the prior distribution of p as $Beta(\alpha_0, \beta_0)$, the prior density is written as,

$$\pi(p) = \frac{\Gamma(\alpha_0+\beta_0)}{\Gamma(\alpha_0)\Gamma(\beta_0)} x^{\alpha_0-1}(1-x)^{\beta_0-1}$$

Posterior density $\pi(p \mid x_1, \ldots, x_n)$ is written as,

$$\begin{aligned}
\pi(p \mid x_1, \ldots, x_n) &\propto f(x_1, \ldots, x_n \mid p)\pi(p) \\
&\propto \prod_{i=1}^{n} \binom{N}{x_i} p^{x_i} (1-p)^{N-x_i} \frac{\Gamma(\alpha_0+\beta_0)}{\Gamma(\alpha_0)\Gamma(\beta_0)} x^{\alpha_0-1}(1-x)^{\beta_0-1} \\
&\propto p^{\alpha_0-1+\sum_i^n x_i} (1-p)^{\beta_0-1+N-\sum_i^n x_i}
\end{aligned}$$

Now let sum of binomial samples be,

$$x = \sum_{i=1}^{n} x_i$$

After substitution, the posterior becomes,

$$\pi(p \mid x_1, \ldots, x_n) \propto p^{\alpha_0+x-1}(1-p)^{\beta_0+N-x-1}$$

So, $\pi(p \mid x_1, \ldots, x_n) \sim Beta(\alpha_n, \beta_n)$ where $\alpha_n = \alpha_0 + x$ and $\beta_n = \beta_0 + N + x$