MTL 106 (Introduction to Probability Theory and Stochastic Processes)
Assignment 1 Report

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1. Basic Probability

2. Random Variable/Function of a Random Variable

   Alice is trying to send X bits of data to Bob, where $X \sim P(5)$. However, during transmission there is a 10% chance for each bit to flip. What is the probability that Bob receives incorrect data? Now suppose Alice also sends a parity bit, which is 0 if there are even number of bits equal to 1, and 1 otherwise; and Bob then checks the data with the parity bit upon receiving i.e. if he receives data with 3 bits set and parity bit 0, he'll know the data is erroneous. What is the probability the Bob receives data which is erroneous and also matches the information given by the parity bit?

   Solution

   <u>Without the parity bit</u>

   If $X = k$, then the probability of successful transmission is $(1 - 0.1)^k = 0.9^k$, which means probability of error in transmission is $1 - 0.9^k$

   $\therefore$ By the total probability rule,

$$\begin{aligned}
P(\text{error in transmission}) &= \sum_{k=0}^{\infty} P(\text{error in transmission} \mid X = k) \times P(X = k) \\
&= \sum_{k=0}^{\infty} \{1 - 0.9^k\} \frac{e^{-5}\, 5^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{e^{-5}\, 5^k}{k!} - \sum_{k=0}^{\infty} 0.9^k\, \frac{e^{-5}\, 5^k}{k!} \\
&= e^{-5} \sum_{k=0}^{\infty} \frac{5^k}{k!} - \sum_{k=0}^{\infty} \frac{e^{-5}\, 4.5^k}{k!} \\
&= 1 - e^{-5}\, e^{0.9 \times 5} \qquad \text{(Using Taylor series of } e^x\text{)} \\
&= 1 - e^{-0.5} \\
&\approxeq 0.39
\end{aligned}$$

   <u>With the parity bit</u>

   Let $p = 0.1$ be the chance that a bit flips

   If $X = k$, then there are two cases to consider, namely, if the parity bit changes during transmission or the parity bit remains the same.

   (a) Parity bit remains the same:

For the data to be changed and still match the information given by the parity bit, we note that an non-zero and even number of bits must be flipped, so the number of bits equal to 1 modulo 2 remains the same.
So, in this case,

$$P(\text{undetectable error}) = \binom{k}{2} p^2 (1-p)^{k-2} + \binom{k}{4} p^4 (1-p)^{k-4} + \cdots$$

(b) Parity bit flips:
For the data to be changed and match the parity bit received by Bob, we observe that an even number of bits in the data must be flipped, so the number of bits equal to 1 modulo 2 changes from the original data
$\therefore$ In this case,

$$P(\text{undetectable error}) = \binom{k}{1} p^1 (1-p)^{k-1} + \binom{k}{3} p^3 (1-p)^{k-3} + \cdots$$

We note that the probability of the parity bit flipping is p, and so combining the cases,

$$\begin{aligned} P(\text{undetectable error}) = (1-p) & \left[\binom{k}{2} p^2 (1-p)^{k-2} + \binom{k}{4} p^4 (1-p)^{k-4} + \cdots\right] \\ + p & \left[\binom{k}{1} p^1 (1-p)^{k-1} + \binom{k}{3} p^3 (1-p)^{k-3} + \cdots\right] \end{aligned} \tag{1}$$

To solve these equations, we use the binomial theorem as follows:

$$\begin{aligned} (x+y)^k &= \binom{k}{0} x^0 y^k + \binom{k}{1} x^1 y^{k-1} + \cdots + \binom{k}{k} x^k y^0 \\ (y-x)^k &= \binom{k}{0} x^0 y^k - \binom{k}{1} x^1 y^{k-1} + \cdots + (-1)^k \binom{k}{k} x^k y^0 \end{aligned}$$

Adding and subtracting the above two equations, and replacing x by p and y by $1-p$, we get

$$\frac{1}{2}\left\{(p+1-p)^k + (1-p-p)^k\right\} = \binom{k}{0} p^0 (1-p)^k + \binom{k}{2} p^2 (1-p)^{k-2} + \cdots \tag{2}$$

$$\frac{1}{2}\left\{(p+1-p)^k - (1-p-p)^k\right\} = \binom{k}{1} p^1 (1-p)^{k-1} + \binom{k}{3} p^3 (1-p)^{k-3} + \cdots \tag{3}$$

From equations 1, 2 and 3, we get

$$\begin{aligned} P(\text{undetectable error}) &= (1-p) \left[\frac{1}{2}\left\{1 + (1-2p)^k\right\} - \binom{k}{0} p^0 (1-p)^k\right] \\ &\quad + p \left[\frac{1}{2}\left\{1 - (1-2p)^k\right\}\right] \\ &= \frac{1}{2} - (1-p)^{k+1} + \frac{1}{2}(1-2p)^{k+1} \end{aligned}$$

Replacing p by 0.1, we get

$$P(\text{undetectable error}) = \frac{1}{2} - 0.9^{k+1} + \frac{1}{2}\,0.8^{k+1}$$

Now by the total probability rule,

$$\begin{aligned}
P(\text{undetected error}) &= \sum_{k=0}^{\infty} P(\text{undetected error} \mid X = k) \times P(X = k) \\
&= \sum_{k=0}^{\infty} \left[\frac{1}{2} - 0.9^{k+1} + \frac{1}{2}\,0.8^{k+1}\right] \frac{e^{-5}\,5^k}{k!} \\
&= \frac{1}{2}e^{-5}\sum_{k=0}^{\infty}\frac{5^k}{k!} - 0.9 \times e^{-5}\sum_{k=0}^{\infty}\frac{4.5^k}{k!} + 0.8 \times \frac{1}{2}e^{-5}\sum_{k=0}^{\infty}\frac{4^k}{k!} \\
&= \frac{1}{2} - 0.9 \times e^{-5}\,e^{4.5} + 0.4 \times e^{-5}\,e^{4} \qquad \text{(Using Taylor series of } e^x\text{)} \\
&= \frac{1}{2} - 0.9 \times e^{-0.5} + 0.4 \times e^{-1} \\
&\approxeq 0.10
\end{aligned}$$

3. Two Dimensional Random Variables
4. Two Dimensional Random Variables
5. Higher Dimensional Random Variables

   (a) Let A, B, C be 3 random variables such that their pdf is given by the function

   $$f(a,b,c) = \begin{cases} 1 & 0 < a < 1, 0 < b < 1, 0 < c < 1 \\ 0 & \text{otherwise} \end{cases}$$

   Find the CDF of $A^2 + B^2 - C$ in terms of f

   (b) Consider the equation $x^2 + y^2 + 2Ax + 2By + C = 0$. Find the probability that this equation represents a real circle (i.e. radius > 0).

Solution

Let $R = A^2 + B^2 - C$

We note that $min(R) = -1$ and $max(R) = 1$ since the maximum and minimum values of A, B, C are 1 and 0 respectively

$\therefore P(R \leq -1) = 0$ and $P(R \leq 1) = 1$

Since $A^2 + B^2 - C <= r \implies A^2 + B^2 <= C + r$, it makes sense to make cases on r with breakpoints $r = -1$ since it gives a lower limit to the value C could take, $r = 0$ since it limits the value of $C + r$, and then finally at $r = 1$

(a) $-1 \leq r < 0$

Since $A^2 + B^2 = C + r$, it must be positive. So C must range from $1 - r$ to 1.

For $C = c$ in the said range, $0 \leq c + r < 1$ and thus A can range from 0 to $c + r$ and similarly for B.

$\therefore$ In this case,

$$P(R \leq r) = \int_{c=-r}^{1} \int_{a=0}^{\sqrt{c+r}} \int_{b=0}^{\sqrt{c+r-a^2}} f(a,b,c)\, db\, da\, dc$$

(b) $0 \leq r < 1$

$A^2 + B^2 \leq C + r$ gives a breakpoint at $1 - r$ for C since for the former, A can go from 0 to $c + r$ but not for the latter since $c + r$ would become greater than 1.

- $0 < c < 1 - r$

  As described above, for this case, A will range from 0 to $\sqrt{c + r}$ and similarly for B. So the required probability is:

$$\int_{c=0}^{1-r} \int_{a=0}^{\sqrt{c+r}} \int_{b=0}^{\sqrt{c+r-a^2}} f(a,b,c)\, db\, da\, dc$$

- $1 - r \leq c < 1$

  For this case $c + r > 1$, and A can range fully from 0 to 1 Then, considering $B^2 \leq c + r - a^2$ we note that when $a > \sqrt{c + r - 1}$, B ranges from 0 to $\sqrt{c + r - a^2}$ and when $0 < aleq\sqrt{c + r - 1}$, B ranges from 0 to 1.

  $\therefore$ The required probability is:

$$\int_{c=1-r}^{1} \int_{a=0}^{\sqrt{c+r-1}} \int_{b=0}^{1} f(a,b,c)\, db\, da\, dc + \int_{c=1-r}^{1} \int_{\sqrt{c+r-1}}^{1} \int_{b=0}^{\sqrt{c+r-a^2}} f(a,b,c)\, db\, da\, dc$$

Totalling up the probabilities for this case, we have

$$\begin{aligned} P(R \leq r) = & \int_{c=0}^{1-r} \int_{a=0}^{\sqrt{c+r}} \int_{b=0}^{\sqrt{c+r-a^2}} f(a,b,c)\, db\, da\, dc \\ & + \int_{c=1-r}^{1} \int_{a=0}^{\sqrt{c+r-1}} \int_{b=0}^{1} f(a,b,c)\, db\, da\, dc \\ & + \int_{c=1-r}^{1} \int_{\sqrt{c+r-1}}^{1} \int_{b=0}^{\sqrt{c+r-a^2}} f(a,b,c)\, db\, da\, dc \end{aligned}$$

(c) $1 \leq r < 2$

$A^2 + B^2 \leq C + r$ gives the breakpoint at $2 - r$ for C since for the part before the breakpoint, A and B can't take full values, but for $c \geq 2 - r$, we get $c + r \geq 2$ which means both A and B can be from 0 to 1.

- $0 < c < 2 - r$
  For this case, $1 \leq c + r < 2$, so A can fully range from 0 to 1. However, from $B^2 \leq c + r - A^2$, when A is from 0 to $\sqrt{c+r-1}$, B can fully range from 0 to 1 but not in the remaining interval.
  - $0 < a < \sqrt{c+r-1}$
    As noted above, B can range from 0 to 1, so the required probability is:
    $$\int_{c=0}^{2-r} \int_{a=0}^{\sqrt{c+r-1}} \int_{b=0}^{1} f(a,b,c)\, db\, da\, dc$$
  - $\sqrt{c+r-1} \leq a < 1$
    Here, $c + r - a^2 \leq 1$, so B will range from 0 to $\sqrt{c+r-a^2}$. So the required probability is:
    $$\int_{c=0}^{2-r} \int_{a=\sqrt{c+r-1}}^{1} \int_{b=0}^{\sqrt{c+r-a^2}} f(a,b,c)\, db\, da\, dc$$
- $2 - r \leq c < 1$
  Here, as we already noted, $c + r \geq 2$, so A and B can fully range from 0 to 1. So the required probability is:
  $$\int_{c=2-r}^{1} \int_{a=0}^{1} \int_{b=0}^{1} f(a,b,c)\, db\, da\, dc$$

Totalling up the probabilities for this case, we have

$$\begin{aligned} P(R \leq r) = &\int_{c=0}^{2-r} \int_{a=0}^{\sqrt{c+r-1}} \int_{b=0}^{1} f(a,b,c)\, db\, da\, dc \\ &+ \int_{c=0}^{2-r} \int_{a=\sqrt{c+r-1}}^{1} \int_{b=0}^{\sqrt{c+r-a^2}} f(a,b,c)\, db\, da\, dc \\ &+ \int_{c=2-r}^{1} \int_{a=0}^{1} \int_{b=0}^{1} f(a,b,c)\, db\, da\, dc \end{aligned}$$

The probabilities as calculated in the different cases can be stated together and thus would define the CDF completely.

Given the equation $x^2+y^2+2Ax+2By+C=0$, we can rearrange it to $(x-A)^2+(y-B)^2 = A^2+B^2-C$, which as we observe is a circle with the centre (A,B) and square of the radius equal to $A^2 + B^2 - C$

We need to find $P(A^2 + B^2 - C > 0)$. Taking the expression of $P(R \leq r)$ corresponding to

$r = 0$ and substituting $r = 0$, we get

$$
\begin{aligned}
P(A^2 + B^2 - C \leq 0) &= \int_{c=0}^{1} \int_{a=0}^{\sqrt{c}} \int_{b=0}^{\sqrt{c-a^2}} f(a,b,c)\, db\, da\, dc \\
&= \int_0^1 \int_0^{\sqrt{c}} \int_0^{\sqrt{c-a^2}} db\, da\, dc \\
&= \int_0^1 \int_0^{\sqrt{c}} \sqrt{c-a^2}\, da\, dc \\
&= \int_0^1 \left[\frac{a}{2}\sqrt{c-a^2} + \frac{c}{2}\sin^{-1}\frac{a}{\sqrt{c}} \right]_0^{\sqrt{c}} dc \\
&= \int_0^1 \frac{c}{2}\,\frac{\pi}{2}\, dc \\
&= \frac{\pi}{8}
\end{aligned}
$$

$\therefore P(A^2 + B^2 - C > 0) = 1 - P(A^2 + B^2 - C \leq 0) = 1 - \dfrac{\pi}{8}$

6. Higher Dimensional Random Variables
7. Cross Moments
8. Cross Moments
9. Limiting Distributions
10. Limiting Distributions