MTL 106 (Introduction to Probability Theory and Stochastic Processes) Assignment 1 Report

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1. Basic Probability

Let the sample space, $\Omega = \mathbb{N}$. Define a σ-field on Ω which has infinite events, and is neither the smallest nor the largest σ-field. Define a probability on that σ-field with an infinite number of non zero values

Solution

Let $\mathcal{F}$ include sets whose elements are powers of 2 and their complements.

Since all elements of $\varnothing$ are powers of 2, $\varnothing \in \mathcal{F}$ and since complements of the set are also included, $\Omega \in \mathcal{F}$

Let $A_i, i = 1, 2, \ldots \in \mathcal{F}$. Then:

   (a) All the sets contain only powers of 2:
       In this case, the union of the sets would also include only powers of 2, and thus $\bigcup_i A_i \in \mathcal{F}$

   (b) There is atleast one set which contains an element which is not a power of 2:
       Since that set does not contain all powers of 2, it must be the complement of a set containing powers of 2 (since those are the only type of sets included in $\mathcal{F}$).
       So, we have one set which is of the form $\mathbb{R} \setminus \{2^{i_1}, 2^{i_2}, \ldots\}$ When we union this with sets which only contain powers of 2, it stays in the form of a complement of a set. Same thing happens when we union sets which are complements of sets containing only powers of 2.
       Therefore, the union of these sets would be of the form $\mathbb{R} \setminus \{2^{i_1}, 2^{i_2}, \ldots\}$ where $i_1, i_2, \ldots \in \mathbb{N} \cup \{0\}$ and thus $\bigcup_i A_i \in \mathcal{F}$

Thus, $\mathcal{F}$ is a σ-field.

Now we define a function $P : \mathcal{F} \to \mathbb{R}$ such that $P(x) = \frac{1}{2x}, \{x\} \in \mathcal{F}$, and for other sets in $\mathcal{F}$, the value of P is defined by $P\left(\bigcup_i \{x_i\}\right) = \sum_i P(x_i)$ when all x_i's are powers of 2, $P(\varnothing) = 0$, and for a set A for which P is defined for A^c, $P(A) = 1 - P(A^c)$

First we note that P is a well defined function since we have defined P for the empty set, for sets containing only powers of 2, and also for sets which are their complements. These being the only types of sets in $\mathcal{F}$ implies P is a well defined function.

To prove P is a probability, we need to show

   (a) $P(\Omega) = 1$
       We have defined $P(\varnothing) = 0$, which implies $P(\Omega) = 1 - P(\varnothing) = 1$

   (b) $P(A) \geq 0 \, \forall A \in \mathcal{F}$
       We note that for all sets consisting of only powers of 2, $P(A) \geq 0$ by definition.

Also, if $A = \bigcup_i \{2^{x_i}\} \neq \varnothing$, then

$$
\begin{aligned}
P(A) = P\left(\bigcup_i \{2^{x_i}\}\right) & \\
= \sum_i P(2^{x_i}) & \qquad \text{(By definition)} \\
\leq \sum_{i=0}^{\infty} P(2^i) & \\
\leq \sum_{i=0}^{\infty} \frac{1}{2 \times 2^i} = 1 &
\end{aligned}
$$
$$
\therefore P(A) \leq 1
$$

So, $P(A^c) = 1 - P(A) \geq 0$ and $P(\varnothing) = 0 \geq 0$

So we have shown that for all type of sets $A \in \mathcal{F}$, $P(A) \geq 0$

(c) $P(\bigcup_i A_i) = \sum_i P(A_i)$, where A_i's are mutually disjoint events

Here, we have two cases:

    i. All A_i's only contain powers of 2:
    Then it is obvious that the union of the A_i's would contain all those powers of 2, and by using the definition of P can be expanded to a sum, and those sums could be grouped into A_i's again. This is possible since A_i's are disjoint.

    ii. There is atleast one A_i which does not contain a multiple of 2:
    We note that there can only be one such set which does not contain a multiple of 2. If there were 2 such sets, then 3 would belong to both of those sets since they have to be complements of sets containing only powers of 2 and that contradicts our assumption of them to be disjoint.
    Let that one set be $\mathbb{R} \setminus \bigcup_i 2^{x_i}$. Then the other sets, which contain only powers of 2 would only have elements which belong to $\bigcup_i 2^{x_i}$ for the sets to be disjoint.
    Then we see that on applying P to the union of these sets, and expanding, we would be able to add an subtract terms of $P(x_i)$'s for x_i's which belonged to other sets, and then group the terms into sum of application of P on all the sets

Thus, P defines a probability on $\mathcal{F}$ and it's easily seen that P has infinite number of non-zero values.

2. Random Variable/Function of a Random Variable

Alice is trying to send X bits of data to Bob, where $X \sim P(5)$. However, during transmission there is a 10% chance for each bit to flip. What is the probability that Bob receives incorrect data? Now suppose Alice also sends a parity bit, which is 0 if there are even number of bits equal to 1, and 1 otherwise; and Bob then checks the data with the parity bit upon receiving i.e. if he receives data with 3 bits set and parity bit 0, he'll know the data is erroneous. What

is the probability the Bob receives data which is erroneous and also matches the information given by the parity bit?

Solution

Without the parity bit

If $X = k$, then the probability of successful transmission is $(1 - 0.1)^k = 0.9^k$, which means probability of error in transmission is $1 - 0.9^k$

$\therefore$ By the total probability rule,

$$P(\text{error in transmission}) = \sum_{k=0}^{\infty} P(\text{error in transmission} \mid X = k) \times P(X = k)$$

$$= \sum_{k=0}^{\infty} \{1 - 0.9^k\} \frac{e^{-5} 5^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{e^{-5} 5^k}{k!} - \sum_{k=0}^{\infty} 0.9^k \frac{e^{-5} 5^k}{k!}$$

$$= e^{-5} \sum_{k=0}^{\infty} \frac{5^k}{k!} - \sum_{k=0}^{\infty} \frac{e^{-5} 4.5^k}{k!}$$

$$= 1 - e^{-5} e^{0.9 \times 5} \qquad \text{(Using Taylor series of } e^x\text{)}$$

$$= 1 - e^{-0.5}$$

$$\approx 0.39$$

With the parity bit

Let $p = 0.1$ be the chance that a bit flips

If $X = k$, then there are two cases to consider, namely, if the parity bit changes during transmission or the parity bit remains the same.

(a) Parity bit remains the same:

For the data to be changed and still match the information given by the parity bit, we note that an non-zero and even number of bits must be flipped, so the number of bits equal to 1 modulo 2 remains the same.

So, in this case,

$$P(\text{undetectable error}) = \binom{k}{2} p^2 (1 - p)^{k-2} + \binom{k}{4} p^4 (1 - p)^{k-4} + \cdots$$

(b) Parity bit flips:

For the data to be changed and match the parity bit received by Bob, we observe that an even number of bits in the data must be flipped, so the number of bits equal to 1 modulo 2 changes from the original data

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∴ In this case,

$$P(\text{undetectable error}) = \binom{k}{1} p^1 (1-p)^{k-1} + \binom{k}{3} p^3 (1-p)^{k-3} + \cdots$$

We note that the probability of the parity bit flipping is p, and so combining the cases,

$$P(\text{undetectable error}) = (1-p) \left[\binom{k}{2} p^2 (1-p)^{k-2} + \binom{k}{4} p^4 (1-p)^{k-4} + \cdots \right]$$
$$+ p \left[\binom{k}{1} p^1 (1-p)^{k-1} + \binom{k}{3} p^3 (1-p)^{k-3} + \cdots \right] \tag{1}$$

To solve these equations, we use the binomial theorem as follows:

$$(x+y)^k = \binom{k}{0} x^0 y^k + \binom{k}{1} x^1 y^{k-1} + \cdots + \binom{k}{k} x^k y^0$$

$$(y-x)^k = \binom{k}{0} x^0 y^k - \binom{k}{1} x^1 y^{k-1} + \cdots + (-1)^k \binom{k}{k} x^k y^0$$

Adding and subtracting the above two equations, and replacing x by p and y by $1-p$, we get

$$\frac{1}{2} \left\{ (p+1-p)^k + (1-p-p)^k \right\} = \binom{k}{0} p^0 (1-p)^k + \binom{k}{2} p^2 (1-p)^{k-2} + \cdots \tag{2}$$

$$\frac{1}{2} \left\{ (p+1-p)^k - (1-p-p)^k \right\} = \binom{k}{1} p^1 (1-p)^{k-1} + \binom{k}{3} p^3 (1-p)^{k-3} + \cdots \tag{3}$$

From equations 1, 2 and 3, we get

$$P(\text{undetectable error}) = (1-p) \left[\frac{1}{2} \left\{ 1 + (1-2p)^k \right\} - \binom{k}{0} p^0 (1-p)^k \right]$$
$$+ p \left[\frac{1}{2} \left\{ 1 - (1-2p)^k \right\} \right]$$
$$= \frac{1}{2} - (1-p)^{k+1} + \frac{1}{2} (1-2p)^{k+1}$$

Replacing p by 0.1, we get

$$P(\text{undetectable error}) = \frac{1}{2} - 0.9^{k+1} + \frac{1}{2} 0.8^{k+1}$$

Now by the total probability rule,

$$P(\text{undetected error}) = \sum_{k=0}^{\infty} P(\text{undetected error} \mid X = k) \times P(X = k)$$

$$= \sum_{k=0}^{\infty} \left[\frac{1}{2} - 0.9^{k+1} + \frac{1}{2} 0.8^{k+1} \right] \frac{e^{-5} 5^k}{k!}$$

$$= \frac{1}{2} e^{-5} \sum_{k=0}^{\infty} \frac{5^k}{k!} - 0.9 \times e^{-5} \sum_{k=0}^{\infty} \frac{4.5^k}{k!} + 0.8 \times \frac{1}{2} e^{-5} \sum_{k=0}^{\infty} \frac{4^k}{k!}$$

$$= \frac{1}{2} - 0.9 \times e^{-5} e^{4.5} + 0.4 \times e^{-5} e^4 \qquad (\text{Using Taylor series of } e^x)$$

$$= \frac{1}{2} - 0.9 \times e^{-0.5} + 0.4 \times e^{-1}$$

$$\approx 0.10$$

3. Two Dimensional Random Variables

4. Two Dimensional Random Variables

5. Higher Dimensional Random Variables

   (a) Let A, B, C be 3 random variables such that their pdf is given by the function

   $$f(a, b, c) = \begin{cases} 1 & 0 < a < 1, 0 < b < 1, 0 < c < 1 \\ 0 & \text{otherwise} \end{cases}$$

   Find the CDF of $A^2 + B^2 - C$ in terms of f

   (b) Consider the equation $x^2 + y^2 + 2Ax + 2By + C = 0$. Find the probability that this equation represents a real circle (i.e. radius > 0).

   Solution

   Let $R = A^2 + B^2 - C$

   We note that $min(R) = -1$ and $max(R) = 1$ since the maximum and minimum values of A, B, C are 1 and 0 respectively

   $\therefore P(R \le -1) = 0$ and $P(R \le 1) = 1$

   Since $A^2 + B^2 - C <= r \implies A^2 + B^2 <= C + r$, it makes sense to make cases on r with breakpoints $r = -1$ since it gives a lower limit to the value C could take, $r = 0$ since it limits the value of $C + r$, and then finally at $r = 1$

(a) $-1 \leq r < 0$

Since $A^2 + B^2 = C + r$, it must be positive. So C must range from $1 - r$ to 1.

For $C = c$ in the said range, $0 \leq c + r < 1$ and thus A can range from 0 to $c + r$ and similarly for B.

$\therefore$ In this case,

$$P(R \leq r) = \int_{c=-r}^{1} \int_{a=0}^{\sqrt{c+r}} \int_{b=0}^{\sqrt{c+r-a^2}} f(a, b, c)\, db\, da\, dc$$

(b) $0 \leq r < 1$

$A^2 + B^2 \leq C + r$ gives a breakpoint at $1 - r$ for C since for the former, A can go from 0 to $c + r$ but not for the latter since $c + r$ would become greater than 1.

- $0 < c < 1 - r$

  As described above, for this case, A will range from 0 to $\sqrt{c + r}$ and similarly for B. So the required probability is:

  $$\int_{c=0}^{1-r} \int_{a=0}^{\sqrt{c+r}} \int_{b=0}^{\sqrt{c+r-a^2}} f(a, b, c)\, db\, da\, dc$$

- $1 - r \leq c < 1$

  For this case $c + r > 1$, and A can range fully from 0 to 1 Then, considering $B^2 \leq c + r - a^2$ we note that when $a > \sqrt{c + r - 1}$, B ranges from 0 to $\sqrt{c + r - a^2}$ and when $0 < a \leq \sqrt{c + r - 1}$, B ranges from 0 to 1.

  $\therefore$ The required probability is:

  $$\int_{c=1-r}^{1} \int_{a=0}^{\sqrt{c+r-1}} \int_{b=0}^{1} f(a, b, c)\, db\, da\, dc + \int_{c=1-r}^{1} \int_{\sqrt{c+r-1}}^{1} \int_{b=0}^{\sqrt{c+r-a^2}} f(a, b, c)\, db\, da\, dc$$

Totalling up the probabilities for this case, we have

$$\begin{aligned}
P(R \leq r) &= \int_{c=0}^{1-r} \int_{a=0}^{\sqrt{c+r}} \int_{b=0}^{\sqrt{c+r-a^2}} f(a, b, c)\, db\, da\, dc \\
&+ \int_{c=1-r}^{1} \int_{a=0}^{\sqrt{c+r-1}} \int_{b=0}^{1} f(a, b, c)\, db\, da\, dc \\
&+ \int_{c=1-r}^{1} \int_{\sqrt{c+r-1}}^{1} \int_{b=0}^{\sqrt{c+r-a^2}} f(a, b, c)\, db\, da\, dc
\end{aligned}$$

(c) $1 \leq r < 2$

$A^2 + B^2 \leq C + r$ gives the breakpoint at $2 - r$ for C since for the part before the breakpoint, A and B can't take full values, but for $c \geq 2 - r$, we get $c + r \geq 2$ which means both A and B can be from 0 to 1.

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- $0 < c < 2 - r$

  For this case, $1 \le c + r < 2$, so A can fully range from 0 to 1. However, from $B^2 \le c + r - A^2$, when A is from 0 to $\sqrt{c+r-1}$, B can fully range from 0 to 1 but not in the remaining interval.

  - $0 < a < \sqrt{c+r-1}$

    As noted above, B can range from 0 to 1, so the required probability is:

    $$\int_{c=0}^{2-r} \int_{a=0}^{\sqrt{c+r-1}} \int_{b=0}^{1} f(a,b,c)\, db\, da\, dc$$

  - $\sqrt{c+r-1} \le a < 1$

    Here, $c + r - a^2 \le 1$, so B will range from 0 to $\sqrt{c+r-a^2}$. So the required probability is:

    $$\int_{c=0}^{2-r} \int_{a=\sqrt{c+r-1}}^{1} \int_{b=0}^{\sqrt{c+r-a^2}} f(a,b,c)\, db\, da\, dc$$

- $2 - r \le c < 1$

  Here, as we already noted, $c + r \ge 2$, so A and B can fully range from 0 to 1. So the required probability is:

  $$\int_{c=2-r}^{1} \int_{a=0}^{1} \int_{b=0}^{1} f(a,b,c)\, db\, da\, dc$$

Totalling up the probabilities for this case, we have

$$
\begin{aligned}
P(R \le r) = & \int_{c=0}^{2-r} \int_{a=0}^{\sqrt{c+r-1}} \int_{b=0}^{1} f(a,b,c)\, db\, da\, dc \\
& + \int_{c=0}^{2-r} \int_{a=\sqrt{c+r-1}}^{1} \int_{b=0}^{\sqrt{c+r-a^2}} f(a,b,c)\, db\, da\, dc \\
& + \int_{c=2-r}^{1} \int_{a=0}^{1} \int_{b=0}^{1} f(a,b,c)\, db\, da\, dc
\end{aligned}
$$

The probabilities as calculated in the different cases can be stated together and thus would define the CDF completely.

Given the equation $x^2 + y^2 + 2Ax + 2By + C = 0$, we can rearrange it to $(x - A)^2 + (y - B)^2 = A^2 + B^2 - C$, which as we observe is a circle with the centre (A, B) and square of the radius equal to $A^2 + B^2 - C$

We need to find $P(A^2 + B^2 - C > 0)$. Taking the expression of $P(R \le r)$ corresponding to

$r = 0$ and substituting $r = 0$, we get

$$P(A^2 + B^2 - C \leq 0) = \int_{c=0}^{1} \int_{a=0}^{\sqrt{c}} \int_{b=0}^{\sqrt{c-a^2}} f(a,b,c)\, db\, da\, dc$$

$$= \int_{0}^{1} \int_{0}^{\sqrt{c}} \int_{0}^{\sqrt{c-a^2}} db\, da\, dc$$

$$= \int_{0}^{1} \int_{0}^{\sqrt{c}} \sqrt{c - a^2}\, da\, dc$$

$$= \int_{0}^{1} \left[\frac{a}{2}\sqrt{c - a^2} + \frac{c}{2}\sin^{-1}\frac{a}{\sqrt{c}} \right]_{0}^{\sqrt{c}} dc$$

$$= \int_{0}^{1} \frac{c}{2}\frac{\pi}{2}\, dc$$

$$= \frac{\pi}{8}$$

$$\therefore P(A^2 + B^2 - C > 0) = 1 - P(A^2 + B^2 - C \leq 0) = 1 - \frac{\pi}{8}$$

6. Higher Dimensional Random Variables

In a buggy multithreaded application, two threads accessing shared state within time $5ns$ of each other will crash the application. Consider three threads which will access the shared state at times T_1, T_2, T_3 which are exponentially distributed random variables with parameters $0.04\ ns^{-1}$, $0.02\ ns^{-1}$, and $0.05\ ns^{-1}$ respectively. What is the probability that the application will not crash?

Solution

Since T_1, T_2, T_3 are continuous random variables, the probability of any two of them being equal is zero. Therefore, T_1, T_2, T_3 are all distinct. Let their parameters be $\lambda_1, \lambda_2, \lambda_3$ respectively.

So, there are six cases based on which thread accesses the variable first. We solve the first one here where $T_1 < T_2 < T_3$

$$\text{Probability} = \int_{0}^{\infty} \int_{5+t_1}^{\infty} \int_{5+t_2}^{\infty} \lambda_1 e^{-\lambda_1 t_1} \lambda_2 e^{-\lambda_2 t_2} \lambda_3 e^{-\lambda_3 t_3}\, dt_3\, dt_2\, dt_1$$

Solving the integral, we get

$$\text{Probability} = \frac{\lambda_2}{\lambda_2 + \lambda_3} \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} e^{-5(\lambda_2 + 2\lambda_3)}$$

Adding up integrals from all the cases, we get

$$\text{Probability of not crashing} = \frac{\lambda_2}{\lambda_2 + \lambda_3} \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} e^{-5(\lambda_2 + 2\lambda_3)}$$

$$+ \frac{\lambda_3}{\lambda_2 + \lambda_3} \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} e^{-5(\lambda_3 + 2\lambda_2)}$$

$$+ \frac{\lambda_3}{\lambda_1 + \lambda_3} \frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} e^{-5(\lambda_3 + 2\lambda_1)}$$

$$+ \frac{\lambda_1}{\lambda_1 + \lambda_3} \frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} e^{-5(\lambda_1 + 2\lambda_3)}$$

$$+ \frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3} e^{-5(\lambda_1 + 2\lambda_2)}$$

$$+ \frac{\lambda_2}{\lambda_1 + \lambda_2} \frac{\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3} e^{-5(\lambda_2 + 2\lambda_1)}$$

Substituting $\lambda_1 = 0.04, \lambda_2 = 0.02, \lambda_3 = 0.05$ we get

$$\text{Probability of not crashing} = 0.410$$

7. Cross Moments

   Two points are chosen randomly on the perimeter of a square with sides of unit length. Find the expected value of the distance between the two points.

   Solution

   We observe that there can be 3 cases for the points with respect to the positions at which they lie on the perimeter, namely, they can lie either on the same side, adjacent sides or on opposite sides.

   In the following cases, we calculate the conditional expectation of the distance between the points given that case:

   (a) Points lie on the same side

       Let E_1 be the event that both points, say A and B, lie on the same side of the square. Let their distance from an endpoint of the edge be X and Y respectively, so distance between them is $|X - Y|$

       We observe that the conditional probability distribution function, $p_{X,Y/E_1} = 1$

       Then,

       $$E(\text{Distance between the points} \,|\, E_1) = \int_0^1 \int_0^1 |x - y| \, dy \, dx = \frac{1}{3}$$

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(b) Points lie on adjacent sides

Let E_2 be the event that both points, say A and B, lie on adjacent sides of the square. Let their distance from the vertex common to both the edges be X and Y respectively, so distance between them is $\sqrt{X^2 + Y^2}$

We observe that the conditional probability distribution function, $p_{X,Y/E_1} = 1$

Then.

$$E(\text{Distance between the points} \mid E_2) = \int_0^1 \int_0^1 \sqrt{x^2 + y^2} \, dy \, dx \approx 0.7652$$

(Note: Calculated using numerical integration tools)

(c) Points lie on opposite sides

Let E_3 be the event that both points, say A and B, lie on opposite sides of the square. Let their distance from the vertices on the same side be X and Y respectively, so distance between them is $\sqrt{(X - Y)^2 + 1}$

We observe that the conditional probability distribution function, $p_{X,Y/E_1} = 1$

Then,

$$E(\text{Distance between the points} \mid E_3) = \int_0^1 \int_0^1 \sqrt{(x - y)^2 + 1} \, dy \, dx \approx 1.0766$$

(Note: Calculated using numerical integration tools)

Now, we calculate the probabilities of the events E_1, E_2, E_3

(a) E_1 is the event that both points lie on the same side of the square. The first point can be on any side and for the second point to be on the same side, the probability is $\frac{1}{4}$

$$\therefore P(E_1) = \frac{1}{4}$$

(b) E_2 is the event that both points lie on adjacent sides of the square. The first point can be on any side and the second point has 2 choices of adjacent sides, so the probability is $\frac{2}{4} = \frac{1}{2}$

$$\therefore P(E_2) = \frac{1}{2}$$

(c) E_3 is the event that both points lie on opposite sides of the square. The first point can be on any side and the second point has 1 choice for an opposite side, so the probability is $\frac{1}{4}$

$$\therefore P(E_3) = \frac{1}{4}$$

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Now, we calculate the expected distance between the two points

$$
\begin{aligned}
E(\text{Distance between the points}) &= P(E_1) \cdot E(\text{Distance between the points} \,|\, E_1) \\
&\quad + P(E_2) \cdot E(\text{Distance between the points} \,|\, E_2) \\
&\quad + P(E_3) \cdot E(\text{Distance between the points} \,|\, E_3) \\
&\approx \frac{1}{4} \times \frac{1}{3} + \frac{1}{2} \times 0.7652 + \frac{1}{4} \times 1.0766 \\
&\approx 0.7351
\end{aligned}
$$

8. Cross Moments

9. Limiting Distributions

A spaceship has an initial velocity of zero and for each interval of 1 second, either accelerates at 1 meter per square second with probability 0.4 or does not accelerate with probability 0.6. Estimate the probability of the total distance covered in 2 minutes being less than 5 kilometers.

Solution

Since the decision of accelerating or not happens at interval of 1 seconds, we will work in these intervals. So, we need to find total distance covered in 120 intervals.

Let X_i denote the increment in speed from the beginning of an interval to the end of the interval. Then, we note that X_i's are independent and

$$
P(X_i = x) = \begin{cases} 0.6 & x = \text{acceleration} \times \text{time} = 1 \\ 0.4 & x = 0 \end{cases}
$$

Therefore, the speed at the starting of the nth interval is $\sum_{i=1}^{n-1} X_i$

Given the increment of speed X_n in then n^{th} interval, the contribution of the acceleration to the distance travelled in the interval is

$$
\frac{1}{2} \times \text{acceleration} \times \text{time}^2 = \frac{1}{2} X_n
$$

Therefore, the distance travelled in the nth interval is given by

$$
d_n = \sum_{i=1}^{n-1} X_i + \frac{1}{2} X_n
$$

Thus, the total distance travelled in 2 minutes i.e. 120 intervals is

$$D = \sum_{n=1}^{120} d_n$$

$$= \sum_{n=1}^{120} \left[\sum_{i=1}^{n-1} X_i + \frac{1}{2} X_n \right]$$

$$= \sum_{n=1}^{120} \sum_{i=1}^{n-1} X_i + \sum_{n=1}^{120} \frac{1}{2} X_n$$

$$= \sum_{n=1}^{119} (120 - n) X_n + \sum_{n=1}^{120} \frac{1}{2} X_n \qquad \text{(Expanding the double summation)}$$

$$= \sum_{n=1}^{119} (120.5 - n) X_n + \frac{1}{2} X_{120}$$

Let $X = \sum_{n=1}^{119} (120.5 - n) X_n$. Since X_n's are independent, then X is a sum of a large number of independent random variables and by the Central Limit Theorem must be normally distributed

$$E \left[\sum_{n=1}^{119} (120.5 - n) X_n \right] = \sum_{n=1}^{119} (120.5 - n) E(X_n)$$

$$= \sum_{n=1}^{119} (120.5 - n)(0.6 \times 1 + 0.4 \times 0)$$

$$= \sum_{n=1}^{119} (120.5 - n) \times 0.6$$

$$= 120.5 \times 0.6 \times 119 - 0.6 \times \frac{119 \times 120}{2}$$

$$= 4319.7$$

$$Var\left[\sum_{n=1}^{119}(120.5-n)X_n\right] = \sum_{n=1}^{119}Var\left[(120.5-n)X_n\right] \qquad \text{(As the rv's are independent)}$$

$$= \sum_{n=1}^{119}(120.5-n)^2\,Var(X_n)$$

$$= \sum_{n=1}^{119}(120.5-n)^2\left[0.6\times(1-0.6)^2+0.4\times(0-0.6)^2\right]$$

$$= 138237.54$$

The total distance travelled in 120 intervals is $D = X + \frac{1}{2}X_{120}$. Then

$$P(D \le 5000) = P\left(X + \frac{1}{2}X_{120} \le 5000\right)$$

Since X is combination of X_1 to X_{119} and X_i's are independent, X and X_{120} are independent

$$\implies P(D \le 5000) = P(X_{120}=1)P(X \le 4995.5) + P(X_{120}=0)P(X \le 2000)$$
$$= 0.6 \times P(X \le 4995.5) + 0.4 \times P(X \le 5000)$$

From above discussion, we know that X is normally distributed with mean 4319.7 and variance 138237.54. This implies that

$$\frac{X - 4319.7}{\sqrt{138237.54}} = \frac{X - 4319.7}{371.8} \sim \mathcal{N}(0,1)$$

In other words, $X = 371.8Z + 4319.7$ where $Z \sim \mathcal{N}(0,1)$

Let $\Phi(z)$ be the CDF of Z i.e. $\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-t^2/2}\,dt$

Then

$$P(D \le 5000) = 0.6 \times P(X \le 4995.5) + 0.4 \times P(X \le 5000)$$
$$= 0.6 \times P(371.8Z + 4319.7 \le 4995.5) + 0.4 \times P(371.8Z + 4319.7 \le 5000)$$
$$= 0.6 \times P(Z \le 1.818) + 0.4 \times P(Z \le 1.830)$$
$$= 0.6 \times 0.96562 + 0.4 \times 0.96638$$
$$\approx 0.97$$

10. Limiting Distributions

For an estimation, take number of people in a factory to be given by the random variable $N \sim P(\mu)$. Total time taken off by a worker is assumed to be independent of each other and

following the distribution $Exp(\lambda(an+b))$. Suppose the factory owner decides the downsize the workforce and fire all the workers who have taken time off for time more than t, estimate the expected number of workers who will be fired. Assume μ to be large, i.e. practically the factory has a large number of workers

Solution

Let T be the time off taken by an employee, then

$$P(T > t) = 1 - P(T \le t) = 1 - (1 - e^{-\lambda t(an+b)}) = e^{-\lambda t(an+b)}$$

For a given $N = n$, we assume that n is large, and applying the Bernoulli law of large numbers, we get that $\frac{n_f}{n} \approx e^{-\lambda t(an+b)}$, where n_f is the number of people fired.

So, $n_f = ne^{-\lambda t(an+b)}$ when factory has n workers

$$
\begin{aligned}
E(n_f) &= E(E(n_f|N)) \\
&= E(Ne^{-\lambda t(aN+b)}) \\
&= \sum_{n=0}^{\infty} ne^{-\lambda t(an+b)} \frac{e^{-\mu}\mu^n}{n!} \\
&= \mu e^{-\mu} \sum_{n=1}^{\infty} e^{-\lambda t(an+b)} \frac{\mu^{n-1}}{(n-1)!} \\
&= \mu e^{-\mu} \sum_{n=0}^{\infty} e^{-\lambda t(an+(a+b))} \frac{\mu^n}{n!} \\
&= \mu e^{-\mu} e^{-\lambda t(a+b)} \sum_{n=0}^{\infty} \frac{(\mu e^{-a\lambda t})^n}{n!} \\
&= \mu e^{-\mu} e^{-\lambda t(a+b)} e^{\mu e^{-a\lambda t}} \\
&= \mu e^{\mu(e^{-a\lambda t}-1) - \lambda t(a+b)}
\end{aligned}
$$