MTL 106 (Introduction to Probability Theory and Stochastic Processes)
Assignment 1 Report

Name: Arpit Saxena   Entry Number: 2018MT10742

1. Basic Probability

2. Random Variable/Function of a Random Variable

   Alice is trying to send X bits of data to Bob, where $X \sim P(5)$. However, during transmission there is a 10% chance for each bit to flip. What is the probability that Bob receives incorrect data? Now suppose Alice also sends a parity bit, which is 0 if there are even number of bits equal to 1, and 1 otherwise; and Bob then checks the data with the parity bit upon receiving i.e. if he receives data with 3 bits set and parity bit 0, he'll know the data is erroneous. What is the probability the Bob receives data which is erroneous and also matches the information given by the parity bit?

   Solution

   Without the parity bit

   If $X = k$, then the probability of successful transmission is $(1 - 0.1)^k = 0.9^k$, which means probability of error in transmission is $1 - 0.9^k$

   $\therefore$ By the total probability rule,

$$\begin{aligned}
P(\text{error in transmission}) &= \sum_{k=0}^{\infty} P(\text{error in transmission} \mid X = k) \times P(X = k) \\
&= \sum_{k=0}^{\infty} \{1 - 0.9^k\} \frac{e^{-5}\, 5^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{e^{-5}\, 5^k}{k!} - \sum_{k=0}^{\infty} 0.9^k\, \frac{e^{-5}\, 5^k}{k!} \\
&= e^{-5} \sum_{k=0}^{\infty} \frac{5^k}{k!} - \sum_{k=0}^{\infty} \frac{e^{-5}\, 4.5^k}{k!} \\
&= 1 - e^{-5}\, e^{0.9 \times 5} && \text{(Using Taylor series of } e^x\text{)} \\
&= 1 - e^{-0.5} \\
&\approxeq 0.39
\end{aligned}$$

   With the parity bit

   Let $p = 0.1$ be the chance that a bit flips

   If $X = k$, then there are two cases to consider, namely, if the parity bit changes during transmission or the parity bit remains the same.

   (a) Parity bit remains the same:

For the data to be changed and still match the information given by the parity bit, we note that an non-zero and even number of bits must be flipped, so the number of bits equal to 1 modulo 2 remains the same.
So, in this case,

$$P(\text{undetectable error}) = \binom{k}{2} p^2 (1-p)^{k-2} + \binom{k}{4} p^4 (1-p)^{k-4} + \cdots$$

(b) Parity bit flips:
For the data to be changed and match the parity bit received by Bob, we observe that an even number of bits in the data must be flipped, so the number of bits equal to 1 modulo 2 changes from the original data
$\therefore$ In this case,

$$P(\text{undetectable error}) = \binom{k}{1} p^1 (1-p)^{k-1} + \binom{k}{3} p^3 (1-p)^{k-3} + \cdots$$

We note that the probability of the parity bit flipping is p, and so combining the cases,

$$\begin{aligned} P(\text{undetectable error}) = (1-p) & \left[\binom{k}{2} p^2 (1-p)^{k-2} + \binom{k}{4} p^4 (1-p)^{k-4} + \cdots\right] \\ + p & \left[\binom{k}{1} p^1 (1-p)^{k-1} + \binom{k}{3} p^3 (1-p)^{k-3} + \cdots\right] \end{aligned} \tag{1}$$

To solve these equations, we use the binomial theorem as follows:

$$\begin{aligned} (x+y)^k &= \binom{k}{0} x^0 y^k + \binom{k}{1} x^1 y^{k-1} + \cdots + \binom{k}{k} x^k y^0 \\ (y-x)^k &= \binom{k}{0} x^0 y^k - \binom{k}{1} x^1 y^{k-1} + \cdots + (-1)^k \binom{k}{k} x^k y^0 \end{aligned}$$

Adding and subtracting the above two equations, and replacing x by p and y by $1-p$, we get

$$\frac{1}{2}\left\{(p+1-p)^k + (1-p-p)^k\right\} = \binom{k}{0} p^0 (1-p)^k + \binom{k}{2} p^2 (1-p)^{k-2} + \cdots \tag{2}$$

$$\frac{1}{2}\left\{(p+1-p)^k - (1-p-p)^k\right\} = \binom{k}{1} p^1 (1-p)^{k-1} + \binom{k}{3} p^3 (1-p)^{k-3} + \cdots \tag{3}$$

From equations 1, 2 and 3, we get

$$\begin{aligned} P(\text{undetectable error}) &= (1-p) \left[\frac{1}{2}\left\{1 + (1-2p)^k\right\} - \binom{k}{0} p^0 (1-p)^k\right] \\ &\quad + p \left[\frac{1}{2}\left\{1 - (1-2p)^k\right\}\right] \\ &= \frac{1}{2} - (1-p)^{k+1} + \frac{1}{2} (1-2p)^{k+1} \end{aligned}$$

Replacing p by 0.1, we get

$$P(\text{undetectable error}) = \frac{1}{2} - 0.9^{k+1} + \frac{1}{2}\,0.8^{k+1}$$

Now by the total probability rule,

$$\begin{aligned}
P(\text{undetected error}) &= \sum_{k=0}^{\infty} P(\text{undetected error} \mid X = k) \times P(X = k) \\
&= \sum_{k=0}^{\infty} \left[\frac{1}{2} - 0.9^{k+1} + \frac{1}{2}\,0.8^{k+1}\right] \frac{e^{-5}\,5^k}{k!} \\
&= \frac{1}{2}e^{-5}\sum_{k=0}^{\infty}\frac{5^k}{k!} - 0.9 \times e^{-5}\sum_{k=0}^{\infty}\frac{4.5^k}{k!} + 0.8 \times \frac{1}{2}e^{-5}\sum_{k=0}^{\infty}\frac{4^k}{k!} \\
&= \frac{1}{2} - 0.9 \times e^{-5}\,e^{4.5} + 0.4 \times e^{-5}\,e^{4} \qquad \text{(Using Taylor series of } e^x\text{)} \\
&= \frac{1}{2} - 0.9 \times e^{-0.5} + 0.4 \times e^{-1} \\
&\approxeq 0.10
\end{aligned}$$

3. Two Dimensional Random Variables
4. Two Dimensional Random Variables
5. Higher Dimensional Random Variables
   (a) Let A, B, C be 3 random variables such that their pdf is given by the function

   $$f(a,b,c) = \begin{cases} 1 & 0 < a < 1, 0 < b < 1, 0 < c < 1 \\ 0 & \text{otherwise} \end{cases}$$

   Find the CDF of $A^2 + B^2 - C$ in terms of f

   (b) Consider the equation $x^2 + y^2 + 2Ax + 2By + C = 0$. Find the probability that this equation represents a real circle (i.e. radius > 0).

   Solution

   Let $R = A^2 + B^2 - C$

   We note that $min(R) = -1$ and $max(R) = 1$ since the maximum and minimum values of A, B, C are 1 and 0 respectively

   $\therefore P(R \leq -1) = 0$ and $P(R \leq 1) = 1$

   Since $A^2 + B^2 - C <= r \implies A^2 + B^2 <= C + r$, it makes sense to make cases on r with breakpoints $r = -1$ since it gives a lower limit to the value C could take, $r = 0$ since it limits the value of $C + r$, and then finally at $r = 1$

(a) $-1 \leq r < 0$

Since $A^2 + B^2 = C + r$, it must be positive. So C must range from $1 - r$ to 1.

For $C = c$ in the said range, $0 \leq c + r < 1$ and thus A can range from 0 to $c + r$ and similarly for B.

$\therefore$ In this case,

$$P(R \leq r) = \int_{c=-r}^{1} \int_{a=0}^{\sqrt{c+r}} \int_{b=0}^{\sqrt{c+r-a^2}} f(a, b, c)\, db\, da\, dc$$

(b) $0 \leq r < 1$

$A^2 + B^2 \leq C + r$ gives a breakpoint at $1 - r$ for C since for the former, A can go from 0 to $c + r$ but not for the latter since $c + r$ would become greater than 1.

- $0 < c < 1 - r$

  As described above, for this case, A will range from 0 to $\sqrt{c + r}$ and similarly for B. So the required probability is:

$$\int_{c=0}^{1-r} \int_{a=0}^{\sqrt{c+r}} \int_{b=0}^{\sqrt{c+r-a^2}} f(a, b, c)\, db\, da\, dc$$

- $1 - r \leq c < 1$

  For this case $c + r > 1$, and A can range fully from 0 to 1 Then, considering $B^2 \leq c + r - a^2$ we note that when $a > \sqrt{c + r - 1}$, B ranges from 0 to $\sqrt{c + r - a^2}$ and when $0 < aleq\sqrt{c + r - 1}$, B ranges from 0 to 1.

  $\therefore$ The required probability is:

$$\int_{c=1-r}^{1} \int_{a=0}^{\sqrt{c+r-1}} \int_{b=0}^{1} f(a, b, c)\, db\, da\, dc + \int_{c=1-r}^{1} \int_{\sqrt{c+r-1}}^{1} \int_{b=0}^{\sqrt{c+r-a^2}} f(a, b, c)\, db\, da\, dc$$

Totalling up the probabilities for this case, we have

$$\begin{aligned} P(R \leq r) = &\int_{c=0}^{1-r} \int_{a=0}^{\sqrt{c+r}} \int_{b=0}^{\sqrt{c+r-a^2}} f(a, b, c)\, db\, da\, dc \\ &+ \int_{c=1-r}^{1} \int_{a=0}^{\sqrt{c+r-1}} \int_{b=0}^{1} f(a, b, c)\, db\, da\, dc \\ &+ \int_{c=1-r}^{1} \int_{\sqrt{c+r-1}}^{1} \int_{b=0}^{\sqrt{c+r-a^2}} f(a, b, c)\, db\, da\, dc \end{aligned}$$

(c) $1 \leq r < 2$

$A^2 + B^2 \leq C + r$ gives the breakpoint at $2 - r$ for C since for the part before the breakpoint, A and B can't take full values, but for $c \geq 2 - r$, we get $c + r \geq 2$ which means both A and B can be from 0 to 1.

- $0 < c < 2 - r$
  For this case, $1 \leq c + r < 2$, so A can fully range from 0 to 1. However, from $B^2 \leq c + r - A^2$, when A is from 0 to $\sqrt{c+r-1}$, B can fully range from 0 to 1 but not in the remaining interval.
  - $0 < a < \sqrt{c+r-1}$
    As noted above, B can range from 0 to 1, so the required probability is:
    $$\int_{c=0}^{2-r} \int_{a=0}^{\sqrt{c+r-1}} \int_{b=0}^{1} f(a,b,c)\, db\, da\, dc$$
  - $\sqrt{c+r-1} \leq a < 1$
    Here, $c + r - a^2 \leq 1$, so B will range from 0 to $\sqrt{c+r-a^2}$. So the required probability is:
    $$\int_{c=0}^{2-r} \int_{a=\sqrt{c+r-1}}^{1} \int_{b=0}^{\sqrt{c+r-a^2}} f(a,b,c)\, db\, da\, dc$$
- $2 - r \leq c < 1$
  Here, as we already noted, $c + r \geq 2$, so A and B can fully range from 0 to 1. So the required probability is:
  $$\int_{c=2-r}^{1} \int_{a=0}^{1} \int_{b=0}^{1} f(a,b,c)\, db\, da\, dc$$

Totalling up the probabilities for this case, we have

$$\begin{aligned} P(R \leq r) = &\int_{c=0}^{2-r} \int_{a=0}^{\sqrt{c+r-1}} \int_{b=0}^{1} f(a,b,c)\, db\, da\, dc \\ &+ \int_{c=0}^{2-r} \int_{a=\sqrt{c+r-1}}^{1} \int_{b=0}^{\sqrt{c+r-a^2}} f(a,b,c)\, db\, da\, dc \\ &+ \int_{c=2-r}^{1} \int_{a=0}^{1} \int_{b=0}^{1} f(a,b,c)\, db\, da\, dc \end{aligned}$$

The probabilities as calculated in the different cases can be stated together and thus would define the CDF completely.

Given the equation $x^2 + y^2 + 2Ax + 2By + C = 0$, we can rearrange it to $(x - A)^2 + (y - B)^2 = A^2 + B^2 - C$, which as we observe is a circle with the centre (A, B) and square of the radius equal to $A^2 + B^2 - C$

We need to find $P(A^2 + B^2 - C > 0)$. Taking the expression of $P(R \leq r)$ corresponding to

$r = 0$ and substituting $r = 0$, we get

$$
\begin{aligned}
P(A^2 + B^2 - C \leq 0) &= \int_{c=0}^{1} \int_{a=0}^{\sqrt{c}} \int_{b=0}^{\sqrt{c-a^2}} f(a,b,c)\, db\, da\, dc \\
&= \int_0^1 \int_0^{\sqrt{c}} \int_0^{\sqrt{c-a^2}} db\, da\, dc \\
&= \int_0^1 \int_0^{\sqrt{c}} \sqrt{c-a^2}\, da\, dc \\
&= \int_0^1 \left[\frac{a}{2}\sqrt{c-a^2} + \frac{c}{2}\sin^{-1}\frac{a}{\sqrt{c}} \right]_0^{\sqrt{c}} dc \\
&= \int_0^1 \frac{c}{2}\frac{\pi}{2}\, dc \\
&= \frac{\pi}{8}
\end{aligned}
$$

$\therefore P(A^2 + B^2 - C > 0) = 1 - P(A^2 + B^2 - C \leq 0) = 1 - \frac{\pi}{8}$

6. Higher Dimensional Random Variables

7. Cross Moments

   Two points are chosen randomly on the perimeter of a square with sides of unit length. Find the expected value of the distance between the two points.

   Solution

   We observe that there can be 3 cases for the points with respect to the positions at which they lie on the perimeter, namely, they can lie either on the same side, adjacent sides or on opposite sides.

   In the following cases, we calculate the conditional expectation of the distance between the points given that case:

   (a) Points lie on the same side

   Let E_1 be the event that both points, say A and B, lie on the same side of the square. Let their distance from an endpoint of the edge be X and Y respectively, so distance between them is $|X - Y|$

   We observe that the conditional probability distribution function, $p_{X,Y/E_1} = 1$

   Then,

   $$E(\text{Distance between the points} \mid E_1) = \int_0^1 \int_0^1 |x-y|\, dy\, dx = \frac{1}{3}$$

(b) Points lie on adjacent sides

Let E_2 be the event that both points, say A and B, lie on adjacent sides of the square. Let their distance from the vertex common to both the edges be X and Y respectively, so distance between them is $\sqrt{X^2 + Y^2}$

We observe that the conditional probability distribution function, $p_{X,Y/E_1} = 1$

Then.

$$E(\text{Distance between the points} \,|\, E_2) = \int_0^1 \int_0^1 \sqrt{x^2 + y^2}\, dy\, dx \approx 0.7652$$

(Note: Calculated using numerical integration tools)

(c) Points lie on opposite sides

Let E_3 be the event that both points, say A and B, lie on opposite sides of the square. Let their distance from the vertices on the same side be X and Y respectively, so distance between them is $\sqrt{(X - Y)^2 + 1}$

We observe that the conditional probability distribution function, $p_{X,Y/E_1} = 1$

Then,

$$E(\text{Distance between the points} \,|\, E_3) = \int_0^1 \int_0^1 \sqrt{(x - y)^2 + 1}\, dy\, dx \approx 1.0766$$

(Note: Calculated using numerical integration tools)

Now, we calculate the probabilities of the events E_1, E_2, E_3

(a) E_1 is the event that both points lie on the same side of the square. The first point can be on any side and for the second point to be on the same side, the probability is $\frac{1}{4}$

$$\therefore P(E_1) = \frac{1}{4}$$

(b) E_2 is the event that both points lie on adjacent sides of the square. The first point can be on any side and the second point has 2 choices of adjacent sides, so the probability is $\frac{2}{4} = \frac{1}{2}$

$$\therefore P(E_2) = \frac{1}{2}$$

(c) E_3 is the event that both points lie on opposite sides of the square. The first point can be on any side and the second point has 1 choice for an opposite side, so the probability is $\frac{1}{4}$

$$\therefore P(E_3) = \frac{1}{4}$$

Now, we calculate the expected distance between the two points

$$\begin{aligned}
E(\text{Distance between the points}) &= P(E_1)\cdot E(\text{Distance between the points}\,|\,E_1) \\
&\quad + P(E_2)\cdot E(\text{Distance between the points}\,|\,E_2) \\
&\quad + P(E_3)\cdot E(\text{Distance between the points}\,|\,E_3) \\
&\approx \frac{1}{4}\times\frac{1}{3}+\frac{1}{2}\times 0.7652+\frac{1}{4}\times 1.0766 \\
&\approx 0.7351
\end{aligned}$$

8. Cross Moments
9. Limiting Distributions

A spaceship has an initial velocity of zero and for each interval of 1 second, either accelerates at 1 meter per square second with probability 0.4 or does not accelerate with probability 0.6. Estimate the probability of the total distance covered in 2 minutes being less than 5 kilometers.

Solution

Since the decision of accelerating or not happens at interval of 1 seconds, we will work in these intervals. So, we need to find total distance covered in 120 intervals.

Let X_i denote the increment in speed from the beginning of an interval to the end of the interval. Then, we note that X_i's are independent and

$$P(X_i = x) = \begin{cases} 0.6 & x = \text{acceleration} \times \text{time} = 1 \\ 0.4 & x = 0 \end{cases}$$

Therefore, the speed at the starting of the n^{th} interval is $\sum_{i=1}^{n-1} X_i$

Given the increment of speed X_n in then n^{th} interval, the contribution of the acceleration to the distance travelled in the interval is

$$\frac{1}{2}\times\text{acceleration}\times\text{time}^2 = \frac{1}{2}X_n$$

Therefore, the distance travelled in the n^{th} interval is given by

$$d_n = \sum_{i=1}^{n-1} X_i + \frac{1}{2}X_n$$

Thus, the total distance travelled in 2 minutes i.e. 120 intervals is

$$\begin{aligned}
D &= \sum_{n=1}^{120} d_n \\
&= \sum_{n=1}^{120} \left[\sum_{i=1}^{n-1} X_i + \frac{1}{2} X_n \right] \\
&= \sum_{n=1}^{120} \sum_{i=1}^{n-1} X_i + \sum_{n=1}^{120} \frac{1}{2} X_n \\
&= \sum_{n=1}^{119} (120 - n) X_n + \sum_{n=1}^{120} \frac{1}{2} X_n && \text{(Expanding the double summation)} \\
&= \sum_{n=1}^{119} (120.5 - n) X_n + \frac{1}{2} X_{120}
\end{aligned}$$

Let $X = \sum_{n=1}^{119} (120.5 - n) X_n$. Since X_n's are independent, then X is a sum of a large number of independent random variables and by the Central Limit Theorem must be normally distributed

$$\begin{aligned}
E\left[\sum_{n=1}^{119} (120.5 - n) X_n \right] &= \sum_{n=1}^{119} (120.5 - n) E(X_n) \\
&= \sum_{n=1}^{119} (120.5 - n)(0.6 \times 1 + 0.4 \times 0) \\
&= \sum_{n=1}^{119} (120.5 - n) \times 0.6 \\
&= 120.5 \times 0.6 \times 119 - 0.6 \times \frac{119 \times 120}{2} \\
&= 4319.7
\end{aligned}$$

$$
\begin{aligned}
Var\left[\sum_{n=1}^{119}(120.5-n)X_n\right] &= \sum_{n=1}^{119} Var\left[(120.5-n)X_n\right] \qquad \text{(As the rv's are independent)}\\
&= \sum_{n=1}^{119}(120.5-n)^2\, Var(X_n)\\
&= \sum_{n=1}^{119}(120.5-n)^2\left[0.6\times(1-0.6)^2+0.4\times(0-0.6)^2\right]\\
&= 138237.54
\end{aligned}
$$

The total distance travelled in 120 intervals is $D = X + \frac{1}{2}X_{120}$. Then

$$
P(D \leq 5000) = P\left(X + \frac{1}{2}X_{120} \leq 5000\right)
$$

Since X is combination of X_1 to X_{119} and X_i's are independent, X and X_{120} are independent

$$
\begin{aligned}
\implies P(D \leq 5000) &= P(X_{120}=1)P(X \leq 4995.5) + P(X_{120}=0)P(X \leq 2000)\\
&= 0.6 \times P(X \leq 4995.5) + 0.4 \times P(X \leq 5000)
\end{aligned}
$$

From above discussion, we know that X is normally distributed with mean 4319.7 and variance 138237.54. This implies that

$$
\frac{X-4319.7}{\sqrt{138237.54}} = \frac{X-4319.7}{371.8} \sim \mathcal{N}(0,1)
$$

$$
\text{In other words, } X = 371.8Z + 4319.7 \text{ where } Z \sim \mathcal{N}(0,1)
$$

Let $\Phi(z)$ be the CDF of Z i.e. $\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-t^2/2}\,dt$

Then

$$
\begin{aligned}
P(D \leq 5000) &= 0.6 \times P(X \leq 4995.5) + 0.4 \times P(X \leq 5000)\\
&= 0.6 \times P(371.8Z + 4319.7 \leq 4995.5) + 0.4 \times P(371.8Z + 4319.7 \leq 5000)\\
&= 0.6 \times P(Z \leq 1.818) + 0.4 \times P(Z \leq 1.830)\\
&= 0.6 \times 0.96562 + 0.4 \times 0.96638\\
&\approx 0.97
\end{aligned}
$$

10. Limiting Distributions