MTL 106 (Introduction to Probability Theory and Stochastic Processes)
Assignment 1 Report

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1. Basic Probability
2. Random Variable/Function of a Random Variable

Alice is trying to send X bits of data to Bob, where $X \sim P(5)$. However, during transmission there is a 10% chance for each bit to flip. What is the probability that Bob receives incorrect data? Now suppose Alice also sends a parity bit, which is 0 if there are even number of bits equal to 1, and 1 otherwise; and Bob then checks the data with the parity bit upon receiving i.e. if he receives data with 3 bits set and parity bit 0, he'll know the data is erroneous. What is the probability the Bob receives data which is erroneous and also matches the information given by the parity bit?

Solution

Without the parity bit

If $X = k$, then the probability of successful transmission is $(1 - 0.1)^k = 0.9^k$, which means probability of error in transmission is $1 - 0.9^k$

$\therefore$ By the total probability rule,

$$\begin{aligned} P(\text{error in transmission}) &= \sum_{k=0}^{\infty} P(\text{error in transmission} \mid X = k) \times P(X = k) \\ &= \sum_{k=0}^{\infty} \{1 - 0.9^k\} \frac{e^{-5}\, 5^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{e^{-5}\, 5^k}{k!} - \sum_{k=0}^{\infty} 0.9^k\, \frac{e^{-5}\, 5^k}{k!} \\ &= e^{-5} \sum_{k=0}^{\infty} \frac{5^k}{k!} - \sum_{k=0}^{\infty} \frac{e^{-5}\, 4.5^k}{k!} \\ &= 1 - e^{-5}\, e^{0.9 \times 5} && \text{(Using Taylor series of } e^x\text{)} \\ &= 1 - e^{-0.5} \\ &\approxeq 0.39 \end{aligned}$$

With the parity bit

Let $p = 0.1$ be the chance that a bit flips

If $X = k$, then there are two cases to consider, namely, if the parity bit changes during transmission or the parity bit remains the same.

(a) Parity bit remains the same:

For the data to be changed and still match the information given by the parity bit, we note that an non-zero and even number of bits must be flipped, so the number of bits equal to 1 modulo 2 remains the same.

So, in this case,

$$P(\text{undetectable error}) = \binom{k}{2} p^2 (1-p)^{k-2} + \binom{k}{4} p^4 (1-p)^{k-4} + \cdots$$

(b) Parity bit flips:

For the data to be changed and match the parity bit received by Bob, we observe that an even number of bits in the data must be flipped, so the number of bits equal to 1 modulo 2 changes from the original data

$\therefore$ In this case,

$$P(\text{undetectable error}) = \binom{k}{1} p^1 (1-p)^{k-1} + \binom{k}{3} p^3 (1-p)^{k-3} + \cdots$$

We note that the probability of the parity bit flipping is p, and so combining the cases,

$$\begin{aligned} P(\text{undetectable error}) = (1-p) & \left[\binom{k}{2} p^2 (1-p)^{k-2} + \binom{k}{4} p^4 (1-p)^{k-4} + \cdots\right] \\ + p & \left[\binom{k}{1} p^1 (1-p)^{k-1} + \binom{k}{3} p^3 (1-p)^{k-3} + \cdots\right] \end{aligned} \tag{1}$$

To solve these equations, we use the binomial theorem as follows:

$$\begin{aligned} (x+y)^k &= \binom{k}{0} x^0 y^k + \binom{k}{1} x^1 y^{k-1} + \cdots + \binom{k}{k} x^k y^0 \\ (y-x)^k &= \binom{k}{0} x^0 y^k - \binom{k}{1} x^1 y^{k-1} + \cdots + (-1)^k \binom{k}{k} x^k y^0 \end{aligned}$$

Adding and subtracting the above two equations, and replacing x by p and y by $1-p$, we get

$$\frac{1}{2}\left\{(p+1-p)^k + (1-p-p)^k\right\} = \binom{k}{0} p^0 (1-p)^k + \binom{k}{2} p^2 (1-p)^{k-2} + \cdots \tag{2}$$

$$\frac{1}{2}\left\{(p+1-p)^k - (1-p-p)^k\right\} = \binom{k}{1} p^1 (1-p)^{k-1} + \binom{k}{3} p^3 (1-p)^{k-3} + \cdots \tag{3}$$

From equations 1, 2 and 3, we get

$$\begin{aligned} P(\text{undetectable error}) &= (1-p) \left[\frac{1}{2}\left\{1 + (1-2p)^k\right\} - \binom{k}{0} p^0 (1-p)^k\right] \\ &\quad + p \left[\frac{1}{2}\left\{1 - (1-2p)^k\right\}\right] \\ &= \frac{1}{2} - (1-p)^{k+1} + \frac{1}{2}(1-2p)^{k+1} \end{aligned}$$

Replacing p by 0.1, we get

$$P(\text{undetectable error}) = \frac{1}{2} - 0.9^{k+1} + \frac{1}{2}\,0.8^{k+1}$$

Now by the total probability rule,

$$\begin{aligned}
P(\text{undetected error}) &= \sum_{k=0}^{\infty} P(\text{undetected error} \mid X = k) \times P(X = k) \\
&= \sum_{k=0}^{\infty} \left[\frac{1}{2} - 0.9^{k+1} + \frac{1}{2}\,0.8^{k+1}\right] \frac{e^{-5}\,5^k}{k!} \\
&= \frac{1}{2}e^{-5}\sum_{k=0}^{\infty}\frac{5^k}{k!} - 0.9 \times e^{-5}\sum_{k=0}^{\infty}\frac{4.5^k}{k!} + 0.8 \times \frac{1}{2}e^{-5}\sum_{k=0}^{\infty}\frac{4^k}{k!} \\
&= \frac{1}{2} - 0.9 \times e^{-5}\,e^{4.5} + 0.4 \times e^{-5}\,e^{4} \qquad \text{(Using Taylor series of } e^x\text{)} \\
&= \frac{1}{2} - 0.9 \times e^{-0.5} + 0.4 \times e^{-1} \\
&\approxeq 0.10
\end{aligned}$$

3. Two Dimensional Random Variables
4. Two Dimensional Random Variables
5. Higher Dimensional Random Variables
6. Higher Dimensional Random Variables
7. Cross Moments
8. Cross Moments
9. Limiting Distributions
10. Limiting Distributions