MTL 106 (Introduction to Probability Theory and Stochastic Processes)
Assignment 2 Report

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1. Basic Probability

2. Random Variable/Function of a Random Variable

   Alice is trying to send data to Bob using the Hamming codes for error resilience. There is 10% chance for each bit to flip during flip. Let X be a random variable with distribution $P(5)$. $2^X - 1$ bits are sent, and when they are received, Bob takes an xor of the binary representation of all the positions in which the bits are set (after arranging them in a square with positions 0 to $2^X - 1$ and starting arranging data from position 1) and if it is 0, there is no error, otherwise there is an error. Find the probability that given ≤ 3 bits are flipped in transmission, the error is not detected by Bob.

   Answer

   When the data is sent, the xor of all the positions in which bits are set must be 0, since that is the requirement of Hammond codes. So, the xor of the location of all the bits which flip would be the calculation that Bob makes.

   We have 4 cases here: that of 1, 2, 3 bits flipping in transmission

   (a) 1 bit flips

   Here, the location of the bit that flip would appear as the result of the xor operation. Since the bits are all placed in non zero locations, the result would be non zero. Hence such an error would be detected with certainty.

   (b) 2 bits flip

   Suppose two bits in distinct locations flip. If Bob were to not be able to detect them, the xor of the location of both these bits would be 0. But $A \oplus B = 0 \implies A = B$ which contradicts our assumption. Thus the result would be non-zero and be detected by Bob with complete certainty.

   (c) 3 bit flips

   Suppose 3 bits flip. Since the address of has X bits, then at each of the positions, xor should be zero for Bob to not be able to detect it.

   Thus, if $A_{X-1} \dots A_1 A_0$, $B_{X-1} \dots B_1 B_0$ and $C_{X-1} \dots C_1 C_0$ represent the addresses of the 3 bits which flipped, for the error to go undetected, we must have:

   $$A_i \oplus B_i \oplus C_i = 0, i = 0, \dots, X - 1$$

   For a single i, from the total number of possibilities, there are $\binom{3}{2} + 1 = 4$ possibilities of error going undetected, either two of them could be equal to 1 and the other is 0, or all 3 bits could be 0.

   Since these are independent, there are $4^X - 1$ possibilities for error to go undetected (1 is subtracted since we don't have the address containing all zeroes). Total number of 3 bit errors possible is $\binom{2^X-1}{3}$ Thus probability of an error going undetected given 3 bits were flipped is

$$\frac{6(2^X+1)}{(2^X-2)(2^X-3)}$$

Now, we have:

$$P(\text{Error going undetected}| \leq 4 \text{ bits flipped}) = \frac{P(\text{Error going undetected}, \leq 4 \text{ bits flipped})}{P(\leq 4 \text{ bits flipped})}$$

Now using total probability law, we have

$$\begin{aligned}
P(\text{Error going undetected}, \leq 4 \text{ bits flipped}) &= P(\text{Undetected}, 1 \text{ bit flipped}) \\
&+ P(\text{Undetected}, 2 \text{ bits flipped}) \\
&+ P(\text{Undetected}, 3 \text{ bits flipped}) \\
&= 0 + 0 + \frac{6(2^X+1)}{(2^X-2)(2^X-3)} \\
P(\leq 4 \text{ bits flipped}) &= 0.9^{2^X-1} + \binom{2^X}{1} 0.1 \cdot 0.9^{2^X-2} \\
&+ \binom{2^X}{2} 0.1^2 \cdot 0.9^{2^X-3} + \binom{2^X}{3} 0.1^3 \cdot 0.9^{2^X-4} \\
\therefore P(\text{Error going undetected}| \leq 4 \text{ bits flipped}) &= \frac{6(2^X+1)}{(2^X-2)(2^X-3)} \cdot \frac{1}{P(\leq 4 \text{ bits flipped})}
\end{aligned}$$

3. Stochastic Processes

For a game theory demonstration, people are asked to pick any number between 0 to 100 (both inclusive) and the one whose guess is closest to two-thirds of the average wins. However, due to a mistake, people guess one by one. After n people have guessed, the next person to guess chooses a random number between 1 to two-thirds of the running average. The first person to guess chose a number uniformly between 1 to 100.

Model the running average as a stochastic process and find out which properties it satisfies. What is the expected running average as more and more people guess?

Answer

Let X_i be the random variable denoting the guess made by the i'th person. Define S_i as:

$$S_i = \begin{cases} 0 & i = 0 \\ S_{i-1} + \frac{X_i - S_{i-1}}{i} & i = 1, 2, \ldots \end{cases}$$

Then S_i is the running average after i people have guessed.

As per the information given in the question, $X_i \sim U(0, 100)$

For $i = 2, 3, \ldots$, we have been given that

$$X_i/S_{i-1} = s \sim U\left(0, \frac{2s}{3}\right)$$

$$\begin{aligned} S_1 &= X_1 \\ \implies E(S_1) = E(X_1) &= \frac{0+100}{2} = 50 \end{aligned}$$

Now, for $n \geq 2$,

$$\begin{aligned} E(X_n) &= E(E(X_n/S_{n-1})) \\ &= E\left(\frac{S_{n-1}}{3}\right) && \text{since } X_n/S_{n-1} \sim U\left(0, \tfrac{2S_{n-1}}{3}\right) \\ &= \frac{E(S_{n-1})}{3} \\ \implies E(S_n) &= E\left(S_{n-1} + \frac{X_n - S_{n-1}}{n}\right) \\ &= E(S_{n-1})\left(1 - \frac{1}{n}\right) + \frac{E(X_n)}{n} \\ &= E(S_{n-1})\left(1 - \frac{2}{3n}\right) \\ &= E(S_1)\prod_{i=0}^{n-1}\left(1 - \frac{2}{3(n-i)}\right) \end{aligned}$$

We note that $n-i \leq n$, $i = 0, 1, \ldots, n-1$, which gives us $1 - \frac{2}{3(n-i)} \leq 1 - \frac{2}{3n} < 1$. This implies that $\prod_{i=0}^{n-1}\left(1 - \frac{2}{3(n-i)}\right) \leq \left(1 - \frac{2}{3n}\right)^n$. From this, we can see that $E(S_n) \to 0$ as $n \to \infty$, which means that the expected running average goes to zero as more and more people guess.

We check for the following properties for this process:

(a) Independent Increments

Since the numbers guessed by the people influence the running average on which the next people make their decision, the increments are not independent. An increment (i.e. a person's guess) depends on the sum of the guessed of the people who came before him.

(b) Wide-sense stationarity

We have shown that $E(S_n) = \left(\frac{2}{3}\right)^{n-1} 50$, which is a function of n, thus this stochastic process is not wide-sense stationary.

(c) Strict-sense stationarity

The process does not satisfy weak-sense stationarity property and hence does not satisfy strong sense stationarity property since it is a stronger form.

(d) Markov property

The process satisfies Markov property as S_n only depends on S_{n-1} and X_n, and not on the previous S_i's and thus the process satisfies Markov property

4. Stochastic Processes

5. DTMC

(a) The transition matrix P of a Discrete Time Markov Chain is given as:

$$P = \begin{pmatrix} 0.6 & 0.2 & 0.1 & 0 & 0 & 0.1 & 0 & 0 \\ 0.1 & 0.1 & 0.2 & 0.2 & 0 & 0 & 0.4 & 0 \\ 0.2 & 0.3 & 0 & 0 & 0 & 0.1 & 0.2 & 0.2 \\ 0.5 & 0.2 & 0.1 & 0.1 & 0 & 0.1 & 0 & 0 \\ 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.1 & 0.3 \\ 0.3 & 0 & 0.3 & 0.2 & 0.1 & 0 & 0 & 0.1 \\ 0 & 0 & 0 & 0 & 0.5 & 0.4 & 0 & 0.1 \\ 0 & 0.1 & 0.1 & 0.1 & 0 & 0.2 & 0.2 & 0.3 \end{pmatrix}$$

Without calculating the stationary probabilities, find out whether stationary distribution exists for this system or not.

(b) Consider a DTMC of n states. We know that the transition matrix for the DTMC is diagonalisable and absolute value of all of its eigenvalues is less than 1. Will the limiting distribution of this system exist?

Use the result to find the existence of limiting distribution of a DTMC with the following transition matrix:

$$\begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

Answer

(a) Let π be the stationary probabilities for the system. i.e. $\pi = (\pi_0, \pi_1, \ldots, \pi_{n-1})$

Also, we have $\pi P = \pi$ since the distribution is taken as stationary

$$\begin{aligned} \pi P &= \pi \\ \implies \pi(P - I) &= 0 \end{aligned}$$

Thus we have a homogenous system of linear equations. Since we are after a non-trivial solution (as row sum of π= 1), we have the condition

$$det(P - I) = 0$$

Note that we have an overconstrained system of linear equations, given by $\pi P = \pi, \sum_{i=0}^{n-1} \pi_i = 1$ along with the condition that $\pi_i \geq 0 \, \forall \, i = 0, 1, \ldots, n-1$

Thus, the condition that $det(P - I) = 0$ is necessary for stationary distribution to exist but not sufficient.

For the given transition matrix:

$$\det(P - I) = \begin{vmatrix} -0.4 & 0.2 & 0.1 & 0 & 0 & 0.1 & 0 & 0 \\ 0.1 & -0.9 & 0.2 & 0.2 & 0 & 0 & 0.4 & 0 \\ 0.2 & 0.3 & -1 & 0 & 0 & 0.1 & 0.2 & 0.2 \\ 0.5 & 0.2 & 0.1 & -0.9 & 0 & 0.1 & 0 & 0 \\ 0.1 & 0.1 & 0.1 & 0.1 & -0.9 & 0.1 & 0.1 & 0.3 \\ 0.3 & 0 & 0.3 & 0.2 & 0.1 & -1 & 0 & 0.1 \\ 0 & 0 & 0 & 0 & 0.5 & 0.4 & -1 & 0.1 \\ 0 & 0.1 & 0.1 & 0.1 & 0 & 0.2 & 0.2 & -0.7 \end{vmatrix}$$
$$\neq 0$$

This tells us that the given DTMC will not have a stationary distribution.

(b) Let P denote the transition probability matrix for the DTMC. Then we know that P is a $n \times n$ matrix. Given that it is diagonalisable, $\exists$ invertible $A_{n \times n}$ such that $P = A \Lambda A^{-1}$ where

$$\begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & \lambda_n \end{pmatrix}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the matrix P. We are given that $|\lambda_i| < 1, i = 1, 2, \ldots, n$

$$\begin{aligned} P &= A \Lambda A^{-1} \\ \implies P^2 &= A \Lambda A^{-1} A \Lambda A^{-1} \\ &= A \Lambda^2 A^{-1} \end{aligned}$$

Continuing like this, we get

$$P^m = A \Lambda^m A^{-1}$$

Since Λ is a diagonal matrix, Λ^m can be easily calculated and is given by:

$$\Lambda^m = \begin{pmatrix} \lambda_1^m & 0 & \dots & 0 \\ 0 & \lambda_2^m & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_n^m \end{pmatrix}$$

$$\implies \lim_{m\to\infty} \Lambda^m = \begin{pmatrix} \lim_{m\to\infty} \lambda_1^m & 0 & \dots & 0 \\ 0 & \lim_{m\to\infty} \lambda_2^m & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & \lim_{m\to\infty} \lambda_n^m \end{pmatrix}$$

$|\lambda_i| \leq 1 \implies \lim_{m\to\infty} \lambda_i^m = 0$ thus we have

$$\lim_{m\to\infty} \Lambda^m = 0$$

$$\implies \lim_{m\to\infty} P^m = \lim_{m\to\infty} A\Lambda^m A^{-1} = 0$$

Thus, the m-step transition probability matrix $P^{(m)} = P^m$ goes to 0 as $m \to \infty$ which implies that the limiting distribution does not exist.

Now, given the transition matrix

$$\begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

The eigenvalues of the given matrix are $\pm\frac{1}{2}, 1$. Since it has 3 distinct eigenvalues, it must be diagonalisable. Two diagonal entries of the eigenvalue matrix will go to zero, and the other would be 1. Thus $\lim_{m\to\infty} P^m$ exists and thus the limiting distribution for the given DTMC exists.

6. DTMC

   Quantum computers can be modelled using Quantum automata, which are a quantum analog of probabilistic automata. A finite probabilistic automaton consists of a set of states and the probability of transition from one state to the other depends on the state itself as well as the input symbol. Consider an automata with input symbol set as $\{0, 1\}$ and the transition probabilities as:

   (a) For input 0:

$$\begin{pmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \end{pmatrix}$$

(b) For input 1:

$$\begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}$$

The probability for states 0, 1, 2 to accept are 0.3, 0.6, 0.2 respectively. What is the probability that the automata accepts the string 0101? Assume the starting state is 0.

Note: While this is not a DTMC in itself, it is a very useful extension of the concept and has wide ranging applications in quantum computing, machine learning, etc.

Answer

We can tabulate the states the automata might be in along with the probabilities.

Input	P(state 0)	P(state 1)	P(state 2)
Initial	1	0	0
0	$0.5 \times 1 = 0.5$	$0.5 \times 1 = 0.5$	$0 \times 0 = 0$
1	$0.5 \times 0.5 + 0.5 \times 0$ $= 0.25$	$0.5 \times 0.5 + 0.5 \times 0$ $= 0.25$	$0.5 \times 0.5 + 0.5 \times 0.5$ $= 0.5$
0 0	$0.5 \times 0.25 + 0.5 \times 0.25$ $= 0.25$	$0.5 \times 0.25 + 0.5 \times 0.5$ $= 0.375$	$0.5 \times 0.25 + 0.5 \times 0.5$ $= 0.375$
1 1	$0.5 \times 0.375 + 0.5 \times 0.375$ $= 0.375$	$0.5 \times 0.25 + 0.5 * 0.375$ $= 0.3125$	$0.5 \times 0.25 + 0.5 \times 0.375$ $= 0.3125$

Taking into account the acceptance probability, we get the probability that the given string is accepted by the automata as $0.375 \times 0.3 + 0.3125 \times 0.6 + 0.3125 \times 0.2 = 0.3625$

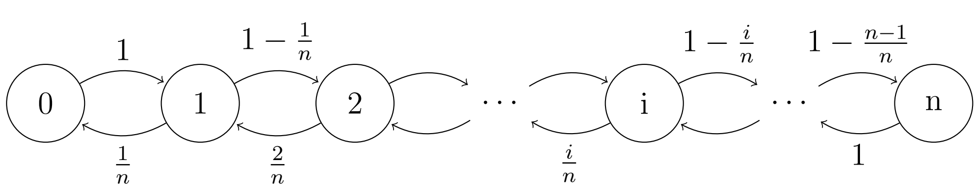
7. CTMC

Consider a modified version of Conway's game of life. In this simplified version, we model using a birth death process where the maximum number of entities allowed in the system is n. When the number of entities in the system is i, the birth rate is given as $1 - \frac{i}{n}$ and the death rate is given as $\frac{i}{n}$. These model the qualities that birth rate decreases with increasing population and death rate increases due to overpopulation.

Give the diagrammatic representation of the process. Find the steady state distribution of the system, and the expected number of entities in steady state.

Answer

The diagrammatic representation is given below:



Let X_t denote the number of entities in the system at time t. Let π denote the stationary probabilities of the system, i.e., $\pi = (\pi_0, \pi_1, \ldots, \pi_n)$ where $\pi_i = \lim_{t\to\infty} P(X_t = i)$

Using the general formula for a birth death process, we have

$$\begin{aligned}
\pi_i &= \frac{\lambda_0\lambda_1 \ldots \lambda_{i-1}}{\mu_1\mu_2 \ldots \mu_i}\pi_0 \\
&= \frac{1(1-\frac{1}{n})\ldots(1-\frac{i-1}{n})}{\frac{1}{n}\frac{2}{n}\ldots\frac{i}{n}}\pi_0 \\
&= \frac{n(n-1)\ldots(n-i+1)}{i!}\pi_0 \\
&= \frac{n!}{i!(n-i)!}\pi_0 \\
\implies \pi_i &= \binom{n}{i}\pi_0 \, , \; i = 1, 2, \ldots, n
\end{aligned}$$

Now we use the normalising condition:

$$\begin{aligned}
\sum_{i=0}^{n} \pi_i &= 1 \\
\implies \pi_0 + \sum_{i=1}^{n} \pi_i &= 1 \\
\implies \pi_0 + \sum_{i=1}^{n} \binom{n}{i}\pi_0 &= 1 \\
\implies \sum_{i=0}^{n} \binom{n}{i}\pi_0 &= 1 \\
\implies 2^n\pi_0 &= 1 \\
\implies \pi_0 &= 2^{-n} \\
\implies \pi_i &= \binom{n}{i}2^{-n} \, , \; i = 0, 1, \ldots n
\end{aligned}$$

Let X be the random variable denoting the number of entities in the system in steady state. Then $P(X = i) = \pi_i \, , \; i = 0, 1, \ldots, n$

$$
\begin{aligned}
E(X) &= \sum_{i=0}^{n} i\pi_i \\
&= \sum_{i=0}^{n} i\binom{n}{i}2^{-n} \\
&= 2^{-n}\sum_{i=0}^{n} i\binom{n}{i}
\end{aligned}
$$

To calculate $\sum_{i=0}^{n} i\binom{n}{i}$ we use the binomial formula as:

$$(1+x)^n = \sum_{i=0}^{n}\binom{n}{i}x^i$$

Differentiating wrt x on both sides, we get

$$n(1+x)^{n-1} = \sum_{i=1}^{n} i\binom{n}{i}x^{i-1}$$

Now putting x = 1 implies

$$
\begin{aligned}
n2^{n-1} &= \sum_{i=1}^{n} i\binom{n}{i} \\
n2^{n-1} &= \sum_{i=0}^{n} i\binom{n}{i}
\end{aligned}
$$

Substituting in the above equation, we get

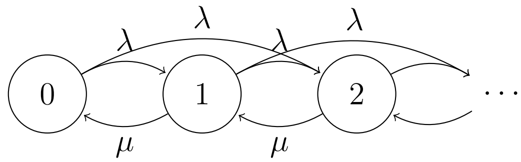
$$E(X) = 2^{-n}\cdot n2^{n-1} = \frac{n}{2}$$

8. CTMC

Consider a Markov process where states are $0, 1, 2, \ldots$ and any state i can move to state $i+1$ and $i+2$ both with rate λ. Also, a state i can move to $i-1$ with rate μ. Give a diagrammatic representation of the process. Find the expected state in which the process would be in steady state

Answer

The diagrammatic representation of the process can be shown as:



At steady state, assume the probability vector to be π, then using the Kolmogorov forward equations and setting rate of change of P to be 0, we get

$$\begin{aligned}
0 &= -2\lambda\pi_0 + \mu\pi_1 \\
0 &= \lambda\pi_0 + \mu\pi_2 - (2\lambda + \mu)\pi_1 \\
0 &= \lambda\pi_{i-2} + \lambda\pi_{i-1} + \mu\pi_{i+1} - (2\lambda + \mu)\pi_i
\end{aligned}$$

Let us define $P(x) = \sum_{n=0}^{\infty} \pi_n x^n$. Then, we have

$$\begin{aligned}
P(x) &= \pi_0 + \pi_1 x + \pi_2 x^2 + \cdots + \pi_{i-2}x^{i-2} + \cdots \\
\lambda x^3 P(x) &= \lambda\pi_0 x^3 + \lambda\pi_1 x^4 + \lambda\pi_2 x^5 + \cdots + \lambda\pi_{i-2}x^{i+1} + \cdots \\
\lambda x^2 P(x) &= \lambda\pi_0 x^2 + \lambda\pi_1 x^3 + \lambda\pi_2 x^4 + \cdots + \lambda\pi_{i-1}x^{i+1} + \cdots \\
-(2\lambda + \mu)xP(x) &= -(2\lambda + \mu)\pi_0 x - (2\lambda + \mu)\pi_1 x^2 - \cdots - (2\lambda + \mu)\pi_i x^{i+1} + \cdots \\
\mu P(x) &= \mu\pi_0 + \mu\pi_1 x + \mu\pi_2 x^2 + \cdots + \mu\pi_{i+1}x^{i+1} + \cdots
\end{aligned}$$

Adding the last 4 equations and using the equations obtained earlier we get

$$\begin{aligned}
&\left\{\lambda x^3 + \lambda x^2 - (2\lambda + \mu)x + \mu\right\}P(x) = \mu\pi_0 - \mu\pi_0 x \\
&\implies P(x) = \frac{\mu\pi_0(1-x)}{\lambda x^3 + \lambda x^2 - (2\lambda + \mu)x + \mu} \\
&\implies P(x) = \frac{\mu\pi_0(1-x)}{(x-1)(\lambda x^2 + 2\lambda x - \mu)} \\
&\implies P(x) = \frac{-\mu\pi_0}{\lambda x^2 + 2\lambda x - \mu}
\end{aligned}$$

Using the definition of $P(x)$, we get $P(1) = \sum_{n=0}^{\infty} \pi_n = 1$ and let X be the distribution in steady state, then $P'(1) = \sum_{n=0}^{\infty} n\pi_n = E(X)$

Using these conditions, we have

$$\begin{aligned}
P(1) &= 1 \\
\implies \frac{-\mu\pi_0}{3\lambda - \mu} &= 1 \\
\implies \pi_0 &= 1 - \frac{3\lambda}{\mu} \\
P'(x) &= \frac{\mu\pi_0}{(\lambda x^2 + 2\lambda x - \mu)^2}(2\lambda x + 2\lambda) \\
\implies P'(1) &= \frac{\mu\pi_0}{(3\lambda - \mu)^2} \cdot 4\lambda
\end{aligned}$$

Substituting $\pi_0 = \frac{\mu - 3\lambda}{\mu}$, we get

$$\begin{aligned}
P'(1) &= \frac{\mu - 3\lambda}{(3\lambda - \mu)^2} \cdot 4\lambda \\
P'(1) &= \frac{4\lambda}{\mu - 3\lambda} \\
\implies E(X) &= \frac{4\lambda}{\mu - 3\lambda}
\end{aligned}$$

Obviously, this only exists when $\mu - 3\lambda > 0 \implies \frac{\lambda}{\mu} < \frac{1}{3}$

9. Queueing Models

   A company has one 16-core machine, two 8-core machines and two 4-core machines. They want to use them as servers. The inter arrival time of queries is exponentially distributed with mean 0.1ms. They estimate that the time taken by a core per query would be exponentially distributed with mean time 3, 2, 4 milliseconds for the 16-core, 8-core and 4-core machines respectively. They want to set up a simple static load balancer in front of these machines, which will schedule the queries on a core of a machine with a probability to assign the query to each core.

   Determine the probabilities by which the load balancer should schedule queries on each type of core to minimise the maximum expected waiting time for a query. Note that one query would occupy the core on which it's running for the entire time it's running.

   Answer

   Basically, what we want to do here is to divide the incoming queries into 3 different types of cores (they are different since they have different service time distributions) We can tabulate the information as follows:

Number of machines	Number of cores	Total number of cores	Mean service time(ms)
1	16	16	3
2	8	16	2
2	4	8	4

   Number the types of cores given in the above table as 1, 2, 3 and let p_1, p_2, p_3 denote the probabilities by which a query will be sent to a core of that type by the load balancer.

   Given that the incoming queries form a Poisson process with parameter $\frac{1}{0.1 \text{ ms}} = 10 \text{ ms}^{-1}$ and the load balancer is decomposing this Poisson process into separate streams. Then, the queries going to cores of type 1, 2, 3 will form a Poisson process with parameters $10p_1, 10p_2, 10p_3$ respectively and $16p_1 + 16p_2 + 8p_3 = 1$ since each query will be routed to one of the given cores.

   We now model each core as a M/M/1 queue. Generically, let's take the arrival process parameter as λ and the service time parameter as μ.

Let W be the waiting time for a query, then $P(W < 0) = 0$ and $P(W = 0) = \rho$ where $\rho = \frac{\lambda}{\mu}$.

For $W > 0$, there has to be atleast one person in the system. Assuming there are n people in the system, we have $W = \widetilde{S_1} + S_2 + \cdots + S_n$, where $\widetilde{S_1}$ is the time left for query that is already running, and $S_2 \cdots S_n$ are the running times for the remaining queued queries.

Since service times are exponentially distributed, which has memoryless property, $\widetilde{S_1}$ is also exponentially distributed with parameter μ

Therefore, $W/N = n \sim Gamma(n, \mu)$ since it is a sum of n independent exponentially distributed random variables with parameter μ.

For $t > 0$,

$$
\begin{aligned}
P(W \leq t) &= \sum_{n=1}^{\infty} P(W \leq t/N = n) P(N = n) \\
&= 1 - \rho + \sum_{n=1}^{\infty} \int_0^t \frac{\mu^n x^{n-1} e^{-\mu x}}{(n-1)!} \, dx \, (1-\rho)\rho^n
\end{aligned}
$$

Now, taking summation inside the integral, we get

$$
\begin{aligned}
P(W \leq t) &= 1 - \rho + \int_0^t \sum_{n=1}^{\infty} \frac{(\mu x \rho)^{n-1}}{(n-1)!} e^{-\mu x} \, dx \, (1-\rho)\mu\rho \\
&= 1 - \rho + \int_0^t \mu\rho e^{\mu x \rho} e^{-\mu x} \, dx \, (1-\rho) \\
&= 1 - \rho + \mu\rho(1-\rho) \frac{e^{-\mu(1-\rho)t} - 1}{\mu(\rho - 1)} \\
&= 1 - \rho + \rho(1 - e^{-(\mu-\lambda)t}) \\
&= 1 - \rho e^{-(\mu-\lambda)t}
\end{aligned}
$$

Therefore, the CDF of W in the steady state is given by

$$
P(W \leq t) = \begin{cases} 0 & t < 0 \\ 1 - \rho & t = 0 \\ 1 - \rho e^{-(\mu-\lambda)t} & 0 < t < \infty \end{cases}
$$

where $\rho = \frac{\lambda}{\mu}$ and the steady state solution is only possible when $\rho < 1 \implies \lambda < \mu$

Then the pdf of W is given by $f_W(t) = \rho(\mu - \lambda)e^{-(\mu-\lambda)t}$ when $0 < t < \infty$ and 0 otherwise.

$$
\begin{aligned}
E(W) &= \int_0^{\infty} t\rho(\mu-\lambda)e^{-(\mu-\lambda)t} \, dt \\
&= \frac{\rho}{\mu - \lambda} \int_0^{\infty} e^{-(\mu-\lambda)t}(\mu-\lambda)t \, d[(\mu-\lambda)t]
\end{aligned}
$$

Since $\mu - \lambda > 0$, $(\mu - \lambda)t \to \infty$ as $t \to \infty$

$$\begin{aligned} E(W) &= \frac{\rho}{\mu - \lambda} \int_0^\infty te^{-t}\, dt \\ &= \frac{\rho}{\mu - \lambda} \\ &= \frac{\lambda}{\mu^2 - \lambda\mu} && \left(\text{Since } \rho = \tfrac{\lambda}{\mu}\right) \end{aligned}$$

Now substituting the values of μ as $\frac{1}{3}, \frac{1}{2}, \frac{1}{4}$ for cores 1, 2, 3 respectively and also using the query process parameters as calculated above, we get the expected waiting times for cores 1, 2, 3 respectively as:

$$\frac{90p_1}{1 - 30p_1}, \frac{40p_2}{1 - 20p_2}, \frac{160p_3}{1 - 40p_3}$$

We also need to have $\rho < 1$ for all the cores, i.e. $\frac{10p_i}{\mu_i} < 1$ for all the cores, which gives $p_1 < \frac{1}{30}, p_2 < \frac{1}{20}, p_3 < \frac{1}{40}$

So our problem reduces to the following optimisation problem:

$$\text{Minimise } max \left\{ \frac{90p_1}{1 - 30p_1}, \frac{40p_2}{1 - 20p_2}, \frac{160p_3}{1 - 40p_3} \right\}$$
$$\text{In the domain } 16p_1 + 16p_2 + 8p_3 = 1, p_1 < \frac{1}{30}, p_2 < \frac{1}{20}, p_3 < \frac{1}{40}$$

Solving the equations with the aid of computational tools available, we find that the minimum expected waiting time is approximately 4.77 ms, when $p_1 \approx 0.020, p_2 \approx 0.035, p_3 \approx 0.013$

So, we have the probabilities by which the load balancer should send the queries to cores of type 1 as $16p_1 \approx 0.327$, cores of type 2 as $16p_2 \approx 0.564$ and cores of type 3 as $8p_3 \approx 0.109$ for minimisation of the maximum expected waiting time for each query.

10. Queueing Models

A restaurant on a very busy main road has a parking capacity of 6 cars. People whose car does not get a parking space can't go to eat in the restaurant due to the busyness of the road and the unavailability of any other parking nearby. People in a car are binomially distributed as B(5, 0.5) (0 included since some people just park their car and not eat). Arrival of cars follows a Poisson process with average inter arrival time 10 minutes. Time spent in the restaurant by one cars passengers is exponentially distributed with average as 40 minutes.

Draw the state diagram of the underlying process, and derive the steady state equations. What is the expected number of people in the restaurant in steady state?

Answer

By the description, we can observe the cars follow a M/M/6/6 queue. Let λ be the parameter of the arrival process and μ be the parameter of the service time exponential distribution. Then, by the given information, $\lambda = \frac{1}{10}\ \mathrm{min}^{-1}$ and $\mu = \frac{1}{40}\ \mathrm{min}^{-1}$

The state diagram of the underlying birth death process is then given by:

$$\begin{tikzcd}
0 \arrow[r, bend left, "\lambda"] & 1 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & 2 \arrow[l, bend left, "2\mu"] \arrow[r, bend left, "\lambda"] & 3 \arrow[l, bend left, "3\mu"] \arrow[r, bend left, "\lambda"] & 4 \arrow[l, bend left, "4\mu"] \arrow[r, bend left, "\lambda"] & 5 \arrow[l, bend left, "5\mu"] \arrow[r, bend left, "\lambda"] & 6 \arrow[l, bend left, "6\mu"]
\end{tikzcd}$$

Here the numbers in each state indicate the number of cars parked in the parking lot. Let $\{X_t \,|\, t \in \mathbb{R}\}$ be the underlying stochastic process where X_t denotes the number of cars parked in the parking lot at time t.

Define $\pi_n = \lim_{t\to\infty} \mathrm{Prob}(X_t = n)$. Then using Kolmogorov, forward equations and letting $t \to \infty$, we get:

$$\begin{aligned}
0 &= -\lambda\pi_0 + \mu\pi_1 \\
0 &= \lambda\pi_{i-1} + (i+1)\mu\pi_{i+1} - (\lambda + i\mu)\pi_i, 1 \le i \le 5 \\
0 &= \lambda\pi_5 - 5\mu\pi_6
\end{aligned}$$

From the first equation, we get $\pi_1 = \frac{\lambda}{\mu}\pi_0$. For $i = 1$:

$$\begin{aligned}
0 &= \lambda\pi_0 + 2\mu\pi_2 - (\lambda+\mu)\pi_1 \\
\implies 0 &= \lambda\pi_0 + 2\mu\pi_2 - (\lambda+\mu)\frac{\lambda}{\mu}\pi_0 \\
\implies 2\mu\pi_2 &= \frac{\lambda^2}{\mu}\pi_0 \\
\implies \pi_2 &= \frac{\lambda^2}{2\mu^2}\pi_0
\end{aligned}$$

By induction, we can show that

$$\pi_i = \frac{\lambda^i}{i!\,\mu^i}\pi_0,\ 1 \le i \le 5$$

And then using the last equation, we get:

$$\begin{aligned}
0 &= \lambda\pi_5 - 5\mu\pi_6 \\
\implies 0 &= \lambda\frac{\lambda^5}{5!\,\mu^5}\pi_0 - 6\mu\pi_6 \\
\implies 6\mu\pi_6 &= \frac{\lambda^6}{5!\mu^5}\pi_0 \\
\implies \pi_6 &= \frac{\lambda^6}{6!\mu^6}\pi_0
\end{aligned}$$

Define $\rho = \frac{\lambda}{\mu}$ Using the fact that the total probability must be 1, which is also called the normalising condition, we get:

$$\begin{aligned}
\sum_{n=0}^{6}\frac{\rho^n}{n!}\pi_0 &= 1 \\
\implies \pi_0 &= \frac{1}{\sum_{n=0}^{6}\frac{\rho^n}{n!}}
\end{aligned}$$

Let X be the random variable denoting the number of cars in the parking lot at steady state. Then,

$$\begin{aligned}
E(X) &= \sum_{n=0}^{6} n\cdot\frac{\rho^n}{n!}\pi_0 \\
&= \sum_{n=0}^{6} n\cdot\frac{\frac{\rho^n}{n!}}{\sum_{n=0}^{6}\frac{\rho^n}{n!}} \\
&= \rho\cdot\frac{\sum_{n=1}^{6}\frac{\rho^{n-1}}{(n-1)!}}{\sum_{n=0}^{6}\frac{\rho^n}{n!}} \\
&= \rho\cdot\frac{\sum_{n=0}^{5}\frac{\rho^n}{n!}}{\sum_{n=0}^{6}\frac{\rho^n}{n!}} \\
&= \rho\left(1 - \frac{\frac{\rho^6}{6!}}{\sum_{n=0}^{6}\frac{\rho^n}{n!}}\right)
\end{aligned}$$

Using $\lambda = \frac{1}{10}, \mu = \frac{1}{40}$ we get $\rho = 4$. Then we can solve the solution using a calculator and get $E(X) \approx 3.531$

Now, we have the expected number of cars in steady state, but we need to calculate the expected number of customers in the restaurant in steady state.

Let N be the random variable denoting the number of customers in the restaurant in steady state. Let N_i be the number of passengers in car i, then we have $N = \sum_{i=0}^{X} N_i$, where X is the random variable denoting the number of cars in steady state. We know that N_i's are iid random variables which are distributed as $B(5, 0.5)$ from which we get $E(N_i) = 5 \times 0.5 = 2.5$ Using Wald's equation since N_i's are iid and also independent from X, we get

$$\begin{aligned} E(N) &= E(N_1) \times E(X) \\ &\approx 2.5 \times 3.531 \\ &= 8.828 \end{aligned}$$

Therefore, the expected number of customers in the restaurant in steady state is approximately 9.