MTL 390 (Statistical Methods)
Major Examination Assignment 2 Report

Name: Arpit Saxena Entry Number: 2018MT10742

1. Testing of Hypothesis
2. Testing of Hypothesis
3. Analysis of correlation and regression

   Give the general formula for Spearman's rank correlation coefficient. Using it, derive the simpler formula when distinct ranks are assumed. Emphasise where the distinctness is assumed.

   Consider the following data. Find the Spearman's rank correlation coefficient using the general formula and the other formula with the distinct rank assumption. What is the difference in the results?

X	Y
106	7
100	27
86	2
101	50
99	29
103	29
99	20
113	12
112	6
110	17

Answer

Let us consider a sample of size n. The n raw scores X_i, Y_i are converted to ranks rg_{X_i}, rg_{Y_i}, and r_s is computed as

$$r_s = \rho_{rg_X, rg_Y} = \frac{cov(rg_X, rg_Y)}{\sigma_{rg_x}\sigma_{rg_Y}} \tag{1}$$

where,

- ρ denotes the Pearson correlation coefficient, but applied to rank variables,
- $cov(rg_X, rg_Y)$ is the covariance of the rank variables
- ρ_{rg_X} and ρ_{rg_Y} are the standard deviations of the rank variables

Let us directly work with ranks to derive the formula. We denote X_i and Y_i as the ranks. Then for each of these

$$
\begin{aligned}
\sum_{i=1}^{n} X_i &= 1 + 2 + \cdots + n \\
&= \frac{n(n+1)}{2}
\end{aligned}
$$

We note that the assumption of distinct ranks doesn't come into play here. This is because in case of ties an average of the tied ranks is given to both the variables and the sum becomes the same in that case as well

$$
\begin{aligned}
\overline{X} &= \frac{1}{n} \sum_{i=1}^{n} X_i \\
&= \frac{n+1}{2} \\
\text{Similarly, } \overline{Y} &= \frac{n+1}{2}
\end{aligned}
$$

Now we calculate the variance.

$$
\begin{aligned}
\sigma_X^2 &= \frac{1}{n} \sum_{i=1}^{n} (X_i - \overline{X})^2 \\
&= \frac{1}{n} \sum_{i=1}^{n} (X_i^2 - 2X_i\overline{X} + \overline{X}^2) \\
&= \frac{1}{n} \left[\sum_{i=1}^{n} X_i^2 - 2\overline{X} \sum_{i=1}^{n} X_i + n\overline{X^2} \right] \\
&= \frac{1}{n} \left[\sum_{i=1}^{n} X_i^2 - 2\overline{X}(n\overline{X}) + n\overline{X^2} \right] \\
&= \frac{1}{n} \left[\sum_{i=1}^{n} X_i^2 - n\overline{X}^2 \right]
\end{aligned}
$$

We use the assumption of distinct ranks here to get

$$
\sum_{i=1}^{n} X_i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}
$$

Plugging in this value and the mean calculated earlier, we get

$$\begin{aligned}
\sigma_X^2 &= \frac{1}{n}\left[\frac{n(n+1)(2n+1)}{6} - n\left\{\frac{n+1}{2}\right\}^2\right] \\
&= \frac{1}{n}\left[\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)^2}{4}\right] \\
&= \frac{1}{n}\frac{n(n+1)}{12}\left[2(2n+1) - 3(n+1)\right] \\
&= \frac{n+1}{12}(n-1) \\
&= \frac{n^2-1}{12} \\
\text{Similarly, we get } \sigma_Y^2 &= \frac{n^2-1}{12}
\end{aligned}$$

Having calculated the variance of X_i, Y_i, we calculate their covariance.

$$\begin{aligned}
Cov(X,Y) &= \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y}) \\
&= \frac{1}{n}\sum_{i=1}^{n}[X_iY_i - X_i\overline{Y} - \overline{X}Y_i + \overline{XY}] \\
&= \frac{1}{n}\left\{\sum_{i=1}^{n} X_iY_i - \overline{Y}\sum_{i=1}^{n} X_i - \overline{X}\sum_{i=1}^{n} Y_i + n\overline{XY}\right\} \\
&= \frac{1}{n}\left\{\sum_{i=1}^{n} X_iY_i - n\overline{YX} - n\overline{XY} + n\overline{XY}\right\} \\
&= \frac{1}{n}\left\{\sum_{i=1}^{n} X_iY_i - n\overline{XY}\right\} \\
&= \frac{1}{n}\sum_{i=1}^{n} X_iY_i - \overline{XY} \\
&= -\frac{1}{2n}\sum_{i=1}^{n}\left\{X_i^2 + Y_i^2 - 2X_iY_i - (X_i^2 + Y_i^2)\right\} - \overline{XY} \\
&= \frac{1}{2n}\sum_{i=1}^{n}\left\{X_i^2 + Y_i^2\right\} - \frac{1}{2n}\sum_{i=1}^{n}\left\{X_i^2 + Y_i^2 - 2X_iY_i\right\} - \overline{XY}
\end{aligned}$$

We are once again invoking the assumption of distinct ranks to calculate the sum of X_i^2 to get

$$
\begin{aligned}
Cov(X,Y) &= \frac{1}{2n} \times 2 \cdot \frac{n(n+1)(2n+1)}{6} - \frac{1}{2n}\sum_{i=1}^{n}\{X_i - Y_i\}^2 - \overline{XY} \\
&= \frac{(n+1)(2n+1)}{6} - \frac{1}{2n}\sum_{i=1}^{n} d_i^2 - \left\{\frac{n+1}{2}\right\}^2 \\
&= \frac{n+1}{12}\{2(2n+1) - 3(n+1)\} - \frac{1}{2n}\sum_{i=1}^{n} d_i^2 \\
&= \frac{n+1}{12} \cdot (n-1) - \frac{1}{2n}\sum_{i=1}^{n} d_i^2 \\
&= \frac{n^2-1}{12} - \frac{1}{2n}\sum_{i=1}^{n} d_i^2
\end{aligned}
$$

Where $d_i = X_i - Y_i$ is the difference of ranks.

Now we calculate the Spearman rank correlation coefficient using (1)

$$
\begin{aligned}
r_s &= \frac{Cov(X,Y)}{\sigma_X \sigma_Y} \\
&= \frac{\frac{n^2-1}{12} - \frac{1}{2n}\sum_{i=1}^{n} d_i^2}{\frac{n^2-1}{12}} \\
\therefore \; & \boxed{r_s = 1 - \frac{6}{n(n^2-1)}\sum_{i=1}^{n} d_i^2} \qquad (2)
\end{aligned}
$$

Thus we have obtained the formula for calculating the Spearman rank correlation coefficient in case of distinct ranks.

Now for the given data, we first convert the raw scores into ranks. These are tabulated below:

X	Y	rank X_i	rank Y_i	d_i	d_i^2
106	7	7	3	4	16
100	27	4	7	-3	9
86	2	1	1	0	0
101	50	5	10	-5	25
99	29	2.5	8.5	-6	36
103	29	6	8.5	-2.5	6.25
99	20	2.5	6	-3.5	12.25
113	12	10	4	6	36
112	6	9	2	7	49
110	17	8	5	3	9

Note that due to ties, there are some fractional ranks. Suppose there was a tie between values at ranks 2 and 3. Then we assign both of these rank 2.5 which is the mean of 2 and 3.

Now, we calculate the spearman rank correlation coefficient using the general formula (1).

$$
\begin{aligned}
Var(\text{rank } X) &= 8.2 \\
Var(\text{rank } Y) &= 8.2 \\
Cov(\text{rank } X, \text{rank } Y) &= -1.725 \\
\therefore r_s &= \frac{Cov(\text{rank } X, \text{rank } Y)}{\sqrt{Var(\text{rank } X)\ Var(\text{rank } Y)}} \\
&= \frac{-1.725}{\sqrt{8.2 \times 8.2}} \\
&= -0.210
\end{aligned}
$$

Now, using the Spearman rank correlation coefficient distinct rank formula (2).

$$
\begin{aligned}
r_s &= 1 - \frac{6}{n(n^2-1)} \sum_{i=1}^{n} d_i^2 \\
&= -0.203
\end{aligned}
$$

We note that there is only an absolute difference of 0.007 in these calculated values which is about 3.5% relative difference. We note that it's a small difference and the distinct rank formula may be useful even in case of ties for approximation purposes.

4. Analysis of correlation and regression

5. Time Series Analysis

   Show that every stationary MA(2) process without a unit root can be converted to an invertible process by suitably changing the coefficients and white noise random variables. Note that this can actually be done for all MA(q) processes. Is the following MA(2) process invertible?

   $$X_t = e_t - 3.2e_{t-1} + 0.6e_{t-2}$$

   Here e_t's are white noise random variables normally distributed with mean 0 and variance 2. If not, convert it to an invertible MA(2) process.

   Answer

   A general MA(2) process looks like the following:

   $$X_t = e_t + \lambda_1 e_{t-1} + \lambda_2 e_{t-2}$$

We can use the backshift operator B to write

$$\begin{aligned} X_t &= e_t + \lambda_1 B(e_t) + \lambda_2 B^2(e_t) \\ \implies X_t &= (1 + \lambda_1 B + \lambda_2 B^2) e_t \end{aligned}$$

We factorise the polynomial into linear factors to get

$$X_t = (1 - \mu_1 B)(1 - \mu_2 B) e_t$$

Thus we get the general characteristic polynomial of the MA(2) process as $(1-\mu_1 B)(1-\mu_2 B)$. The zeros of the polynomial are $\frac{1}{\mu_1}$ and $\frac{1}{\mu_2}$ respectively.

We note that the way to find if the process is invertible is that the absolute value of the zeros of the characteristic polynomial are greater than 1. If not, there would be atleast one zero which is less than 1. We note that the absence of unit roots means that no zero is exactly equal to 1.

We show that a process with the characteristic polynomial with one root μ taken as the reciprocal and variance of the white noise multiplied by μ^2 yields a process with the same autocovariance function.

Let us define $\lambda_0 = 1$. Then,

$$\begin{aligned} Cov(X_t, X_{t+\tau}) &= E(X_t X_{t+\tau}) - E(X_t)E(X_{t+\tau}) \\ &= E(X_t X_{t+\tau}) && \text{(since } E(X_t) = 0 \, \forall t) \\ &= E\left[\left\{\sum_{j=0}^{q} \lambda_j e_{t-j}\right\}\left\{\sum_{k=0}^{q} \lambda_k e_{t-k+\tau}\right\}\right] \\ &= \sum_{j=0}^{q}\sum_{k=0}^{q} \lambda_j \lambda_k E(e_{t-j} e_{t-k+\tau}) \end{aligned}$$

Since e_t's are independent and identically distributed with variance as σ^2

$$Cov(X_t, X_{t+\tau}) = \sum_{j=0}^{q}\sum_{k=0}^{q} \lambda_j \lambda_k \sigma^2 \delta_{t-j,t-k+\tau} \qquad \text{(Where } \delta_{ij} \text{ is the Kronecker delta)}$$

To remove the Kronecker delta, We make the following observations:

- When $\tau = 0$, the (j, k) pairs which will be included are $(0, 0), (1, 1), \ldots, (q, q)$
- For $\tau = 1$, we'll have $(0, 1), (1, 2), \ldots, (q - 1, q)$
- For $\tau = 2$, we'll have $(0, 2), (1, 3), \ldots, (q - 2, q)$
- $\vdots$
- For $\tau = q - 1$, we'll have $(0, q - 1), (1, q)$

– For $\tau = q$, we'll have $(0, q)$

– For $\tau > q$, $t - j \neq t - k + \tau \, \forall j, k = 0, \ldots, q$

So we'll have the following simplification:

$$Cov(X_t, X_{t+\tau}) = \begin{cases} \sigma^2 \sum_{j=0}^{q-\tau} \lambda_j \lambda_{j+\tau} & \text{if } \tau = 0, 1, \ldots, q \\ 0 & \tau > q \end{cases} \tag{3}$$

Now for a process $X_t = (1 - \mu_1 B)(1 - \mu_2 B)e_t$, we have $\lambda_1 = -\mu_1 - \mu_2$ and $\lambda_2 = \mu_1\mu_2$, then we get

$$\begin{aligned}
Cov(X_t, X_t) &= \sigma^2[1 + (\mu_1 + \mu_2)^2 + \mu_1^2\mu_2^2] \\
Cov(X_t, X_{t+1}) &= \sigma^2[(-\mu_1 - \mu_2) + (-\mu_1 - \mu_2)(\mu_1\mu_2)] \\
&= -\sigma^2(\mu_1 + \mu_2)(1 + \mu_1\mu_2) \\
Cov(X_t, X_{t+2}) &= \sigma^2\mu_1\mu_2 \\
Cov(X_t, X_{t+\tau}) &= 0 \, \forall \tau > 2
\end{aligned}$$

If we replace μ_1 by $\frac{1}{\mu_1}$ in the above equations, we'll get

$$\begin{aligned}
Cov(X_t, X_t) &= \sigma^2 \left[1 + \left(\frac{1}{\mu_1} + \mu_2\right)^2 + \frac{1}{\mu_1^2}\mu_2^2\right] \\
&= \frac{\sigma^2}{\mu_1^2}\left[\mu_1^2 + (1 + \mu_1\mu_2)^2 + \mu_2^2\right] \\
&= \frac{\sigma^2}{\mu_1^2}\left[\mu_1^2 + 1 + 2\mu_1\mu_2 + \mu_1^2\mu_2^2 + \mu_2^2\right] \\
&= \frac{\sigma^2}{\mu_1^2}\left[1 + (\mu_1 + \mu_2)^2 + \mu_1^2\mu_2^2\right] \\
Cov(X_t, X_{t+1}) &= -\sigma^2 \left(\frac{1}{\mu_1} + \mu_2\right)\left(1 + \frac{1}{\mu_1}\mu_2\right) \\
&= -\frac{\sigma^2}{\mu_1^2}\left(1 + \mu_1\mu_2\right)\left(\mu_1 + \mu_2\right) \\
Cov(X_t, X_{t+2}) &= \sigma^2 \frac{1}{\mu_1}\mu_2 \\
&= \frac{\sigma^2}{\mu_1^2} \, \mu_1\mu_2 \\
Cov(X_t, X_{t+\tau}) &= 0 \, \forall \tau > 2
\end{aligned}$$

We observe that we have the exact same autocovariance equations with the variance σ^2 divided by μ^2.

Thus, we can change a root of the characteristic polynomial to its reciprocal by suitably changing the variance of the white noise variables to get the same autocovariance function. This way, we can change all roots which are less than 1 to become more than 1 and obtain an invertible process this way.

For the process $X_t = e_t - 3.2e_{t-1} + 0.6e_{t-2}$, the characteristic polynomial is $\theta(B) = 1 - 3.2B + 0.6B^2 = (1 - 3B)(1 - 0.2B)$. The zeros of this polynomial are $\frac{1}{3}, 5$. Since one of them is less than 1, this process is not invertible.

We can make this invertible by the procedure outlined above. We'll replace the root $\frac{1}{3}$ by 3 and change the variance of white noise which is currently 2 to $\frac{2}{\frac{1}{3}^2} = 18$. Thus, we obtain the process

$$
\begin{aligned}
X_t &= (1 - \frac{1}{3}B)(1 - 0.2B)e_t \\
&= e_t - \frac{8}{15}e_{t-1} + \frac{1}{15}e_{t-2}
\end{aligned}
$$

where e_t's are white noise variables normally distributed with mean 0 and variance 18.

6. Time Series Analysis

Consider the stationary ARMA(p, q) process. Elaborate the general method of finding the variance and covariances. Use the method to find the variance and covariances & correlations of time difference upto 3 (i.e. ρ_1, ρ_2, ρ_3) of the stationary ARMA(1, 1) process.

Answer

Let $\{X_t\}_{t \in \mathbb{N}}$ be a time series following the stationary ARMA(p, q) process. Then we have the following recurrence relation:

$$X_t = \sum_{j=1}^{p} \alpha_j X_{t-j} + \sum_{j=1}^{q} \beta_j e_{t-j} + e_t$$

where e_t's are independent and identically distributed random variables following a normal distribution with mean 0 and variance 1. We also take the boundary condition as $X_t = 0 \ \forall t < max(p, q)$

Since this is a stationary process, it is implied that it will also satisfy wide sense stationarity, which means that

$$
\begin{aligned}
E(X_t) &= \mu \ \forall t \\
Cov(X_t, X_s) &= f(t - s)
\end{aligned}
$$

i.e. the mean of all the random variables of the process is constant and the covariance of the random variables at two time instances of the process depends only on the time difference between them.

We first find the mean of the random variables of the process:

$$X_t = \sum_{j=1}^{p} \alpha_j X_{t-j} + \sum_{j=1}^{q} \beta_j e_{t-j} + e_t$$

$$\implies E(X_t) = E\left(\sum_{j=1}^{p} \alpha_j X_{t-j} + \sum_{j=1}^{q} \beta_j e_{t-j} + e_t\right)$$

$$\implies E(X_t) = \sum_{j=1}^{p} \alpha_j E(X_{t-j}) + \sum_{j=1}^{q} \beta_j E(e_{t-j}) + E(e_t)$$

Now we use the fact that e_t's have a mean of 0 to get

$$E(X_t) = \sum_{j=1}^{p} \alpha_j E(X_{t-j})$$

$$\implies \mu = \sum_{j=1}^{p} \alpha_j \mu \qquad \text{(Since the process is stationary)}$$

$$\implies \left(\sum_{j=1}^{p} \alpha_j - 1\right) \mu = 0$$

Now assuming that $\sum_{j=1}^{p} \alpha_j \neq 0$, we get

$$\mu = 0$$

Therefore we have the following result:

$$E(X_t) = 0 \; \forall t \tag{4}$$

We denote γ_τ as the covariance of random variables in this process at time τ apart. Then,

$$X_t = \sum_{j=1}^{p} \alpha_j X_{t-j} + \sum_{j=1}^{q} \beta_j e_{t-j} + e_t$$

Multiplying by $X_{t-\tau}$ on both sides, we get

$$X_t X_{t-\tau} = \sum_{j=1}^{p} \alpha_j X_{t-j} X_{t-\tau} + \sum_{j=1}^{q} \beta_j e_{t-j} X_{t-\tau} + e_t X_{t-\tau}$$

$$\implies E\left(X_t X_{t-\tau}\right) = E\left(\sum_{j=1}^{p} \alpha_j X_{t-j} X_{t-\tau} + \sum_{j=1}^{q} \beta_j e_{t-j} X_{t-\tau} + e_t X_{t-\tau}\right)$$

$$\implies E\left(X_t X_{t-\tau}\right) = \sum_{j=1}^{p} \alpha_j E\left(X_{t-j} X_{t-\tau}\right) + \sum_{j=1}^{q} \beta_j E\left(e_{t-j} X_{t-\tau}\right) + E\left(e_t X_{t-\tau}\right)$$

(Since expectation is a linear operator)

Using (4), we have $Cov(X_t, X_s) = E(X_t X_s) - E(X_t)E(X_s) = E(X_t X_s)$

$$\implies Cov\left(X_t, X_{t-\tau}\right) = \sum_{j=1}^{p} \alpha_j Cov\left(X_{t-j}, X_{t-\tau}\right) + \sum_{j=1}^{q} \beta_j E\left(e_{t-j} X_{t-\tau}\right) + E\left(e_t X_{t-\tau}\right)$$

$$\implies \gamma_\tau = \sum_{j=1}^{p} \alpha_j \gamma_{\tau-j} + \sum_{j=1}^{q} \beta_j E\left(e_{t-j} X_{t-\tau}\right) + E\left(e_t X_{t-\tau}\right)$$

Now we note that $X_{t-\tau}$ is a function of the white noise variables $e_1, \ldots, e_{t-\tau}$ and since they are independent from each other, e_t is independent from all of $e_1, \ldots, e_{t-\tau}$ and thus $E(e_t X_{t-\tau}) = Cov(e_t, X_{t-\tau}) + \cancelto{0}{E(e_t)} E(X_{t-\tau}) = Cov(e_t, X_{t-\tau}) = 0$. Therefore, we get

$$\gamma_\tau = \sum_{j=1}^{p} \alpha_j \gamma_{\tau-j} + \sum_{j=1}^{q} \beta_j E\left(e_{t-j} X_{t-\tau}\right) \tag{5}$$

Using the previous logic, we can see that $E(e_t X_s) = 0$ whenever $t > s$ i.e. e_t and X_s are independent whenever $t > s$. In lieu of this observation, we split into the following two cases:

- **Case 1:** $\tau > q$
  Here, $j < \tau \implies t - j > t - \tau$ for all $j = 1, \ldots, q$. This implies, by our previous observation, that $E(e_{t-j} X_{t-\tau}) = 0$ for all $j = 1, \ldots, q$
  Thus the equation simplifies to:

$$\gamma_\tau = \sum_{j=1}^{p} \alpha_j \gamma_{\tau-j} \tag{6}$$

- **Case 2:** $0 < \tau \leq q$
  In this case, we split up the summation into two parts as:

$$\begin{aligned}\sum_{j=1}^{q} \beta_j E(e_{t-j} X_{t-\tau}) &= \cancelto{0}{\sum_{j=1}^{\tau-1} \beta_j E(e_{t-j} X_{t-\tau})} + \sum_{j=\tau}^{q} \beta_j E(e_{t-j} X_{t-\tau}) \\ &= \sum_{j=\tau}^{q} \beta_j E(e_{t-j} X_{t-\tau})\end{aligned}$$

  For each of the terms we'll need to expand $X_{t-\tau}$ to find the coefficient of e_{t-j} in it, which will give us the expectation.
  Thus we get the equation as:

$$\gamma_\tau = \sum_{j=1}^{p} \alpha_j \gamma_{\tau-j} + \sum_{j=\tau}^{q} \beta_j E(e_{t-j} X_{t-\tau}) \tag{7}$$

Now, we find the variance of X_t.

$$X_t = \sum_{j=1}^{p} \alpha_j X_{t-j} + \sum_{j=1}^{q} \beta_j e_{t-j} + e_t$$

Multiplying by X_t on both sides, we get

$$\begin{aligned}
X_t^2 &= \sum_{j=1}^{p} \alpha_j X_{t-j} X_t + \sum_{j=1}^{q} \beta_j e_{t-j} X_t + e_t X_t \\
\implies E(X_t^2) &= \sum_{j=1}^{p} \alpha_j E(X_{t-j} X_t) + \sum_{j=1}^{q} \beta_j E(e_{t-j} X_t) + E(e_t X_t) \\
\implies Var(X_t) &= \sum_{j=1}^{p} \alpha_j \gamma_j + \sum_{j=1}^{q} \beta_j E(e_{t-j} X_t) + E\left(e_t \left\{ \sum_{j=1}^{p} \alpha_j X_{t-j} + \sum_{j=1}^{q} \beta_j e_{t-j} + e_t \right\} \right) \\
\implies Var(X_t) &= \sum_{j=1}^{p} \alpha_j \gamma_j + \sum_{j=1}^{q} \beta_j E(e_{t-j} X_t) + \sum_{j=1}^{p} \alpha_j \cancelto{0}{E(e_t X_{t-j})} + \sum_{j=1}^{q} \beta_j \cancelto{0}{E(e_t e_{t-j})} + E(e_t^2)
\end{aligned}$$

Therefore, we have the variance equation as:

$$Var(X_t) = \sum_{j=1}^{p} \alpha_j \gamma_j + \sum_{j=1}^{q} \beta_j E(e_{t-j} X_t) + 1 \tag{8}$$

Now we consider a stationary ARMA(1, 1) process with the equation

$$X_t = \alpha X_{t-1} + \beta e_{t-1} + e_t$$

To find the variance, we use (8).

$$\begin{aligned}
Var(X_t) &= \sum_{j=1}^{p} \alpha_j \gamma_j + \sum_{j=1}^{q} \beta_j E(e_{t-j} X_t) + 1 \\
&= \alpha\gamma_1 + \beta E(e_{t-1} X_t) + 1 \\
&= \alpha\gamma_1 + \beta E\left(e_{t-1} \{\alpha X_{t-1} + \beta e_{t-1} + e_t\}\right) + 1 \\
&= \alpha\gamma_1 + \alpha\beta E(e_{t-1} X_{t-1}) + \beta^2 E(e_{t-1}^2) + \beta \cancelto{0}{E(e_{t-1} e_t)} + 1
\end{aligned}$$

Now using $E(e_{t-1}^2 = Var(e_{t-1}) = 1)$ and $E(e_t X_t) = 1 \; \forall t$, we get

$$Var(X_t) = \alpha\gamma_1 + \alpha\beta + \beta^2 + 1 \tag{9}$$

Note we have yet to find γ_1 which we'll do next. Note that since $0 < \tau = 1 \leq q = 1$, we'll use (7) to calculate.

$$\begin{aligned}
\gamma_\tau &= \sum_{j=1}^{p} \alpha_j \gamma_{\tau-j} + \sum_{j=\tau}^{q} \beta_j E(e_{t-j} X_{t-\tau}) \\
\implies \gamma_1 &= \alpha\gamma_0 + \beta E(e_{t-1} X_{t-1}) \\
\implies \gamma_1 &= \alpha\gamma_0 + \beta \\
\implies \gamma_1 &= \alpha Var(X_t) + \beta \qquad (10)
\end{aligned}$$

Now using (9), we get

$$\begin{aligned}
\gamma_1 &= \alpha(\alpha\gamma_1 + \alpha\beta + \beta^2 + 1) + \beta \\
\implies \gamma_1 &= \alpha^2\gamma_1 + \alpha^2\beta + \alpha\beta^2 + \alpha + \beta \\
\implies (1 - \alpha^2)\gamma_1 &= \alpha^2\beta + \alpha\beta^2 + \alpha + \beta \\
\implies \gamma_1 &= \frac{\alpha^2\beta + \alpha\beta^2 + \alpha + \beta}{1 - \alpha^2} \qquad (11)
\end{aligned}$$

Now that we have found γ_1, we'll describe the other results using it since they get very messy otherwise. From (10), we have

$$\begin{aligned}
\gamma_1 &= \alpha Var(X_t) + \beta \\
\implies \gamma_1 &= \alpha\gamma_0 + \beta
\end{aligned}$$

Now dividing both sides by γ_0 and setting $\frac{\gamma_1}{\gamma_0}$ to ρ_1, we get

$$\begin{aligned}
\rho_1 &= \alpha + \frac{\beta}{\gamma_0} \\
\implies \rho_1 &= \alpha + \frac{\beta}{\alpha\gamma_1 + \alpha\beta + \beta^2 + 1} \qquad \text{(Using (9))}
\end{aligned}$$

For $\tau > 1 = q$, we can use the equation for case 1 i.e. (6). We have

$$\begin{aligned}
\gamma_\tau &= \sum_{j=1}^{p} \alpha_j \gamma_{\tau-j} \\
\implies \gamma_\tau &= \alpha\gamma_{\tau-1}
\end{aligned}$$

Now dividing both sides by the variance to the correlations,

$$\begin{aligned}
\rho_\tau &= \alpha\rho_{\tau-1} \\
\therefore\ \rho_2 &= \alpha\rho_1 = \alpha^2 + \frac{\alpha\beta}{\alpha\gamma_1 + \alpha\beta + \beta^2 + 1} \\
\rho_3 &= \alpha\rho_2 = \alpha^3 + \frac{\alpha^2\beta}{\alpha\gamma_1 + \alpha\beta + \beta^2 + 1}
\end{aligned}$$

Therefore, we have found the values of ρ_1, ρ_2, ρ_3 as

$$\rho_1 = \alpha + \frac{\beta}{\alpha\gamma_1 + \alpha\beta + \beta^2 + 1}$$
$$\rho_2 = \alpha^2 + \frac{\alpha\beta}{\alpha\gamma_1 + \alpha\beta + \beta^2 + 1}$$
$$\rho_3 = \alpha^3 + \frac{\alpha^2\beta}{\alpha\gamma_1 + \alpha\beta + \beta^2 + 1}$$

where,

$$\gamma_1 = \frac{\alpha^2\beta + \alpha\beta^2 + \alpha + \beta}{1 - \alpha^2}$$