Inverse Theorem by Linear Combination for Certain Baskakov-Durrmeyer Type Operators

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Abstract

Very recently, I have studied several direct results for certain Baskakov-Durrmeyer type operators. Here, I extend the studies to obtain the inverse theorem of unbounded continuous functions by a linear combination.

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1 Introduction

Recently Gupta et al. [1] introduced the following type of Baskakov-Durrmeyer type operators as for $f \in C_a[0,\infty) \equiv f \in C[0,\infty)$:

$$|f(t)| \leq M(1+t)^{\alpha} \text{ for some } M > 0, \alpha > 0.$$

The operators introduced in [?] are defined as:

$$\begin{aligned} B_n f(t)x &= \sum_{v=1}^{\infty} p_{nv}(x) \int_0^{\infty} b_{nv}(t) f(t)\, dt + p_{n0}(x) f(0) \\ &= \int_0^{\infty} W_n(x,t)\, dt \end{aligned}$$

where

$$p_{nv}(x) = \binom{n+v-1}{v} \frac{x^v(1+x)^{-(n+v)}}{,} \qquad b_{nv} = \frac{1}{B(n+1,v)} t^{v-1}(1+t)^{-(n+v+1)},$$

And

$$W_n(x,t) = \sum_{v=1}^{\infty} p_{nv}(x) b_{nv}(t) + p_{n0}(x)\delta(t),$$

with $\delta(t)$ being the Dirac-delta function.

The space $C^\alpha[0,\infty)$ is normed by

$$\|f\|_{C^\alpha} = \sup_{0 \le t < \infty} \frac{|f(t)|}{(1+t)^\alpha}.$$

Let $d_0, d_1, \ldots, d_k$ be $(k+1)$ arbitrary but fixed distinct positive integers. Then the linear combination $B_n(f,k,x)$ of $B_{d_j,n} f(x)$ for $j = 0, 1, \ldots, k$ is given by:

$$B_n(f,k,x) = \sum_{j=0}^{k} C_{jk} B_{d_j,n} f(x),$$

where

$$C_{jk} = \prod_{\substack{i=0 \\ i \neq j}}^{k} \frac{d_i}{d_i - d_k}, \quad \text{and} \quad C_{00} = 1.$$

These linear combinations were first used by May [?] and later by Kasana [?] to improve the order of approximation of exponential type operators.

2 Basic Results

[**Lemma 2.1**] For $N \subset \mathbb{N} \cup \{0\}$, if we have

$$\mu_{nm}(x) = \sum_{v=0}^{\infty} p_{nv}(x) v^n - x^m,$$

then for $m > 1$ we have:

$$n\mu_{n,m+1}(x) = x(x+1)\mu_{n,m}(x) + m\mu_{n,m-1}(x).$$

Consequently, for all $x \in [0,\infty)$, one has

$$\mu_{nm}(x) = O(n^{-(m+1)/2}),$$

where α denotes the integral part.

[**Lemma 2.2** [1] Let the function $T_{n,m}(x)$, $m \in \mathbb{N} \cup \{0\}$, be defined as

$$T_{n,m}(x) = B_n((t-x)^m, x) = \sum_{v=1}^{\infty} p_{nv}(x) \int_0^{\infty} b_{nv}(t)(t-x)^m dt + (1+x)^{-m}(-x)^m.$$

Then

$$T_{n,0}(x) = 1, \quad T_{n,1}(x) = 0, \quad T_{n,2}(x) = \frac{2x(1+x)}{n-1},$$

and also there holds the recurrence relation:

$$n - mT_{n,m+1}(x) = x(1+x)T_{n,m-1}(x) + 2mT_{n,m-1}(x).$$

From the above recurrence relation, it is easily verified that

$$T_{n,m}(x) = O(n^{-(m+1)/2}) \quad \text{for all } x \in [0, \infty).$$

[**Lemma 2.3**] Consequently, using Hölder's inequality in Lemma 2.2, we have

$$B_n(t-x)^r x = O(n^{-r/2}),$$

for each $r > 0$ and every fixed $x \in [0, \infty)$.

[**Theorem 2.4**] [Direct theorem] Let $f \in C^{2k+2}[0, \infty)$ admitting a derivative of order $(2k+2)$ at a point $x \in [0, \infty)$. Then

$$\lim_{n\to\infty} n^{k+1}\left[B_n(f,k,x) - f(x)\right] = \sum_{r=1}^{2k+2} \frac{f^{(r)}(x)}{r!} Q(r,k,x)$$
$$\lim_{n\to\infty} n^{k+1}\left[B_n(f,k,x) - f(x)\right] = 0$$

and where $Q(r,k,x)$ are certain polynomial in k of degree r.

Further, $f^{(2k+1)}$ exists and absolutely continuous over $[a,b]$ and $f^{(2k+1)} \in L_\infty[a,b]$, then for any $[c,d] \subset [a,b]$, there hold

$$||B_n(f,k,x) - f(x)||_{C[c,d]} \leq B_n^{-(k+1)} \left\{||f||_{C_\alpha} + ||f^{(2k+2)}||_{L_\infty[a,b]}\right\}$$

[**Theorem 2.5**] [Direct theorem] Let $f \in C_\alpha[0, \infty)$. Then, for sufficiently large n, there exist a constant M independent of f and n such that

$$||B_n(f,k,\cdot) - f||_{C[a_2,b_2]} \leq M \left\{\omega_{2k+2}(f, \frac{1}{n}, a_1, b_1) + n^{-(k+1)}||f||_{C_\alpha}\right\}.$$

3 Inverse Theorem

[Theorem 3.1] Let $f \in C_\alpha[0, \infty)$ and $B_n(f,x)$ be as previously defined. Then, in the following statements the implications $(i) \Rightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv) hold:$

(i)

$$\|B_n(f;k,x) - f\|_{C[a,b]} = O\left(n^{-\dfrac{\alpha(k+1)}{2}}\right)$$

(ii) f $\in Li_2(x, k+1, a, b)$

(iii) $\begin{cases} (a)\text{For } m < \alpha(k+1) < m+1, \quad m = 0,1,2,\ldots,2k+1, \quad f^{(m)} \text{ exist and belong to the class } Lip \\ (b)\text{For } \alpha(k+1) = m+1, \quad m = 0,1,2,\ldots,2k+1, \quad f^{(m)} \text{ exist and belong to the class } Lip(1, a_2 \end{cases}$

(iv) ——$\mathrm{B}_n(f,k,x) - f||_{C[a_n,b_1]} = O\left(n^{-\frac{\alpha(k+1)}{2}}\right)$, where $Liz(\alpha,k,a,b)$ denotes the class of functions for which $\omega_2(f,h,a,b) \leq Mh^\alpha$, when $k=1$, $Liz(\alpha,1)$ reduces to the Zygmund class $Lip^*\alpha$.

Proof. The implications (i) and (iii) are equivalent (see [?]). The implication (iii) $\Rightarrow$ (iv) follows from Theorem 2.4. Therefore, to prove the theorem, we have to show (i) $\Rightarrow$ (ii). We proceed as follows:

Choose points a', a", b', b" in such a way that $a_1 < a' < a < a'' < a_2 < b' < b'' < b < b_1$.

and a function $g \in C_0^\infty$ such that:

$\in (a', b') g(t) = 1$

$g(t) = 1 \quad \text{for } t \in [a, b]$.

Hence, to prove the assertion, it is sufficient to show that

$$(3.1) \quad ||B_n(fg, k, x) - fg||_{C[a;b]} = O\left(n^{-\frac{\alpha(k+1)}{2}}\right) \implies (ii). \qquad (1)$$

Writing $f(g) = \bar{f}$ for all small values of r.

$$||\Delta_r^{2k+2} \bar{f}||_{C[a'',b'']} \leq ||\Delta_r^{2k+2}||\bar{f} - B_n(\bar{f}, k, x)||_{C[a'',b'']} + ||\Delta_r^{2k+2}||B_n(\bar{f}, k, x)||_{C[a'',b'']}||.$$

Therefore, by definition of Δ_r^{2k+2}

$$|\Delta_r^{2k+2} B_n(\bar{f}, k, x)||_{C[a'',b'']} = \left| \int_0^r \int_0^r \cdots \int_0^r B_n^{(2k+2)}(\bar{f}, k, x + \sum_{i=1}^{2k+2} dt_i) dt_1 dt_2 \cdots dt_{2k+2} \right| |_{C[a'',b'']} \leq r^{2k+2} |B_n^{(2$$

i.e., $\Delta_r^{2k+2} B_n(\bar{f}, k, x)||_{C[a',b']}$

$\leq r^{2k+2} \left\{ ||B_n^{(2k+2)}(f - f_{n,2k+2}, k, x)||_{C[a'',b''+(2k+2)r]} \right.$

$\left. + |\Delta_r^{2k+2} B_n B_n^{(2k+2)}(f_{n,2k+2}, k, x)||_{C[a'',b''+(2k+2)r]} \right\}$.

where $\overline{f}_{n,2k+2}$ is the Steklov mean of (2k+2)-th order corresponding to $\overline{f}$.

Consequently by Lemma 2.3[5], we have

$\int_0^\infty |\frac{\partial^{2k+2}}{\partial x^{2k+2}} W_n(x, t)\Big\| dt$

$\leq \sum_{2i+l \leq 2k+2i, l \geq 0} \sum_{v=1}^\infty n^i |v - nx^l| \left| \frac{q_{i,l,2k+2}(x)}{x(1+x)^{2k+2}} p_{n,v} \int_0^\infty b_{n,v}(t) dt - \frac{(n+2k+1)}{(n-1)!} (1+x)^{-(n+2k+2)} \right.$

Now applying Schwarz inequality and Lemma 2.1[5], we get

——$B_n^{(2k+2)}(\overline{f} - \overline{f}_{n,2k+2}, k, x)||_{C[a'',b''+(2k+2)r]}$

$\leq K_1 n^{k+1} ||\overline{f} - \overline{f}_{n,2k+2}||_{C[a'',b'']}$

on the other hand, by Lemma 2.2[5], we have

By Taylor's expansion of $\bar{f}_{n,2k+2}(t)$, we have

$$(3.3) \quad \overline{f}_{n,2k+2}(t) = \sum_{i=0}^{2k+1} \frac{\overline{f}_{\eta,2k+2}^{(i)}(x)}{l!}(t-x)^i + f_{n,2k+2}^{-(2k+2)}(\xi) \frac{(t-x)^{2k+2}}{(2k+2)!}. \qquad (2)$$

Using (3.2) and (3.3), we obtain

$$\left\| \frac{\partial^{2k+2}}{\partial x^{2k+2}} B_n\left(\overline{f}_{n,2k+2}, k, x\right) \right\|_{C[a'',b''+(2k+2)r]} \leq \sum_{j=0}^k \frac{|C(j,k)|}{(2k+2)!} \left\| \overline{f}_{n,2k+2}^{(2k+2-j)} \right\|_{C[a'',b'']} \left\| \int_0^\infty \frac{\partial^{2k+2}}{\partial x^{2k+2}} W_{d_j(n)}(x,t)(t -$$

Next, applying Lemma 2.1 and Lemma 2.2 [5] and the Cauchy-Schwarz inequality, we get

$$I = \left| \int_0^\infty \left\| \frac{\partial^{2k+2}}{\partial x^{2k+2}} W_{d_j n}(x,t)(t-x)^{2k+2} \right\|_{C[a'',b'']} dt \right| \leq \sum_{i,j=0}^{2k+2} n^i \left| \frac{q_{i,j,2k+2}(x)}{M_{n,2k+2}} \right| \left| O(n^{-i}) O(n^{-\frac{\alpha(k+1)}{2}}) \right|$$

Hence

$$\left\| B_n^{(2k+2)} \left(\overline{f}_{n,2k+2}, k, x \right) \right\|_{C[a'',b''+(2k+2)r]} \leq K_n \left\| \overline{f}_{n,2k+2}^{(2k+2)} \right\|_{C[a'',b'']}$$

(3.3)
combining the estimates of Theorem 2.4and (3.3), we get

$$\left\| \Delta_r^{2k+2} \overline{f} \right\|_{C[a',b']} \leq \Delta_r^{2k+2} \left\| \overline{f} - B_n(\overline{f}, k, x) \right\|_{C[a',b']} + K_3 r^{2k+2} \left\{ \left\| \overline{f} - \overline{f}_{n,2k+2} \right\|_{C[a'',b'']} + \left\| \overline{f}_{n,2k+2}^{(2k+2)} \right\|_{C[a'',b'']} \right\}$$

Since the above holds for sufficiently small values of r, it follows from Theorem 2.4 and by the property of the Steklov mean $\bar{f}_{n,2k+2}$that

$$\omega_{2k+1}(\overline{f}, h, [a,b]) \leq K_4 \left\{ n^{\frac{\alpha(k+1)}{2}} + h^{2k+2}(n^{k+1} + n^{-(2k+2)}) \omega_{2k+2} \left(\overline{f}_{n,2k+2}, h, [a'', b''] \right) \right\}.$$

Finally, choosing n such that $n < n^{-2} < 2n$ and following Berens and Lorentz [7], we obtain (3.4)

$$\omega_{2k+2}(\overline{f}, h, [a_2, b_2]) = o(h^{\alpha(k+1)}) \tag{3}$$

For $t \in [a_2, b_2]$, $\overline{f}(t) = f(t)$, the result follows from (3.4).

To complete the proof of $(i) \Rightarrow (ii)$, we have to prove the validity of Theorem 2.4 under the hypothesis

Let $r = a(k+1)$. First, we consider the case $0 < r \leq 1$. For $t \in [a', b']$, we have

$$B_n(fg,k,x) - f(x)g(x) = g(x) \left[B_n((f(t)-f(x)), k, x) \right] + \sum_{j=0}^{i} C(j,k) \left| \int_a^b W_{d,n}(x,t) f^{(j)}(t)(g(t)-g(x)) dx \right| \tag{4}$$

where the o-term holds uniformly for $x \in [a', b']$ (by Lemma 2.4 [5]). From the assumption

$$||B_n(f,k,x) - f||_{C[a,b]} = O(n^{-1})$$

we have

$$||J_1||_{C[a,b]} \leq ||g||_{C[a,b]} ||B_n(f,k,x) - f||_{C[a,b]} \leq K_3 n^{-2} \tag{5}$$

By the use of the Mean Value Theorem, we get

$$J_2 = \sum_{j=0}^{i} C(i,j) \left| \int_a^b W_{d,n}(x,t) f^{(j)}(t) \left(g\left(\frac{x+t}{2} \right) - g(x) \right) dt \right|$$

Also, by using Lemma 2.2 [5] and the Cauchy-Schwarz inequality, we obtain

$$—|J_2||_{C[a,b]} \leq ||f||_{C[a,b]} ||g||_{C[a,b]} \left| \sum_{j=0}^{k} C(j,k) \max_{t \in [a'',b'']} \left| \int_0^\infty W_{d,n}(x,t)(t-x)^j dt \right| \right|$$

$\leq K_6||f||_{C[a_1,b_1]}||g||_\infty \left(\sum_{j=2}^k |C(j,k)|\right) n^{-1/2} \leq O(n^{-1/2})(6)$

Combining the estimates of (3.6) and (3.7), we have in (3.5) that

$$||B_n(fg,k,x) - fg||_{C[a',b']=O(n^{-r/2})}$$

Hence, the result holds for $0 < r \leq 1$.

To prove the result for $0 < r \leq 2(k+1)$, we assume it for ${}^{r\in(m-1,m)}$ and prove it for ${}_{r\in(m+1,m),m=1,2.....,2k+1}$. We assume that ${}^{r\in(m+1,m)}$ and (i) hold. Choosing the points x_i, y_i such that $a_1 < x_1 < a' < b' < y_1 < b_1$. Then, in view of the assumption for the interval (ii) and (iii), it follows that $f^{(k+1)} \in C_\alpha[0,\infty)$. Then, with ${}^{f^{m-1}}$ exists and belongs to ${}_{Lip(1-\delta,x_1,y_1),\delta>0}$ denoting the characteristic function of the interval $[x_1, y_1]$, we have

(3.8)

$$||B_n(fg,k,x) - fg||_{C[a,b]} \leq ||B_n(g(x)(f(t)-f(x)),k,x||_{C[a'',b'']}||B_n((f(t)-f(x)),k,x)||_{C[a'',b'']+o(n^{-(k+1)})}(7)$$

Now,

(3.9)

$$||B_n(g(x)(f(t)-f(x)),k,x)||_{C[a,b]}$$

$\leq ||g||_{C[a,b]}||B_n(f,k,x) - f||_{C[a,b]}$

$=$ O($\mathrm{n}^{-r/2}$)(8)

By the use of Taylor's expansion of f, we get

$$f(t) = \sum_{i=0}^{m-1} \frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{f^{(m-1)}(\xi)}{(m-1)!}(t-x)^{m-1} \quad {}^{\epsilon}\text{(lying between } t \text{ and } x).$$

Thus,

$$J_1 = ||B_n(f(t)(g(t) - g(x)), k, x)||_{C[a,b]}$$

$= \left|\sum_{i=0}^{m-1} \frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{f^{(m-1)}(\xi)-f^{(m-1)}(x)}{(m-1)!}(t-x)^{m-1}\right| |g(t) - g(x)|_{C[a,b]}$

Since

$$f^{(m-1)} \in Lip(1-\delta, x_1, y_1)$$

—f$^{(m-1)}(\xi) - f^{(m-1)}(x)| \leq K_{-} \left|\frac{\xi-x}{t-x}\right|^{1-\delta} \leq K_{-}|t-x|^{1-\delta}$ where ^{K_7} is the Lip constant for $f^{(m-1)}$. We have

(3.10)

$$J_1 \leq ||B_n \left(\sum_{i=0}^{m-1} \frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{f^{(m-1)}(\xi) - f^{(m-1)}(x)}{(m-1)!}(t-x)^{m-1}\right)(g(t)-g(x)), k, x)||_{C[a,b]} +$$

$\mathrm{K}_{-} \frac{1}{(m-1)!}||g||_{C[a,b]} \left(\sum_{j=0}^i C(i,j)\right) ||B_{n,p}(t-x)^i y^{(i)}(t,x)||_{C[a,b]}$

$=$ J$_1$ + J_2,say(9)

Making use of Theorem 2.5 and Taylor's expansion of g, we get (3.11)

$$J_2 = O(n^{-(k+1)}) \tag{10}$$

Next, using Hölder's inequality and Lemma 2.1[5], we have

$$J_3 \leq K_7 ||g||_{C[a,b]} \left(\sum_{j=0}^{k} |C(j,k)| \right) \left| \max_{t \in [a'',b'']} \left| \int_0^\infty W_{d,n}(x,t)(t-x)^j dt \right| \right|$$

(3.12) $\mathrm{J}_3 \leq K_7 \max_{x \in [a',b']} \left| \int_0^\infty W_{d,n}(x,t) |t-x|^{2(m-1)} dt \right|$

$\mathrm{O}\left(n^{-\frac{(m+1-j)}{2}} \right) = O(n^{-r/2})(11)$
Combining (3.8)–(3.12), we obtain

$$||B_n(fg,k,x) - fg||_{C[a,b]} = O(n^{-1})$$

This completes the proof. □

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