MATHS 7027
MATHEMATICAL FOUNDATION OF
DATA SCIENCE

ASSIGNMENT-2

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MASTER IN DATA SCIENCE

ASSIGNMENT-2

Answer-1

To proof: $\sum_{i=1}^{n} i^2 = \frac{(2n+1)(n+1)(n)}{6}$ by Mathematical Induction

By mathematical induction, it include Basic step and Induction hypothesis.

Step 1: Basic step $(n=1)$

Left Hand Side

$$\sum_{i=1}^{1} i^2 = 1^2 = 1$$

Right hand side

$$\frac{(2\times1+1)(1+1)(1)}{6} = \frac{3\times2\times1}{6} = \frac{\cancel{6}}{\cancel{6}}\ 1$$

So we get that $LHS = RHS$

Step 2: Induction Hypothesis

Let it be true for $n=k$

$$\sum_{i=1}^{k} i^2 = \frac{(2k+1)(k+1)(k)}{6} \quad -(1)$$

Using this (1), we must show for $n = k+1$

WMS:

$n = k+1$

LHS: $\sum_{i=1}^{k+1} i^2$ — (2)

RHS: $\frac{(2(k+1)+1)((k+1)+1)(k+1)}{6}$ — (3)

$= \frac{(2k+3)(k+2)(k+1)}{6}$

Taking (2) into consideration:

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2 \quad \text{— (4)}$$

Putting eq (1) in eq (4)

$$= \frac{(2k+1)(k+1)k}{6} + (k+1)^2$$

Solving the above equation:

$$= \frac{(k+1)}{6}\left[(2k+1)k + 6(k+1)\right]$$

$$= \frac{(k+1)}{6}\left[2k^2 + k + 6k + 1\right]$$

$$= \frac{(k+1)}{6}\left[2k^2 + 7k + 1\right]$$

Now $2k^2 + 7k + 1 = (2k+3)(k+2)$ — (5)

Using 5

$$= \frac{(k+1)(2k+3)(k+2)}{6}$$

$$\boxed{\sum_{i=1}^{k+1} i^2 = \frac{(2k+3)(k+2)(k+1)}{6}} \quad -6$$

Now eq 3 and eq 6 are equal
So
this is proved by Hypothesis of Mathematical Induction.

Hence Proved.

Answer 2 a-1

Given: Maclaurin polynomial for $\cos(x)$

To find: Maclaurin polynomial of degree $n = 2k$ for $f(x) = \cos(2x)$.

Bounded by $[-1, 1]$

$f(x) = \cos(2x)$ $\qquad\qquad$ $f(0) = 1$

So the Maclaurin ~~series~~ polynomial for $f(x)$ is

$$P_n(x) = f(0) + f'(0)(x) + \frac{f''(0)}{2}x^2 + \ldots + \frac{f^{n}(0)}{n!}x^n \quad -(1)$$

$$\left.\begin{aligned}
f(x) &= \cos 2x \\
f(0) &= 1 \\
f'(x) &= -2\sin(2x) \\
f'(0) &= 0 \\
f''(x) &= -4\cos(2x) \\
f''(0) &= -4 \\
f'''(x) &= 8\sin(2x) \\
f'''(0) &= 0
\end{aligned}\right\} \quad -(2)$$

$\vdots$

$$f^{n}(0) = (-1)^k (2)^{2k}$$

P4

Using (1) and (2), we get

$$P_n(x) = 1 + 0 \times x + \frac{(-4)\,x^2}{2!} + 0 + \dots + \frac{(-1)^k (2)^{2k} x^{2k}}{(2k)!}$$

$$\boxed{P_n(x) = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots + \frac{(-1)^k (2x)^{2k}}{(2k)!} = \sum_{n=0}^{\infty} \frac{(-1)^k (2x)^{2R}}{(2k)!}}$$

Answer 2-b)

How many terms needed to approximate $\cos(2)$ to within 0.001.

formula: $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1}$

Now $f(x) = \cos x$,

$f^n(x)$ is bounded by $[-1, 1]$

$f(x) = \cos(x)$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

for all values of $f^n(x)$ it is between $[-1, 1]$

P5

Range of $f^{n+1} z \in [-1, 1]$

Now, $R_n(x) \leq 0.001$

We are given with $2x$, so $n = 2$

$$\frac{c}{(n+1)!} 2^{n+1} \leq 0.001$$

By the bounded value it lie between $[-1, 1]$

So maximum value of C is 1

$$\frac{2^{n+1}}{(n+1)!} \leq 0.001$$

$$\frac{2^n \cdot 2}{(n+1)!} \leq 0.001$$

$$\frac{2^n}{(n+1)!} \leq \frac{0.001}{2}$$

$$\frac{2^n}{(n+1)!} \leq 0.0005 \quad (1)$$

The equation (1) satisfied for $n = 9$.

So

$$\boxed{n = 9}$$

$$\frac{2^n}{0.0005} \leq (n+1)!$$

$$(n+1)! < \frac{2^n}{0.0005}$$

Answer-3

Taylor Series for $f(x) = \ln(x)$ at $a = 3$ along with interval of convergence

$$\text{Taylor Series} = \sum_{i=0}^{\infty} \frac{f^{(i)}(a)(x-a)^i}{i!} \quad (1)$$

$$P_n(a) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \dots$$

$$\dots + \frac{f^n(a)(x-a)^n}{n!}$$

$f(x) = \ln x$ so $f(3) = 1.099$

$f'(x) = 1/x$ so $f'(3) = \frac{1}{3}$

$f''(x) = -1/x^2$ so $f''(3) = -\frac{1}{3^2}$

$f'''(x) = \frac{2}{x^3}$ so $f'''(3) = \frac{2}{3^3}$

$f^{iv}(x) = -\frac{6}{x^4}$ so $f^{iv}(3) = -\frac{6}{3^4}$

Substituting in equation (1).

$$P_n(a) = 1.099 + \frac{1}{3}(x-3) + \frac{-1}{3^2} \times \frac{(x-3)^2}{2!} + \frac{2}{3^3} \times \frac{(x-3)^3}{3!}$$

$+ \frac{-6}{3^4} \times \frac{(x-3)^4}{4!} + \ldots$

so

$P_n(a) = 1.099 + \frac{1}{3}(x-3) - \frac{1}{3^2}\frac{(x-3)^2}{2} + \frac{1}{3^3}\frac{(x-3)^3}{3}$

$- \frac{(x-3)^4}{3^4 \times 4} + \ldots$

so n^{th} term will be $\frac{(-1)^{n-1}(x-3)^n}{3^n \times n}$

so our Taylor series for $\ln x$ centered at $a=3$ is

$= 1.099 + \frac{1}{3}(x-3) - \frac{1}{3^2} \times \frac{(x-3)^2}{2} + \frac{1}{3^3} \times \frac{(x-3)^3}{3} + \ldots$

$\frac{(-1)^{n-1}(x-3)^n}{(3^n) \times (n)}$

OR

$= \boxed{1.099 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-3)^n}{(3^n) \times (n)}}$ (2)

Finding with the interval of convergence

Ratio test $r = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$ (3)

Using 2 and 3

$$a_{n+1} = \frac{(-1)^{n-1+1} (x-3)^{n+1}}{3^{n+1} \times (n+1)}$$

$$a_n = \frac{(-1)^{n-1} (x-3)^n}{3^n \times n}$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{(-1)^n (x-3)^{\cancel{n}+1}}{3^{\cancel{n}+1} \times (n+1)} \times \frac{\cancel{3^n} \times n}{\cancel{(-1)^{n-1}} \cancel{(x-3)^n}}$$

$$= \left| \frac{(-1)\ (x-3) \times n}{3 \times (n+1)} \right|$$

$$= |(-1)| \left| \frac{(x-3) \times n}{3 \times (n+1)} \right|$$

Dividing by n both numerator and denomenator.

$$\lim_{n\to\infty} \left(\frac{(x-3)\times \cancel{n}}{\cancel{n}} \Big/ \frac{3\cancel{n}}{\cancel{n}} + \frac{3}{n} \right)$$

Applying limit $\lim_{n\to\infty}$ so $\frac{1}{n} = 0$ so $\frac{3}{n} = 0$

$$\left(\frac{x-3}{3} \right)$$

So its interval of convergence is

$$\boxed{0 < x < 6}$$ $0<x<6$

(P9)

Answer 4 a)

Given $a_0 = 2$

$$a_{n+1} = \frac{4n+1}{kn+3} a_n$$

for some $k \in N$

Write out the first six terms of the series (i.e. upto $n=5$)!

$$\sum_{n=0}^{\infty} a_n$$

When $n=0$

$a_0 = 2$ $\quad a_{0+1} = \frac{4\times 0+1}{k\times 0+3} a_0$

$$a_1 = \frac{0+1}{0+3} \times 2$$

$$a_1 = \frac{2}{3}$$

When $n=1$

$$a_{1+1} = \frac{4n+1}{kn+3} a_1$$

$$a_2 = \frac{4+1}{k+3} \times \frac{2}{3}$$

$$a_2 = \frac{10}{3(k+3)}$$

When $n = 2$

$$a_{2+1} = \frac{4n+1}{kn+3} a_2$$

$$a_3 = \frac{8+1}{2k+3} \times \frac{10}{3(k+3)}$$

$$a_3 = \frac{90}{(2k+3)\ (3k+9)} = \frac{90}{6k^2+27k+27} = \frac{30}{2k^2+9k+9}$$

When $n = 3$

$$a_{3+1} = \frac{4n+1}{kn+3} a_3$$

$$a_4 = \frac{4(3)+1}{k(3)+3} \times \frac{30}{2k^2+9k+9}$$

$$a_4 = \frac{13}{3k+3} \times \frac{30}{2k^2+9k+9}$$

$$a_4 = \frac{390}{(3k+3)\ (2k^2+9k+9)}$$

$$a_4 = \frac{390}{6k^3+33k^2+54k+27}$$

When $n = 4$

$$a_5 = \frac{4n+1}{kn+3} a_4$$

$$a_5 = \frac{4(4)+1}{k\times 4+3} \times \frac{390}{6k^2+33k^2+54k+27}$$

$$a_5 = \frac{17}{4k+3} \times \frac{390}{6k^3+33k^2+54k+27}$$

$$a_5 = \frac{6630}{24k^4+132k^3+216k^2+108k+18k^3+99k^2+162k+81}$$

$$a_5 = \frac{6630}{24k^4+249k^3+216k^2+270k+81}$$

$$a_5 = \frac{\cancel{6630}^{2210}}{\cancel{3}(8k^4+83k^3+72k^2+90k+27)}$$

$$a_5 = \frac{2210}{8k^4+83k^3+72k^2+90k+27}$$

When $n=5$

$$a_{5+1} = \frac{4n+1}{kn+3}\, a_5$$

$$a_6 = \frac{20+1}{5k+3} \times a_5$$

$$a_6 = \frac{21}{(5k+3)} \times \frac{2210}{(8k^4+83k^3+72k^2+90k+27)}$$

$$a_6 = \frac{46410}{(5k+3)(8k^4+83k^3+72k^2+90k+27)}$$

Answer 4(b)

$$\sum_{n=0}^{\infty} a_n$$

$a_0 = 2 \qquad a_{n+1} = \frac{4n+1}{kn+3} a_n$ (1)

Ratio test $= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ (2)

from eqn (1) we get

$$\frac{a_{n+1}}{a_n} = \frac{4n+1}{kn+3}$$

Use this by equation (2)

$$\lim_{n\to\infty} \left| \frac{4n+1}{kn+3} \right|$$

Divide the numerator and denominator by n.

$$\lim_{n\to\infty} \left| \frac{4\cancel{n}/\cancel{n} + 1/n}{k\cancel{n}/\cancel{n} + 3/n} \right|$$

We get

$$\lim_{n\to\infty} \left| \frac{4 + 1/n}{k + 3/n} \right| \quad (3)$$

We know that $\lim n\to\infty$ so $\frac{1}{n} = 0$

Using this in equation 3

We left with

$$R = \frac{4}{k}$$

So if $k \leq 3$ then value of Ratio test R will be greater than 1 so it diverges.

If $k \geq 5$ then value of Ratio test R will be less than 1 so it converges.

If $k = 4$ then value of Ratio test R will be equal to 1 then it is ~~incol~~ inconclusive.

Answer-5

Maclaurin Series to compute the limit

$$\lim_{x \to 0} \frac{1-\cos(x)}{1+x+-e^x}$$

Maclaurin Series of $\cos(x) = 1 - \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ — ①

Maclaurin Series of $e^{+x} = 1 + \frac{x}{1!} + \dots + \frac{x^n}{n!}$ ②

$$\lim_{x \to 0} \frac{1-\cos(x)}{1+x+-e^x} \quad \text{—③}$$

Using ①, ② and ③

$$1-\cos(x) = \cancel{1} - \cancel{1} + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \text{—⑤}$$

$$1+x+-e^x = \cancel{1} + \cancel{x} - \cancel{1} - \cancel{x} - \sum_{n=2}^{\infty} \frac{x^n}{n!} \quad \text{—⑥}$$

Now our equation using ⑤ and ⑥ is

$$\text{Numerator} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n)!} x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^{n+2}}{(2(n+1))!} x^{2(n+1)}$$

—⑦

Denomenator

$$\sum_{n=2}^{\infty} -\frac{x^n}{n!} = \sum_{n=0}^{\infty} -\frac{x^{n+2}}{(n+2)!} \quad (8)$$

Equation 7 / Equation 8 =

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+2} x^{2(n+1)}}{(2n+2)!} \Big/ \sum_{n=0}^{\infty} -\frac{x^{n+2}}{(n+2)!}$$

$$\lim_{x \to 0} \left(x^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+2)!} x^{2n} \Big/ x^2 \sum_{n=0}^{\infty} -\frac{1}{(n+2)!} x^n \right)$$

$\cancel{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+2)!}}$ Expanding the series

$$\lim_{n \to 0} \frac{\frac{(-1)^0}{(2 \cdot 0+2)!} x^0 + \frac{(-1)^1}{(2+2)!} x^2 + \ldots}{-\frac{1}{2!} x^0 + -\frac{1}{3!} \cdot x + \ldots}$$

Putting $x = 0$, we have left with

$$\frac{1/2!}{-1/2!} = \boxed{-1}$$