MATHS 7027
MATHEMATICAL FOUNDATION OF
DATA SCIENCE

ASSIGNMENT-1

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MASTER IN DATA SCIENCE

ASSIGNMENT-1

1.) Consider the function

$$f(x) = 1 + \frac{1}{x} \; ; \quad g(x) = \frac{x+1}{x+2}$$

a) Domain of $f(x)$

$$f(x) = 1 + \frac{1}{x}$$

$$f(x) = \frac{x+1}{x}$$

So, x can't be equal to zero

$$\boxed{\text{Domain of } f(x) = \{x \in \mathbb{R} \; ; \; x \neq 0\}}$$

Domain of $g(x)$

$$g(x) = \frac{x+1}{x+2}$$

So, $x+2$ can't be equal to zero

$$x + 2 \neq 0$$
$$x \neq -2$$

$$\boxed{\text{Domain of } g(x) = \{x \in \mathbb{R} \; ; \; x \neq -2\}}$$

Answer 1 b) Find:

fog:

$f(n) = 1 + \frac{1}{n}$; $g(n) = \frac{n+1}{n+2}$

$fog(n) = 1 + \frac{1}{\frac{n+1}{n+2}}$

$fog(n) = \frac{n+2}{n+1} + 1$ ∴ Taking LCM

$fog(n) = \frac{n+2+n+1}{n+1}$

$\boxed{fog(n) = \frac{2n+3}{n+1}}$

gof:

$f(n) = 1 + \frac{1}{n}$; $g(n) = \frac{n+1}{n+2}$

$f(n) = \frac{n+1}{n}$

$gof(n) = \frac{\frac{n+1}{n} + 1}{\frac{n+1}{n} + 2}$ ∴ Taking LCM

$gof(n) = \frac{\frac{n+1+n}{n}}{\frac{n+1+2n}{n}}$

Cutting the common terms

$$gof(x) = \frac{2x+1}{\cancel{x}} \times \frac{\cancel{x}}{3x+1}$$

$$\boxed{gof(x) = \frac{2x+1}{3x+1}}$$

fof :

fof(x) :

$$f(x) = 1 + \frac{1}{x}$$

$$fof(x) = 1 + \frac{1}{1+\frac{1}{x}}$$ ∴ Taking LCM

$$fof(x) = 1 + \frac{1}{\frac{x+1}{x}}$$

$$fof(x) = 1 + \frac{x}{x+1}$$ ∴ Taking LCM

$$fof(x) = \frac{x+1+x}{x+1}$$

$$\boxed{fof(x) = \frac{2x+1}{x+1}}$$

gog:

$$g(x) = \frac{x+1}{x+2}$$

$$gog(x) = \frac{\frac{x+1}{x+2} + 1}{\frac{x+1}{x+2} + 2}$$

$$gog(x) = \frac{\frac{x+1+x+2}{x+2}}{\frac{x+1+2x+4}{x+2}}$$ ∵ Taking LCM

Cutting the common terms

$$gog(x) = \frac{2x+3}{\cancel{x+2}} \times \frac{\cancel{x+2}}{3x+5}$$

$$\boxed{gog(x) = \frac{2x+3}{3x+5}}$$

Answer 1 c)

Inverse of f(x)

$$f(x) = 1 + \frac{1}{x}$$

$$y = 1 + \frac{1}{x}$$

$$\frac{1}{x} = y - 1$$

$$x = \frac{1}{y-1}$$

Now change 'y' with 'x'

$$\boxed{f^{-1}(x) = \frac{1}{x-1}} \quad -(i)$$

Inverse of $g(x)$

$$g(x) = \frac{x+1}{x+2}$$

for inverse

Step-1: $y = \frac{x+1}{x+2}$

$$xy + 2y = x + 1$$

$$xy - x = 1 - 2y$$

$$x(y-1) = (1-2y)$$

Step-2: $x = \frac{1-2y}{y-1}$

Step-3: Replace 'y' with x

$$\boxed{g^{-1}(x) = \frac{1-2x}{x-1}}$$

Checking inverse of $f(x)$

$$fof^{-1}(x) = x$$

$$f(x) = 1 + \frac{1}{x} \quad -(i)$$

$$f^{-1}(x) = \frac{1}{x-1} \quad -(ii)$$

$$fof^{-1}(x) = 1 + \frac{1}{\frac{1}{x-1}}$$

Solving

$$fof^{-1}(x) = \cancel{1} + x - \cancel{1}$$

$$\boxed{fof^{-1}(x) = x}$$

So there inverse exist.

Checking inverse of $g(x)$

$$gog^{-1}(x) = x$$

$$g(x) = \frac{x+1}{x+2} \quad -(i)$$

$$g^{-1}(x) = \frac{1-2x}{x-1} \quad -(ii)$$

Using (i) and (ii)

$$gog^{-1}(x) = \frac{\frac{1-2x}{x-1}+1}{\frac{1-2x}{x-1}+2}$$

$$gog^{-1}(x) = \frac{\frac{1-2x+x-1}{x-1}}{\frac{1-2x+2x-2}{x-1}}$$

$$= \cancel{\frac{1-2x+x-1}{x-1}} \times \cancel{\left(\frac{1-2x+2x-2}{}\right)}$$

$$gog^{-1}(x) = \frac{1-2x+x-1}{\cancel{x-1}} \times \frac{\cancel{x-1}}{1-2x+2x-2}$$

$$gog^{-1}(x) = -x \times \frac{1}{-1}$$

$$\boxed{gog^{-1}(x) = x}$$

So its <u>inverse exist</u>

Answer 1(d)

Q-) find the range of the function(s) for which the inverse exists

A-) As for both the funtion inverse exists. So we need to find range for both funtion.

Range of $f(n)$

So Range of $f(n)$ = Domain of $f^{-1}(n)$

So, finding domain of $f^{-1}(n)$

$$f^{-1}(n) = \frac{1}{n-1}$$

So $n-1 \neq 0$

$n \neq 1$

So $\text{Domain}_1^{(f^{-1}(n))} = \{n \in R ; n \neq 1\}$

$$\boxed{\text{Range}(f(n)) = \{n \in R ; n \neq 1\}}$$

Range of $g(n)$

So Range of $g(n)$ = Domain of $g^{-1}(n)$

So, finding domain of $g^{-1}(n)$

$$g^{-1}(n) = \frac{1-2n}{n-1}$$

So $n-1 \neq 0$

$n \neq 1$

So $\text{Domain}(g^{-1}(n)) = \{n \in \mathbb{R} ; n \neq 1\}$

$$\boxed{\text{Range}(g(n)) = \{n \in \mathbb{R} ; n \neq 1\}}$$

Answer 2 a)

$g(n) = 2n+1$

$h(n) = 4n^2 + 6n + 9$

find a function f such that $f o g = h$

$f o g(n) = h(n)$

Replacing n with $g^{-1}(n)$

$f o g o (g^{-1}(n)) = h o (g^{-1}(n))$ — 1

finding inverse of $g(n)$

$g(n) = 2n+1$

$g^{-1}(n) = ?$

$y = 2n+1$

$y-1 = 2n$

$n = \frac{y-1}{2}$

Replacing y with n

$g^{-1}(n) = \frac{n-1}{2}$ — 2

Using 1 and 2

Verifying $g^{-1}(n)$ so,

$$gog^{-1}(n) = n$$

$$g(n) = 2n + 1$$
$$g^{-1}(n) = \frac{n-1}{2}$$

$$gog^{-1}(n) = \cancel{2} \times \frac{n-1}{\cancel{2}} + 1$$

$$gog^{-1}(n) = n$$

So inverse verified

Putting 2 in 1

$$gog^{-1}(n) = n$$

$$\text{so } fo(gog^{-1}(n)) = ho(g^{-1}(n))$$

$$f(n) = ho(g^{-1}(n))$$

$$h(n) = 4n^2 + 6n + 9$$

$$g^{-1}(n) = \frac{n-1}{2}$$

Putting or finding $hog^{-1}(n)$

$$f(n) = \overset{.}{4}\left(\frac{n-1}{2}\right)^2 + \overset{3}{\cancel{6}}\left(\frac{n-1}{2}\right) + 9$$

$$f(n) = n^2 + 1 - 2n + 3n - 3 + 9$$

$$\boxed{f(n) = n^2 + n + 7}$$

Verifying:

$$fog(n) = h(n)$$

$$fog(n) = (2n+1)^2 + (2n+1) + 7$$

$$4n^2 + 1 + 4n + 2n + 1 + 7$$

$$fog(n) = 4n + 6n + 9$$

$$fog(n) = h(n)$$

$\therefore$ Hence verified

So, $f(n) = n^2 + n + 7$

Verified $f(n)$

Answer 2(b)

$f(n) = 3n+5$
$h(n) = 3n^2 + 3n + 2$

find $g(n)$ such that $fog = h$

~~So~~ if

$f^{-1} o f o g = g(n)$

~~So~~

$f^{-1} o f o g = f^{-1} o h$

$f(n) = 3n+5$

~~So~~ finding inverse of f

$y = 3n+5$

$3n = y-5$
$n = y-5/3$

Replacing n with y

$$f^{-1}(n) = \frac{n-5}{3}$$

~~for~~ $f^{-1} o f o g(n)$

~~So~~

$f(n) = 3n+5$

$$f \circ g(n) = 3g(n) + 5$$

$$f^{-1} \circ f \circ g(n) = \frac{3g(n) + 5 - 5}{3}$$

$$f^{-1} \circ f \circ g(n) = \frac{\cancel{3}g(n)}{\cancel{3}}$$

$$f^{-1} \circ f \circ g(n) = g(n)$$ Verified

Now solving Right hand side

$$f^{-1} \circ h(n)$$

$$f^{-1}(n) = \frac{n-5}{3}$$

$$h(n) = 3n^2 + 3n + 2$$

$$f^{-1} \circ h = \frac{3n^2 + n + 2 - 5}{3}$$

$$f^{-1} \circ h = \frac{3n^2 + 3n - 3}{3}$$

$$f^{-1} \circ h(n) = n^2 + n - 1$$

$$\boxed{g(n) = n^2 + n - 1}$$

Answer 3a)

$$\sum_{i=2}^{97} \left(\frac{1}{i} - \frac{1}{i+1} \right)$$

$$\cancel{= \sum_{i=2}^{97} \left(\frac{i+1-i}{i(i+1)} \right)}$$

$$\cancel{= \sum}$$

$\Rightarrow$ Using Telescoping series

$$\sum_{n=0}^{N} a_n - a_{n+1} = a_0 - a_{N+1}$$

So applying this formula we get,

$$\sum_{i=2}^{97} a \left(\frac{1}{i} - \frac{1}{i+1} \right)$$

So $N = 97$

$a_n = \frac{1}{i}$

$a_{n+1} = \frac{1}{i+1}$

So putting the values in the formula

$$= \frac{1}{2} - \frac{1}{98}$$

$$= \frac{96}{196}$$

$$= \boxed{\frac{24}{49}}$$

Answer 3b)

$$\sum_{i=1}^{m}\left(\sum_{j=1}^{n}(i+j)\right)$$

Solving the inner part first

$$\sum_{j=1}^{n}(i+j)$$

$$\sum_{j=1}^{n}(i+j) = (i+1) + (i+2) + \ldots + (i+n)$$

$$= ni + \sum_{j=1}^{n} j \quad -①$$

The formula of first n terms of arithmetic expression is

$$\sum_{i=1}^{n} a_i = \left(\frac{n}{2}\right)(a_1 + a_n)$$

$$\left(\frac{n}{2}\right)(1+n)$$

$$\frac{n}{2}(n+1) \quad -②$$

Putting ② in ①

$$= ni + \frac{n}{2}(n+1)$$

$$\sum_{i=1}^{m} \left(ni + \frac{n}{2}(n+1) \right)$$

$$= n\sum_{i=1}^{m} i + \sum_{i=1}^{m} \frac{n(n+1)}{2}$$

$$= n\frac{(m+1)m}{2} + \frac{nm(n+1)}{2}$$

$$= \boxed{\frac{nm}{2}(m+n+2)}$$

we know

$\therefore \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$\therefore \sum_{i=1}^{n} c = nc$

Answer 3 c)

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n} \left(\frac{i}{n} \right)^2$$

$$= \sum_{i=1}^{n} \frac{1}{n} \times \frac{i^2}{n^2}$$

$$= \sum_{i=1}^{n} \frac{i^2}{n^3}$$

$$= \frac{1}{n^3} \sum_{i=1}^{n} i^2 \quad ①$$

We know that

$$\sum_{i=1}^{N} i^2 = \frac{n(n+1)(2n+1)}{6} \quad (2)$$

Putting (2) in (1)

$$= \frac{1}{n^{\cancel{3}2}} \times \frac{\cancel{n}(n+1)(2n+1)}{6}$$

$$= \frac{(n+1)(2n+1)}{6n^2}$$

Now limit $n \to \infty$

$$\lim_{n\to\infty}\left[\frac{(n+1)(2n+1)}{6n^2}\right]$$

$$\lim_{n\to\infty}\left[\frac{2n^2+n+2n+1}{6n^2}\right]$$

$$\lim_{n\to\infty}\left[\frac{2n^2+3n+1}{6n^2}\right]$$

$$\lim_{n\to\infty}\left[\frac{\cancel{2n^2}}{{}_3\cancel{6n^2}}+\frac{\cancel{3n}}{{}_2\cancel{6n^2}}+\frac{1}{6n^2}\right]$$

$$\lim_{n\to\infty}\left[\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}\right]$$

$$= \lim_{n\to\infty}\left(\frac{1}{3}\right) + \lim_{n\to\infty}\left(\frac{1}{2n}\right) + \lim_{n\to\infty}\left(\frac{1}{6n^2}\right) \quad -③$$

$$\lim_{n\to\infty}\left(\frac{1}{3}\right) = \frac{1}{3} \quad -④$$

$$\lim_{n\to\infty}\left(\frac{1}{2n}\right) \Rightarrow \frac{1}{2}\left(\lim_{n\to\infty}\frac{1}{n}\right)$$

$$\text{so } \lim_{n\to\infty}\frac{1}{n} = 0$$

$$= 0 \quad -⑤$$

$$\lim_{n\to\infty}\left(\frac{1}{6n^2}\right) = \frac{1}{6}\left(\lim_{n\to\infty}\frac{1}{n^2}\right)$$

$$\text{so } \lim_{n\to\infty}\frac{1}{n^2} = 0$$

$$= 0 \quad -⑥$$

Putting ④, ⑤ and ⑥ in equation ③

$$= \frac{1}{3} + 0 + 0$$

$$= \boxed{1/3}$$

Answer -4)

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{3^n}$$

Test whether converges.

We can find converges or diverges by ratio test.

$$a_n = \frac{(-1)^n n^3}{3^n}$$

$$a_{n+1} = \frac{(-1)^{n+1} (n+1)^3}{3^{n+1}}$$

Ratio test

$$r = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{\frac{(-1)^{n+1} (n+1)^3}{3^{n+1}}}{\frac{(-1)^n n^3}{3^n}} \right|$$

$$= \lim_{n \to \infty} \left| \frac{(-1)^{n+1} (n+1)^3}{3^{n+1}} \times \frac{3^n}{(-1)^n n^3} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}\cdot(n+1)^3}{3^{n+1}} \times \frac{3^n}{(-1)^n \times n^3} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\cancel{(-1)^n} \times (-1) \times (n+1)^3}{\cancel{3^n}\cdot 3} \times \frac{\cancel{3^n}}{\cancel{(-1)^n}\cdot n^3} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1) \times (n+1)^3}{3 \cdot n^3} \right|$$

~~so~~ $(a+b)^3 = a^3 + b^3 + 3ab(a+b)$

~~so~~ using this

$$(n+1)^3 = n^3 + 1^3 + 3n(n+1)$$

$$= \lim_{n\to\infty} \left| \frac{-1 \times (n^3 + 1 + 3n(n+1))}{3n^3} \right|$$

$$\lim_{n\to\infty} \left| \frac{-n^3 - 1 - 3n(n+1)}{3n^3} \right|$$

$$\lim_{n\to\infty} \left| \frac{-\cancel{n^3}}{3\cancel{n^3}} \right| - \lim_{n\to\infty} \left| \frac{1}{3n^3} \right| - \lim_{n\to\infty} \left| \frac{\cancel{3}\cancel{n}(n+1)}{\cancel{3}n^{\cancel{3}\,2}} \right| \quad \text{①}$$

$$\therefore \lim_{n\to\infty} \frac{1}{n^c} = 0 \quad \{\text{if } c>0\} \quad \text{-②}$$

Using ② in ① so

$$= \frac{1}{3} - 0 - 0 = \boxed{\frac{1}{3}}$$

Test of convergence

So answer is ~~g~~ $\frac{1}{3}$ which is less than 1 so it converges.