Terminology and Representations of Graphs

This post covers basic definition in terminologies associated with graphs and covers Adjacency list and adjacency matrix graph representations of the graph.

What is a Graph?

A **graph** is an ordered pair G = (V, E) comprising a set V of vertices or nodes and a collection of pairs of vertices from V called edges of the graph.

For example for above graph.

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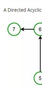
V = { 1, 2, 3, 4, 5, 6 }
E = { (1, 4), (1, 6), (2, 6), (4, 5), (5, 6) }
```

Types of Graphs:

1. Undirected graph

An *undirected graph* (graph) is a graph in which edges have no orientation. The edge (x, y) is identical to the edge (y, x), i.e., they are not ordered pairs. The maximum number of edges possible in an undirected graph without a loop is $n(n - 1)/2$.



2. Directed graph

A Directed graph (di-graph) is a graph in which edges have orientations. i.e. The edge (x, y) is not identical to the edge (y, x).

3. Directed Acyclic Graph (DAG)

A Directed Acyclic Graph (DAG) is a directed graph that contains no cycles.

4. Multigraph

A multigraph is an undirected graph in which multiple edges (and sometimes loops) are allowed. Multiple edges are two or more edges that connect the same two vertices. A loop is an edge (directed or undirected) that connects a vertex to itself. It may be permitted or not, according to the application.

5. Simple graph

A simple graph, as opposed to a multigraph, is an undirected graph in which both multiple edges and loops are disallowed. In a simple graph with n vertices, the degree of every vertex is at most n − 1.

6. Weighted and Unweighted graph

A weighted graph associates a value (weight) with every edge in the graph. Words cost or length can also be used instead of weight.

A *unweighted graph* do not have any a value (weight) associated with every edge in the graph. In other words, weighted graph is a weighted graph with all edge weight as 1. Unless specified otherwise, all graphs are assumed to be unweighted by default.

7. Complete graph

A complete graph is one in which every two vertices are adjacent: all edges that could exist are present.

8. Connected graph

A Connected graph has path between every pair of vertices. In other words, there are no unreachable vertices. A **disconnected graph** is a graph which is not connected.

Most commonly used terms in Graphs:

- An **edge** is (together with vertices) one of the two basic units out of which graphs are constructed. Each edge has two vertices to which it is attached, called its **endpoints**.
- Two vertices are called **adjacent** if they are endpoints of the same edge.
- **Outgoing edges** of a vertex are directed edges that the vertex is the origin.
- **Incoming edges** of a vertex are directed edges that the vertex is the destination.
- The **degree** of a vertex in a graph is the number of edges incident to it.
- In a directed graph, **outdegree** of a vertex is the number of outgoing edges from it and **indegree** is the number of incoming edges.
- A vertex with indegree zero is called a source vertex, while a vertex with outdegree zero is called sink vertex.
- An isolated vertex is a vertex with degree zero; that is, a vertex that is not an endpoint of any edge.
- **Path** is a sequence of alternating vertices and edges such that each successive vertex is connected by the edge.
- **Cycle** is a path that starts and end at the same vertex.
- **Simple path** is a path with distinct vertices.
- A graph is **Strongly Connected** if it contains a directed path from u to v and a directed path from v to u for every pair of vertices u, v.
- A directed graph is called **Weakly Connected** if replacing all of its directed edges with undirected edges produces a connected (undirected) graph. The vertices in a weakly connected graph have either outdegree or indegree of at least 1.
- **Connected component** is the maximal connected sub-graph of a unconnected graph.
- A **bridge** is an edge whose removal would disconnect the graph.
- **Forest** is a graph without cycles.
- **Tree** is a connected graph with no cycles. If we remove all the cycles from DAG(Directed acyclic graph) it becomes tree and if we remove any edge in a tree it becomes forest.
- **Spanning tree** of an undirected graph is a subgraph that is a tree which includes all of the vertices of the graph.

Relationship between number of edges and vertices:

For a simple graph with m edges and n vertices, if graph is

- directed, then $m = n(n-1)$
- undirected, then $m = n(n-1)/2$
- connected, then $m \geq n - 1$
- a tree, then $m = n - 1$
- a forest, then $m \leq n - 1$
- complete, then $m = n(n-1)/2$

Therefore, O(m) may vary between O(1) and O(n^2), depending on how dense the graph is.

Graphs Representations:

1. Adjacency Matrix Representation:

An adjacency matrix is a square matrix used to represent a finite graph. The elements of the matrix indicate whether pairs of vertices are adjacent or not in the graph.

Definition –

For a simple unweighted graph with vertex set *V*, the adjacency matrix is a square $|V| \times |V|$ matrix *A* such that its element

$A_{ij} = 1$, when there is an edge from vertex *i* to vertex *j*, and

$A_{ij} = 0$, when there is no edge.

Each row in the matrix represent source vertices and each column represent destination vertices. The diagonal elements of the matrix are all zero, since edges from a vertex to itself i.e. loops are not allowed in simple graphs. If the graph is undirected, the adjacency matrix will be symmetric. Also for a weighted graph, A_{ij} can represent edge weights.

An adjacency matrix keeps a value (1/0/edge-weight) for every pair of vertices, whether the edge exists or not, so it requires n^2 space. They can be efficiently used only when the graph is sparse (or dense).

2. Adjacency List Representation:

An adjacency list representation for a graph associates each vertex in the graph with the collection of its neighboring vertices or edges i.e every vertex stores a list of adjacent vertices. There are many variations of adjacency list representation depending upon the implementation. This data structure allows the storage of additional data on the vertices but practically very efficient when the graph contains only few edges.