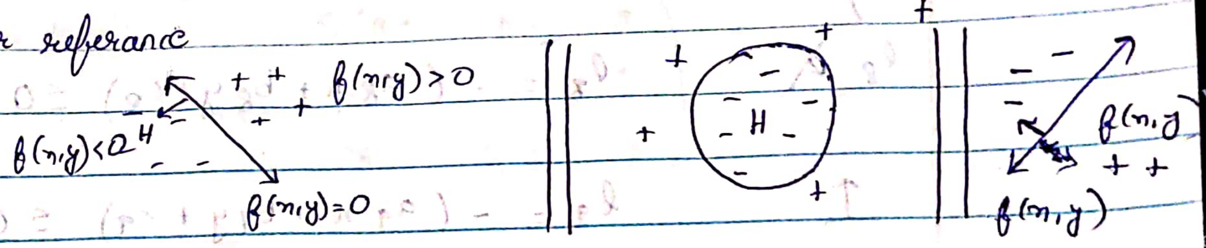
NAME: ARPIT SAVARKAR

COURSE: ALGORITHMIC Motion Planning

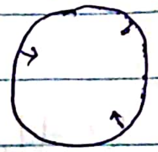
HW 2

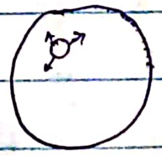
Q.1

Ans For referance



Consider

(i)  $\rightarrow H_{head} = x^2 + y^2 - r^2 \le 0$

(ii)  $\rightarrow H_{eye} = -\left((x-x_e)^2 + (y-y_e)^2 - r_e^2\right) \le 0$

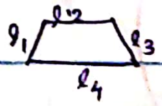
(iii)

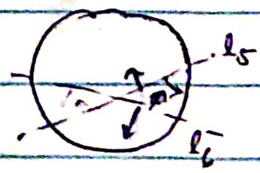
l_1 $\rightarrow l_1 = -(a_1x + b_1y + c_1) \le 0$

l_2 $\rightarrow l_2 = -(a_2x + b_2y + c_2) \le 0$

l_3 $\rightarrow l_3 = a_3x + b_3y + c_3 \le 0$

l_4 $\rightarrow l_4 = a_4x + b_4y + c_4 \le 0$

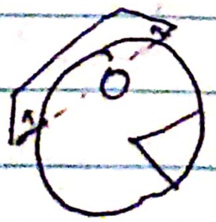
i.e,  $\equiv H_{heap} = l_1 \cap l_2 \cap l_3 \cap l_4$

(iv) 

$l_5 = +(a_5x + b_5y + c_5) \le 0$

$l_6 = -(a_6x + b_6y + c_6) \le 0$

$H_{mouth} = l_5 \cup l_6$

(v) 

$H_{half_1} = H_{head} \cap H_{eye} \cap H_{heap} \cap H_{mouth}$

(vi) $l_7 = l_7 = +(a_7 x + b_7 y + c_7) \leq 0$

$l_8 = -(a_8 x + b_8 y + c_8) \leq 0$

$l_9 = -(a_9 x + b_9 y + c_9) \leq 0$

$H_{cap} = \Delta = l_7 \cap l_8 \cap l_9$

$\therefore$

$H = H_{half_1} \cancel{H_{cap}} \cup H_{cap}$

Q.2

Ans

$$R(\alpha, \beta, \gamma) = R_z(\gamma) \times R_y(\beta) \times R_z(\alpha)$$

$$R_z(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$$

$$R_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

a) $$R(\overset{\alpha,\beta}{\cancel{\alpha,\beta,\alpha}}) = \begin{pmatrix} c_\alpha c_\beta - & -c_\beta s_\alpha & s_\beta \\ s_\alpha & c_\alpha & 0 \\ -c_\alpha s_\beta & s_\alpha s_\beta & c_\beta \end{pmatrix}$$

$$R(\alpha,\beta,\gamma) = \begin{pmatrix} c_\alpha c_\beta c_\gamma - s_\alpha s_\gamma & -c_\alpha s_\gamma - c_\beta c_\gamma s_\alpha & c_\gamma s_\beta \\ c_\gamma s_\alpha + c_\alpha c_\beta s_\gamma & c_\alpha c_\gamma - c_\beta s_\alpha s_\gamma & s_\beta s_\gamma \\ -c_\alpha s_\beta & s_\alpha s_\beta & c_\beta \end{pmatrix}$$

b) To prove : $R(\alpha,\beta,\gamma) = R(\alpha-\pi, -\beta, \gamma-\pi)$

As : $\sin(a+b) = \sin a \cos b + \cos a \sin b$

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$\sin(a-b) = \sin a \cos b - \cos a \sin b$

$\cos(a-b) = \cos a \cos b + \sin a \sin b$

$\sin(\pi) = 0$, $\cos(\pi) = -1$

$$\therefore R_z(\gamma-\pi) = \begin{pmatrix} \cos(\gamma-\pi) & -\sin(\gamma-\pi) & 0 \\ \sin(\gamma-\pi) & \cos(\gamma-\pi) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -c_\gamma & s_\gamma & 0 \\ -s_\gamma & -c_\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_y(-\beta) = \begin{pmatrix} \cos(-\beta) & 0 & \sin(-\beta) \\ 0 & 1 & 0 \\ -\sin(-\beta) & 0 & \cos(-\beta) \end{pmatrix} = \begin{pmatrix} c_\beta & 0 & -s_\beta \\ 0 & 1 & 0 \\ s_\beta & 0 & c_\beta \end{pmatrix}$$

$$R_z(\alpha-\pi) = \begin{pmatrix} \cos(\alpha-\pi) & -\sin(\alpha-\pi) & 0 \\ \sin(\alpha-\pi) & \cos(\alpha-\pi) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -c_\alpha & s_\alpha & 0 \\ -s_\alpha & -c_\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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$$R(\alpha-\pi, -\beta) = \begin{pmatrix} -c_\alpha c_\beta & c_\beta s_\alpha & -s_\beta \\ -s_\alpha & -c_\alpha & 0 \\ -c_\alpha s_\beta & s_\alpha s_\beta & c_\beta \end{pmatrix}$$

$$R(\alpha-\pi, -\beta, \gamma-\pi) = \begin{pmatrix} c_\alpha c_\beta c_\gamma - s_\alpha s_\beta & -c_\alpha s_\gamma - c_\beta c_\gamma s_\alpha & c_\gamma s_\beta \\ c_\gamma s_\alpha + c_\alpha c_\beta s_\gamma & c_\alpha c_\gamma - c_\beta s_\alpha s_\gamma & s_\beta s_\gamma \\ -c_\alpha s_\beta & s_\alpha s_\beta & c_\beta \end{pmatrix}$$

Thus, $R(\alpha, \beta, \gamma) = R(+\alpha-\pi, -\beta, \gamma-\pi)$

c)

Ans "Since it is specifically mentioned R' & not R as given in the question $R' \neq R$"

Thus $R' = R_z(\alpha) R_y(\beta) R_x(\gamma)$

$$R'(\alpha, \beta, \gamma) = \begin{pmatrix} c_\alpha c_\beta & c_\alpha s_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta c_\gamma + s_\alpha s_\gamma \\ s_\alpha c_\beta & s_\alpha s_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta c_\gamma - c_\alpha s_\gamma \\ -s_\beta & c_\beta s_\gamma & c_\beta c_\gamma \end{pmatrix} \quad \text{-} (I)$$

$$= \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \quad \text{-} (II)$$

Since $r_{31} = -s_\beta \Rightarrow \beta = s^{-1}(-r_{31})$

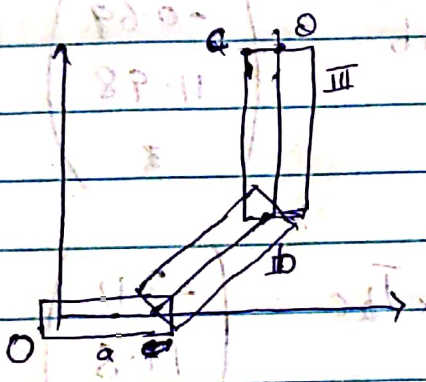
Additionally Upon Comparison of (I) & (II)

$$\alpha = \tan^{-1}\left(\frac{r_{31}}{r_{11}}\right) \quad \& \quad \gamma = \tan^{-1}\left(\frac{r_{32}}{r_{33}}\right)$$

Furthermore, $\tan 2^{-1}$ should be used for quadrant inverse analysis Since, there was ambiguity if $R' = R$, this θ is resolved later

0.3

Ans



T_{ab} = Transformation of 'b' in frame of 'a'

$$T_{oa} = \begin{pmatrix} \cos \pi/4 & -\sin \pi/4 & \cancel{0}4 \\ \sin \pi/4 & \cos \pi/4 & \cancel{0}0 \\ 0 & 0 & \cancel{1}1 \\ \cancel{0} & \cancel{0} & \cancel{0}\ 1 \end{pmatrix}_{3\times 3} \times \begin{pmatrix} 1 \\ 1 \\ \cancel{0}1 \end{pmatrix}^{T}_{3\times 1} = \begin{pmatrix} 0 \\ 1.4142 \\ 1 \end{pmatrix}_{3\times 1}$$

$$T_{ob} = T_{oa} \times \underbrace{\begin{pmatrix} \cos \pi/2 & -\sin \pi/2 & 9 \\ \sin \pi/2 & \cos \pi/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{T_{ab}} = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

Based on the forward kinematics coded, for Q.7,

$\Rightarrow$ Point A = $\begin{pmatrix} 2.8 \\ 2.8 \\ z \end{pmatrix}$

Point B = $T_{oa} \times T_{ab}$ = $\begin{pmatrix} -0.69 \\ 11.98 \\ z \end{pmatrix}$

Point C = $T_{oa} \times T_{ab} \times T_{bc}$ = $\begin{pmatrix} -3.17 \\ 19.8 \\ z \end{pmatrix}$

Calculated based off code in Q.7, ~~an~~ logical analytical solution is discussed further

Q.4

b) ~~Answer)~~ The analysis ~~use~~ done for this part using Matlab and using Newton Raphson convergence is implemented in Q.7, further reasoning is discussed next.

Q.2 c) Since, there was a confusion if R' is generic or same as part (a) both solutions methods are solved

Ans
$$R(\alpha,\beta,\gamma) = \begin{pmatrix} c_\alpha c_\beta c_\gamma - s_\alpha s_\gamma & -c_\alpha s_\gamma - c_\beta c_\gamma s_\alpha & c_\gamma s_\beta \\ c_\gamma s_\alpha + c_\alpha c_\beta s_\gamma & c_\alpha c_\gamma - c_\beta s_\alpha s_\gamma & s_\beta s_\gamma \\ -c_\alpha s_\beta & s_\alpha s_\beta & c_\beta \end{pmatrix}$$

~~If~~

Since, $r_{31} = \cos(\beta) \Rightarrow \beta = \cos^{-1}(r_{31})$

Additionally upon comparison, ~~cot~~ ~~tan^{-1}(~~

$$\frac{r_{32}}{r_{31}} = \frac{s_\alpha s_\beta}{-c_\alpha c_\beta} = -\tan(\alpha)$$

$$\therefore \alpha = \tan^{-1}\left(\frac{-r_{32}}{r_{31}}\right)$$

$\therefore$ Also,
$$\frac{r_{13}}{r_{23}} = \frac{c_\gamma s_\beta}{s_\beta s_\gamma}$$

$$\gamma = \tan^{-1}\left(\frac{r_{23}}{r_{13}}\right)$$

Q.3 (a)

Ans
$$\text{Point } A = \begin{pmatrix} \cos\pi/4 & -\sin\pi/4 & 0 \\ \sin\pi/4 & \cos\pi/4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.82 \\ 2.82 \\ 1 \end{pmatrix} \text{ Coordinates}$$

$$\text{Point } B = \begin{pmatrix} \cos\pi/4 & -\sin\pi/4 & 0 \\ \sin\pi/4 & \cos\pi/4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\pi/2 & -\sin\pi/2 & 8 \\ \sin\pi/2 & \cos\pi/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 9 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -0.707 \\ 12.02 \\ 1 \end{pmatrix}$$

Point C

$$= \begin{pmatrix} \cos\pi/4 & -\sin\pi/4 & 0 \\ \sin\pi/4 & \cos\pi/4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\pi/2 & -\sin\pi/2 & 8 \\ \sin\pi/2 & \cos\pi/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(-\pi/6) & -\sin(-\pi/6) & 8 \\ \sin(-\pi/6) & \cos(\pi/6) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 9 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3.287 \\ 19.739 \\ 1 \end{pmatrix}$$

b)

Ans By equations of inverse kinematics for 2 link robot)

$$\cos\theta_2 = \frac{1}{2a_1 a_2}\left((x^2+y^2) - (a_1^2+a_2^2)\right)$$

$$\sin\theta_2 = \pm\sqrt{1-\cos^2\theta_2}$$

$$\cos\theta_1 = \frac{1}{x^2+y^2}\left(x(a_1+a_2\cos\theta_2) \pm y a_2\sqrt{1-\cos^2\theta_2}\right)$$

$$\sin\theta_1 = \frac{1}{x^2+y^2}\left(y(a_1+a_2\cos\theta_2) \mp x a_2\sqrt{1-\cos^2\theta_2}\right)$$

Q.3 b)

Ans According to Newton Raphson Method to solve a closed form equation $g(\theta)=0$

Consider, $g(\theta) = g(\theta^0) + \frac{\partial g}{\partial \theta}(\theta^0)(\theta-\theta^0) + HOT$

$$\Rightarrow \theta = \theta^0 - \left(\frac{\partial g}{\partial \theta}(\theta^0)\right)^{-1} g(\theta^0)$$

i.e, $$\theta^{k+1} = \theta^k - \left(\frac{\partial g}{\partial \theta}(\theta^k)\right)^{-1} g(\theta^k)$$

The above forms the iteration criteria until convergence less than some ϵ is observed

In our 3 link case:

$$\frac{\partial g}{\partial \theta}(\theta^k) = \text{Jacobian} = \begin{bmatrix} \frac{\partial g_1}{\partial \theta_1} & \frac{\partial g_1}{\partial \theta_2} & \frac{\partial g_1}{\partial \theta_3} \\ \frac{\partial g_2}{\partial \theta_1} & \frac{\partial g_2}{\partial \theta_2} & \frac{\partial g_2}{\partial \theta_3} \end{bmatrix}_{3\times 2}$$

From forward kinematics,

consider if x_d is the desired eef coordinates

then $g(\theta_d) = x_d - f(\theta_d) = 0$

Closed form analysis

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Applying taylor series

$$J(\theta^0)\,\Delta\theta = x_d - f(\theta^0)$$

i.e, $\Delta\theta = J^{-1}(\theta^0)\,(x_d - f(\theta^0))$

This is because forward kinematics is linear in θ & thus HOT go to zero

usually J^+ i.e pseudo inverse is needed

i.e, $J^+ = J^T (J J^T)^{-1}$

i.e, update steps are

$$\Delta\theta = J^+(\theta^0)\,(x_d - f(\theta^0))$$

The same exact steps are implemented with additional constraints, choosing an initialization closer since, this would be hand calculated

let $\theta^0 = [\,62^\circ,\ 72^\circ,\ 142^\circ\,]$

& link-lengths = [8, 8, 9]

Only 1 ~~step~~ iteration of process / formulaic implementation is showed here, details are analyzed in code.

From

1. Forward - Kinematic Analysis ~~done~~ similar to

Point 2

$$\begin{pmatrix} \cos 62^\circ & -\sin 62^\circ & 0 \\ \sin 62^\circ & \cos 62^\circ & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos 72^\circ & -\sin 72^\circ & 8 \\ \sin 72^\circ & \cos 72^\circ & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos 142^\circ & -\sin 142^\circ & 8 \\ \sin 142^\circ & \cos 142^\circ & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 9 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0.104 & 0.994 & -1.801 \\ -0.994 & 0.104 & 12.818 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 9 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} -0.86 \\ 3.86 \\ 1 \end{pmatrix}$$

~~Initially~~ = Initiate Jacobian to $\cancel{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \end{bmatrix}}$ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Jacobian now becomes $\frac{\partial}{\partial\theta}\sin\theta = \cos\theta$, $\frac{\partial \cos\theta}{\partial\theta} = -\sin\theta$

$$\begin{bmatrix} \; \\ \; \end{bmatrix}$$

x_i ~~J[0,0] = J[0][0]~~ $- 8 \times \sin\left(\sum_{i=1}^{3} \theta_i\right)$

~~By the following~~

P.T.O

~~For joint 1~~

~~$x_1 = x_1$~~ Jacobian are calculated as per following updates

~~$\sum x_i = \sum x_i - \delta$~~

~~$x_2 =$~~ ~~$\theta\delta$~~ δx ~~J~~

~~$x_2 =$~~

$$J[0, joint_i] -= link\_length_i \times \sin\left(\sum_{j=0}^{n-1} \theta_i\right)$$

$$J[1, joint_i] += link\_length_i \times \sin\left(\sum_{j=0}^{n-1} \theta_i\right)$$

Upon implementing this in ~~matlab~~ python

$$J = \begin{bmatrix} -7.219 & -7.219 & 0.181 \\ 20.03 & 12.03 & 8.97 \end{bmatrix}$$

Thus θ_i update steps becomes

$$\theta \quad J^{+} = \begin{pmatrix} 0.0063 & 0.0343 \\ -0.1404 & -0.0323 \\ 0.1739 & 0.0781 \end{pmatrix}$$

$$\therefore \theta = \theta_i + J^{+}(\theta_i) \cdot epsilon$$

gives $J^{+}\theta_i(\epsilon_i) = \begin{bmatrix} -0.05 \\ 2.53 \\ -3.06 \end{bmatrix}$

$x_d - \theta^i$

thus $\Theta = \Theta_i + J^{+}(\Theta_i) \times \epsilon$

$$\Theta_i = \begin{bmatrix} 64.7^\circ \\ 77.2^\circ \\ 151.\cancel{8}^\circ \end{bmatrix}$$

From Forward Kinematic analysis as before

$$\begin{pmatrix} x_{new} \\ y_{new} \end{pmatrix} = \begin{pmatrix} 0.385 & 0.922 & -2.859 \\ -0.922 & 0.385 & 12.19 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.63 \\ 3.89 \\ 1 \end{pmatrix}$$

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Q.4

Ans

a) As the two tracks are independent, and each train can be oriented along R^1

$$\therefore \quad Cspace = \mathbb{R}^1 \times \mathbb{R}^1$$

b) As the space craft can translate and yaw in a 2D plane

$$C\ Space = \mathbb{R}^2 \times S^1$$

c) As both the robots are independent and can rotate and translate.

$$CSpace : SE(2) \times SE(2)$$

d) There are two cases, here

if ~~they are~~ both the robots can rotate in place (

$$C\text{-}Space = S^1 \times S^1 \times SE(2)$$

If the robots are rigidly connected, and not dformable link is attached in between

$$C\text{-}Space = \times \mathbb{R}^2 \times S^1$$

e) $\mathbb{R}^3 \times SO(2)$, due to the redundant degree of freedom minimum params that define the configuration space

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f)
Ans $SE(3) \times T^3$ as the spacecraft can translate & rotate in 3D & T^3 for 3 link robot arm.

g)
Ans Since each ~~rotate~~ joint is a revolute joint, and there are 7 joints

$$\therefore \text{C-Space} = T^7$$

for ~~ea~~ 'n'-revolute joint $= T^n$,
an assumption is made here that the joint can rotate complete $360°$, if it cannot be projective C-spaces will need to be considered.

Q.5

Ans Attached the code at the end.

Q.6

Ans Subset $X \subset \mathbb{R}^n$ is called convex iff for any pair of points in X, all points along the line segment that connect them are contained in X i.e,

$$(\lambda x_1 + (1-\lambda)x_2) \in X \quad \forall x_1, x_2 \in X, \ \forall \lambda \in [0,1]$$

If Workspace 'W' has all convex obstacles

~~If~~ i.e, let ~~A~~, ~~B~~ be sets $\subset \mathbb{R}^n$, ~~pr~~ to prove

if A & B are convex sets, then $C = A+B$ is also convex

consider, $m, n \in C$, let $a, b \in A$ & $c, d \in B$ such that

$$m = a+c \ \& \ n = b+d$$

$$\lambda m + (1-\lambda) n = \lambda(a+c) + (1-\lambda)(b+d)$$
$$= (\lambda a + (1-\lambda) b) + (\lambda c + (1-\lambda) d)$$
$$\in A+B = C$$

$$\forall \lambda \in [0,1]$$

$\therefore$ For general sets A, B, the sum $\text{convex}(A) + \text{convex}(B)$ is convex, since it contains $A+B$

i.e, $\lambda(A+B) + (1-\lambda)(A+B) = (\lambda A + (1-\lambda) A) + (\lambda B + (1-\lambda) B)$

$$= A+B$$

Additionally, let us consider a pt in the C-space of convex hull of (A+B). Therefore it is a convex combination of some points of A+B i.e,

$$= \Sigma \lambda_k (a_k + b_k).$$
$$= \Sigma \lambda_k a_k + \Sigma \lambda_k b_k \in \text{convex}(A) + \text{convex}(B)$$

Thus the C-space obstacles are also convex for a convex robot with translational motion in W.

Q7

Ans Attached Code

Q.8

Ans Attached Code

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