Problem Set 8

November 26, 2015

28.1-2

Find an LU decomposition of the matrix

$$\begin{pmatrix} 4 & -5 & 6 \\ 8 & -6 & 7 \\ 12 & -7 & 12 \end{pmatrix}$$

Solution :

A = $\begin{pmatrix} 4 & -5 & 6 \\ 8 & -6 & 7 \\ 12 & -7 & 12 \end{pmatrix}$

Step 1 : Pivot = $a_{11} = 4$

$$\begin{pmatrix} \mathbf{4} & -5 & 6 \\ 2 & 4 & -5 \\ 3 & 8 & -6 \end{pmatrix}$$

Step 2 : Pivot = $a_{22} = 4$

$$\begin{pmatrix} \mathbf{4} & -5 & 6 \\ 2 & 4 & -5 \\ 3 & 2 & 4 \end{pmatrix}$$

Therefore,

$$\underset{\boldsymbol{A}}{\begin{pmatrix} 4 & -5 & 6 \\ 8 & -6 & 7 \\ 12 & -7 & 12 \end{pmatrix}} = \underset{\mathbf{L}}{\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}} \underset{\mathbf{U}}{\begin{pmatrix} 4 & -5 & 6 \\ 0 & 4 & -5 \\ 0 & 0 & 4 \end{pmatrix}}$$

28.1-3

Solve the equation

$$\begin{pmatrix} 1 & 5 & 4 \\ 2 & 0 & 3 \\ 5 & 8 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 12 \\ 9 \\ 5 \end{pmatrix}$$

Solution :

$$\boxed{\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}} \begin{pmatrix} 1 & 5 & 4 \\ 2 & 0 & 3 \\ \mathbf{5} & 8 & 2 \end{pmatrix}$$

Since 5 in column 1 and Row 3 is the largest elemement, we interchange Row 1 with Row 3 and update the permutation matrix.

$$\boxed{\begin{matrix} 3 \\ 2 \\ 1 \end{matrix}} \begin{pmatrix} \mathbf{5} & 8 & 2 \\ 2 & 0 & 3 \\ 1 & 5 & 4 \end{pmatrix}$$

Now, with $Pivot = a_{11} = 5$, updating the matrix using Schur complement.

$$\boxed{\begin{matrix} 3 \\ 2 \\ 1 \end{matrix}} \begin{pmatrix} \mathbf{5} & 8 & 2 \\ 0.4 & -3.2 & 2.2 \\ 0.2 & \mathbf{3.4} & 3.6 \end{pmatrix}$$

Now, 3.4 is the largest absolute value in column 2, we swap Row 2 and Row 1 and update the permutation matrix.

$$\boxed{\begin{matrix} 3 \\ 1 \\ 2 \end{matrix}} \begin{pmatrix} 5 & 8 & 2 \\ 0.2 & \mathbf{3.4} & 3.6 \\ 0.4 & -3.2 & 2.2 \end{pmatrix}$$

Now, with $Pivot = a_{22} = 3.4$, updating the matrix using Schur complement.

$$\boxed{\begin{matrix} 3 \\ 1 \\ 2 \end{matrix}} \begin{pmatrix} 5 & 8 & 2 \\ 0.2 & \mathbf{3.4} & 3.6 \\ 0.4 & -0.94 & 5.58 \end{pmatrix}$$

Now, we have the LUP Decompostion as follows :

$$\underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{\mathbf{P}} \underbrace{\begin{pmatrix} 1 & 5 & 4 \\ 2 & 0 & 3 \\ 5 & 8 & 2 \end{pmatrix}}_{\mathbf{A}} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0.2 & 1 & 0 \\ 0.4 & -0.94 & 1 \end{pmatrix}}_{\mathbf{L}} \underbrace{\begin{pmatrix} 5 & 8 & 2 \\ 0 & 3.4 & 3.6 \\ 0 & 0 & 5.58 \end{pmatrix}}_{\mathbf{U}}$$

Using forward substitution, we solve $Ly = Pb$ for y :

$$\begin{pmatrix} 1 & 0 & 0 \\ 0.2 & 1 & 0 \\ 0.4 & -0.94 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 12 \\ 9 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0.2 & 1 & 0 \\ 0.4 & -0.94 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 12 \\ 9 \end{pmatrix}$$

On solving, we have

$$y = \begin{pmatrix} 5 \\ 11 \\ 17.34 \end{pmatrix}$$

Using back substitution, we solve $Ux = y$ for x :

$$\begin{pmatrix} 5 & 8 & 2 \\ 0 & 3.4 & 3.6 \\ 0 & 0 & 5.58 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \\ 17.34 \end{pmatrix}$$

Therefore, solution for x is as follows:

$$x = \begin{pmatrix} -0.15 \\ -0.05 \\ 3.10 \end{pmatrix}$$

$x_1 = -0.15$, $x_2 = -0.05$ and $x_3 = 3.10$.

28.1-7

In LU-DECOMPOSITION, is it necessary to perform the outermost **for** loop iteration when $k = n$? How about in LUP-Decomposition?

Solution :

For LU decomposition, it is indeed necessary. If we didn't run the last run of the outermost for loop, u_{nn} would be left its initial value of zero instead of being set equal to a_{nn}. This can clearly produce incorrect results, because the LU decomposition of any non-singular matrix must have both L and U having positive determinant. However, if $u_{nn} = 0$, the determinant of U will be zero.

For LUP-decomposition, the iteration of the outermost for loop that occurs with $k = n$ will not change the final answer. Since π would have to be a permutation on a single element, it cannot be non-trivial. and the for loop on line 16 will not run at all.

28.3-6

Find the function of the form

$F(x) = c_1 + c_2 x \lg x + c_3 e^x$

that is the best least-squares fit to the data points

(1,1),(2,1),(3,3),(4,8).

Solution :

Let, the matrix of basic function values be A

$$A = \begin{pmatrix} 1 & x_1 \lg x_1 & e^{x_1} \\ 1 & x_2 \lg x_2 & e^{x_2} \\ 1 & x_3 \lg x_3 & e^{x_3} \\ 1 & x_4 \lg x_4 & e^{x_4} \end{pmatrix} = \begin{pmatrix} 1 & 0 & e^1 \\ 1 & 2\lg 2 & e^2 \\ 1 & 3\lg 3 & e^3 \\ 1 & 4\lg 4 & e^4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 2.718 \\ 1 & 2 & 7.389 \\ 1 & 4.752 & 20.085 \\ 1 & 8 & 54.598 \end{pmatrix}$$

The Pseudoinverse of the matrix is A^+, where

$A^+ = ((A^T A)^{-1} A^T)$

$$A^T = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 4.752 & 8 \\ 2.718 & 7.389 & 20.085 & 54.598 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 4 & 14.752 & 84.790 \\ 14.752 & 90.582 & 547.006 \\ 84.790 & 547.006 & 3446.334 \end{pmatrix}$$

$$(A^T A)^{-1} = \begin{pmatrix} 0.681 & -0.234 & 0.020 \\ -0.234 & 0.347 & -0.049 \\ 0.020 & -0.049 & 0.008 \end{pmatrix}$$

$$((A^T A)^{-1} A^T) = A^+ = \begin{pmatrix} 0.736 & 0.363 & -0.022 & -0.078 \\ -0.368 & 0.095 & 0.424 & -0.150 \\ 0.041 & -0.022 & -0.061 & 0.042 \end{pmatrix}$$

$$c = A^{+}y = \begin{pmatrix} 0.409 \\ -0.201 \\ 0.172 \end{pmatrix}$$

$$Therefore, F(x) = 0.409 - 0.201x \lg x + 0.172e^{x}$$