Problem Set 9

November 27, 2015

29.1-1

If we express the linear program in (29.24) -(29.28) in the compact notation of (29.19)-(29.21), what are n, m, A, b, and c ?

Solution :

$$c = \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & -2 & 2 \end{pmatrix}$$

$$b = \begin{pmatrix} 7 \\ -7 \\ 4 \end{pmatrix}$$

with $m = n = 3$

29.1-5

Convert the following linear program into slack form:

maximize $\quad 2x_1 \quad - 6x_3$

subject to

$$\begin{array}{rcrcrl} x_1 & + & x_2 & - & x_3 & \leq 7 \\ 3x_1 & - & x_2 & & & \geq 8 \\ -x_1 & + & 2x_2 & + & 2x_3 & \geq 0 \\ & & x_1, x_2, x_3 & & & \geq 0 \end{array}$$

What are the basic and nonbasic variables?

Solution :

First, we will multiply the second and third inequalities by minus one to make it so that they are all $\leq$ inequalities. We will introduce the three new variables x_4, x_5, x_6, and perform the usual procedure for rewriting in slack form

$$\begin{aligned} x_4 &= 7 - x_1 - x_2 + x_3 \\ x_5 &= -8 + 3x_1 - x_2 \\ x_6 &= -x_1 + 2x_2 + 2x_3 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0 \end{aligned}$$

where we are sill tring to maximize $2x_1 - 6x_3$. The basic variables are x_4, x_5, x_6 and the nonbasic variables are x_1, x_2, x_3.

29.2-2

Write out explicitly the linear program corresponding to finding the shortest path from node s to node y in Figure 24.2(a).

Solution :

The Linear program can be written as follows :

$$\begin{array}{ll} \text{maximize} & d_y \\ \text{subject to} & d_t \leq d_s + 3 \\ & d_x \leq d_t + 6 \\ & d_y \leq d_s + 5 \\ & d_y \leq d_t + 2 \\ & d_z \leq d_x + 2 \\ & d_t \leq d_y + 1 \\ & d_x \leq d_y + 4 \\ & d_z \leq d_y + 1 \\ & d_s \leq d_z + 1 \\ & d_x \leq d_z + 7 \\ & d_z = 0 \end{array}$$

29.2-6

Write a linear program that given a bipartite graph G = (V,E), solves the maximum-bipartite-matching problem.

Solution :

We can solve the maximum-bipartite-matching problem by viewing it as a network flow problem, where we append a source s and sink t, each connected to every vertex is L and R respectively by an edge with capacity 1, and we give ever edge already in the bipartite graph capacity 1. The integral maximum flows are in correspondence with maximum bipartite matchings. In this setup, the linear programming problem to solve is as follows:

$$\begin{aligned}
\text{maximize} \quad & \sum_{v \in L} f_{sv} \\
\text{subject to} \quad & f_{uv} \leq 1 \text{ for each } u, v \in \{s\} \cup L \cup R \cup \{t\} = V \\
& \sum_{v \in V} f_{vu} = \sum_{v \in V} f_{uv} \text{ for each } u \in L \cup R \\
& f_{uv} \geq 0 \text{ for each } u, v \in V
\end{aligned}$$

29.5-5

Solve the following linear program using SIMPLEX:

$$\begin{array}{lrcrl}
\text{maximize} & x_1 & + & 3x_2 & \\
\text{subject to} & & & & \\
& x_1 & - & x_2 & \leq 8 \\
& -x_1 & - & x_2 & \leq -3 \\
& -x_1 & + & 4x_2 & \leq 2 \\
& & x_1, x_2 & & \geq 0
\end{array}$$

Solution :

The initial basic solution isn't feasible, so we will write the linear program in slack form :

$$\begin{aligned}
z - x_1 - 3x_2 &= 0 \\
x_1 - x_2 + s_1 &= 8 \\
-x_1 - x_2 + s_2 &= -3 \\
-x_1 + 4x_2 + s_3 &= 2 \\
x_1, x_2 &\geq 0
\end{aligned}$$

Now, we solve the equation using SIMPLEX method creating a table :

Iteration	Basic Variables	x_1	x_2	s_1	s_2	s_3	RHS	Ratio
0	z	-1	-3	0	0	0	0	
	s_1	1	-1	1	0	0	8	-8
s_3 leaves	s_2	-1	-1	0	1	0	-3	3
x_2 enters	s_3	-1	4	0	0	1	2	1/2

Dividing key row by 4 to make key element 1 and then perform row transformations to make elements in key column as 0.

$$R_3 = R_3 + R_4$$

$$R_2 = R_2 + R_4$$

$$R_1 = R_1 + 3R_4$$

Iteration	Basic Variables	x_1	x_2	s_1	s_2	s_3	RHS	Ratio
1	z	-7/4	0	0	0	3/4	3/2	
	s_1	3/4	0	1	0	1/4	17/2	34/3
s_2 leaves	s_2	-5/4	0	0	1	1/4	-5/2	2
x_1 enters	x_2	-1/4	1	0	0	1/4	1/2	-2

Dividing key row by -5/3 to make key element 1 and then perform row transformations to make elements in key column as 0.

$$R_1 = R_1 + 7/4R_3$$

$$R_2 = R_2 - 3/4R_3$$

$$R_4 = R_4 + 1/4R_3$$

Iteration	Basic Variables	x_1	x_2	s_1	s_2	s_3	RHS	Ratio
2	z	0	0	0	-7/5	2/5	5	
	s_1	0	0	1	3/5	2/5	7	35/3
s_1 leaves	x_1	1	0	0	-4/5	-1/5	2	
s_2 enters	x_2	0	1	0	-1/5	1/5	1	

Dividing key row by 3/5 to make key element 1 and then perform row transformations to make elements in key column as 0.

$$R_1 = R_1 + 7/5R_2$$

$$R_3 = R_3 + 4/5R_2$$

$$R_4 = R_4 + 1/5R_2$$

Iteration	Basic Variables	x_1	x_2	s_1	s_2	s_3	RHS
3	z	0	0	7/3	0	4/3	64/3
	s_2	0	0	5/3	1	2/3	35/3
	x_1	1	0	4/3	0	1/3	34/3
	x_2	0	1	1/3	0	1/3	10/3

The Optimal solution for the equation is:

$x_1 = 34/3$
$x_2 = 10/3$
$z_{max} = 64/3$

29.5-7

Solve the following linear program using SIMPLEX:

$$\begin{array}{lllll} \text{maximize} & x_1 & + & 3x_2 & \\ \text{subject to} & & & & \\ & -x_1 & + & x_2 & \leq -1 \\ & -x_1 & - & x_2 & \leq -3 \\ & -x_1 & + & 4x_2 & \leq 2 \\ & & x_1, x_2 & & \geq 0 \end{array}$$

Solution :

The initial basic solution isn't feasible, so we will write the linear program in slack form :

$$z - x_1 - 3x_2 = 0$$
$$-x_1 + x_2 + s_1 = -1$$
$$-x_1 - x_2 + s_2 = -3$$
$$-x_1 + 4x_2 + s_3 = 2$$
$$x_1, x_2 \geq 0$$

Now, we solve the equation using SIMPLEX method creating a table :

Iteration	Basic Variables	x_1	x_2	s_1	s_2	s_3	RHS	Ratio
0	z	-1	-3	0	0	0	0	
	s_1	-1	1	1	0	0	-1	-1
s_3 leaves	s_2	-1	-1	0	1	0	-3	3
x_2 enters	s_3	-1	4	0	0	1	2	1/2

Dividing key row by 4 to make key element 1 and then perform row transformations to make elements in key column as 0.

$$R_3 = R_3 + R_4$$

$$R_2 = R_2 - R_4$$

$$R_1 = R_1 + 3R_4$$

Iteration	Basic Variables	x_1	x_2	s_1	s_2	s_3	RHS	Ratio
1	z	0	0	-7/3	0	4/3	5	
	x_1	1	0	-4/3	0	1/3	2	
s_1 leaves	s_2	0	0	-5/3	1	2/3	0	0
s_2 enters	x_2	0	1	-1/3	0	1/3	1	

Dividing key row by -5/3 to make key element 1 and then perform row transformations to make elements in key column as 0.

$$R_1 = R_1 + 7/4R_2$$

$$R_3 = R_3 + 5/4R_2$$

$$R_4 = R_4 + 1/4R_2$$

Iteration	Basic Variables	x_1	x_2	s_1	s_2	s_3	RHS	Ratio
2	z	0	0	-7/3	0	4/3	5	
	x_1	0	0	1	3/5	2/5	7	35/3
s_1 leaves	s_2	1	0	0	-4/5	-1/5	2	
s_2 enters	x_2	0	1	0	-1/5	1/5	1	

Dividing key row by 3/5 to make key element 1 and then perform row transformations to make elements in key column as 0.

$$R_1 = R_1 + 7/3R_3$$

$$R_2 = R_2 + 4/3R_3$$

$$R_4 = R_4 + 1/3R_3$$

Iteration	Basic Variables	x_1	x_2	s_1	s_2	s_3	RHS	Ratio
3	z	0	0	0	-7/5	2/5	5	
	x_1	1	0	0	-4/5	-1/5	2	
s_1 leaves	s_1	0	0	1	-3/5	-2/5	0	0
s_2 enters	x_2	0	1	0	-1/5	1/5	1	

After repeating the same operation on the above table it again yields a negative value in Row 1. So, the Row 1 always has a negative value.

Hence, we can say that the objective function does not have a solution with the given constraints.