Machine Vision
Lecture 3

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Course Outline

- Introduction and low-level processing
  - Physics of digital images, histogram equalization, segmentation, edge detection, linear filters.
- Model-based Vision
  - Hough transform, pattern representation, matching
- **Geometric methods**
  - **Camera model, calibration, pose estimation**
- Neural network for machine vision
  - Basics, training algorithms, and applications
- Color images and selected topics
  - Physics, perception, processing and applications

Reasons for Camera Calibration

- Need to recover 3D quantitative measures about the observed scene from 2D images.
- Model and predict the performance or accuracy of any machine vision algorithms
- Determine the camera location relative to the calibration board (or the working plane)
- Basis for other calibration; robot kinematics, hand-eye relationship, geometric calibrations.

Definition:

- The problem of determining the elements that govern the relationship or transformation between the 2D image that a camera sees and the 3D of the observed scene.
- Two kinds of parameters defining this 2D/3D relationship:
    - Intrinsic
    - Extrinsic

Intrinsic parameters

- Parameters that characterize the inherent geometric properties of the camera and optics:
  - Image center
  - Image X and Y scale factors
  - Lens principal distance (effective focus length)
  - Lens distortion coefficients

Externsic parameters

- Parameters that indicates the position and orientation of the camera with respect to the world coordinate system:
  - Translation (T_x, T_y and T_z)
  - Rotation about X, Y and Z axes.

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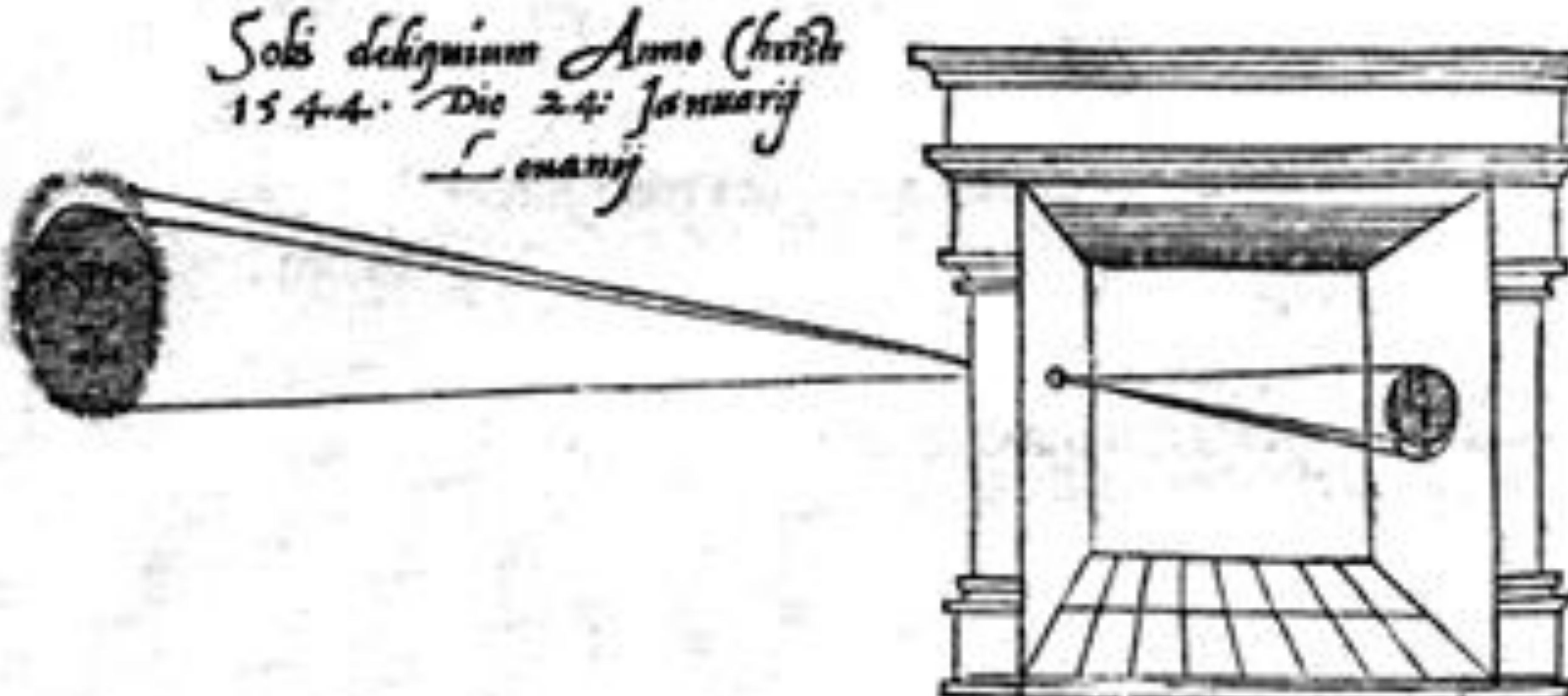

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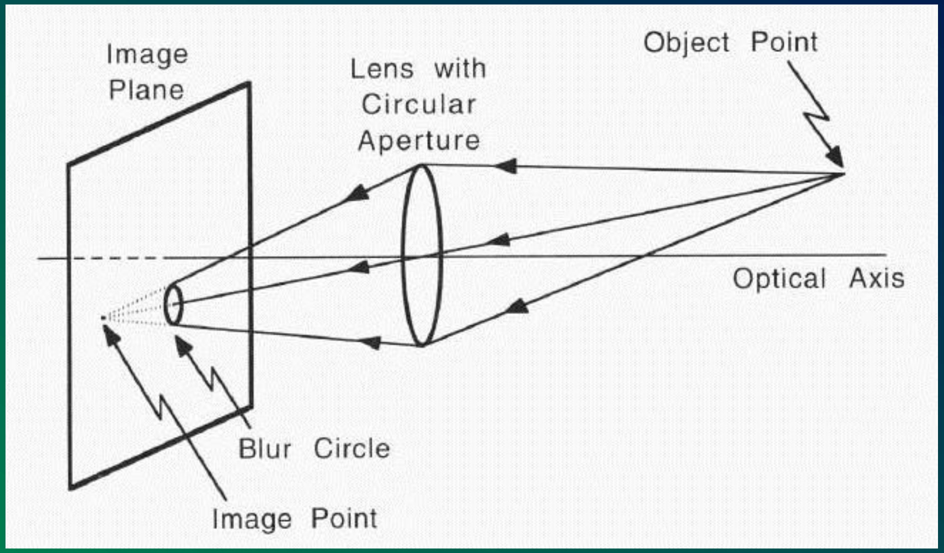
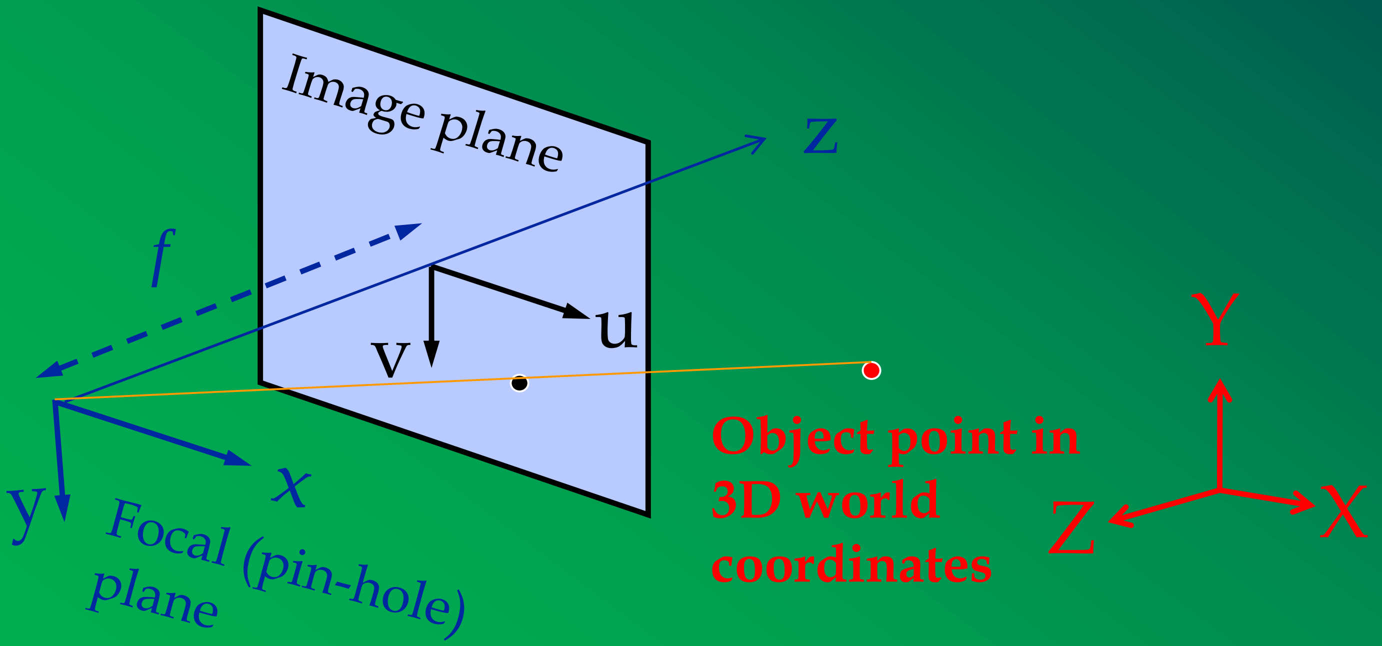
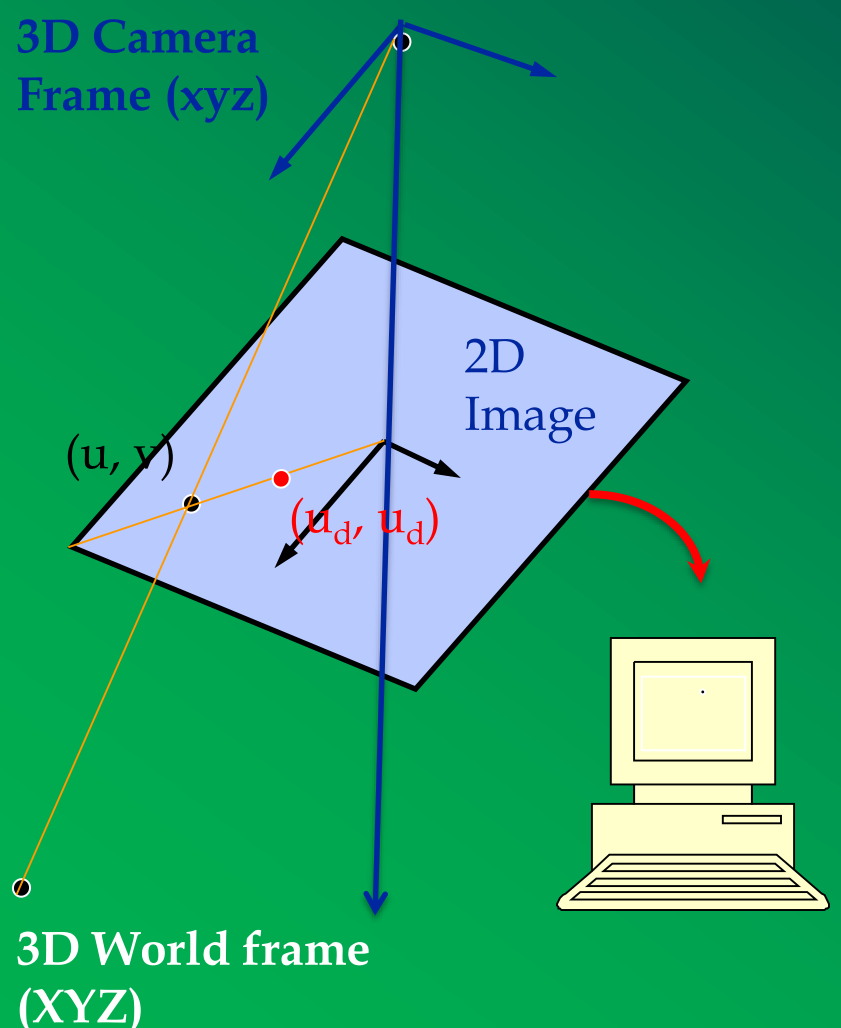
Camera Model

XYZ word coordinates
xyz camera coordinates
uv image coordinates





Tsai's Camera Model *(Radial lens distortion assumption)*



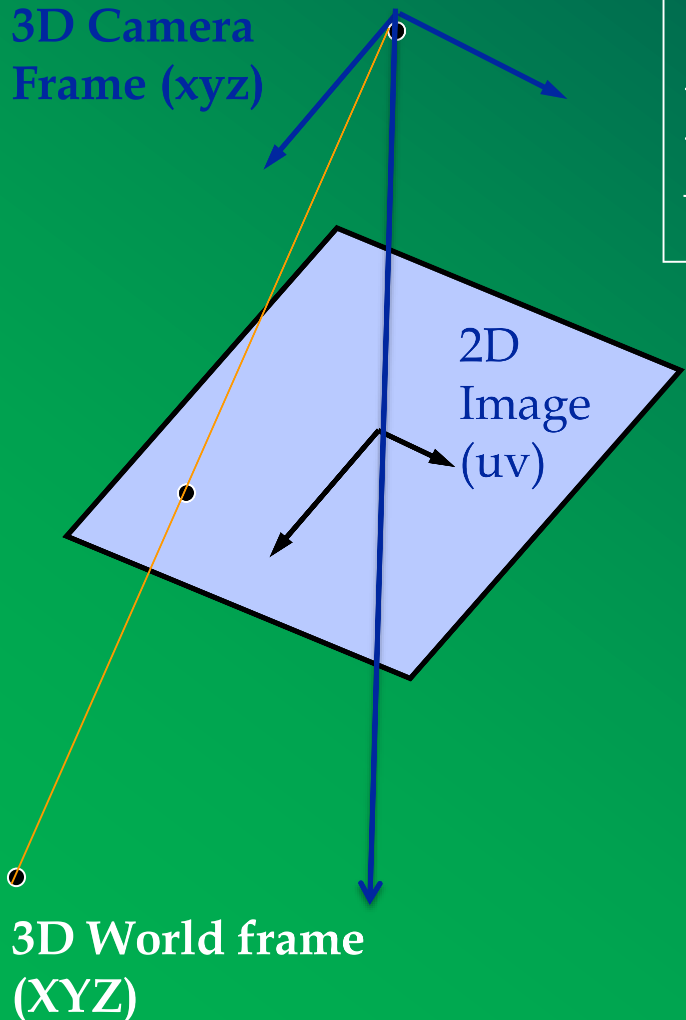
Step 1
Rigid body transformation from **world frame** to camera frame
Parameters to be calibrated: R, T

Step 2
Perspective projection with pin hole geometry
Parameters to be calibrated: *f*

Step 3
Radial lens distortion correction
Parameters to be calibrated: k

Step 4
Relation between physical and computer coordinates
Parameters to be calibrated: scaling

Tsai's Camera Model

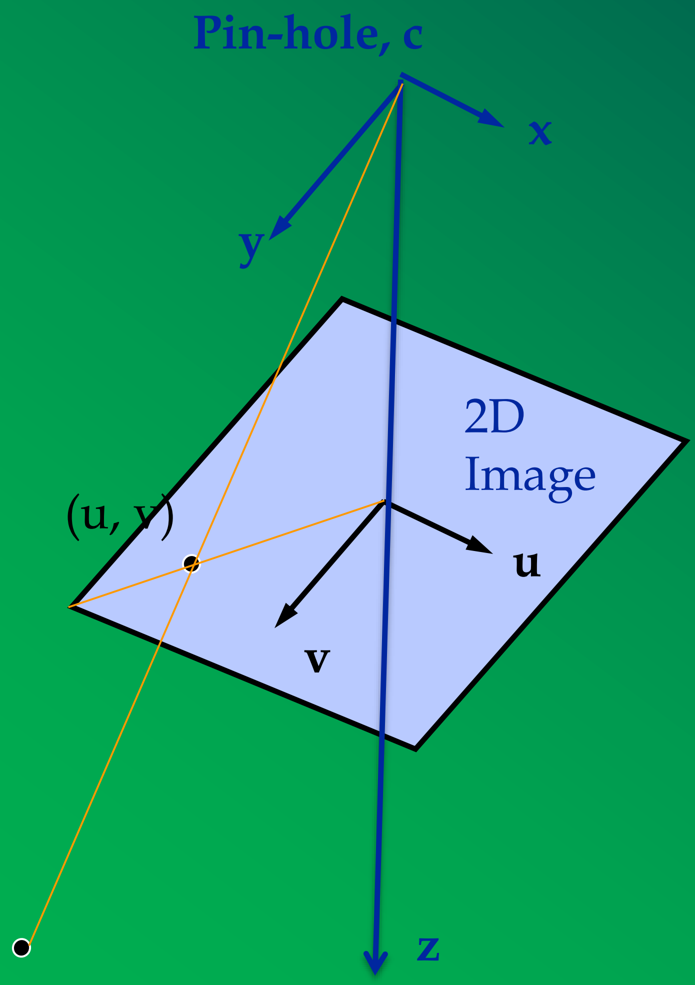


Step 1
Rigid body transformation from world frame to camera frame

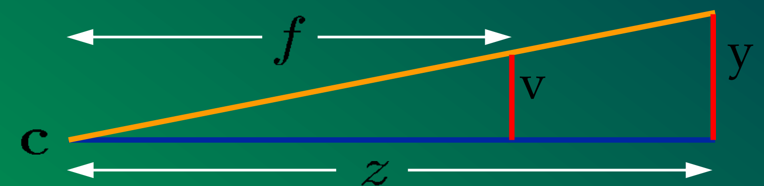
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}}_{[\mathbf{R}]} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \underset{\mathbf{T}}{\begin{pmatrix} T_x \\ T_y \\ T_z \end{pmatrix}}$$

Parameters to be calibrated:
[R], T

Tsai's Camera Model



Step 2
Perspective projection with pin- hole geometry



Similarly,

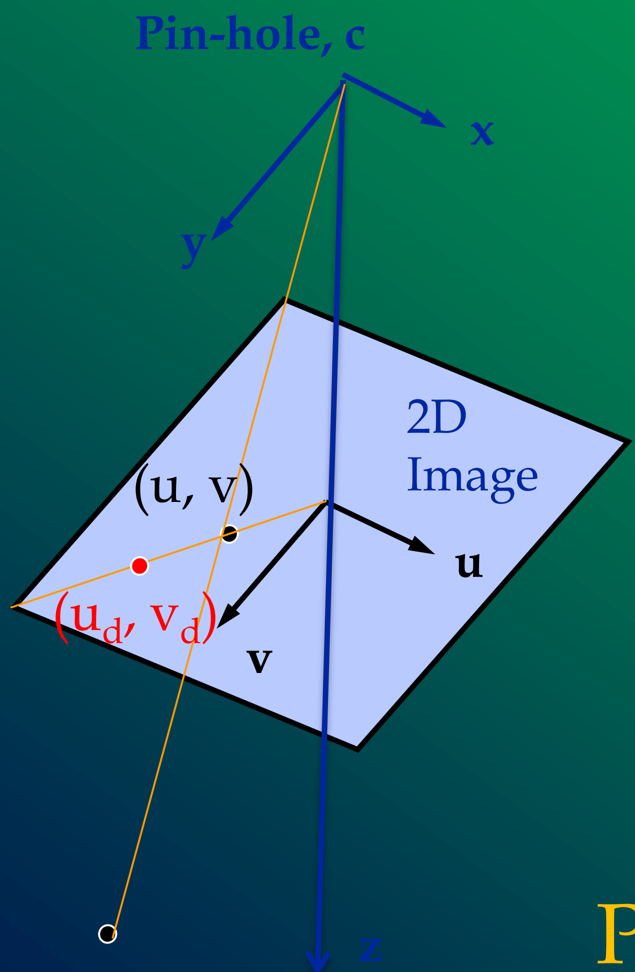
$$u = f\frac{x}{z} \qquad v = f\frac{y}{z}$$

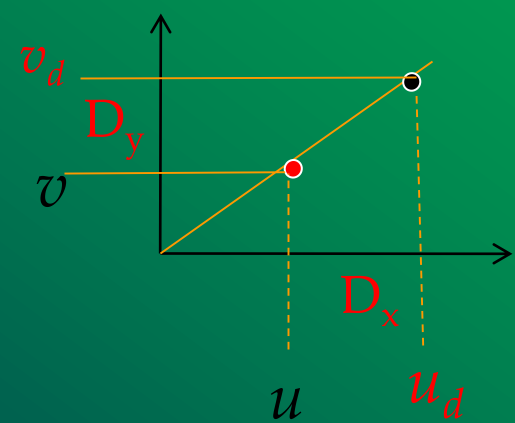
Parameter to be calibrated: *f*

Tsai's Camera Model

Step 3: Radial lens distortion correction

$$\begin{bmatrix} u_d \\ v_d \end{bmatrix} = L(r) \begin{bmatrix} u \\ v \end{bmatrix} \quad \text{where } L = 1 + k_1 r^2 + \cdots$$





$$u_d = u(1 + k_1 r_d^2)$$

$$v_d = v(1 + k_1 r_d^2)$$

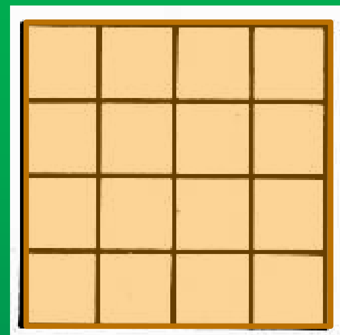
$$\text{where } r_d^2 = u_d^2 + v_d^2$$

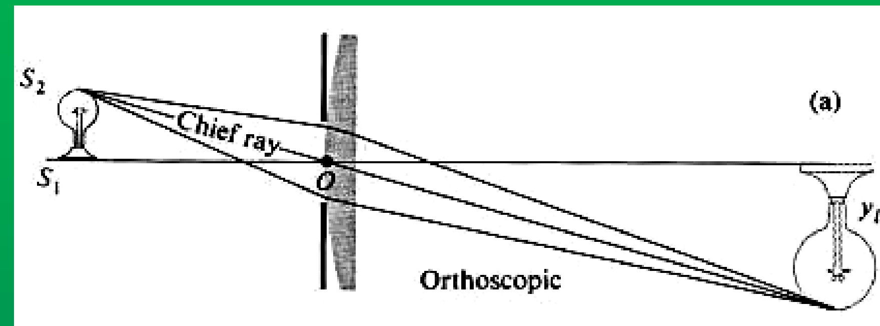
Parameter to be calibrated: k_1

Radial Distortion *(straight lines curve around the image)*

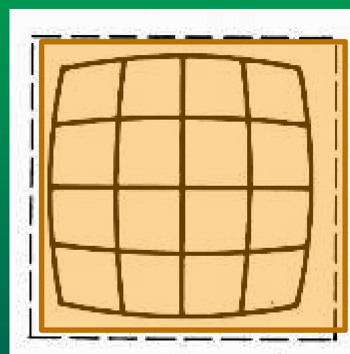
Radial Distortion *(straight lines curve around the image)*

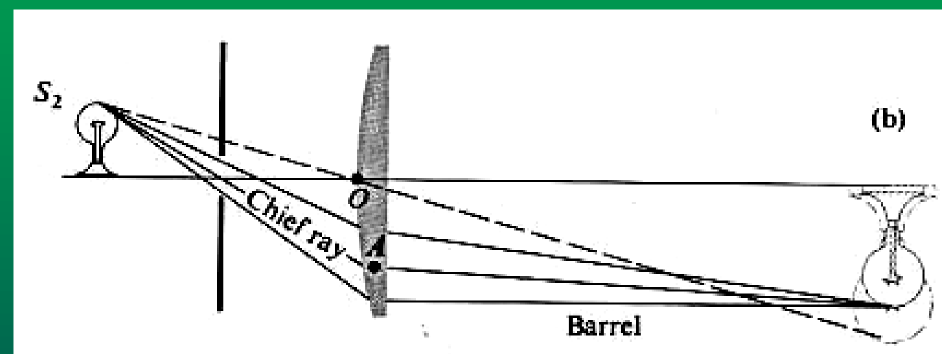
No distortion



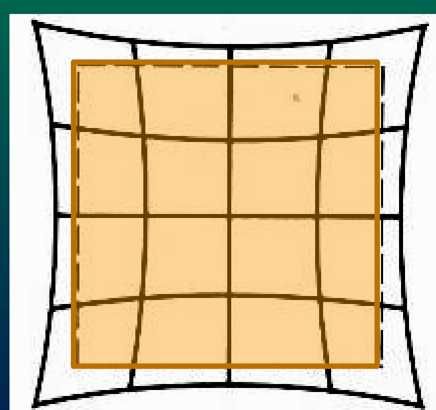


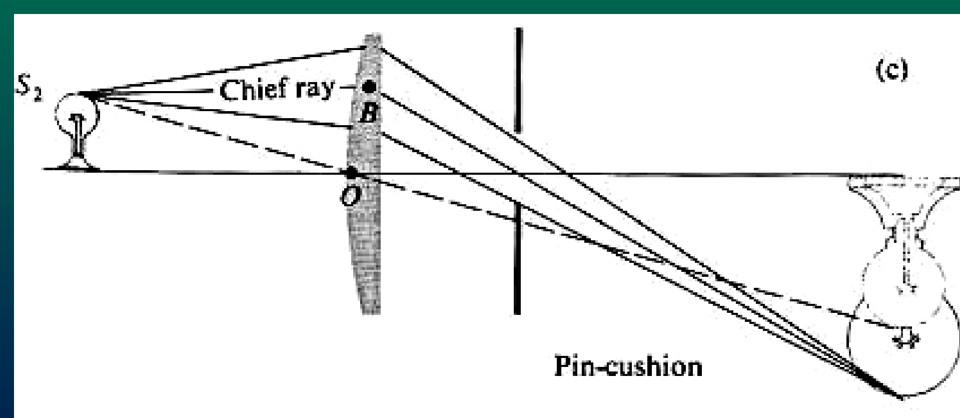
Barrel





Pin cushion





Correcting Radial Distortion (example)

Radial distortion of the image

- Caused by imperfect lenses
- Deviations are most noticeable for rays that pass through the edge of the lens

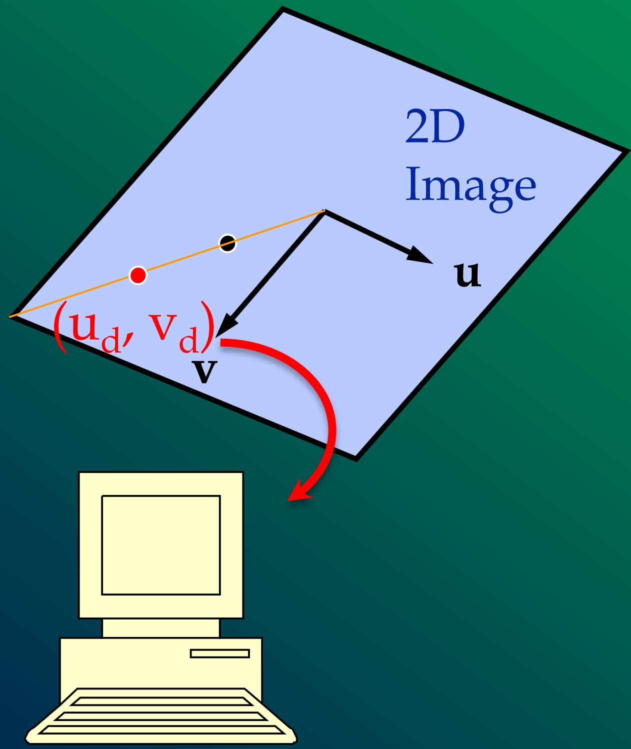
Distorted

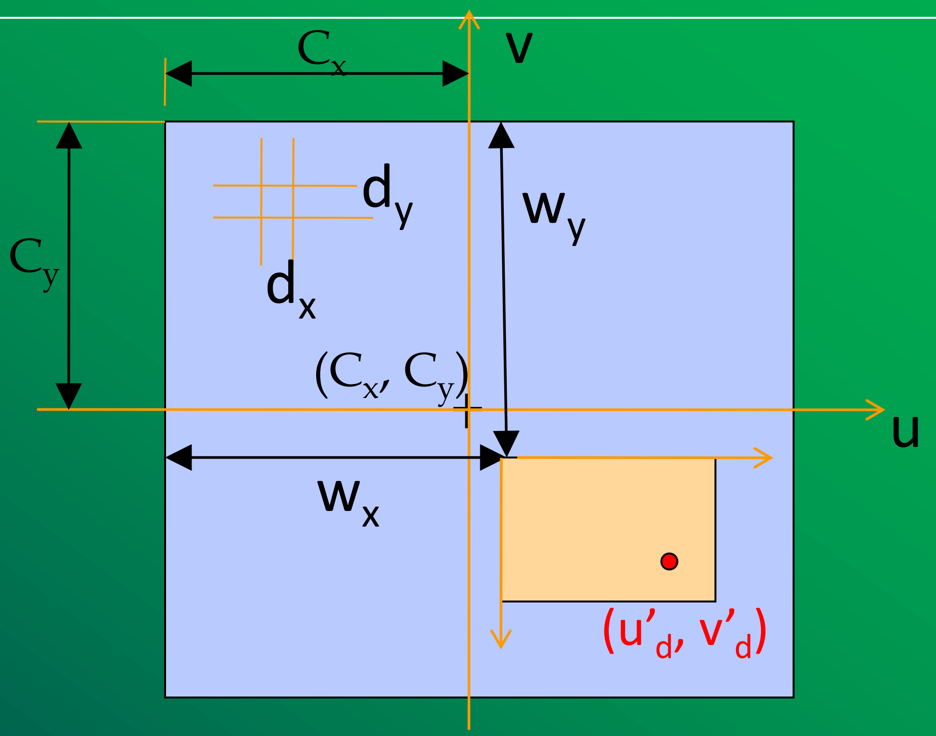
After correction

courtesy of Shawn Becker

Tsai's Camera Model

Step 4: Relation between physical and computer coordinates.





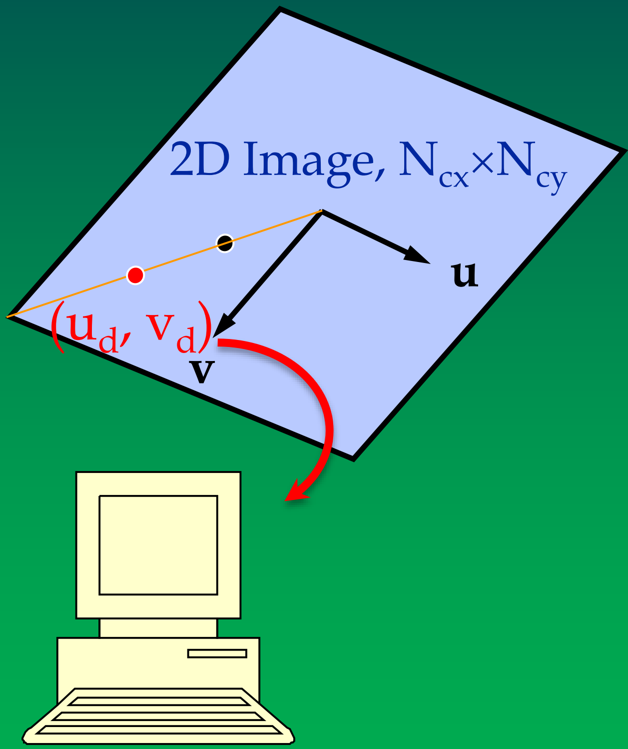
Parameter scaling

$$u_d = \left(w_x + u'_d - c_x\right) dx$$

$$v_d = \left(w_y + v'_d - c_y\right) dy$$

Tsai's Camera Model (Step 4: cont.)

If the frame grabber is used, which has $N_{fx} \times N_{fy}$ pixels spaced at dx' and dy' respectively.



$$u_d = \left(w_x + u'_d - c_x\right) dx'$$

$$v_d = \left(w_y + v'_d - c_y\right) dy'$$

where $dx' = \dfrac{N_{cx}}{N_{fx}} dx; \quad dy' = \dfrac{N_{cy}}{N_{fy}} dy$

Parameter scaling

Tsai's Camera Model (Step 4 example)

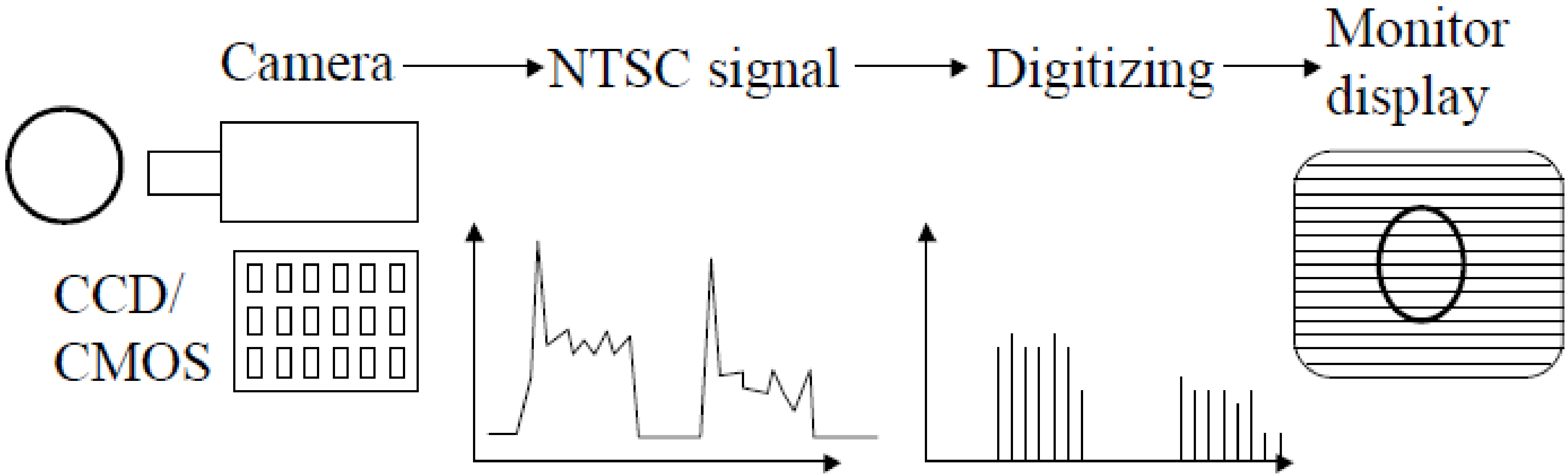
Table 1 PULNIX Camera Properties

Imager	2/3 inch progressive scanning interline transfer CCD
No. Pixels	768 (H) x 484 (V)
Cell Size	11.6 µm x 13.6 µm progressive scanning
Scanning	525 lines, 30 Hz or 60 Hz 2:1 interlace Internal/External autoswitch
Sync	HD/VD 4.0 ∀p-p impedance 4.7 kΩ VD=interlace/non-interlace, HD=15.734 kHz +/- 5%
Dataclock Output	14.31818 MHz
TV Resolution	470 (H) x 484 (V) analog 760(H) x 484 (V) digital sampling
S/N Ratio	50 dB min. (AGC=off)
Min. Illumination	10.0 lux. f=1.4 (no shutter) sensitivity 10 µV/e-
Size (WxHxL)	46 x 51 x 171.7 mm 1.81 x 2.0 x 6.766 inches
Weight	225 grams (4.3 oz)
Power Requirement	12 V DC 500 mA
Lens Mount	C Mount
Gamma	0.45 or 1.0 (0.45) std
Operating Temperature	-10° C to 50° C

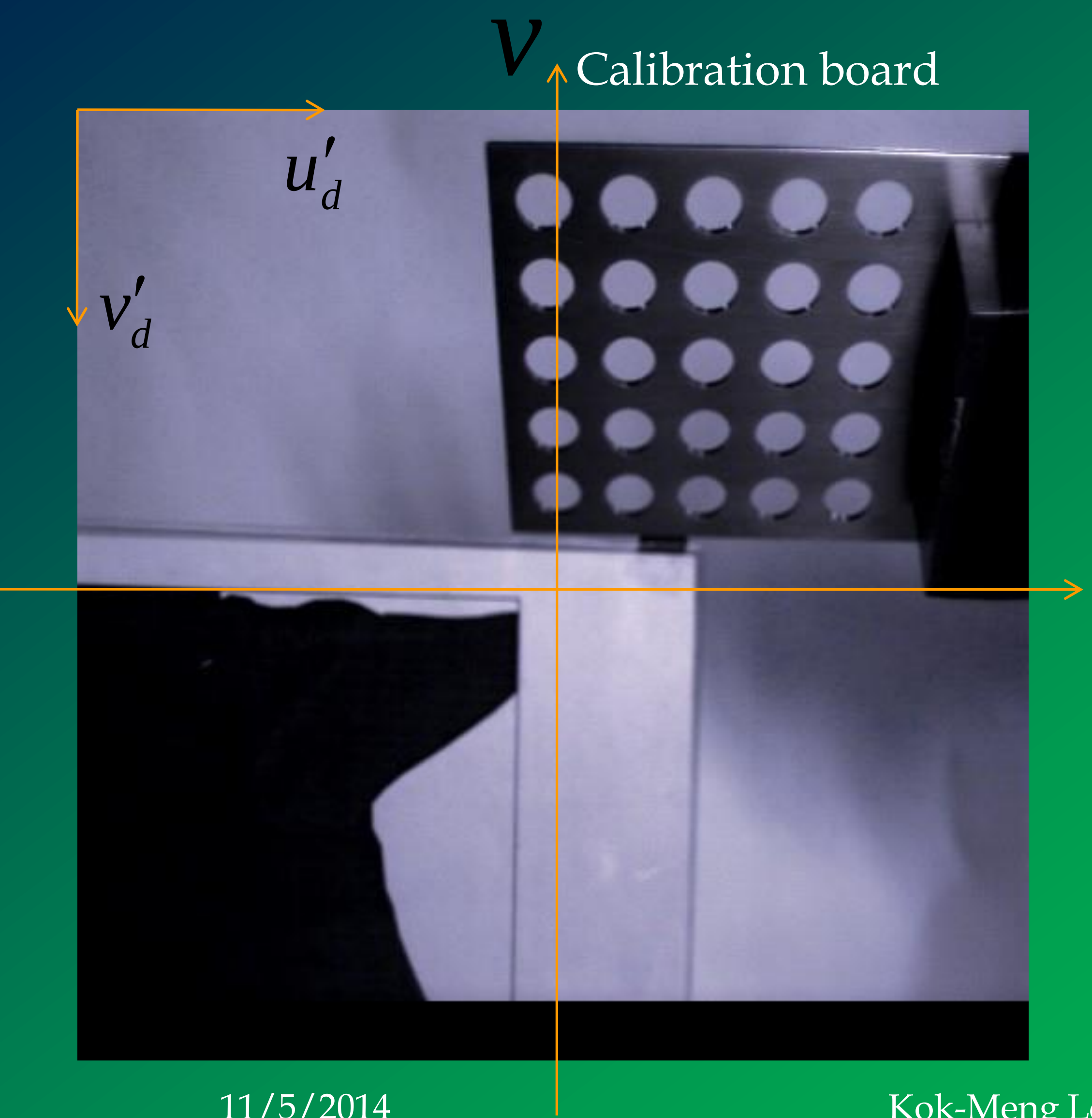
Tsai's Camera Model **(Step 4 example)**

Table 2 Camera Sensor and Variables, Definitions, and Values

Variable	Definition	Value
dx	Center to center distance between adjacent sensor elements in the x scanline	11.6 µm
dy	Center to center distance between adjacent sensor elements in the y scanline	13.6 µm
N_{cx}	Number of sensor elements in the x direction	768 pixels
N_{cy}	Number of sensor elements in y direction	484 pixels
N_{fx}	Number of pixels in a line as sampled by the computer	512 pixels
N_{fy}	Number of rows (sensor elements plus blank rows) in y direction	512 pixels
C_x	Camera center x-coordinate taken to be the center of the camera sensor	768/2 = 384 pixels
C_y	Camera center y-coordinates taken to be the center of the camera sensor	484/2 = 242 pixels
w_x	X-coordinate of top-left window element defined by user	varies
w_y	Y-coordinate of top-left window element defined by user	varies



Tsai's Camera Model
(Step 4 example)



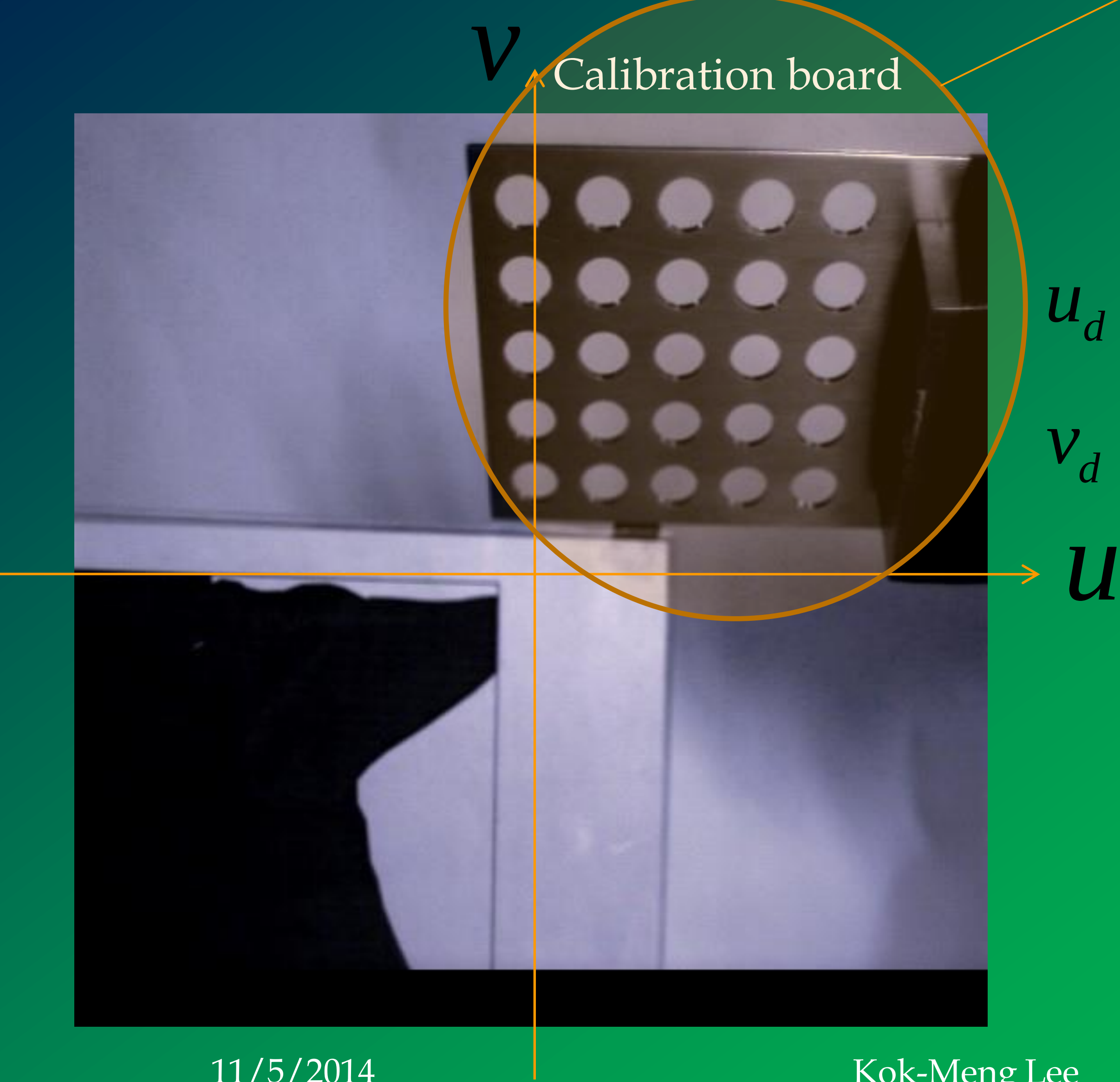
$$u_d = (u'_d - c_x)\,dx'$$

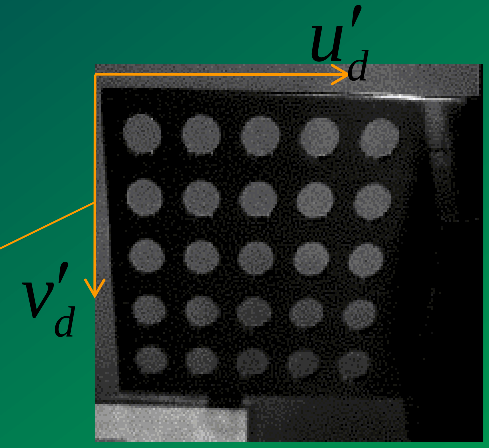
$$v_d = (v'_d - c_y)\,dy'$$

$$dx' = dx\frac{N_{cx}}{N_{fx}} = 11.6\left(\frac{768}{512}\right)\mu m$$

$$dy' = dy\frac{N_{cy}}{N_{fy}} = 13.6\left(\frac{484}{512}\right)\mu m$$

Tsai's Camera Model
(Step 4 example)





Cropped image

$$u_d = \left(w_x + u'_d - c_x\right) dx'$$

$$v_d = \left(w_y + v'_d - c_y\right) dy'$$

Two-stage approach

Stage 1: Radial Alignment Constraint

(1a) Solve for five parameters:

r_{11}/T_y, r_{21}/T_y , r_1/T_y, r_{22}/T_y, T_x/T_y

(1b) Find T_y

(1c) Solve for [R] T_x, T_y

Stage 2: Perspective Constraint

Given [R] T_x, T_y solve for f, k, T_z

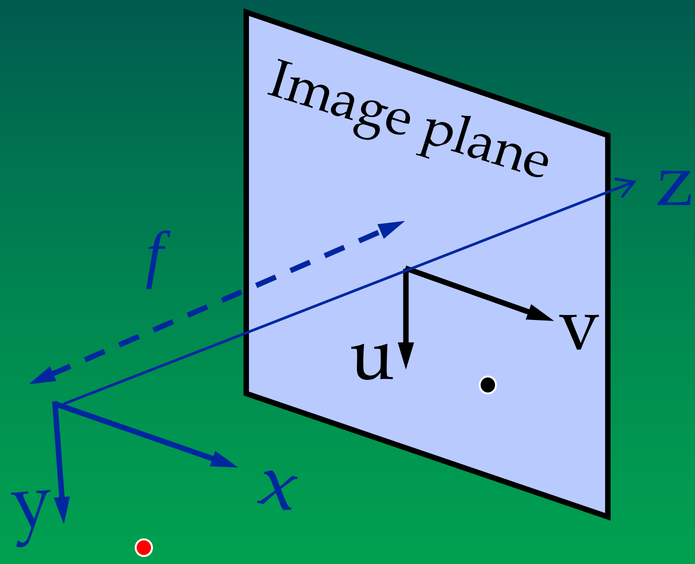
Note: Only three of nine r_{ij} are independent.

$$[R]^T=[R]^{-1}$$

Two-stage approach *(Stage 1a)*

(1a) Solve for five parameters:

r_{11}/T_y, r_{21}/T_y , r_1/T_y, r_{22}/T_y, T_x/T_y



xy plane // *uv* plane,

$$xv_d - yu_d = 0 \qquad (1)$$

From Camera Model Step 1:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \begin{pmatrix} T_x \\ T_y \\ T_z \end{pmatrix} \qquad (2)$$

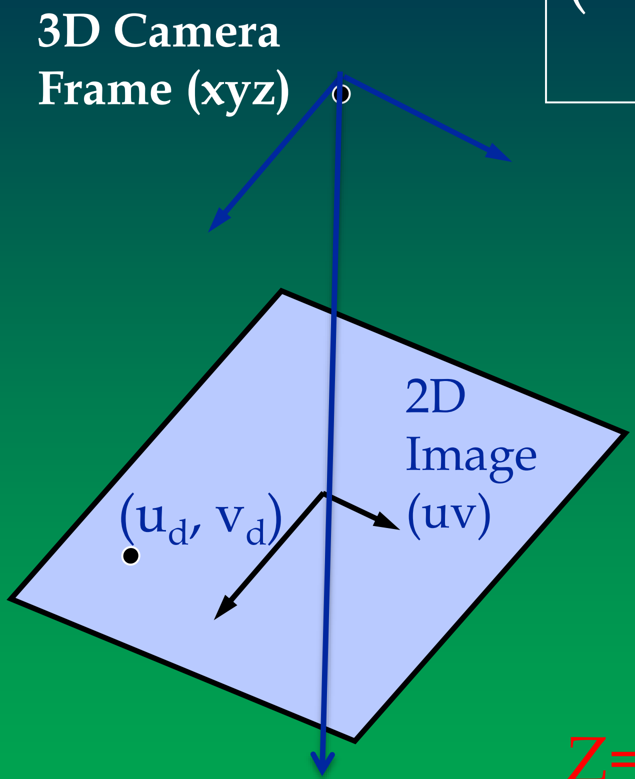
Substituting (2) into (1):

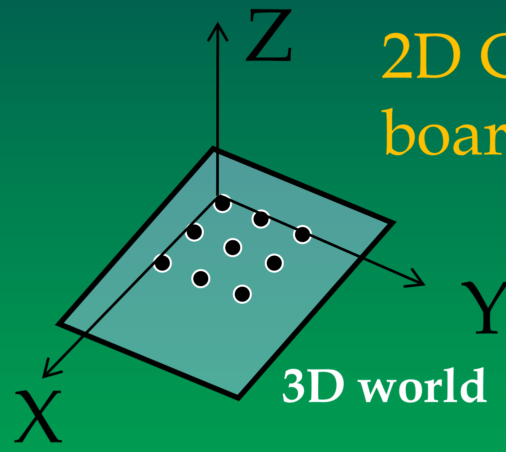
$$(r_{11}X + r_{12}Y + r_{13}Z + T_x)v_d - (r_{21}X + r_{22}Y + r_{23}Z + T_y)u_d = 0$$

Two-stage approach *(Stage 1a)*

(1a) Solve for five parameters:

$r_{11}/T_y,\ r_{21}/T_y,\ r_1/T_y,\ r_{22}/T_y,\ T_x/T_y$





2D Calibration board, Z=0

$$(r_{11}X_i + r_{12}Y_i + \cancelto{Z=0}{r_{13}Z_i} + T_x)v_{di} - (r_{21}X_i + r_{22}Y_i + \cancelto{Z=0}{r_{23}Z_i} + T_y)u_{di} = 0$$

6 unknowns: $r_{11},\ r_{12},\ r_{21},\ r_{22},\ T_x,\ T_y$

Two-stage approach (Stage 1a)

(1a) Solve for five parameters:
r_{11}/T_y, r_{21}/T_y , r_1/T_y, r_{22}/T_y, T_x/T_y

$$(r_{11}X_i + r_{12}Y_i + T_x)v_{di} - (r_{21}X_i + r_{22}Y_i + T_y)u_{di} = 0$$

$$\underset{n\times 6}{[\mathbf{A}]}\underset{6\times 1}{\begin{bmatrix} r_{11} \\ r_{12} \\ r_{21} \\ r_{22} \\ T_x \\ T_y \end{bmatrix}} = 0$$

6 unknowns: r_{11}, r_{12} , r_{21}, r_{22}, T_x, T_y

Homogeneous equation: $[\mathbf{A}]\mathbf{x}^* = 0$

For calibration, over-determined system

$n \geq 6$ (trivial solutions, $\mathbf{x}^* = 0$)

Two-stage approach *(Stage 1)*

1a) Solve for five parameters using least square (pseudo-inverse) :

r_{11}/T_y, r_{21}/T_y , r_1/T_y, r_{22}/T_y, T_x/T_y

$$\left[X_i\left(\frac{r_{11}}{T_y}\right)+Y_i\left(\frac{r_{12}}{T_y}\right)+\left(\frac{T_x}{T_y}\right)\right]v_{di}-\left[X_i\left(\frac{r_{21}}{T_y}\right)+Y_i\left(\frac{r_{22}}{T_y}\right)+1\right]u_{di}=0$$

$$\underbrace{\begin{bmatrix} X_1 v_{d1} & Y_1 v_{d1} & -X_1 u_{d1} & -Y_1 u_{d1} & v_{d1} \\ X_2 v_{d2} & Y_2 v_{d2} & -X_2 u_{d2} & -Y_2 u_{d2} & v_{d2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ X_n v_{dn} & Y_2 v_{d2} & -X_n u_{dn} & -Y_n u_{dn} & v_{dn} \end{bmatrix}}_{\substack{\mathbf{[A]} \\ n\times 5}} \underset{\substack{\boldsymbol{\mu} \\ 5\times 1}}{\begin{bmatrix} r_{11}/T_y \\ r_{12}/T_y \\ r_{21}/T_y \\ r_{22}/T_y \\ T_x/T_y \end{bmatrix}} = \underset{\substack{\mathbf{b} \\ n\times 1}}{\begin{bmatrix} u_{d1} \\ u_{d2} \\ \vdots \\ u_{dn} \end{bmatrix}}$$

$$[\mathbf{A}]\boldsymbol{\mu}=\mathbf{b}$$

$$\boldsymbol{\mu}=\mathbf{A}^{+}\mathbf{b}$$

Two-stage approach (Stage 1b, 1c)

Find T_y using the orthogonally of [R]

Given $\mu_{ij} = {r_{ij}}/{T_y}$ $(i, j=1,2)$ and $\mu_5 = {T_x}/{T_y}$

$$[R] = \begin{bmatrix} \mu_{11}T_y & \mu_{12}T_y & r_{13} \\ \mu_{21}T_y & \mu_{22}T_y & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$\alpha^T =$ first row, $\beta^T =$ second row

Step1

$$\boldsymbol{\alpha} \bullet \boldsymbol{\beta} = 0$$

Step2

$$\boldsymbol{\alpha} \bullet \boldsymbol{\alpha} = 1 \Rightarrow r_{13}^2 = 1 - T_y^2\left(\mu_{11}^2 + \mu_{12}^2\right)$$

$$\boldsymbol{\beta} \bullet \boldsymbol{\beta} = 1 \Rightarrow r_{23}^2 = 1 - T_y^2\left(\mu_{21}^2 + \mu_{22}^2\right)$$

Step 2: square the result to eliminate r_{13} and r_{23}.

Let $U = \sum_{j=1}^{4} \mu_j^2$

$$\begin{bmatrix} r_{13} \\ r_{23} \\ r_{33} \end{bmatrix} = \begin{bmatrix} r_{11} \\ r_{12} \\ r_{13} \end{bmatrix} \times \begin{bmatrix} r_{21} \\ r_{22} \\ r_{23} \end{bmatrix}$$

$$T_y^2 = \begin{cases} \dfrac{1}{U} & \text{if } \mu_1\mu_4 = \mu_2\mu_3 \\[2ex] \dfrac{U - \left[U^2 - 4(\mu_1\mu_4 - \mu_2\mu_3)^2\right]^{1/2}}{2(\mu_1\mu_4 - \mu_2\mu_3)^2} & \text{if } \mu_1\mu_4 \neq \mu_2\mu_3 \end{cases}$$

Two-stage approach *(Stage 1)*

Let $T_y = \left(T_y^2\right)^{1/2}$

then $r_{ij} = \mu_{ij} T_y$ $(i, j=1,2)$ and $T_x = \mu_5 T_y$

To determine the sign of T_y, select one object point $P(X_0, Y_0, 0)$:

$$\begin{bmatrix} \xi_x \\ \xi_y \end{bmatrix} = \begin{bmatrix} r_{11} X_o + r_{12} Y_o + T_x \\ r_{21} X_o + r_{22} Y_o + T_y \end{bmatrix}$$

If ξ_x must have the sign as u_{do}, and ξ_y must have the sign as v_{do} then T_y has the correct sign.

Otherwise, negate it.

Two-stage approach *(Stage 1c)*

1c) Solve for [R] T_x, T_y

$$[\mathbf{R}]=\begin{bmatrix} r_{11} & r_{12} & s_1\sqrt{1-r_{11}^2-r_{12}^2} \\ r_{21} & r_{22} & s_2\sqrt{1-r_{21}^2-r_{22}^2} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

The unknown signs s_1 and s_2 are determined from the orthogonal property of [R].

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \end{bmatrix}\begin{bmatrix} r_{21} \\ r_{22} \\ r_{23} \end{bmatrix}=0$$

$$\Rightarrow \underbrace{\left(r_{11}r_{21}+r_{12}r_{22}\right)}_{a}+s_1s_2\underbrace{\sqrt{1-r_{11}^2-r_{12}^2}\sqrt{1-r_{21}^2-r_{22}^2}}_{b}=0$$

The two factors, a and b, must be equal and opposite.

Thus, choose $s_1 = s_2 = +1$, check $\operatorname{sgn}\left(r_{11}r_{21}+r_{12}r_{22}\right)\overset{?}{=}\text{negative}$

If yes, keep the signs; otherwise $s_2 = -1$.

Two-stage approach (Stage 2)

Stage 2: Perspective Constraint

Given [R] T_x, T_y solve for f, k, T_z

Recall Camera Model Step 1:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} + \begin{pmatrix} T_x \\ T_y \\ T_z \end{pmatrix}$$

$$\boxed{\frac{x_i}{z_i} = \frac{r_{11}X_i + r_{12}Y_i + T_x}{r_{31}X_i + r_{32}Y_i + T_z}}$$

From Camera Model Steps 2 and 3: $u = f\dfrac{x}{z}$ $\quad u = \dfrac{u_d}{\left(1 + k_1 r_d^{\,2}\right)}$

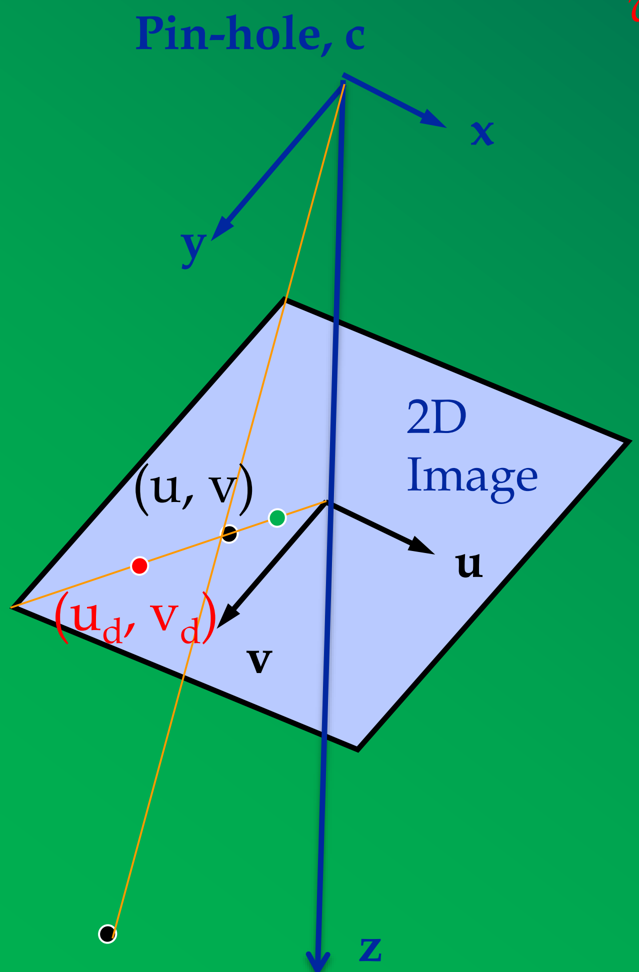
Eliminating u,

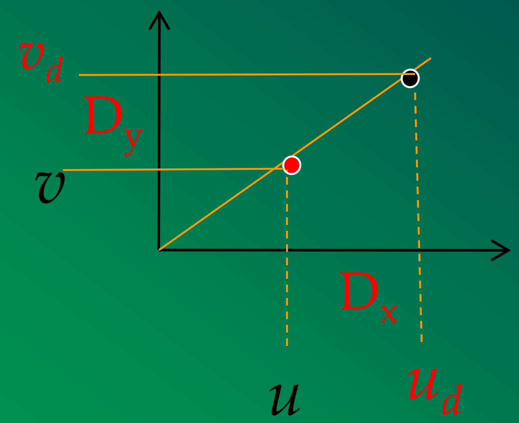
$$\frac{x}{z} = \frac{u_d}{f\,(1 + k_1 r_d^2)}$$

$$\boxed{u_d = f\,(1 + k_1 r_d^2)\frac{r_{11}X_i + r_{12}Y_i + T_x}{r_{31}X_i + r_{32}Y_i + T_z}}$$

Notes on Tsai's Camera Model

Recall Step 3: Radial lens distortion correction





$$u_d = u(1 + k_1 r_d^2)$$

$$v_d = v(1 + k_1 r_d^2)$$

where $r_d^2 = u_d^2 + v_d^2$

$$u = \frac{u_d}{\left(1 + k_1 {r_d}^2\right)} \qquad \textit{(Modified)}$$

Parameter to be calibrated: k_1

$$u = u_d\left(1 + \kappa_1 {r_d}^2\right) \qquad \textit{(Tsai)}$$

Notes on Tsai's Radial lens distortion

If Tsai's model were used,

$$u_i = u_{di}\left(1+\kappa_1 r_{di}^{\ 2}\right)$$

Recall Steps 1 and 2

$$x_i = r_{11}X_i + r_{12}Y_i + T_x = \text{known}$$
$$z_i = r_{31}X_i + r_{32}Y_i + T_z$$
$$u_i = f\,\frac{x_i}{z_i}$$

$$\frac{x_i}{z_i} = \frac{u_{di}}{f}\left(1+\kappa_1 r_{di}^{\ 2}\right)$$

$$\frac{u_{di}}{f}\left(1+\kappa_1 r_{di}^{\ 2}\right) = \frac{x_i}{r_{31}X_i + r_{32}Y_i + T_z}$$

$$u_{di}\left(r_{31}X_i + r_{32}Y_i + T_z\right)\left(1+\kappa_1 r_{di}^{\ 2}\right) = f x_i$$

Tsai's model for solving T_z, k_1 and f are non-linear!

Notes on Tsai's Radial lens distortion

Recall Steps 1 and 2

$$x_i = r_{11}X_i + r_{12}Y_i + T_x = \text{known}$$

$$z_i = r_{31}X_i + r_{32}Y_i + T_z$$

$$u_i = f\,\frac{x_i}{z_i}$$

Modified: $$u_i = \frac{u_{di}}{\left(1 + k_1 r_{di}^{\;2}\right)}$$

$$\frac{u_{di}}{f\left(1 + \kappa_1 r_{di}^{\;2}\right)} = \frac{x_i}{r_{31}X_i + r_{32}Y_i + T_z}$$

$$u_{di}\left(r_{31}X_i + r_{32}Y_i + T_z\right) = x_i f\left(1 + \kappa_1 r_{di}^{\;2}\right)$$

$$x_i f + x_i r_{di}^{\;2} f k_1 - u_{di}T_z = u_{di}\left(r_{31}X_i + r_{32}Y_i\right)$$

Two-stage approach (Stage 2)

Stage 2: Perspective Constraint

Given [R] T_x, T_y solve for f, k, T_z

$$\begin{bmatrix} x_1 & r_{d1}^2 x_1 & -u_{d1} \\ x_2 & r_{d2}^2 x_2 & -u_2 \\ \vdots & \vdots & \vdots \\ x_n & r_{dn}^2 x_n & -u_{dn} \end{bmatrix} \begin{bmatrix} f \\ fk_1 \\ T_z \end{bmatrix} = \begin{bmatrix} (r_{31}X_1 + r_{32}Y_1)u_{d1} \\ (r_{31}X_2 + r_{32}Y_2)u_{d2} \\ \vdots \\ (r_{31}X_n + r_{32}Y_n)u_{dn} \end{bmatrix}$$

where $x_i = r_{11}X_i + r_{12}Y_i + T_x$

$$[\mathbf{A}']\mathbf{x}' = \mathbf{b}'$$

$$\mathbf{x}' = \mathbf{A}^{+}\mathbf{b}'$$

Example

- The following Table gives five point correspondences input to the Calibration system.
- The units for both the world coordinate system and the u-v image coordinate system are centimeters.
- Assume that this is no lens distortion, compute (f, [R], $\underline{T}$)

	Object points			**Image points**	
i	X_i	Y_i	Z_i	u_i	v_i
1	0.00	5.00	0.00	-0.58	0.00
2	10.00	7.50	0.00	1.73	1.00
3	10.00	5.00	0.00	1.73	0.00
4	5.00	10.00	0.00	0.00	1.00
5	5.00	0.00	0.00	0.00	-1.00

Example (cont.)

Stage 1

points	Object			Image	
i	X_i	Y_i	Z_i	u_i	v_i
1	0.00	5.00	0.00	-0.58	0.00
2	10.00	7.50	0.00	1.73	1.00
3	10.00	5.00	0.00	1.73	0.00
4	5.00	10.00	0.00	0.00	1.00
5	5.00	0.00	0.00	0.00	-1.00

$$[\mathbf{A}]\boldsymbol{\mu} = \mathbf{b}$$

$$\boldsymbol{\mu} = \begin{bmatrix} \dfrac{r_{11}}{T_y} & \dfrac{r_{12}}{T_y} & \dfrac{r_{21}}{T_y} & \dfrac{r_{22}}{T_y} & \dfrac{T_x}{T_y} \end{bmatrix}^{\mathrm{T}}$$

$$a_i = \begin{bmatrix} v_i X_i & v_i Y_i & -u_i X_i & -u_i Y_i & v_i \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 2.9 & 0 \\ 10 & 7.5 & -17.3 & -12.9 & 1 \\ 0 & 0 & -17.3 & -8.65 & 0 \\ 5 & 10 & 0 & 0 & 1 \\ -5 & 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\mathbf{b} = \begin{array}{c} u_i \\ \begin{bmatrix} -0.58 \\ 1.73 \\ 1.73 \\ 0 \\ 0 \end{bmatrix} \end{array}$$

$$\boldsymbol{\mu} = \mathbf{A}^{+}\mathbf{b} = \begin{bmatrix} -0.173 \\ 0 \\ 0 \\ -0.2 \\ 0.865 \end{bmatrix}$$

Example (cont.)

points	Object			Image	
i	X_i	Y_i	Z_i	u_i	v_i
1	0.00	5.00	0.00	-0.58	0.00
2	10.00	7.50	0.00	1.73	1.00
3	10.00	5.00	0.00	1.73	0.00
4	5.00	10.00	0.00	0.00	1.00
5	5.00	0.00	0.00	0.00	-1.00

Stage 1: Calculate T_y

$$S_r = \sum_{j=1}^{4} \mu_j^2 = 0.0699$$

$$T_y^2 = \frac{U - \left[U^2 - 4(\mu_1\mu_4 - \mu_2\mu_3)^2\right]^{1/2}}{2(\mu_1\mu_4 - \mu_2\mu_3)^2} = 25$$

$$\boldsymbol{\mu} = \begin{bmatrix} r_{11}/T_y \\ r_{12}/T_y \\ r_{21}/T_y \\ r_{22}/T_y \\ T_x/T_y \end{bmatrix} = \begin{bmatrix} -0.173 \\ 0 \\ 0 \\ -0.2 \\ 0.865 \end{bmatrix}$$

Try $T_y = +5$

$r_{11} =$ -0.865; $r_{12} = r_{21} = 0$; $r_{22} = -1$; $T_x =$ 4.325

Check Point 2: $\xi_x = r_{11}x + r_{12}y + T_x = -4.325$

$$\xi_y = r_{21}x + r_{22}y + T_y = -2.5$$

Wrong sign $\Rightarrow$ $T_y = -5$.

Example *Stage 1 (cont.)*

$T_y = -5$ Recalculate $r_{11} = 0.865$; $r_{12} =$ r21 = 0; $r_{22} = 1$; $T_x =$ -4.325

Try $S_1 = S_2 = +1$

$$r_{13} = S_1\sqrt{1 - r_{11}^2 - r_{12}^2} = 0.5018$$

$$r_{23} = S_2\sqrt{1 - r_{21}^2 - r_{22}^2} = 0$$

$$sign(r_{11}r_{21} + r_{12}r_{22}) = 0 \quad \text{non-positve, OK}$$

$$r_{31} = r_{12}r_{23} - r_{13}r_{22} = -0.5018$$

$$r_{32} = r_{13}r_{21} - r_{11}r_{23} = 0$$

$$r_{33} = r_{11}r_{22} - r_{12}r_{21} = 0.8650$$

$$\Rightarrow \mathbf{R} = \begin{bmatrix} 0.865 & 0 & 0.5018 \\ 0 & 1 & 0 \\ -0.5018 & 0 & 0.865 \end{bmatrix}$$

Example *(cont.)*

Stage 2:

Solving for the unknown x',

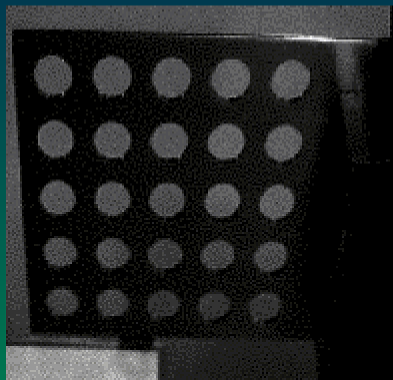
points	Object			Image	
i	X_i	Y_i	Z_i	u_i	v_i
1	0.00	5.00	0.00	-0.58	0.00
2	10.00	7.50	0.00	1.73	1.00
3	10.00	5.00	0.00	1.73	0.00
4	5.00	10.00	0.00	0.00	1.00
5	5.00	0.00	0.00	0.00	-1.00

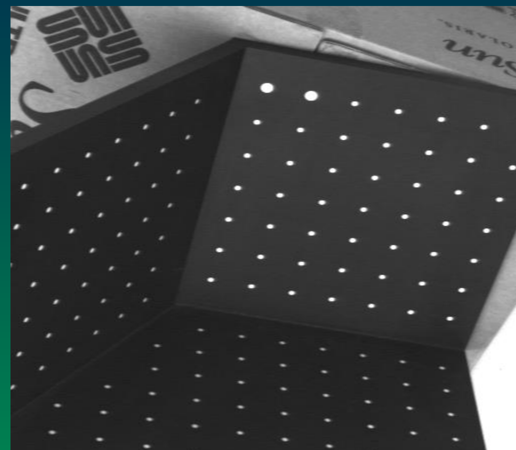
$$[\mathbf{A}']\underbrace{\begin{bmatrix} f \\ T_z \end{bmatrix}}_{\mathbf{x}'} = \mathbf{b}'$$

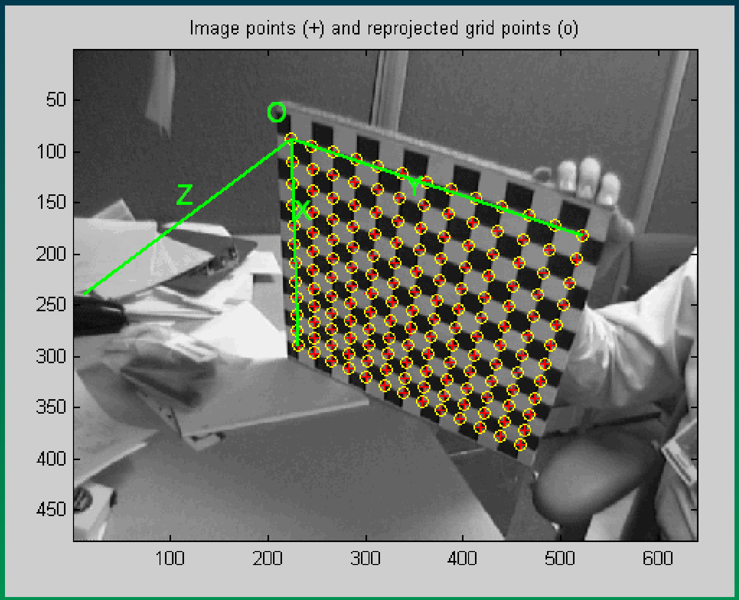
$$\mathbf{x}' = \mathbf{A}^{+}\mathbf{b}'$$

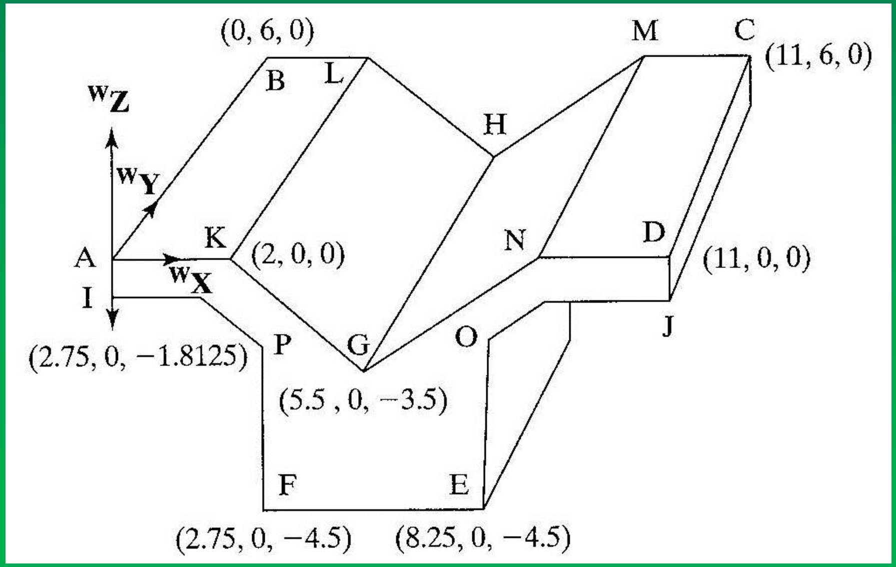
Hence, f **=1.0123** $\qquad \mathbf{T} = \begin{bmatrix} -4.325 \\ -5 \\ 7.5484 \end{bmatrix}$

Example calibration boards

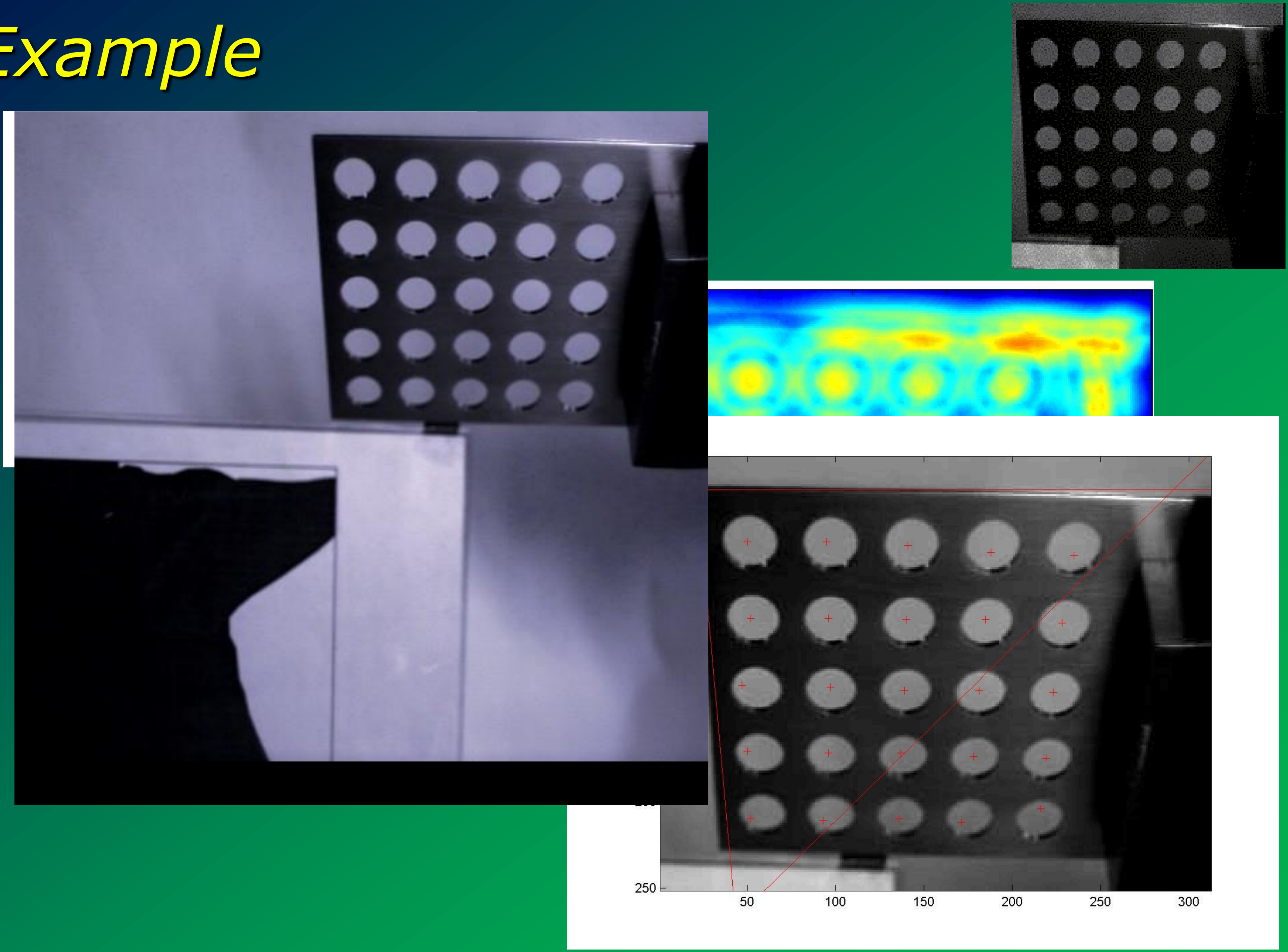








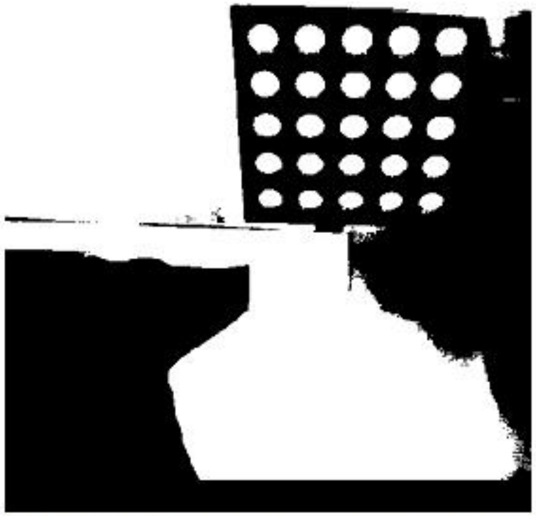
Example

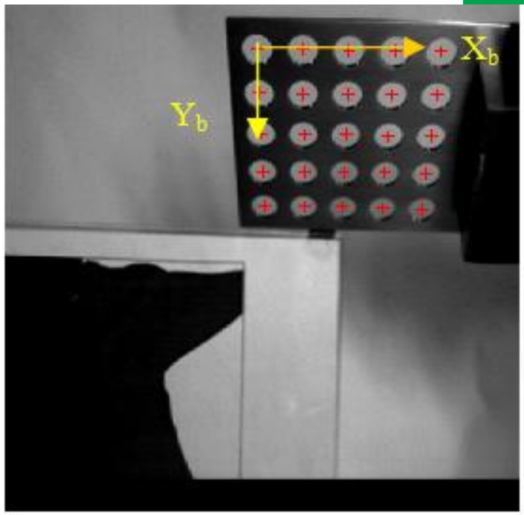


Example

Getting feature points from image (using regionprops):

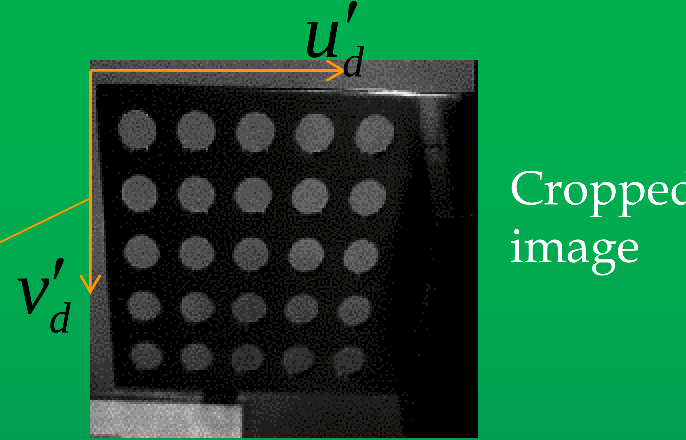
```
img_lab = bwlabel(img_bin);
props_struc = regionprops(img_lab, 'centroid','area'); %obtaining the
structure datatype with info
centroid = cat(1, centr_v.Centroid); %converting desired structured data
into numerical
count = 0;
%%using area to filter features
for ii = 1:length(areas)
    areav = areas(ii);
   if (areav > 200)  & (areav < 800)
        count = count+1;
        area_t(count) = areav;
        centr_t(count,:) = centroid(ii,:);
   end
end
```

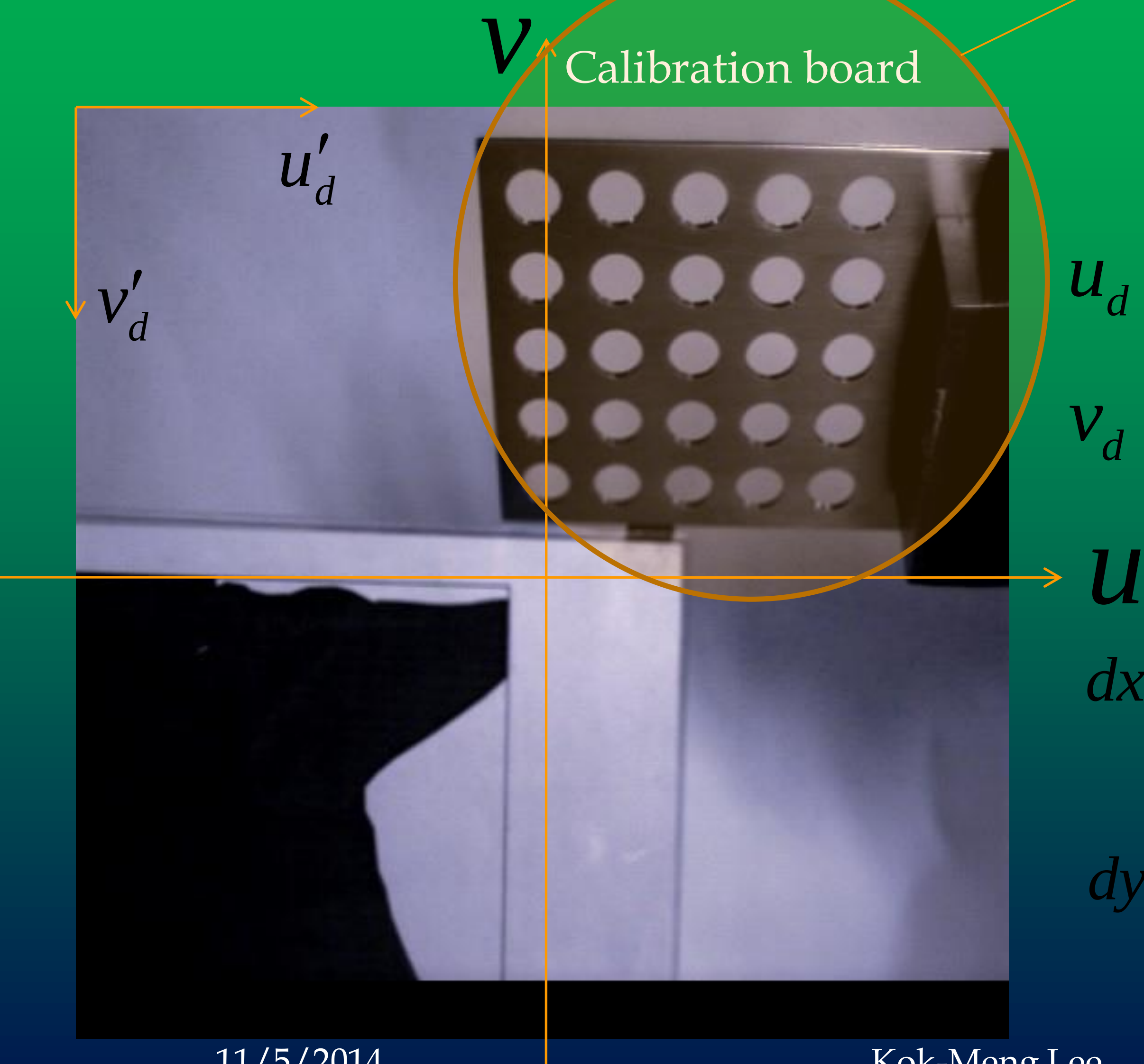




Binarized image and calibration board image with 25 feature points labeled with '+'.

Use Camera Model Step 4 to convert from pixels to physical units





$$u_d = \left(u'_d - c_x\right) dx'$$

$$u_d = \left(w_x + u'_d - c_x\right) dx'$$

$$v_d = \left(v'_d - c_y\right) dy'$$

$$v_d = \left(w_y + v'_d - c_y\right) dy'$$

$$dx' = dx \frac{N_{cx}}{N_{fx}} = 11.6\left(\frac{768}{512}\right)\mu m$$

$$dy' = dy \frac{N_{cy}}{N_{fy}} = 13.6\left(\frac{484}{512}\right)\mu m$$

Example (cont., Results)

Camera Model Step 4:

```
%%%NOTE: IN THIS CASE, Cfx = 256, Cfy = Cfx = 256
Xd_phy =  (Xd_pix(:,1) - Cfx)*dxp; %use computer centers!!!
Yd_phy =  (Xd_pix(:,2) - Cfy)*dyp;  %use computer centers!!!
```

Stage I:

R **- Rotation matrix**			***T*** **- Translation** *mm (in)*	
0.9979	0.0412	0.0492	T_x	– 1.8719 (–0.0737)
0.0147	0.5992	-0.8004	T_y	–63.5523 (–2.5011)
-0.0624	0.7995	0.5974		

Stage II:

f, κ_1, T_z – Stage 2 parameters		
f = 14.1 *mm* (0. 5568 *in*)	k_1= 1228.8689 mm^{-2} (0.7766 in^{-2})	T_z =336.5 *mm* (13.2457 *in*)

Example (cont., Results)

Solving for parameters using distortion *model 1* for intrinsic parameters.

***R* - Rotation matrix**			***T* - Translation *mm* (*in*)**	
0.9979	0.0412	0.0492	T_x	−1.8719 (−0.0737)
0.0147	0.5992	-0.8004	T_y	−63.5523 (−2.5011)
-0.0624	0.7995	0.5974	T_z	340.6396 (13.4057)

Intrinsic parameters

f = 14.3 *mm* (0.5622 *in*) k_1 = −879.7752 mm^{-2} (-0.5560 in^{-2})

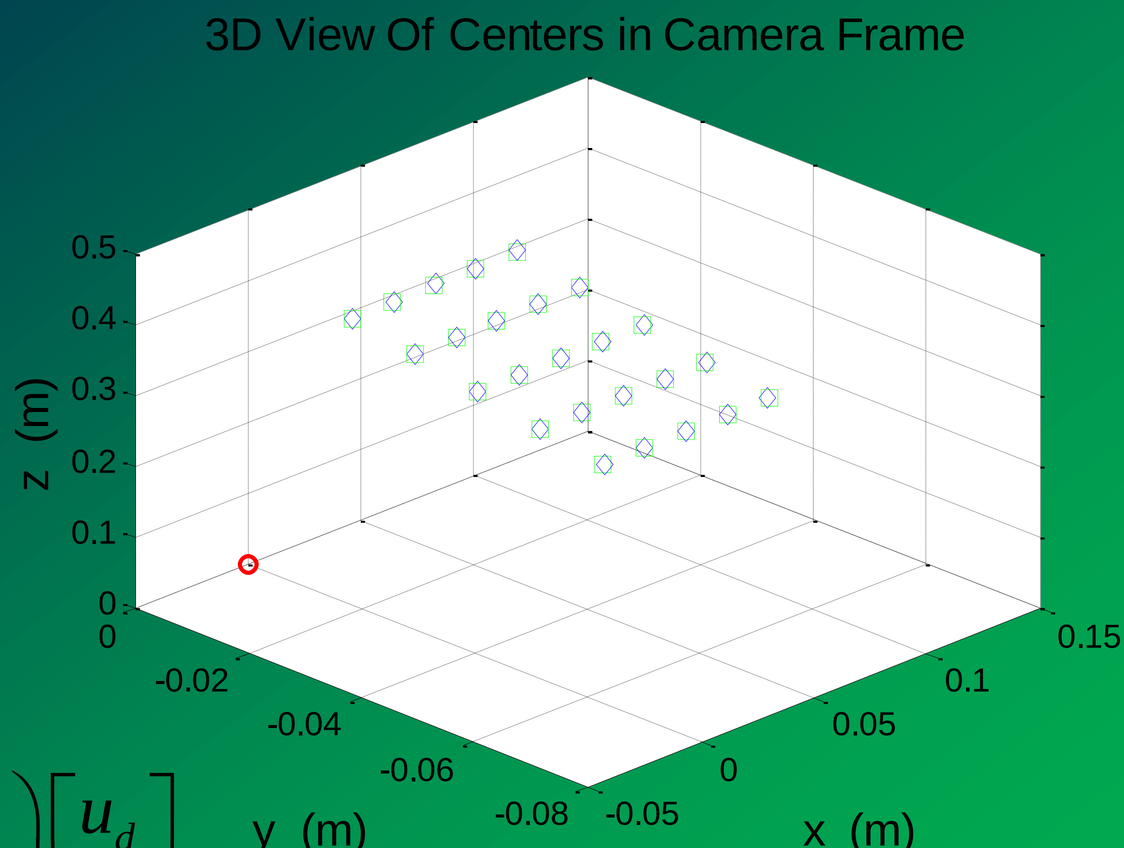
In terms of Euler angles yaw θ, pitch ϕ, and tilt ψ for rotation,

$$R = \begin{bmatrix} \cos\psi\cos\theta & \sin\psi\cos\theta & -\sin\theta \\ -\sin\psi\cos\phi + \cos\psi\sin\theta\cos\phi & \cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi & \cos\theta\sin\phi \\ \sin\psi\sin\phi + \cos\psi\sin\theta\cos\phi & -\cos\psi\sin\phi + \sin\psi\sin\theta\cos\phi & \cos\theta\cos\phi \end{bmatrix}$$

θ (x-axis rotation)	ϕ (y-axis rotation)	ψ (z-axis rotation)
$\theta = \tan^{-1}(r_{23} / r_{33})$	$\phi = -\sin^{-1}(r_{13})$	$\psi = \tan^{-1}(r_{12} / r_{11})$
-53°	-3°	2°

The results satisfy $[R]^T = [R]^{-1}$

Example
(validating results)



$$\boxed{\mathbf{x_i} = [\mathbf{R}]\mathbf{X_i} + \mathbf{T}}$$

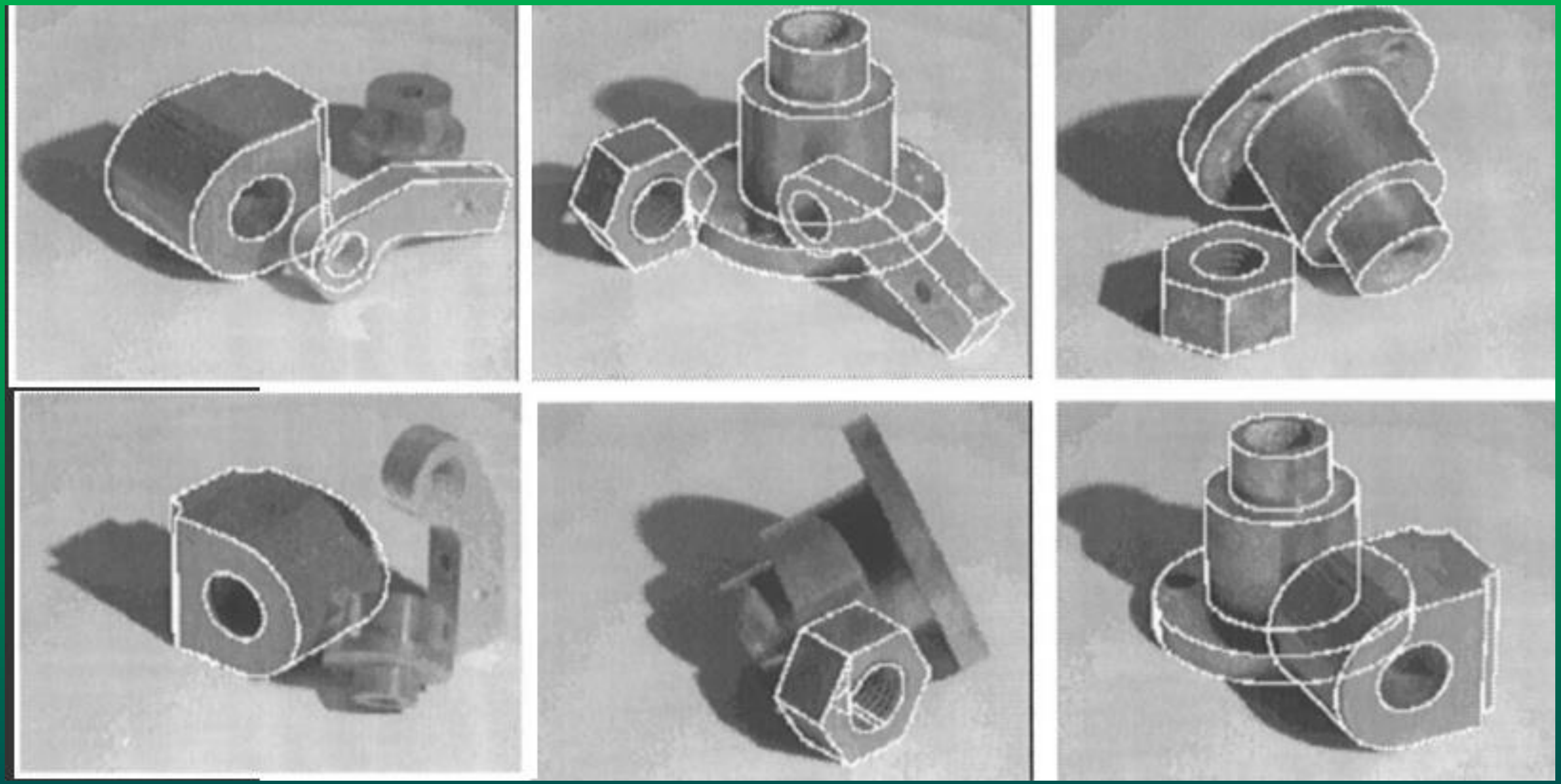
$$\mathrm{RHS} = [\mathbf{R}]\mathbf{X_i} + \mathbf{T}$$

$$\text{LHS:} \quad \begin{bmatrix} x_i \\ y_i \end{bmatrix} = \frac{z_i}{f}\left(\frac{1}{1+k_1 r_d^2}\right)\begin{bmatrix} u_d \\ v_d \end{bmatrix}$$

3 sample points from the figure above

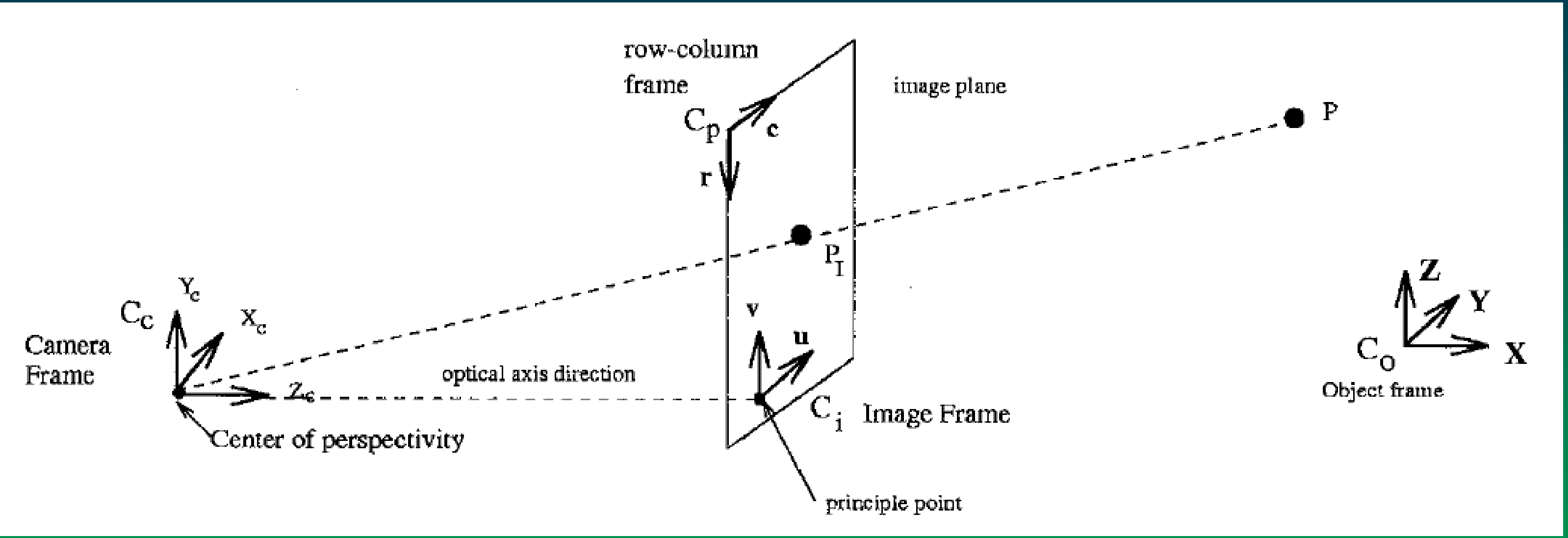
x coordinates (*m*)		*y* coordinates (*m*)		*z* coordinates (*m*)	
LHS	RHS	LHS	RHS	LHS	RHS
-0.0018444	-0.0018719	-0.0637924	-0.0635523	0.3388329	0.3388329
-0.0011130	-0.0010866	-0.0521947	-0.0521322	0.3540779	0.3540779
-0.0003378	-0.0003012	-0.0407762	-0.0407121	0.3693227	0.3693227

Pose estimation



- Using point, line, circle-ellipse correspondences
- Allow for 3D objects

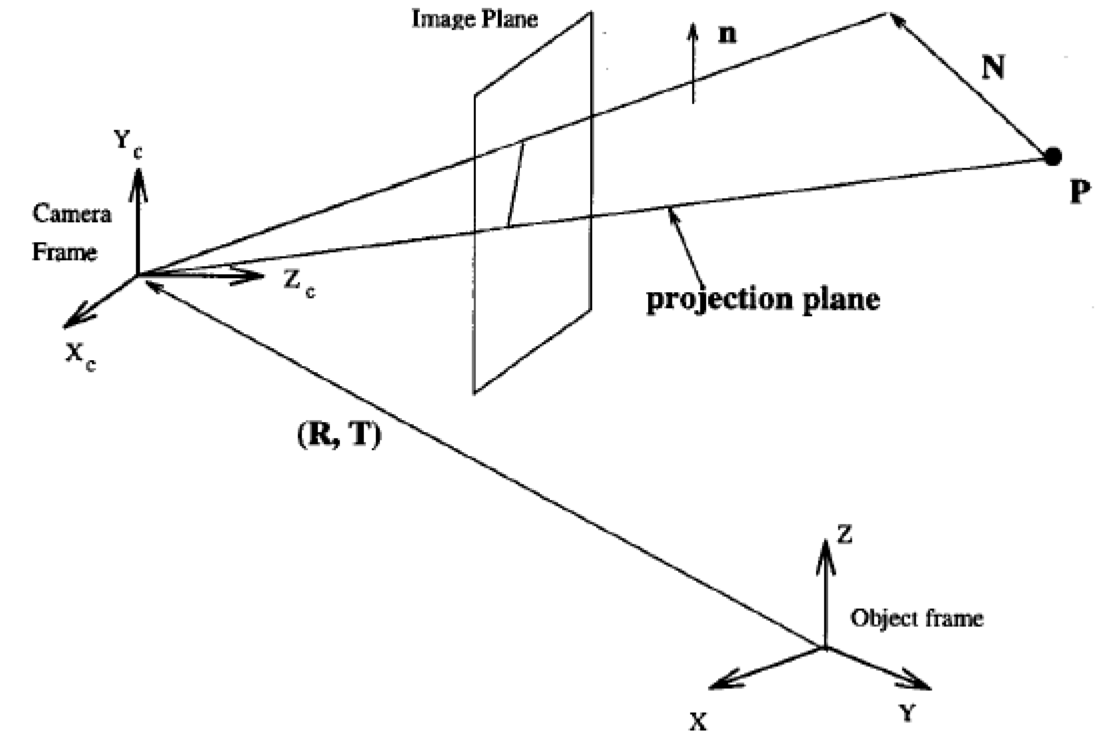
Point correspondences
(Neglect lens distortion and f is known)



$$u_n = f\frac{r_{11}x_n + r_{12}y_n + r_{13}z_n + t_x}{r_{31}x_n + r_{32}y_n + r_{33}z_n + t_z}$$

$$v_n = f\frac{r_{21}x_n + r_{22}y_n + r_{23}z_n + t_y}{r_{31}x_n + r_{32}y_n + r_{33}z_n + t_z}.$$

Line correspondences



Known with respect to object frame:

$$L: \quad X = \lambda N + P$$

Known from 2D image:

$$\ell: \quad au + bv + c = 0$$

3D camera frame:

$$\ell: \quad afx_c + bfy_c + cz_c = 0$$

Relative to the camera frame, the equation of the projection plane can be derived from the 2D line equation as

$$afx_c + bfy_c + cz_c = 0$$

where f is the focal length. Since the 3D line lies on the projection plane, the plane normal must be perpendicular to the line.

Line correspondences *(cont.)*

Known with respect to the object frame:

$$L: \quad X = \lambda N + P$$

Known from 2D image:

$$\ell: \quad au + bv + c = 0$$

Line and its normal in 3D camera frame:

$$\ell: \quad afx_c + bfy_c + cz_c = 0$$

$$n = \frac{(af, bf, c)}{\sqrt{(af)^2 + (bf)^2 + c^2}}$$

Co planarity equations:

$$\mathbf{n}^{\mathrm{T}}[\mathbf{R}]\mathbf{N} = 0$$

$$\mathbf{n}^{\mathrm{T}}\left([\mathbf{R}]\mathbf{P} + \mathbf{T}\right) = 0$$

Denote the plane normal by $n = \frac{(af, bf, c)^t}{\sqrt{a^2f^2+b^2f^2+c^2}}$; then given an ideal projection, we have

$$n^t RN = 0\,.$$

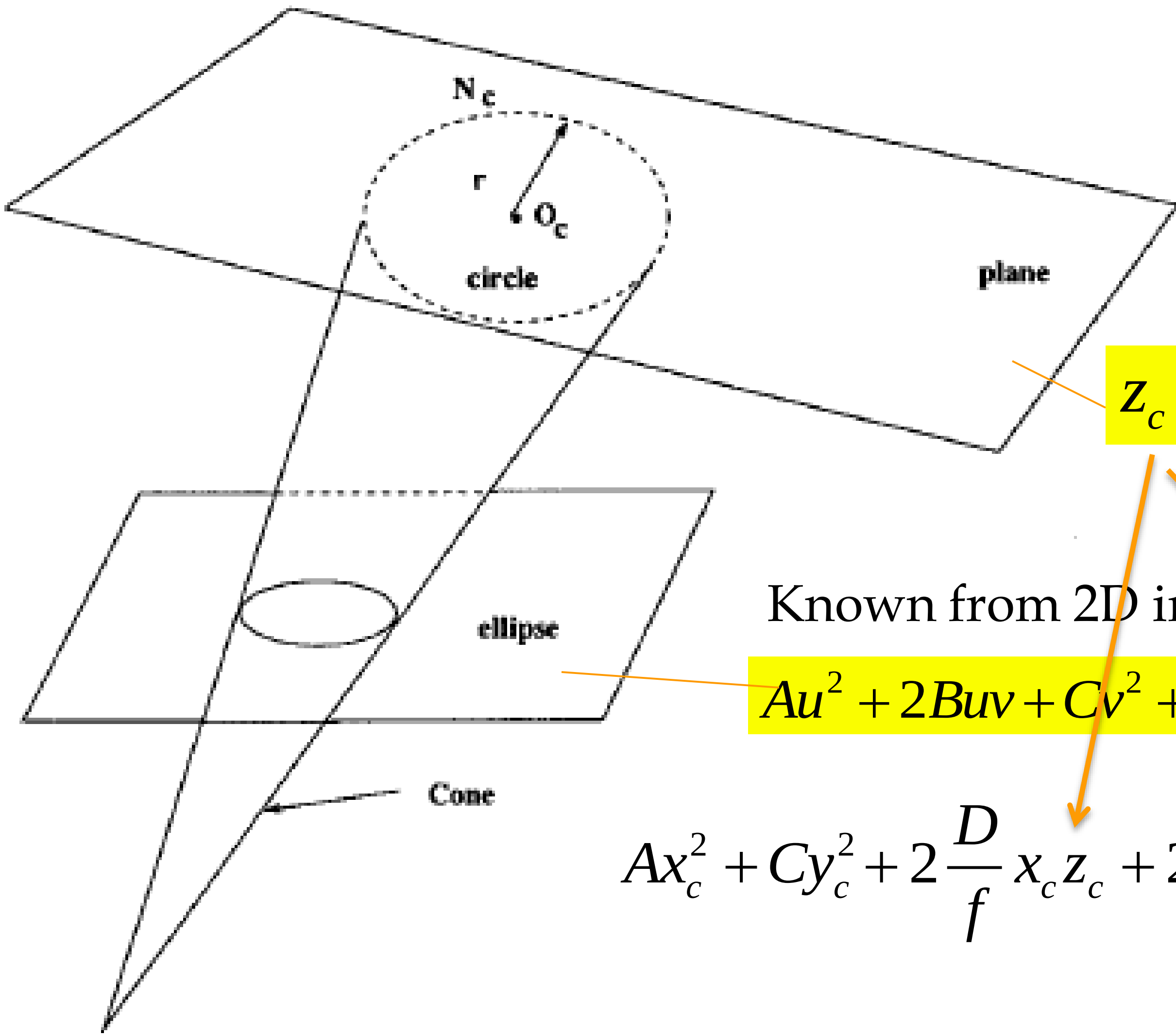
Similarly, since point P is also located on the projection plane, this leads to

$$n^t(RP + T) = 0\,.$$

Unknown to be solved: **[R], T**

Circle-ellipse correspondences

Find object circle from imaged ellipse



Recall CM Step 2:

$$u = f\frac{x_c}{z_c} \qquad v = f\frac{y_c}{z_c}$$

$$z_c = \alpha x_c + \beta y_c + \gamma$$

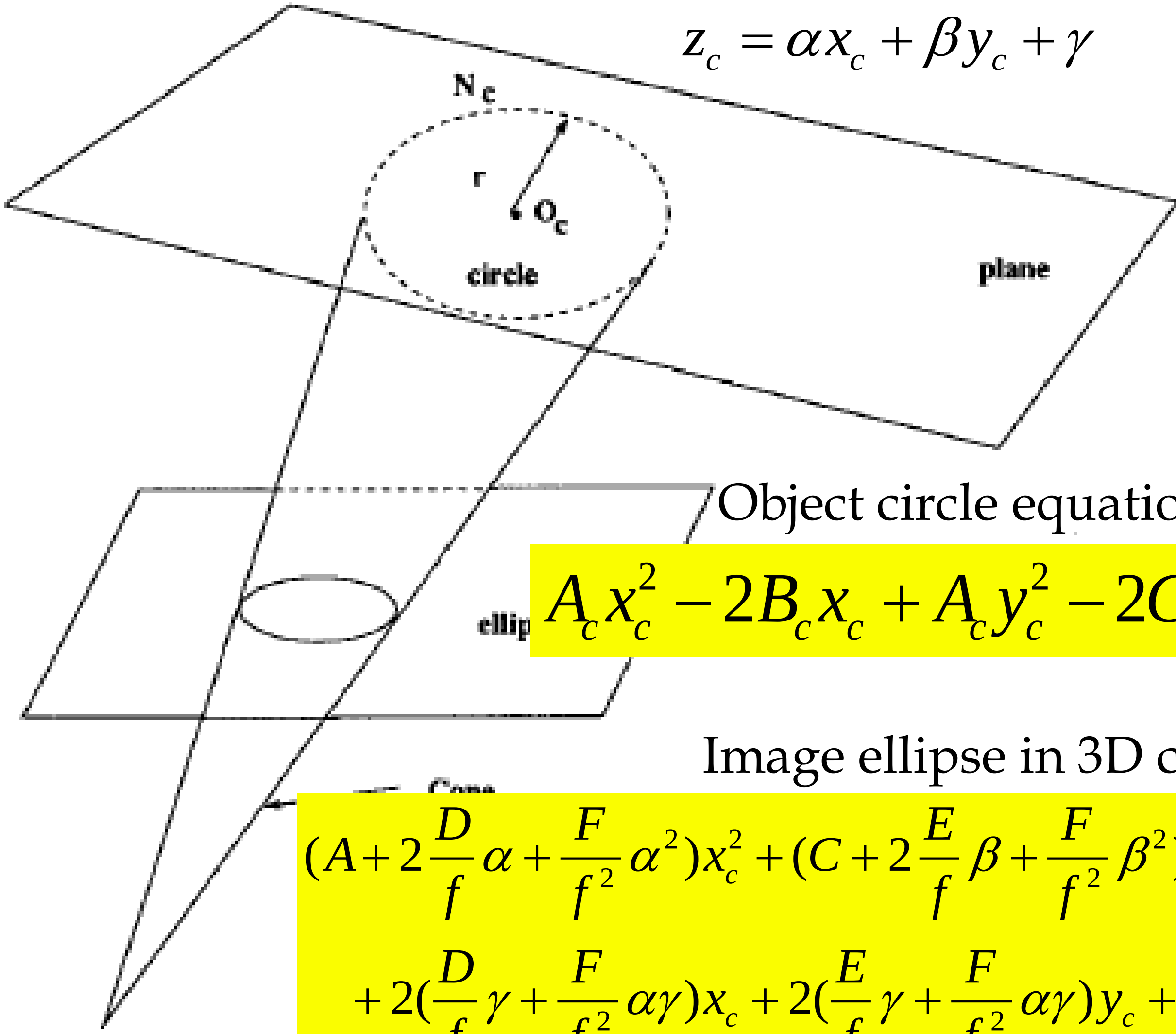
Known from 2D image:

$$Au^2 + 2Buv + Cv^2 + 2Du + 2Ev + F = 0$$

$$Ax_c^2 + Cy_c^2 + 2\frac{D}{f}x_c z_c + 2\frac{E}{f}y_c z_c + \frac{F}{f^2}z_c^2 = 0$$

Circle-ellipse correspondences

Find object circle from imaged ellipse



$$z_c = \alpha x_c + \beta y_c + \gamma$$

Object circle equation in 3D camera frame:

$$A_c x_c^2 - 2B_c x_c + A_c y_c^2 - 2C_c y_c + D_c = 0, \quad A_c \neq 0$$

Image ellipse in 3D camera frame:

$$\left(A + 2\frac{D}{f}\alpha + \frac{F}{f^2}\alpha^2\right)x_c^2 + \left(C + 2\frac{E}{f}\beta + \frac{F}{f^2}\beta^2\right)y_c^2 + 2\left(\frac{D}{f}\beta + \frac{E}{f}\alpha + \frac{F}{f^2}\alpha\beta\right)x_c y_c$$
$$+ 2\left(\frac{D}{f}\gamma + \frac{F}{f^2}\alpha\gamma\right)x_c + 2\left(\frac{E}{f}\gamma + \frac{F}{f^2}\alpha\gamma\right)y_c + \frac{F}{f^2}\gamma^2 = 0$$

Circle-ellipse correspondences

Find object circle from imaged ellipse

Image ellipse in 3D camera frame:

$$(A+2\frac{D}{f}\alpha+\frac{F}{f^2}\alpha^2)x_c^2+(C+2\frac{E}{f}\beta+\frac{F}{f^2}\beta^2)y_c^2+2(\frac{D}{f}\beta+\frac{E}{f}\alpha+\frac{F}{f^2}\alpha\beta)x_c y_c$$
$$+2(\frac{D}{f}\gamma+\frac{F}{f^2}\alpha\gamma)x_c+2(\frac{E}{f}\gamma+\frac{F}{f^2}\alpha\gamma)y_c+\frac{F}{f^2}\gamma^2=0$$

Object circle equation in 3D camera frame:

$$A_c x_c^2-2B_c x_c+A_c y_c^2-2C_c y_c+D_c=0, \quad A_c\neq 0$$

(1) x_c^2 and y_c^2 have the same coefficient

(2) coefficient of $x_c y_c$ =0

Circle-ellipse correspondences

Find object circle from imaged ellipse

we have $\frac{A}{F} f^2 + 2\left(\frac{D}{F} f\right)\alpha + \alpha^2 = \frac{C}{F} f^2 + 2\left(\frac{E}{F} f\right)\beta + \beta^2 = A_c \frac{f^2}{F}$

$$\Rightarrow \boxed{\left(\alpha + \frac{Df}{F}\right)^2 - \left(\beta + \frac{Ef}{F}\right)^2 = \left(\frac{f}{F}\right)^2 \left(CF - AF + D^2 - E^2\right)}$$

$\frac{Df}{F}\beta + \frac{Ef}{F}\alpha + \alpha\beta = 0$ or $\boxed{\left(\alpha + \frac{Df}{F}\right)\left(\beta + \frac{Ef}{F}\right) = \left(\frac{f}{F}\right)^2 DE}$

$2(Df + F\alpha)\frac{\gamma}{f^2} = -2B_c$ or $\boxed{B_c = -(Df + F\alpha)\frac{\gamma}{f^2}}$

$2(Ef + F\alpha)\frac{\gamma}{f^2} = -2C_c$ or $\boxed{C_c = -(Ef + F\beta)\frac{\gamma}{f^2}}$

$$\boxed{D_c = \frac{F}{f^2}\gamma^2}$$

Circle-ellipse correspondences

The circle equation can be re-written as $\left(x_c - \frac{B_c}{A_c}\right)^2 + \left(y_c - \frac{C_c}{A_c}\right)^2 = \frac{B_c^2 + C_c^2 - A_c D_c}{A_c^2}$

Solve (6a) and (6b) for $\alpha + Df/F$ and $\beta + Ef/F$, and then γ from $\boxed{\left(B_c^2 + C_c^2 - A_c D_c\right)/A_c^2 = r^2}$.

Once α, β and γ are found, the circle center and its plane normal are given by

$$\boxed{\mathbf{O}_c = \frac{1}{A_c}\begin{bmatrix} B_c \\ C_c \\ \gamma A_c + \alpha B_c + \beta C_c \end{bmatrix}} \text{ and } \boxed{\mathbf{N}_c = \frac{1}{\sqrt{\alpha^2 + \beta^2 + 1}}\begin{bmatrix} \alpha \\ \beta \\ -1 \end{bmatrix}}.$$

$$\begin{aligned} N_c &= R N_o \\ &= \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \begin{pmatrix} N_{o_x} \\ N_{o_y} \\ N_{o_z} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} O_c &= R\, O_o + T \\ &= \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} \begin{pmatrix} O_{o_x} \\ O_{o_y} \\ O_{o_z} \end{pmatrix} + \begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix}. \end{aligned}$$

$[R]^T = [R]^{-1}$

$$\begin{aligned} N_o &= R^t N_c \\ &= \begin{pmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{pmatrix} \begin{pmatrix} N_{c_x} \\ N_{c_y} \\ N_{c_z} \end{pmatrix} \end{aligned}$$