

Open problems in interference and proofs for SAT solving

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Reverse unit propagation (RUP) [Goldberg, Novikov '03]

a clause C is a **RUP clause** over a CNF formula F if $F \wedge \neg C$ leads to a conflict via unit propagation

Then, $F \models C$ holds

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this is just reasoning without loss of generality!

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Interpolants from interference-based proofs?

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- Inferences are model-preserving
- Inferences depend only on specific clauses
- Exponentially more compact formulas (*maybe*) (*perhaps*)
- Satisfiability problem is NP-complete

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Craig interpolant [Craig '57]

$$F \wedge G \text{ unsatisfiable} \Rightarrow F \models P \models \neg G \text{ and } \text{var}(P) \subseteq \text{var}(F) \cap \text{var}(G)$$

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Theorem [Reckhow '75] [Krajíček, Pudlák '98] [Kiesl, Rebola, Heule '18] [Heule, Biere, '18]

Interpolants cannot be constructed from SR proofs in polynomial time unless RSA is insecure

Open problem 1 does this result still hold for interpolants in mutation logic?

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Turn $F \vee Q$ into $(F \vee x) \wedge (Q \vee \bar{x})$

forgotten by 2015, enormous resurgence in 2023

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Encode “ C is an SR clause over F upon σ ” in a sub-solver

modest effect, enormous resurgence in 2023 [Oliveras, Li, Wu, Chung, Ganesh '23]

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Idea: splitting-based inprocessing

Perform SDCL/BVA over a subformula $G \subseteq F$, then identify clauses that can be brought back into F .

Can be proven sound in mutation logic, but not even in WSR

Similar problems in incremental solving [Fazekas, Biere, Scholl '19]

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Open problem 2 How does entailment behaves across interference?

are we being artificially limited by how we think of redundancy/interference?

A critical history of proof formats for CDCL SAT

[Goldberg, Novikov '03]

RUP

○

[Malik, Zhang '03]

resolution

○

$$\frac{C \vee x \quad D \vee \bar{x}}{C \vee D}$$

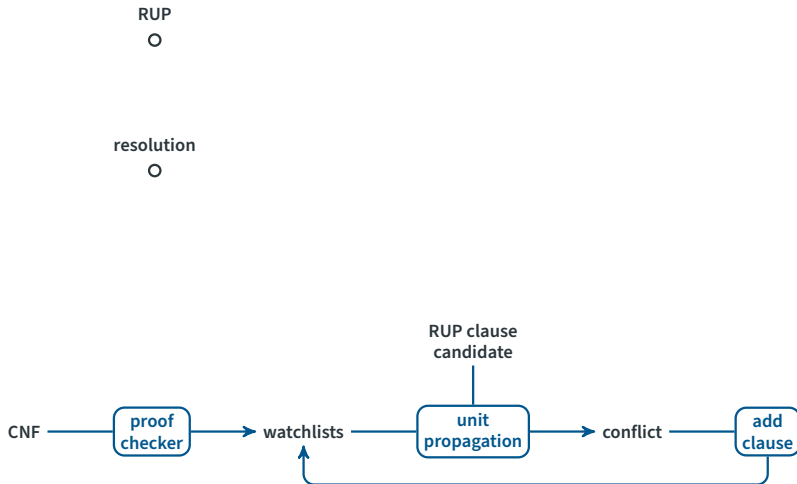
resolution

$$\frac{\frac{E_0}{A_0} \quad E_1}{A_1 \quad \ddots \quad A_{n-1} \quad E_n} \quad \frac{A_n}{A_n}$$

propagate $\overline{A_n}$,
check for conflict

RUP

A critical history of proof formats for CDCL SAT

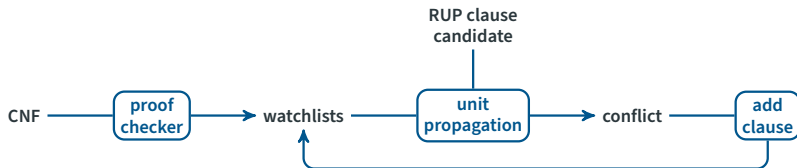


A critical history of proof formats for CDCL SAT

[Heule, Hunt, Wetzler '14]



resolution
○

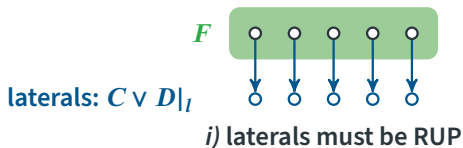


A critical history of proof formats for CDCL SAT

[Wetzler, Heule, Hunt '14]



resolution
○



- C : RAT candidate
- I : witness literal

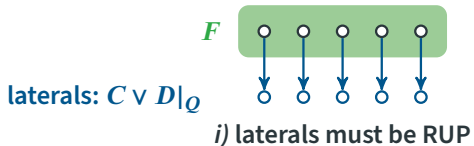
ii) $I \models C$

[Järvisalo, Heule, Biere '12]

A critical history of proof formats for CDCL SAT



resolution
○



○ C : PR candidate
 Q : witness cube

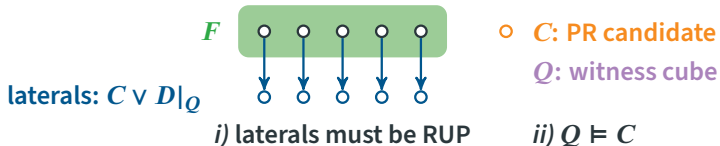
ii) $Q \models C$

A critical history of proof formats for CDCL SAT

[Heule, Kiesl, Biere '17]



resolution



A critical history of proof formats for CDCL SAT



resolution
○



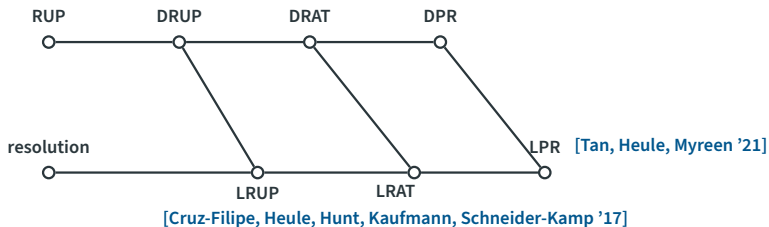
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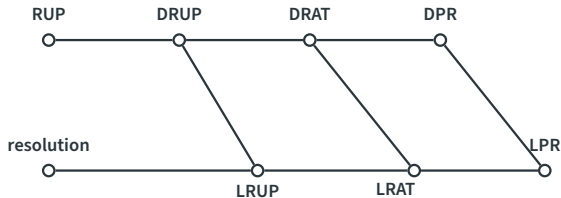
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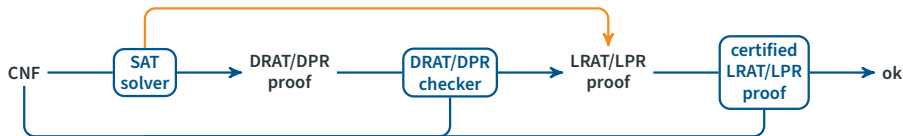


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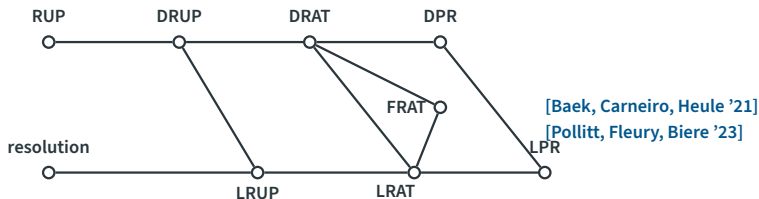


“There are several challenges regarding unsatisfiability proofs. How can one store resolution proofs using much less space on disk and using much less memory overhead?”

[Biere, Heule '15]

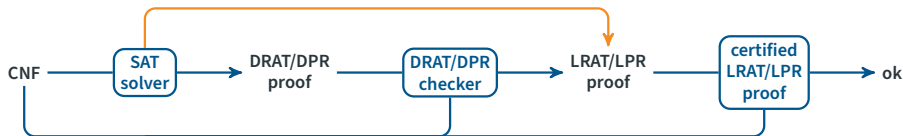


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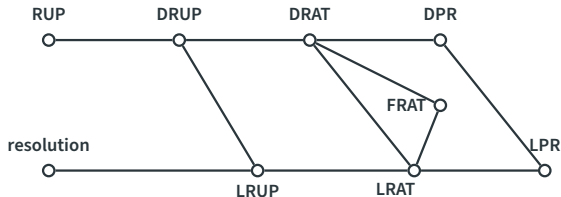


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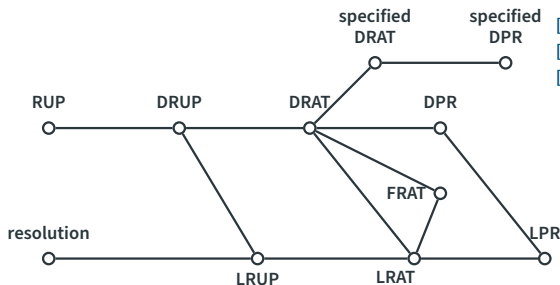


A critical history of proof formats for CDCL SAT



“The main reason to add deletion information to a clausal proof is to reduce the computation costs to validate that proof. However, deletion of unit clauses has the opposite effect. Notice that ignoring deletion steps of unit clauses can turn a valid DRAT proof into an invalid one — and the other way around.” [\[Heule '16\]](#)

A critical history of proof formats for CDCL SAT



[Rebola-Pardo, Biere '18]

[Rebola-Pardo, Cruz-Felipe '18]

[Altmanninger, Rebola-Pardo '20]

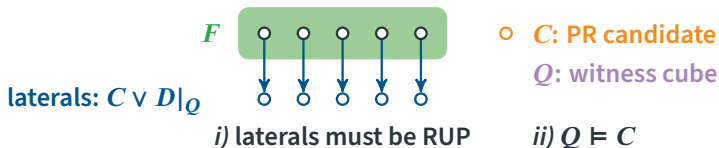
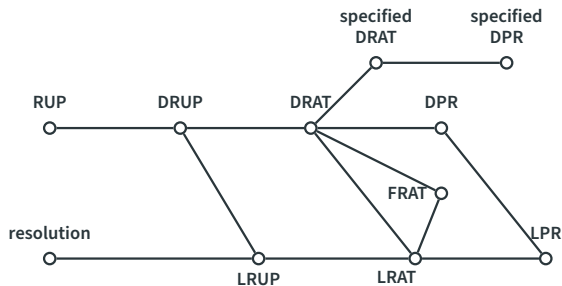
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Problem 1 checking if a clause is unit requires unit propagation

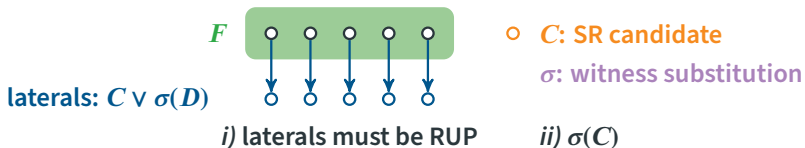
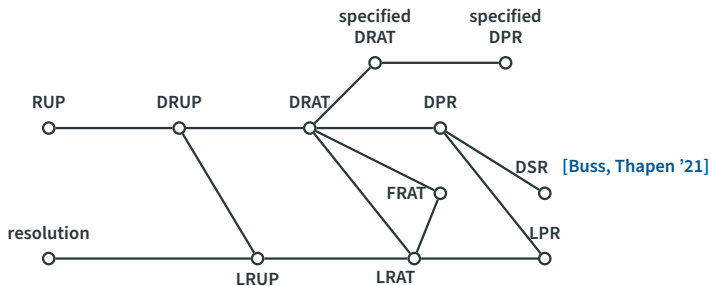
Problem 2 DRAT is non-monotonic, so skipping a deletion gives surprising results

Problem 3 the computational cost is the same (in our implementation, slightly lower)

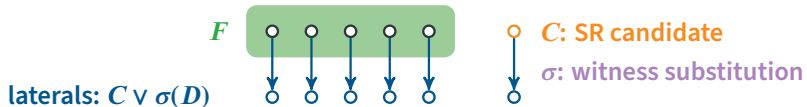
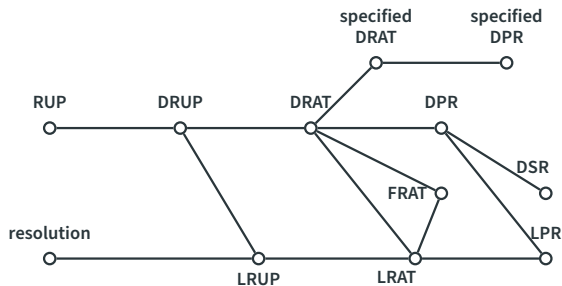
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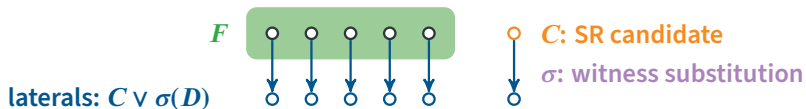
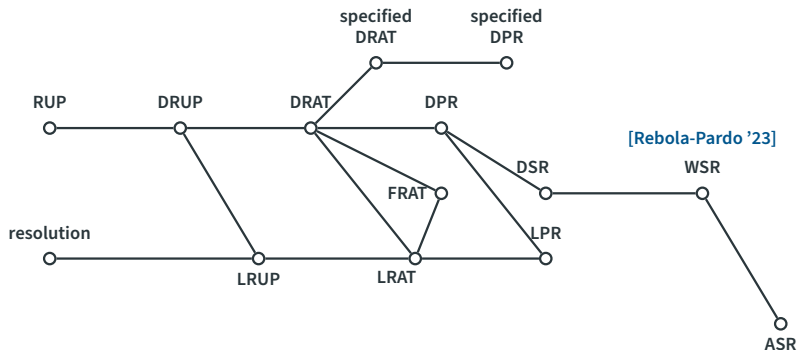


i) laterals must be RUP

[Gocht, Nordström '21]

[Rebola-Pardo '23]

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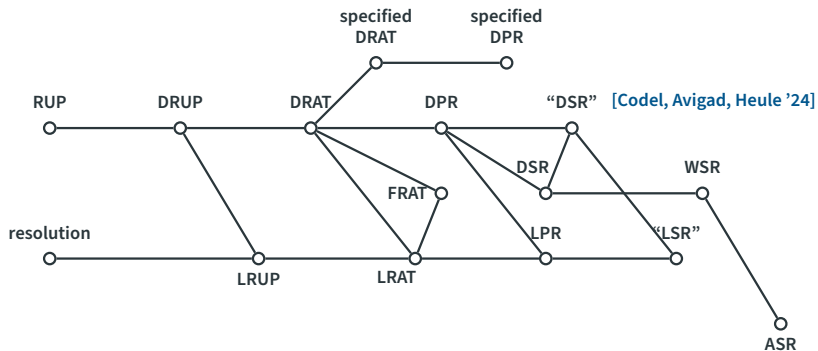


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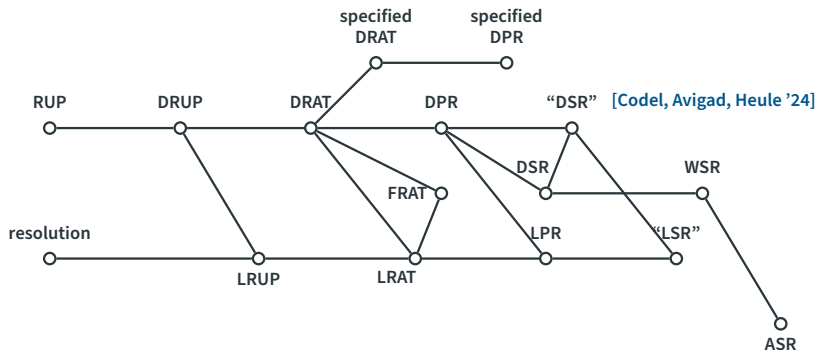
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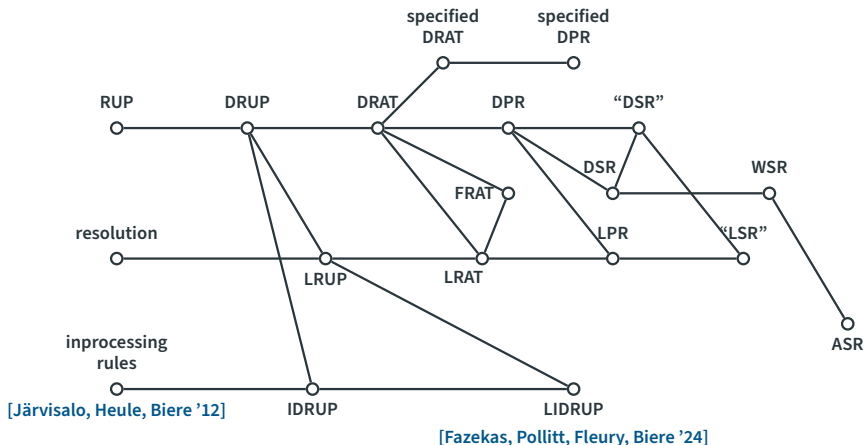
A critical history of proof formats for CDCL SAT



“Currently, our checkers assume that the witness σ satisfies the candidate clause C . However, the DSR and LSR formats can also express proofs where σ causes $C|_{\sigma}$ to be a tautology: the proof can simply map the pivot p to itself or to any other literal in the substitution portion. We plan to support this general case in the future.”

“At the moment, `dsr-trim` can only perform forwards checking, which means that it checks DSR proofs from start to finish and adds hints as it goes. (...) In practice, backwards checking can significantly reduce the size of proofs. Adding backwards checking to `dsr-trim` is ongoing work.” [Codel, Avigad, Heule '24]

A critical history of proof formats for CDCL SAT



$$\frac{\varphi \wedge C[\rho] \sigma}{\varphi[\rho] \sigma \cdot (\omega : C)} \boxed{b}$$

WEAKEN⁺

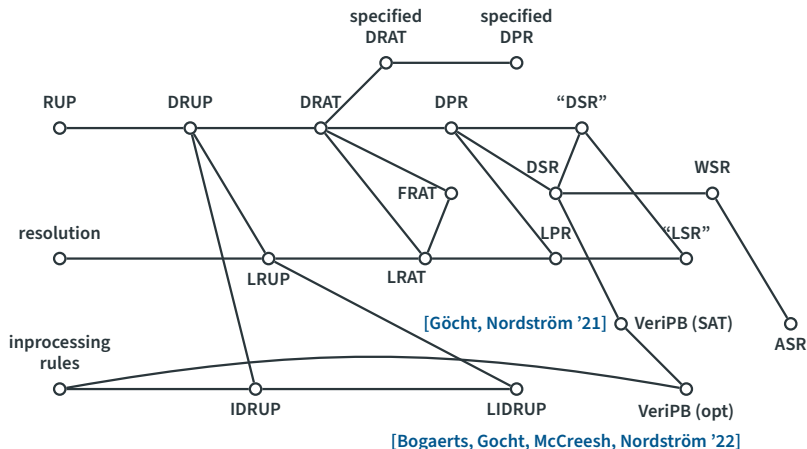
$$\frac{\varphi \wedge C[\rho] \sigma}{\varphi[\rho] \sigma} \boxed{\emptyset}$$

DROP

where \boxed{b} is $\varphi \wedge C \equiv_{sat}^{\omega} \varphi$ and $\boxed{\emptyset}$ is $\varphi \models C$

Fig. 2. New weakening and dropping rules

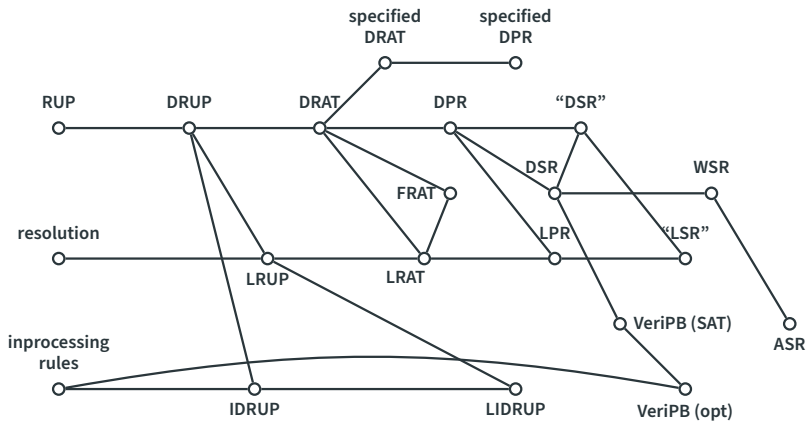
A critical history of proof formats for CDCL SAT



A proof for (F, f) in our proof system consists of a sequence of *proof configurations* $(\mathcal{C}, \mathcal{D}, \theta_{\leq}, \bar{z}, v)$, where

- \mathcal{C} is a set of pseudo-Boolean *core constraints*;
- \mathcal{D} is another set of pseudo-Boolean *derived constraints*;
- θ_{\leq} is a PB formula encoding a preorder and \bar{z} a set of literals on which this preorder will be applied; and
- v is the best value found so far for f .

A critical history of proof formats for CDCL SAT



... plus binary vs text variants!

Proof formats: what do we need, what must go?

Open Problem 3 consolidating proof formats

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Political theory? should provers adapt to checkers or checkers to provers?

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Implicit clause ids? trade-off with solvers! [Matthias Fleury]

Zero-terminated vs length-prefixed lists? is this just about readability?

Explicit SR vs RUP introductions? [Cynthia Peyrer, Ilija Vorontsov]

Beyond CNF reasoning? should the format be extensible?

Unified framework for satisfiable instances? for incremental SAT and optimization

Partial proof checking? unit deletions are tricky here!

Political theory? should provers adapt to checkers or checkers to provers?

What do we certify? what counts as a “correct” proof? [yes, again]

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clauses can be introduced or deleted, but only sequentially

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Open Problem 4 can we find a sensible framework unifying these ideas?