JOHANNES KEPLER
UNIVERSITY LINZ

Boolean Quantifier Shifting as an Optimization Problem

Adrian Rebola-Pardo
July 1, 2024
QUANTIFY 2024, Nancy (France)

Some pictures by HashiCorp, Inc. and The Walt Disney Company.

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The larger context

The work presented here is contained in:

Quantifier Shifting for Quantified Boolean Formulas Revisited
Simone Heisinger, Maximilian Heisinger, Adrian Rebola-Pardo, Martina Seidl

Today theoretical basis

Thursday 14:30 IJCAR talk by **Simone Heisinger**

Friday 17:10 IJCAR talk by **Maximilian Heisinger**

QBF solvers and prenex formulas

QBF solvers are decision procedures for the satisfiability problem on quantified Boolean formulas...

$$\exists a.\, (\forall x.\, a \vee \neg x) \wedge (\forall y.\, \exists b.\, y \vee (\neg a \wedge b))$$

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... on *prenex form*...

$$\exists a. \forall x. \forall y. \exists b. (a \vee \neg x) \wedge (y \vee (\neg a \wedge b))$$

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$$\exists a.\, \forall x.\, \forall y.\, \exists b.\, (a \vee \neg x) \wedge (y \vee (\neg a \wedge b))$$

... and in many cases in *prenex conjunctive normal form (PCNF)*...

$$\exists a.\, \forall x.\, \forall y.\, \exists b.\, (a \vee \neg x) \wedge (y \vee \neg a) \wedge (y \vee b)$$

Do prenexing methods matter?

real-world problem

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$$\text{real-world problem} \xrightarrow{\text{encoder}} \text{QBF encoding}$$

Do prenexing methods matter?

$$
\begin{tikzcd}
\text{real-world problem} \arrow[d, "\text{encoder}"] \\
\text{QBF encoding} \arrow[d, "\text{quantifier shift technique}"] \\
\text{PCNF QBF encoding}
\end{tikzcd}
$$

Do prenexing methods matter?

real-world problem

↓ encoder

QBF encoding

↓ quantifier shift technique

PCNF QBF encoding

↓ QBF solver

solution

Do prenexing methods matter?

real-world problem

negligible ↓ encoder

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does the specific quantifier shift technique affect QBF solver runtime?

Do prenexing methods matter?

real-world problem

negligible $\downarrow$ encoder

QBF encoding

negligible $\downarrow$ quantifier shift technique

PCNF QBF encoding

we are only measuring this!

expensive $\downarrow$ QBF solver

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real-world problem

negligible ↓ encoder

QBF encoding

negligible ↓ quantifier shift technique

PCNF QBF encoding

expensive ↓ QBF solver

solution

... how does one shift quantifiers anyway?

we are only measuring this!

does the specific quantifier shift technique affect QBF solver runtime?

Quantifier shifting

We can assume the target QBF:

- is in *negation normal form* (i.e. negation only occurs in front of variables)
- follows the *Barendregt convention* (i.e. variables are renamed apart)

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$$(\mathsf{Q}x.\, \varphi) \circ \psi \equiv \varphi \circ (\mathsf{Q}x.\, \psi) \equiv \mathsf{Q}x.\, \varphi \circ \psi \qquad \mathsf{Q}x.\, \mathsf{Q}y.\, \varphi \equiv \mathsf{Q}y.\, \mathsf{Q}x.\, \varphi$$

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Lesson 1 Connective structure is irrelevant for quantifier shifting.

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Lesson 1 Connective structure is irrelevant for quantifier shifting.

Lesson 2 Only choices in distinct, branched quantifiers matter:

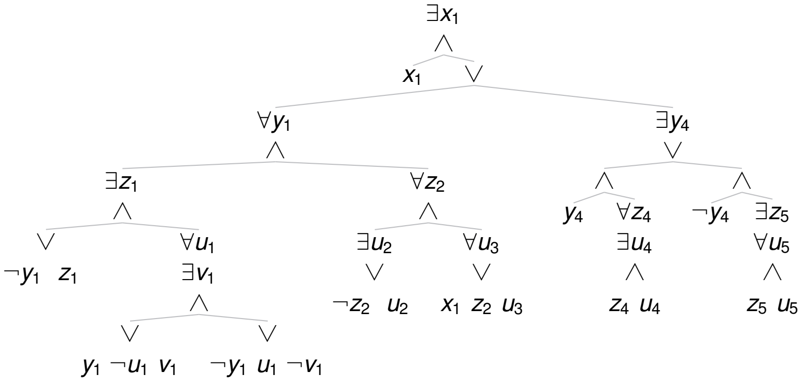
$$(\mathsf{Q}x.\,\varphi) \circ (\overline{\mathsf{Q}}y.\,\psi) \equiv \mathsf{Q}x.\,\overline{\mathsf{Q}}y.\,\varphi \circ \psi \equiv \overline{\mathsf{Q}}y.\,\mathsf{Q}x.\,\varphi \circ \psi$$

Quantifier shifting: how the sausage is made

$$\exists x_1 .\, x_1 \wedge ((\forall y_1 .\, (\exists z_1 .\, (\neg y_1 \vee z_1) \wedge \forall u_1 .\, \exists v_1 .\, (y_1 \vee \neg u_1 \vee v_1) \wedge (\neg y_1 \vee u_1 \vee \neg v_1)) \wedge (\forall z_2 .\, (\exists u_2 .\, \neg z_2 \vee u_2) \wedge (\forall u_3 .\, x_1 \vee z_2 \vee u_3))) \vee (\exists y_4 .\, (y_4 \wedge \forall z_4 .\, z_4 \wedge u_4) \vee (\neg y_4 \wedge \exists z_5 .\, \forall u_5 .\, z_5 \wedge u_5)))$$

QBF formula φ

Quantifier shifting: how the sausage is made



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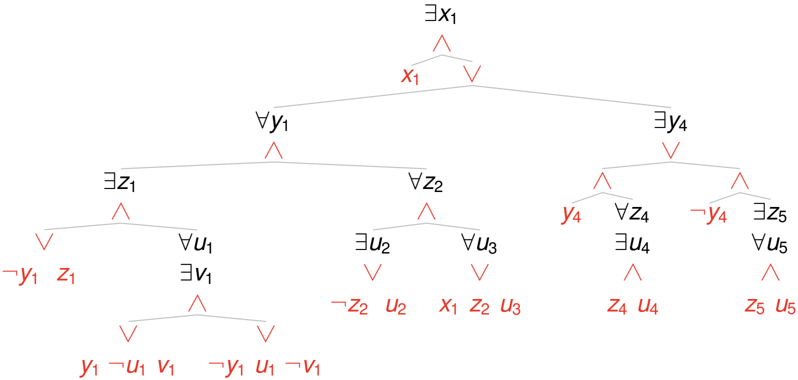

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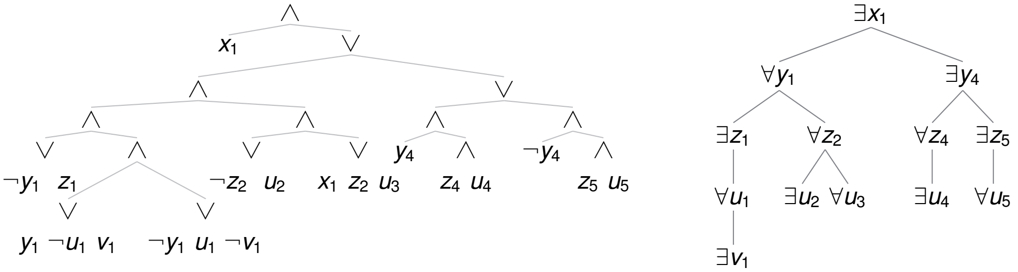

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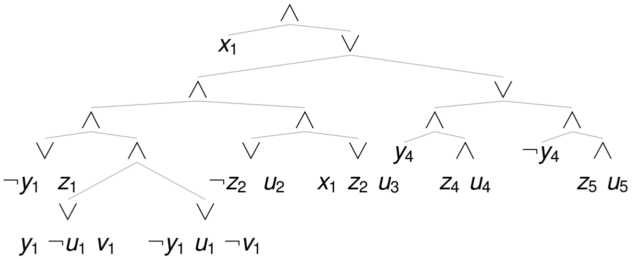

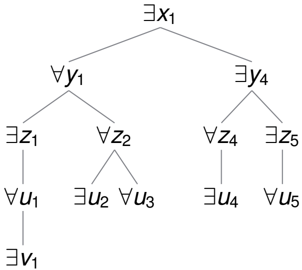

propositional skeleton φ^{psk}

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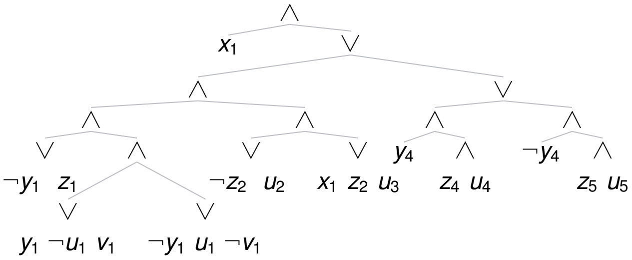

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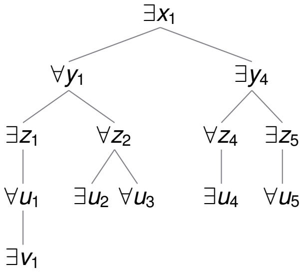

quantifier tree

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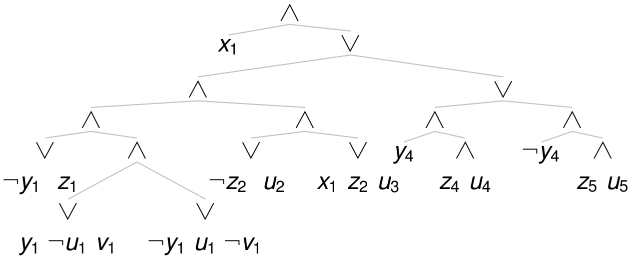

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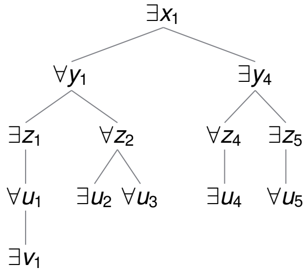
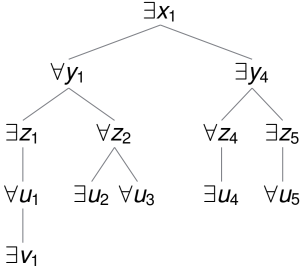


quantifier tree

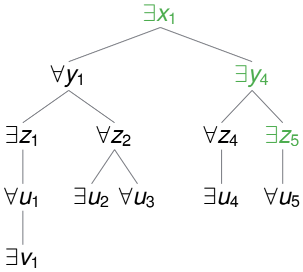
goal: $\exists X_1 . \forall X_2 . \exists X_3 . \forall X_4 . \exists X_5 . \varphi^{psk}$

Quantifier shifting: how the sausage is made



$$\exists \qquad . \forall \qquad . \exists \qquad . \forall \quad . \exists \quad . \varphi^{\mathrm{psk}}$$

Quantifier shifting: how the sausage is made



$$\exists x_1 y_4 z_5 . \forall \qquad . \exists \qquad . \forall \quad . \exists \quad . \varphi^{\mathrm{psk}}$$

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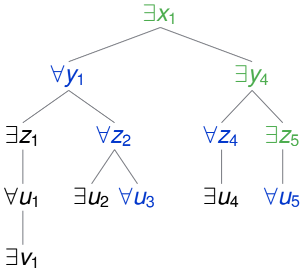

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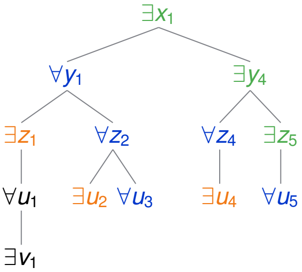

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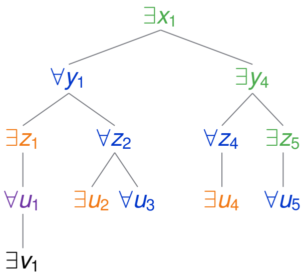

$$\exists x_1\, y_4\, z_5 .\, \forall y_1\, z_2\, u_3\, z_4\, u_5 .\, \exists z_1\, u_2\, u_4 .\, \forall u_1 .\, \exists \quad .\, \varphi^{\mathrm{psk}}$$

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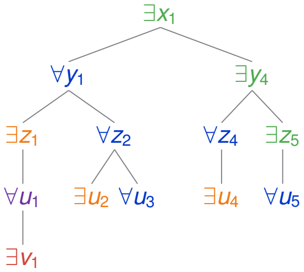

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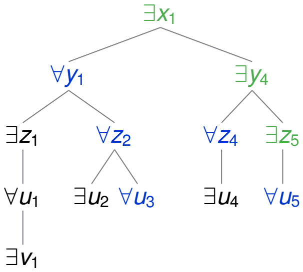

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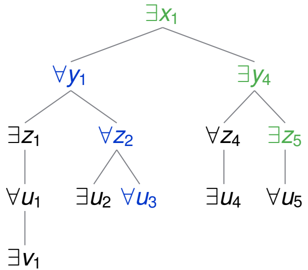

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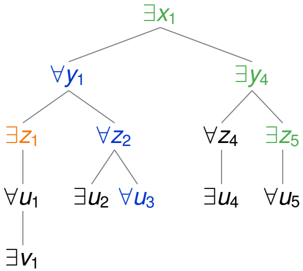

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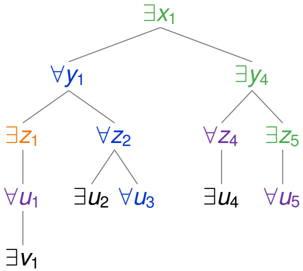

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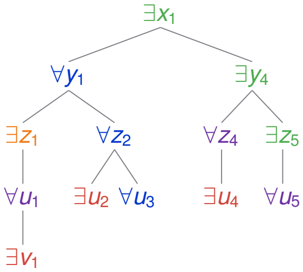

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Digression: Thoralf Skolem would like a word with us

In FOL we can eliminate existential quantifiers via Skolem functions:

$$\forall x.\,(\forall y.\,\exists a.\,\varphi(x, y, a)) \wedge (\forall z.\,\exists b.\,\psi(x, z, b))$$

$$\downarrow$$

$$\forall x.\,\forall y.\,\forall z.\,\varphi(x, y, a(x, y)) \wedge \psi(x, z, b(x, z))$$

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Reality: QBF $\subseteq$ SOL

Skolemization over QBF operates on HOL

Methods for quantifier shifting

- Egly, Seidl, Tompits, Woltran, Zolda. *Comparing different prenexing strategies for quantified Boolean formulas*. SAT 2003.

  Quantifiers are shifted following strategies $\exists^{\dagger}\forall^{\ddagger}$ with $\dagger, \ddagger \in \{\uparrow, \downarrow\}$.
  - $\uparrow$ means "push this quantifier rootwards"
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A systematic study of the effect of quantifier shifting on solver performance requires well-defined quantifier shifting algorithms

Digression: Hamilton would like a word with us

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Newtonian mechanics describe the configuration of a system depending on previous configuration

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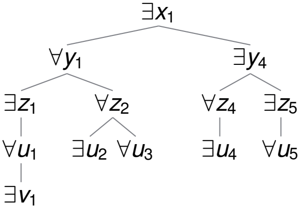
Newton rules modify a configuration until halt
Hamilton rules describe configuration preference

Quantifier trees and linearizations

We can regard quantifier trees as partially ordered sets

$x < y$ iff y is a descendent of x



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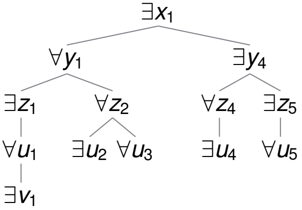

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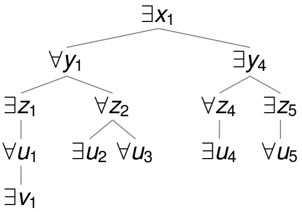
Linearizations: quantifier tree maps that preserve (non-strict) order and quantifiers



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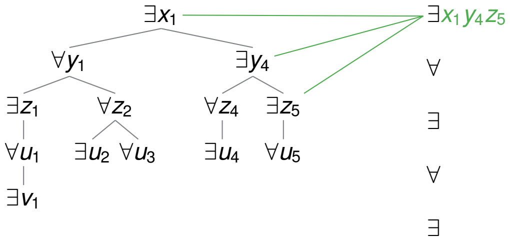

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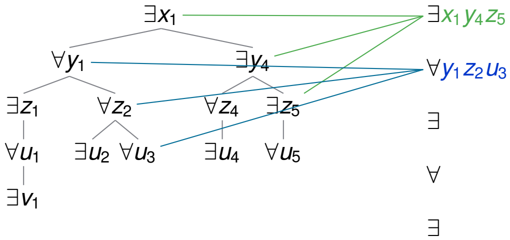

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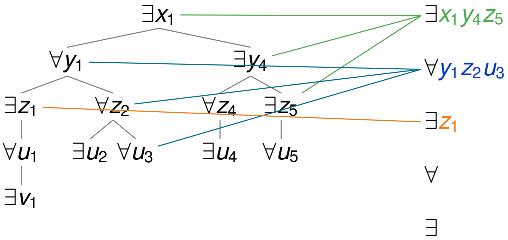

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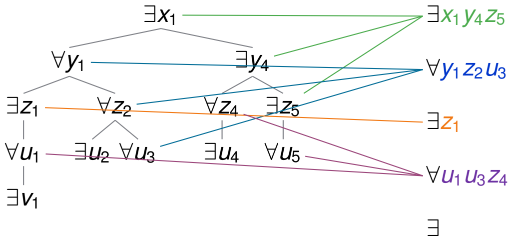

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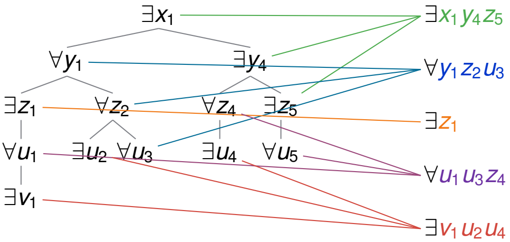

Quantifier trees and linearizations

We can regard quantifier trees as partially ordered sets

$x < y$ iff y is a descendent of x

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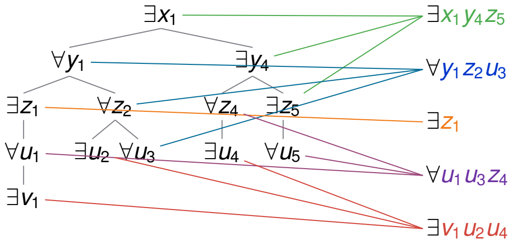
Linearizations: quantifier tree maps that preserve (non-strict) order and quantifiers



linearizations induce semantically equivalent quantifier shifts

Preferences over linearizations

The plan

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- Define a partial order over linearizations (*preferences*)

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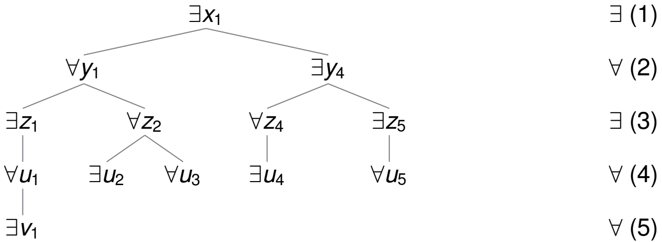

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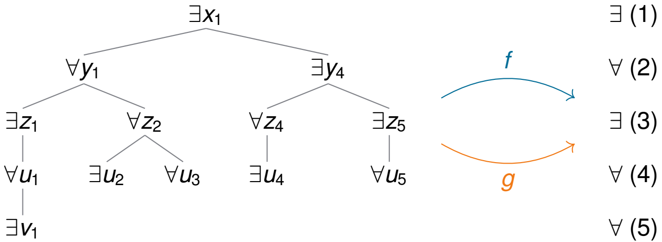

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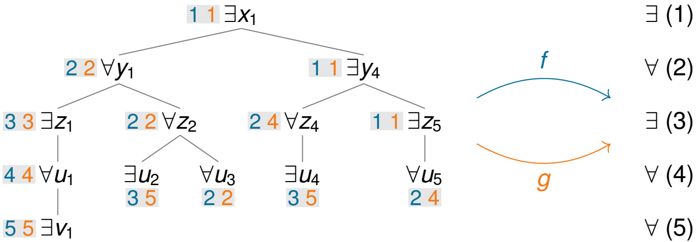

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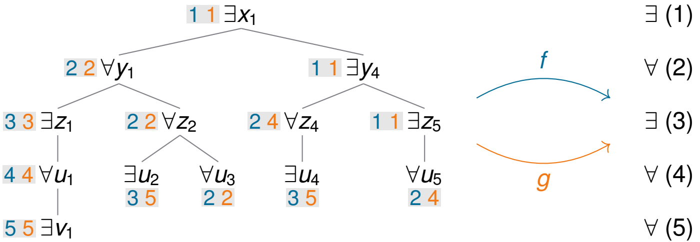

"push ∀ rootwards"

$f > g$

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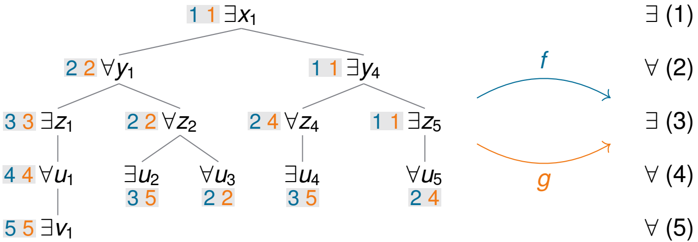

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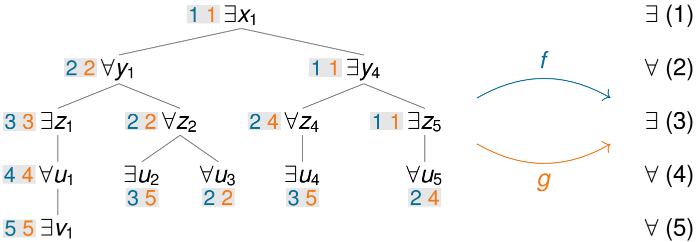

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Preferences over linearizations

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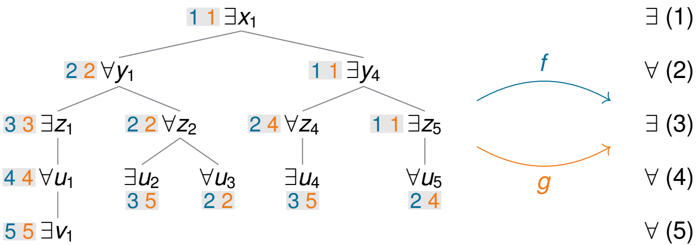
- Define a partial order over linearizations (*preferences*)
- Show that the maximal linearization is unique (and pray that it is easy to compute)



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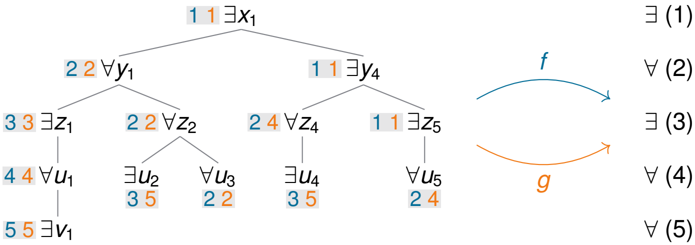
Semipreferences



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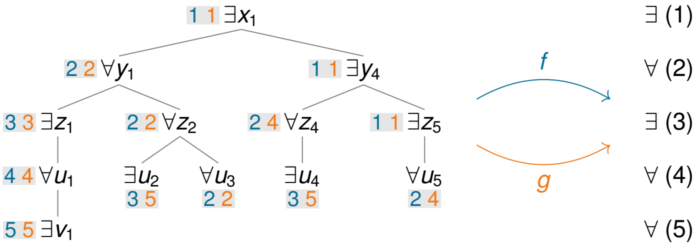

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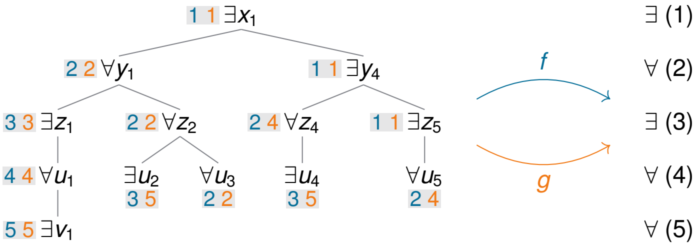

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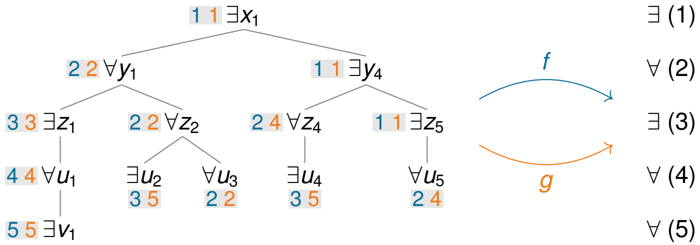

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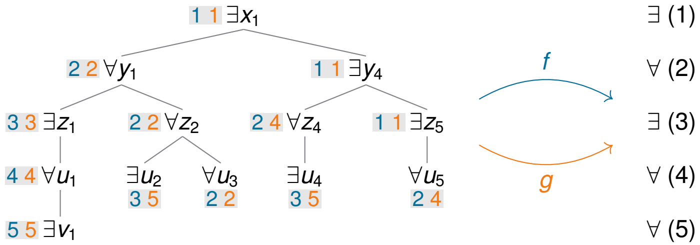

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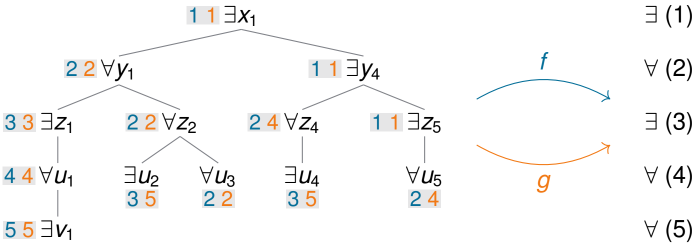

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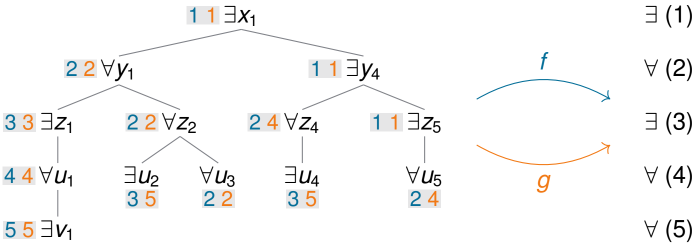

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these are not partial orders, though...

Strategies, but better

We can now express the strategies from [Egly *et al*, 2003] as preference relations:

$$f \leq^{\exists\downarrow\forall\uparrow} g \text{ iff } f \leq^{\exists\downarrow} g \text{ and } f \leq^{\forall\uparrow} g$$

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We propose strategies of the form $Q\dagger\ddagger$, where $Q \in \{\forall, \exists\}$ and $\dagger, \ddagger \in \{\uparrow, \downarrow\}$:

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- In case of a trade-off, Q quantifiers are prioritized over $\overline{Q}$

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- No. We give a single-pass recursive algorithm that computes the maximum with negligible overhead

The algorithm

Theoretical results. Let us consider a quantree $(T, \leq, q)$ and a strategy $\mathsf{Q}\dagger\ddagger$. We define the mapping $\Gamma_\dagger : T \to \{1, \ldots, \mathrm{aht}(T)\}$ given by

$$\Gamma_\dagger(x) = \big\lfloor |\max{}^\dagger\{1, \ldots, \mathrm{aht}(T)\} - \mathrm{aht}(T_x^\dagger)| + 1 \big\rfloor^\dagger_{q^\star(x)} . \tag{1}$$

Furthermore, we define the mapping $[f]^{\mathsf{Q}\ddagger} : T \to \{1, \ldots, \mathrm{aht}(T)\}$ for $f \in \mathrm{Lin}(T)$ given by

$$[f]^{\mathsf{Q}\ddagger}(x) = \big\lfloor \min{}^\ddagger\{f(y) \mid y \in T_x^\ddagger \text{ and } q(y) = \mathsf{Q}\} \big\rfloor^\ddagger_{q^\star(x)} . \tag{2}$$

Example 7 suggests that $[f]^{\mathsf{Q}\ddagger}$ can be computed recursively. Indeed, the rank of a node can be computed based on the ranks of its children or parent.

Corollary 1. *Let $x \in T$ such that $q(x) \neq \mathsf{Q}$. Then,*

$$[f]^{\mathsf{Q}\ddagger}(x) = \Big\lfloor \min{}^\ddagger\{[f]^{\mathsf{Q}\ddagger}(y) \mid x \textit{ is covered by } y \in T \textit{ w.r.t. } \leq^\ddagger\}. \Big\rfloor^\ddagger_{q^\star(x)}$$

Conclusion

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Is there a tool? Yes. Go to **Maximilian Heisinger**'s talk on Friday at 17:10.

JKU

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