Solution to Exercise 19.6(b)

Arpon Basu

February 2023

If σ_1, σ_2 are most general substitutions, then $\sigma_1 = \sigma_2\tau_2$, $\sigma_2 = \sigma_1\tau_1$ so $\sigma_1 = \sigma_1\tau_1\tau_2$. Let y be a free variable in the image of σ_1. Then there exists a variable x such that y belongs to FV($x\sigma_1$). Then note that $y\tau_1\tau_2$ belongs to FV($x\sigma_1\tau_1\tau_2$) = FV($x\sigma_1$), and similarly, $y(\tau_1\tau_2)^n$ belongs to FV($x\sigma_1$) for every n. Now note that for every term t and substitution s, we have len(ts) $\geq$ len(t). Furthermore, y is contained in $y\tau_1\tau_2 = x\sigma_1$. Thus if the length of $y\tau_1\tau_2$ is greater than 1, then the size of $y(\tau_1\tau_2)^n$ goes to infinity as n goes to infinity. Thus $y\tau_1\tau_2$ is of length 1. But since $y\tau_1\tau_2$ contains y, it must be that $y\tau_1\tau_2 = y$. Now note that $y\tau_1\tau_2 = y$ also implies that $y\tau_1$ is a variable (it can't be a constant since that would prevent further substitution by τ_2). Thus τ_1 is a map from Vars to Vars (Note that Vars = Domain(σ_1) $\cup$ Domain(σ_2)). Furthermore, τ_1 is injective: Indeed, if we have two variables y, y' such that $y\tau_1 = y'\tau_1$, then we have $y = y\tau_1\tau_2 = y'\tau_1\tau_2 = y'$. Since τ_1 is an injective map between two finite sets, τ_1 is a bijection. But that means that σ_2 is a renaming of σ_1. Similarly as above, σ_1 is also a renaming of σ_2, and thus σ_1, σ_2 are renamings of each other, as desired.