CS228 Tutorial Solutions

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Contents

1 Tutorial 1

1.1 Q1

Note that FO-definable languages are a (strict) subset of regular languages. Consequently, if a language is FO-definable then it is automatically regular.

(a) The language is $a\Sigma^*a+b\Sigma^*b+a+b+\varepsilon$, which is regular and FO definable by the formula $\varphi := \exists x, y.(\forall z.(x \leq z \leq y) \wedge (Q_a(x) \iff Q_a(y))) \vee \forall x.(x \neq x)$, assuming $\Sigma = \{a, b\}$.

(b) The language is $a^*\#b^*$, which is regular and FO definable by the formula $\varphi := \exists x.(Q_\#(x) \wedge \forall y.(y < x \implies Q_a(y)) \wedge \forall y.(x < y \implies Q_b(y)))$.

(c) The language is a^*b^*, which is regular and FO definable by the formula $\varphi := \forall x, y.(S(x, y) \wedge Q_b(x) \implies Q_b(y))$.

(d) The language is $\Sigma 0\Sigma^* 0\Sigma + \Sigma 0\Sigma + 0^2$, which is regular and FO definable by the formula $\varphi := \exists x.((\forall y.y \geq x) \wedge \exists t.(S(x, t) \wedge Q_0(t))) \wedge \exists x.((\forall y.y \leq x) \wedge \exists t.(S(t, x) \wedge Q_0(t)))$.

(e) Note that our alphabet here is $\Sigma = \{\binom{0}{0}, \binom{0}{1}, \binom{1}{0}, \binom{1}{1}\}$. If the top row is larger than the bottom row, then we must have a $\binom{1}{0}$ somewhere, preceding which all digits should be the same. Thus the language is $(\binom{0}{0}+\binom{1}{1})^*\binom{1}{0}\Sigma^*$, which is FO definable by the formula $\varphi := \exists x.(Q_{\binom{1}{0}}(x) \wedge \forall y.(y < x \implies (Q_{\binom{0}{0}}(y) \vee Q_{\binom{1}{1}}(y))))$.

1.2 Q2

(1)

(a) $\mathcal{L}(\varphi) = \{\varepsilon\}$

(b) $\overline{\mathcal{L}(\varphi)} = \Sigma^* \setminus \{\varepsilon\}$, ie:- the set of all non-empty words.

(c) Yes. Consider a DFA $\mathcal{A}$ with only two states α, β, where α is the start state, and the only accept state. The transitions are $\delta(\alpha, \sigma) = \delta(\beta, \sigma) = \beta$ for every $\sigma \in \Sigma$. $\mathcal{A}$ accepts $\{\varepsilon\}$.

(d) The complement of a regular language is regular, so yes.

(2)

(a) $\mathcal{L}(\varphi) = \Sigma^*(ba^+)\Sigma^*$

(b) $\overline{\mathcal{L}(\varphi)} = a^*b^*$

(c)

(d) The complement of a regular language is regular, so yes.

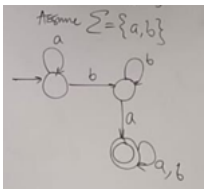


Figure 1: 2(2)(c)

(3)

(a) $\mathcal{L}(\varphi) = \Sigma^* a \Sigma$

(b) $\overline{\mathcal{L}(\varphi)} = \{\varepsilon, a, b\} \cup \Sigma^* b \Sigma$

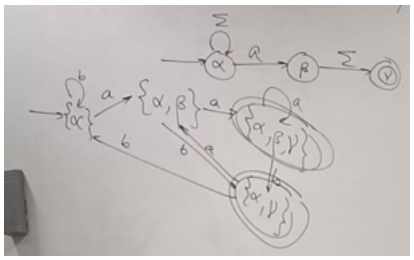


Figure 2: 2(3)(c)

(c)

(d) The complement of a regular language is regular, so yes.

(4)

(a) $\mathcal{L}(\varphi) = (ab)^+$

(b)

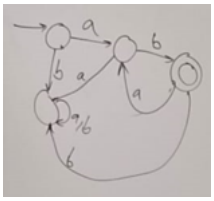


Figure 3: 2(4)(c)

(c)

(d) The complement of a regular language is regular, so yes.

1.3 Q3

The language described by the DFA is $b(a^+b^3)^*$. It is FO describable, with the (conjunction of the) following sentences:-

1. First letter is 'b': $\exists x \forall y (x \leq y \wedge Q_b(x))$
2. If the length of the word is more than 1, then it ends with a "bbb":

$$(\exists x, y.S(x,y)) \implies \exists x.((\forall t.t \leq x) \wedge Q_b(x) \wedge \exists y.(S(y,x) \wedge Q_b(y) \wedge \exists z.(S(z,y) \wedge Q_b(z))))$$

3. For every "bbb", there is a non-empty block of 'a's before it, preceding which there is a 'b':

$$\forall x, y, z.((S(x,y) \wedge S(y,z) \wedge Q_b(x) \wedge Q_b(y) \wedge Q_b(z)) \implies ($$
$$(\exists t.S(t,x) \wedge Q_a(t)) \bigwedge$$
$$(\exists u.u < x \wedge Q_b(u) \wedge (\forall v.(u < v < x) \implies Q_a(v))))$$

4. We don't have 4 consecutive 'b's:

$$\neg \exists x, y, z, w.S(x,y) \wedge S(y,z) \wedge S(z,w) \wedge Q_b(x) \wedge Q_b(y) \wedge Q_b(z) \wedge Q_b(w)$$

5. We don't have "aba":

$$\neg \exists x, y, z.S(x,y) \wedge S(y,z) \wedge Q_a(x) \wedge Q_b(y) \wedge Q_a(z)$$

6. If a 'b' is preceded by an 'a' and succeeded by a 'b', then it is a triplet,

$$\forall x, y, z.(Q_a(x) \wedge S(x,y) \wedge Q_b(y) \wedge S(y,z) \wedge Q_b(z) \implies \exists t.(S(z,t) \wedge Q_b(t)))$$

2 Tutorial 2

Even though ε-NFAs have not been taught in class, they are very useful in practice. In particular, we shall be using them a lot in our tutorial. Please refer to this link for a formal proof of equivalence (of power) of ε-NFAs, NFAs, and DFAs.
We shall represent DFAs as 5-tuples $(Q, q_0, \Sigma, \delta, \mathcal{F})$, where Q denotes the set of states, q_0 is the start state, Σ the alphabet, $\delta : Q \times \Sigma \mapsto Q$ the transition function, and $\mathcal{F}$ is the set of final states.
Similarly, NFAs will be represented as $(Q, Q_0, \Sigma, \delta, \mathcal{F})$, where Q denotes the set of states, Q_0 is the set of start states, Σ the alphabet, $\delta : Q \times \Sigma \mapsto 2^Q$ the transition function, and $\mathcal{F}$ is the set of final states.

2.1 Q1

(a) Yes. Given a DFA $(Q, q_0, \Sigma, \delta, \mathcal{F})$, construct the NFA $(Q, \mathcal{F}, \Sigma, \delta', \{q_0\})$, where $\delta'(i, \sigma) = j$ if and only if $\delta(j, \sigma) = i$, for every $i, j \in Q, \sigma \in \Sigma$, ie:- we reverse all the transitions, and make our final states initial states, and the initial state final. It's easy to see that the NFA only accepts words that are the reverse of some word accepted by the DFA, and thus the reverse of a language is also regular.

(b) Let $\mathcal{A}$ be a DFA representing L. Replace all transitions of a in $\mathcal{A}$ to ε-transitions. The resulting ε-NFA recognizes $L \downarrow \{b, c\}$.

(c) Let our NFA be $(Q, Q_0, \Sigma, \delta, \mathcal{F}), |\mathcal{F}| > 1$. Construct a new NFA $(Q \cup \{f\}, Q_0', \Sigma, \delta', \{f\})$, where, for every $i \in Q$ and every $\sigma \in \Sigma$ we have

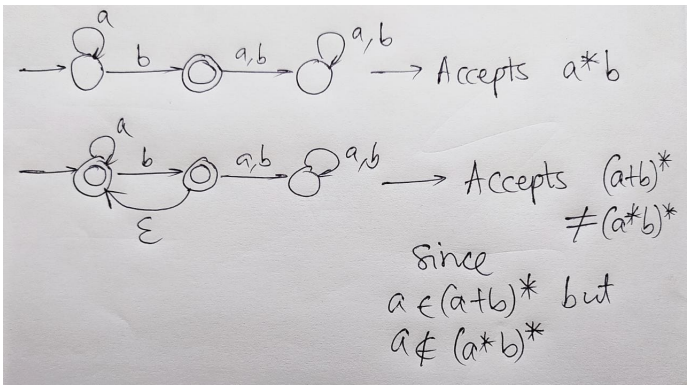
$$\delta'(i, \sigma) = \begin{cases} \delta(i, \sigma), \text{ if } \delta(i, \sigma) \cap \mathcal{F} = \emptyset \\ \delta(i, \sigma) \cup \{f\}, \text{ otherwise} \end{cases}$$

$$\delta'(f, \sigma) = \emptyset$$

$$Q_0' = \begin{cases} Q_0, \text{ if } Q_0 \cap \mathcal{F} = \emptyset \\ Q_0 \cup \{f\}, \text{ otherwise} \end{cases}$$

The new NFA recognizes the same language as the old one but has only one final state.
Challenge:- Demonstrate a regular language L such that any DFA recognizing L has atleast 2 accepting states.

(d) This construction, as specified is not equivalent to L^*, as was pointed out by Krishna. Here's the counterexample of this construction which shows that this construction doesn't work.
Given here is a close variant of the above construction that does work.



(e) Refer this.

2.2 Q2

Fix any $n \geq 2$ (for $n = 1$, $L_1 = \Sigma^* \setminus \{\varepsilon\}$, which is regular). We assume that ε is not divisible by any $n \in \mathbb{N}$.
Construct the DFA $\mathcal{A}_n = (Q, \text{start state}, \Sigma, \delta, \mathcal{F}) = (\{s, q_0, q_1, \ldots, q_{n-1}\}, s, \{0, 1\}, \delta, \{q_0\})$, with the transitions being as follows:

$$\delta(s, 0) = q_0, \delta(s, 1) = q_1$$

$$\delta(q_i, 0) = q_{(2i) \bmod n}, \delta(q_i, 1) = q_{(2i+1) \bmod n}$$

We finish the proof that L_n is regular by noting that $\mathcal{L}(\mathcal{A}_n) = L_n$ [1].

2.3 Q3

Let $\mathcal{A} = (Q, Q_0, \Sigma, \delta, \mathcal{F})$ be our NFA. Let $\mathcal{B} = (2^Q, q_0', \Sigma, \delta', \mathcal{F}')$ be the DFA constructed from our NFA according to the usual powerset construction, assuming that $\mathcal{A}$ is angelic. Note that $\mathcal{F}' = \{S \in 2^Q : S \cap \mathcal{F} \neq \emptyset\}$.
However, if $\mathcal{A}$ were to be now interpreted with a devilish condition, we show that it would still accept a regular language by claiming that the DFA $\tilde{\mathcal{B}}$ is equivalent to $\mathcal{A}_{\text{devilish}}$, where $\tilde{\mathcal{B}} := (2^Q, q_0', \Sigma, \delta', \tilde{\mathcal{F}}')$ where we now define $\tilde{\mathcal{F}}' := \{S \in 2^Q : S \subseteq \mathcal{F}, S \neq \emptyset\}$.
The reason this construction works can be attributed to the meaning of the "subset of states" in the powerset construction: Since NFAs are non-deterministic when NFAs read a word, they can land up in multiple different states, and we collect those states to form one state of our DFA. Thus, when a set of states contains a final state, that represents that *some* run of the word through the NFA lands up in a final state, and the angelic condition then ensures its acceptance.
However, since the devilish acceptance condition requires that *every* run of our NFA ends up in a final state, we correspondingly set our final state condition in the constructed DFA to reflect that.
Also note that the question *does not* claim that $\mathcal{L}(\mathcal{A}_{\text{angelic}}) = \mathcal{L}(\mathcal{A}_{\text{devilish}})$. Indeed, all the question asks us to show is that $\mathcal{L}(\mathcal{A}_{\text{devilish}})$ is regular. Its easy to see that in general, $\mathcal{L}(\mathcal{A}_{\text{angelic}}) \neq \mathcal{L}(\mathcal{A}_{\text{devilish}})$.

2.4 Q4

Let $\mathcal{A} = (Q, q_0, \Sigma, \delta, \mathcal{F})$ be a DFA recognizing L.
Note that a proper prefix of some word w can belong to L if and only if while reading w, we pass through some final state of $\mathcal{A}$ (and don't stop there).
We shall prove L' is regular by constructing a DFA recognizing (exactly) L'. Indeed, consider the DFA $\mathcal{B} := (Q \cup \{d\}, q_0, \Sigma, \delta', \mathcal{F})$, with the transition function being given by

$$\forall \sigma \in \Sigma, \forall n \in Q \setminus \mathcal{F},\ \delta'(n, \sigma) = \delta(n, \sigma)$$

[1] Note that in the exam you would be expected to formally argue why $\mathcal{A}_n$ accepts every word in L_n and doesn't accept any word in $\overline{L_n}$, for full credit.

$$\forall \sigma \in \Sigma, \forall f \in \mathcal{F},\ \delta'(f, \sigma) = \delta'(d, \sigma) = d$$

We finish the proof that L' is regular by noting that $\mathcal{L}(\mathcal{B}) = L'$.
Note:- In this question, we've assumed that ε is a proper prefix of every non-empty word.

2.5 Q5

Assume for the sake of contradiction that there exists $n \in \mathbb{N}$ such that L_n is the language of a DFA $\mathcal{A}$ with $< 2^{n-1}$ states. Let K_n be the set of all strings with length $\mathbf{n-1}$. For any $w \in \Sigma^*$, denote s_w to be the state $\mathcal{A}$ reaches on reading w.
Since $|K_n| = 2^{n-1}$, and since $\mathcal{A}$ has $< 2^{n-1}$ states, by the pigeonhole principle there exist two distinct words $u, v \in K_n$ such that $s_u = s_v$. Since $u \neq v$, there exists $\ell \in [n-1]$ such that $u_\ell \neq v_\ell$. WLOG let $u_\ell = 0, v_\ell = 1$. Now, since $s_u = s_v$, we also have $s_{u0^\ell} = s_{v0^\ell}$. Now, since the n^{th} last bit of $v0^\ell$ is the same as $v_\ell = 1$, we get that $v0^\ell \in L_n$, and consequently $s_{v0^\ell} = s_{u0^\ell}$ is a final state of $\mathcal{A}$, implying that $u0^\ell \in L_n$. However, the n^{th} last bit of $u0^\ell$ is equal to $u_\ell = 0$, which contradicts the definition of L_n.

3 Tutorial 3

3.1 Q1

1. The formulation of a counting language is

$$\forall n_0 \exists n \geq n_0 \exists u, v, w \in \Sigma^*. uv^n w \in L \iff \neg(uv^{n+1}w \in L)$$

2. $(aa)^+$ is a counting language: Indeed, let n_0 be any natural number. Then for $u = \varepsilon, v = a, w = \varepsilon$, $uv^{n_0}w$ and $uv^{n_0+1}w$ can't both belong to $(aa)^+$.

3. We shall prove that $(ab)^+$ is non-counting by showing that for all $u, v, w \in \Sigma^*$, we have for $L := \{uv^n w : n \geq 2\}$, $L \cap (ab)^+ = L$ or $\emptyset$. Then the definition of non-counting would be satisfied by the witness $n_0 = 2$.
   If $v = \varepsilon$, the statement is trivially true, so assume $v \neq \varepsilon$. If $|v| = 1$, $L \cap (ab)^+ = \emptyset$, since $uv^n w$, for every $n \geq 2$, contains two consecutive alphabets which are the same. Similarly, even for any v such that $|v| \geq 2$, if v contains two consecutive alphabets then $L \cap (ab)^+ = \emptyset$. Similarly, for v with $|v| \geq 2$, $L \cap (ab)^+ = \emptyset$ if the start and end letter of v are the same. Consequently, we assume that v is of the form $(ab)^t$ or $(ba)^t$ for some $t \geq 1$.
   If $v = (ab)^t$, then for any $n \in \mathbb{N}$ if we have $uv^n w \in (ab)^+$, we must also have $u, w \in (ab)^*$. But then L is composed of strings of the form $(ab)^*(ab)^{tn}(ab)^*$, which is a subset of $(ab)^+$, thus showing that $L \cap (ab)^+ = L$, as desired.
   If $v = (ba)^t$, then for any $n \in \mathbb{N}$ if we have $uv^n w \in (ab)^+$, we must also have $u \in (ab)^* a, w \in b(ab)^*$. But then L is composed of strings of the

form $(ab)^*a(ba)^{tn}b(ab)^*$, which is a subset of $(ab)^+$, thus showing that $L \cap (ab)^+ = L$, as desired.

3.2 Q2

(a) The flaw with the earlier constructions was that an inductive construction (such as the one we gave, where the base case was s, and we then inductively constructed the set of vertices reachable from s) only specifies what vertices we should *include*, not what we should *exclude*. Consequently, as was pointed out in the tutorial, one can include superfluous elements in our reachable set and defeat the intent of our construction. The solution then is to say that the reachable set of s is the *smallest* possible set satisfying our inductive construction: Which prevents superfluous elements from entering our set. Thus,

$$\text{PossibleReachable}(X) := X(s) \wedge \forall x, y.(X(x) \wedge E(x, y) \implies X(y))$$

$$\text{Subset}(A, B) := \forall x.(A(x) \implies B(x))$$

$$\text{Reachable}(S) := \text{PossibleReachable}(S) \wedge \forall X.(\text{PossibleReachable}(X) \implies \text{Subset}(S, X))$$

Our final sentence then is

$$\exists S.(\text{Reachable}(S) \wedge S(t))$$

(b) We define a few formulae first:

$$\text{NonEmpty}(X) := \exists x.X(x)$$

$$\text{UpperBound}(X, y) := \forall z.(X(z) \implies z \leq y)$$

$$\text{Bounded}(X) := \exists y.\text{UpperBound}(X, y)$$

$$\text{LeastUpperBound}(X, y) := \text{UpperBound}(X, y) \wedge \forall y'((\forall z.(X(z) \implies z \leq y')) \implies y \leq y')$$

Then our desired MSO sentence is

$$\forall X.(\text{NonEmpty}(X) \wedge \text{Bounded}(X) \implies \exists y.\text{LeastUpperBound}(X, y))$$

3.3 Q3

MSO and MSO_0 are equi-expressive.
Consider an MSO_0 formula. Wherever the formula contains the following MSO_0 atomic formulae, replace them with the following MSO formulae [2]. In this conversion, ensure that the variables introduced (free or quantified) are not present anywhere else in the formula, quantified or otherwise.

1. $\text{Sing}(X)$: $\exists x.(X(x) \wedge \forall y.(X(y) \implies (x = y)))$.

[2] these MSO_0 atomic formulae are replaced sequentially, in any arbitrary order

2. $X \subseteq Y$: $\forall x.(X(x) \implies Y(x))$
3. $X < Y$: First replace this by $\text{Sing}(X) \wedge \text{Sing}(Y) \wedge \forall x, y.(X(x) \wedge Y(y) \implies x < y)$, and then replace the Sing predicates according to the first rule.
4. $S(X, Y)$: First replace this by $\text{Sing}(X) \wedge \text{Sing}(Y) \wedge \forall x, y.(X(x) \wedge Y(y) \implies S(x, y))$, and then replace the Sing predicates according to the first rule.
5. $Q_a(X)$: $\forall x.(X(x) \implies Q_a(x))$

If we are given a formula in MSO, we can convert it to MSO_0 as follows:

1. $x = y$: $\text{Sing}(X) \wedge \text{Sing}(Y) \wedge Y \subseteq X$
2. $x < y$: $\text{Sing}(X) \wedge \text{Sing}(Y) \wedge X < Y$
3. $S(x, y)$: $\text{Sing}(X) \wedge \text{Sing}(Y) \wedge S(X, Y)$
4. $X(x)$: $\text{Sing}(Y) \wedge Y \subseteq X$. Here Y plays the role of x in the original formula.
5. $Q_a(x)$: $\text{Sing}(X) \wedge Q_a(X)$
6. $\forall x.\varphi$: $\forall X.(\text{Sing}(X) \implies \varphi)$

3.4 Q4

$$\exists X.\forall Y.((\text{Sing}(X) \wedge \text{Sing}(Y)) \implies (X < Y \implies Q_a(Y)))$$

3.5 Q5

Note that $L(N) = b(a^+b^3)^*$. We shall use two SO variables X_0, X_1 such that $X_0 \xrightarrow{b} X_1, X_1 \xrightarrow{a^+b^2} X_0$. Then, our sentence is

$$\exists X_0, X_1 \cdot$$
$$\forall x.(X_0(x) \iff \neg X_1(x)) \bigwedge$$
$$X_0(\text{first}) \wedge X_1(\text{last}) \wedge Q_b(\text{last}) \bigwedge$$
$$\forall x, y.S(x, y) \wedge X_0(x) \wedge Q_b(x) \implies X_1(y) \bigwedge$$
$$\forall y.X_1(y) \implies \exists x.S(x, y) \wedge X_0(x) \wedge Q_b(x) \bigwedge$$
$$\forall x, y.$$
$$x < y \wedge X_1(x) \wedge \forall z.(x < z < y \implies (\neg X_0(z) \wedge \neg X_1(z))) \bigwedge$$
$$\exists y_1, y_2.S(y_2, y_1) \wedge S(y_1, y) \wedge Q_b(y_2) \wedge Q_b(y_1) \wedge \forall z.(x < z < y_2 \implies Q_a(z)) \wedge ((\exists z.x < z < y_2) \implies X_0(y))$$

A very troll-like answer to this question would be to note that $L(N)$ is FO-definable: Consequently, write the FO formula to describe $L(N)$ (which we already did in a previous tutorial), and introduce two dummy SO variables to sit and do nothing, thus constructing an MSO formula with two SO variables describing $L(N)$.

4 Tutorial 4

4.1 Q1

(a) $\Diamond(p \wedge \neg p)$

(b) $\neg p \bigcup (p \wedge \bigcirc((\neg q \wedge \neg p) \bigcup (p \wedge \bigcirc\Box\neg p)))$

(c) $(\Diamond\Box\neg r) \implies (\Diamond\Box p)$

(d) $r \bigcup (\neg r \wedge p \wedge q \wedge \varphi)$, where

$$\varphi := \Box((q \implies \bigcirc\Diamond r) \wedge (r \implies \bigcirc\Diamond q))$$

Note that it isn't clear in the question what is meant by q, r "alternate infinitely often": I have taken it to mean that if q, r occur anywhere, then they are eventually followed by r, q respectively, and that sets up a chain of an infinite alternation.

(e) $\Box\Diamond(p \wedge \bigcirc p)$

(f) $(p \wedge \bigcirc(\neg p \bigcup p)) \implies (\bigcirc\psi_1)$ where $\psi_1 := \neg(\neg q \bigcup p)$.

4.2 Q2

Note that TS $\models P \iff$ TS $\subseteq P$.
If TS and TS$'$ satisfy the same set of LT properties, then for every $P \subseteq (2^{\text{AP}})^\omega$, we have Traces(TS) $\subseteq P \iff$ Traces(TS$'$) $\subseteq P$. Consequently, taking $P :=$ Traces(TS) $\subseteq (2^{\text{AP}})^\omega$ yields Traces(TS) $\subseteq$ Traces(TS$'$). Similarly one may obtain Traces(TS$'$) $\subseteq$ Traces(TS), thus proving Traces(TS) = Traces(TS$'$).
Conversely, if Traces(TS) = Traces(TS$'$), then it's obvious that Traces(TS) $\subseteq P \iff$ Traces(TS$'$) $\subseteq P$ for every $P \subseteq (2^{\text{AP}})^\omega$, and consequently TS and TS$'$ satisfy the same set of LT properties.

4.3 Q3

Note that

$$\varphi \Delta \psi \equiv \bigcirc\left(\varphi \bigcup \psi\right)$$

Consequently, LTL is at least as powerful as $\mathcal{X}$.
Conversely, note that

$$\bigcirc\psi \equiv (p \wedge \neg p)\Delta\psi$$

[3] since $\bot \bigcup \psi \equiv \psi$. Also,

$$\varphi \bigcup \psi \equiv \psi \vee (\varphi \wedge (\varphi \Delta \psi))$$

Furthermore, note that every other LTL operator such as $\Diamond$ and $\Box$ can be written in terms of $\bigcirc$ and $\bigcup$. Consequently, $\mathcal{X}$ is at least as powerful as LTL.
Thus, LTL and $\mathcal{X}$ have equal expressive power, or in other words, any LTL property can be expressed in $\mathcal{X}$, and any $\mathcal{X}$-property can be expressed in LTL.

[3] as some people pointed out in the tutorial, $\bot\Delta\psi \equiv \psi\Delta\psi$

4.4 Q4

4.4.1 Exercise 5.1

A state s satisfies an LT property P if and only if *every* path originating from s satisfies P. For example, $\bigcirc a$ is satisfied by s_1, s_2, s_3, s_4, since *every* trace originating from these states have their second state satisfying a.

4.4.2 Exercise 5.2

1. φ_1, φ_4 are not satisfied, since Traces(TS) contains $\{a\}(\{b\}\{b, c\})^\omega$.
2. φ_3 is satisfied: Indeed, one can verify that $\bigcirc\neg c$ can occur only if the second state of our trace is s_4. Since every state which is reachable from s_4 in exactly one step contains c, we have $\bigcirc\bigcirc c$.

4.4.3 Exercise 5.5

Let p_i denote the propositional variable that the lift is at the i^{th} floor, where $0 \leq i \leq 3$. Let q denote whether the lift door is open or not. We assume that the door *may* open if and only if the lift has stopped. Finally, let r_i denote the propositional variable that the i^{th} floor is requested. Then

1. $\Box(q \implies \text{ExactlyOneOf}(p_0, p_1, p_2, p_3))$ (unless you're building quantum lifts which can be on two floors simultaneously:))
2. $\bigwedge_{i=0}^{3} \Box(r_i \implies \Diamond(p_i \wedge q))$: By "serving" a floor, we mean that the lift is on the requested floor and its door is open.
3. $\Box\Diamond p_0$
4. $\Box(r_3 \implies (\neg q \bigcup (p_3 \wedge q)))$

4.4.4 Exercise 5.6

For showing non-equivalence, one counterexample is enough. For example, $\Diamond(\varphi \wedge \psi) \not\equiv (\Diamond\varphi) \wedge (\Diamond\psi)$ since for $\varphi := a, \psi := b$, we have that $(\{a\}\{b\})^\omega \in (\Diamond\varphi) \wedge (\Diamond\psi)$, but $(\{a\}\{b\})^\omega \notin \Diamond(\varphi \wedge \psi)$

4.4.5 Exercise 5.7

$$\varphi \mathsf{N} \psi \equiv \bigcirc((\Box\neg\psi) \vee \neg\psi \bigcup (\varphi \wedge \psi))$$

$$\varphi \mathsf{W} \psi \equiv \Box(\varphi \wedge \psi) \vee ((\psi \bigcup \neg\psi) \implies ((\psi \wedge \varphi) \bigcup \neg\psi))$$

$$\varphi \mathsf{B} \psi \equiv \Box(\neg(\neg\varphi \bigcup \psi))$$

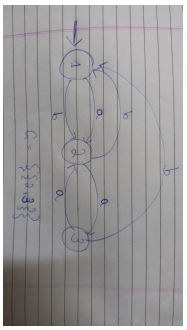


Figure 4: Tutorial 5, Q1(i). Credits to Om Swostik Mishra for this image

5 Tutorial 5

5.1 Q1

1.

2. Take the usual ω-automata $(Q, \Sigma, \delta, q_0, \text{Acc})$ whose $\text{Acc} = \mathcal{G} \subseteq Q$. Then in the new acceptance condition it corresponds to $(Q, \Sigma, \delta, q_0, \text{Acc}')$, where $\text{Acc}' = \mathcal{G}' \subseteq 2^Q$, where $\mathcal{G}' := \{S \subseteq Q : S \cap \mathcal{G} \neq \emptyset\}$.
   Now take any automata $\mathcal{A}$ with the new acceptance condition. Consider any $S \in \mathcal{G} \subseteq 2^Q$. Then $\mathcal{A}$ accepts every run ρ such that $\text{Inf}(\rho) = S$. Also, let s be an arbitrary member of S.
   Now, consider the subgraph of $\mathcal{A}$ induced by S. This subgraph can be interpreted as an NFA, whose start state and final state we specify to be s, and which recognizes some regular language $\mathcal{L}_{S,s}$. Now note that every run ρ such that $\text{Inf}(\rho) = S$ has an infinite suffix that starts from s, and consists only of states from S. Consequently, every such ρ, after reading some states in $\mathcal{A}$, settles down in the subgraph induced by S. Moreover, since $s \in \text{Inf}(\rho)$, ρ keeps returning to s. Consequently, the word read by ρ is of the form $u\mathcal{L}_{S,s}^{\omega}$, where u is some word that is read while traveling from the start state of $\mathcal{A}$ to s.
   We thus obtain a characterization of all words read by ρs such that $\text{Inf}(\rho) = S$: Interpret $\mathcal{A}$ as an NFA with final state s. Let the regular language recognized by this be U_s. Then $U_s\mathcal{L}_{S,s}^{\omega}$ is the exact set of words read by runs ρ such that $\text{Inf}(\rho) = S$.
   Consequently, the language accepted by $\mathcal{A}$ is $\bigcup_{S \in \mathcal{G}} U_s\mathcal{L}_{S,s}^{\omega}$. By Q4, this language is ω-regular.

3. We set $\mathcal{G}' := 2^Q \setminus \mathcal{G}$.

5.2 Q2

False. Consider the ω-word formed by the binary expansion of $\frac{1}{\sqrt{2}}$. The singleton language consisting of this word is not regular, since if it is of the format specified by Q4, it will contain a repeating unit, contradicting the fact that $\frac{1}{\sqrt{2}}$ is irrational.

5.3 Q3

The language $\mathcal{L} := \{a^{2n}b^{\omega} : n \geq 0\}$ is expressible in NBA but not in LTL.
Consider the NBA $\mathcal{A} = (Q, \Sigma, \delta, q_0, \text{Acc}) = (\{\alpha, \alpha', \beta\}, \{a, b\}, \delta, \{\alpha\}, \{\beta\})$, where $\delta(\alpha, a) = \{\alpha'\}, \delta(\alpha', a) = \{\alpha\}, \delta(\alpha, b) = \{\beta\}, \delta(\beta, b) = \{\beta\}$. Then $\mathcal{L}(\mathcal{A}) = \mathcal{L}$.
Since we have not established the technique of EF games yet, we can't show (at this stage) that $\mathcal{L}$ is not LTL-expressible. However, we can think of LTL as being analogous to FO-definable words, and NBAs as being analogous to regular languages. Consequently, for reasons similar to why $(aa)^*$ is regular but not FO-definable, $\mathcal{L}$ is NBA-expressible but not LTL-expressible.

5.4 Q4

This Wikipedia page establishes that all languages accepted by NBAs are of the form $\bigcup_{i=1}^{n} U_i V_i^{\omega}$, and conversely, every language of the form $\bigcup_{i=1}^{n} U_i V_i^{\omega}$ is ω-regular.
This link may also be helpful.