CS228 Tutorial Solutions

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1 Tutorial 1

1.1 Q1

Note that FO-definable languages are a (strict) subset of regular languages. Consequently, if a language is FO-definable then it is automatically regular.

(a) The language is $a\Sigma^*a + b\Sigma^*b$, which is regular and FO definable by the formula $\varphi := \exists x, y.(\forall z.(x \leq z \leq y) \wedge (Q_a(x) \iff Q_a(y)))$, assuming $\Sigma = \{a, b\}$.

(b) The language is $a^*\#b^*$, which is regular and FO definable by the formula $\varphi := \exists x.(Q_\#(x) \wedge (\forall y.y < x \implies Q_a(y)) \wedge (\forall y.x < y \implies Q_b(y)))$.

(c) The language is a^*b^*, which is regular and FO definable by the formula $\varphi := \forall x, y.(S(x, y) \wedge Q_b(x) \implies Q_b(y))$.

(d) The language is $\Sigma 0\Sigma^*0\Sigma + \Sigma 0\Sigma$, which is regular and FO definable by the formula $\varphi := \exists x.((\forall y.y \geq x) \wedge \exists t.(S(x, t) \wedge Q_0(t))) \wedge \exists x.((\forall y.y \leq x) \wedge \exists t.(S(t, x) \wedge Q_0(t)))$.

(e) Note that our alphabet here is $\Sigma = \{\binom{0}{0}, \binom{0}{1}, \binom{1}{0}, \binom{1}{1}\}$. If the top row is larger than the bottom row, then we must have a $\binom{1}{0}$ somewhere, preceding which all digits should be the same. Thus the language is $(\binom{0}{0}+\binom{1}{1})^*\binom{1}{0}\Sigma^*$, which is FO definable by the formula $\varphi := \exists x.(Q_{\binom{1}{0}}(x) \wedge \forall y.(y < x \implies (Q_{\binom{0}{0}}(y) \vee Q_{\binom{1}{1}}(y))))$.

1.2 Q2

(1)

(a) $\mathcal{L}(\varphi) = \{\varepsilon\}$

(b) $\overline{\mathcal{L}(\varphi)} = \Sigma^* \setminus \{\varepsilon\}$, ie:- the set of all non-empty words.

(c) Yes. Consider a DFA $\mathcal{A}$ with only two states α, β, where α is the start state, and the only accept state. The transitions are $\delta(\alpha, \sigma) = \delta(\beta, \sigma) = \beta$ for every $\sigma \in \Sigma$. $\mathcal{A}$ accepts $\{\varepsilon\}$.

(d) The complement of a regular language is regular, so yes.

(2)

(a) $\mathcal{L}(\varphi) = \Sigma^*(ba^+)\Sigma^*$

(b) $\overline{\mathcal{L}(\varphi)} = a^*b^*$

(c)

(d) The complement of a regular language is regular, so yes.

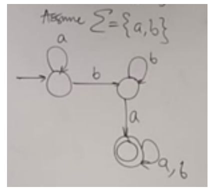

Figure 1: 2(2)(c)

(3)

(a) $\mathcal{L}(\varphi) = \Sigma^* a \Sigma$

(b) $\overline{\mathcal{L}(\varphi)} = \{\varepsilon, a, b\} \cup \Sigma^* b \Sigma$

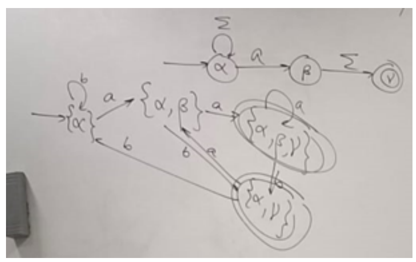

Figure 2: 2(3)(c)

(c)

(d) The complement of a regular language is regular, so yes.

(4)

(a) $\mathcal{L}(\varphi) = (ab)^+$

(b)

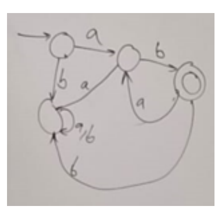

Figure 3: 2(4)(c)

(c)

(d) The complement of a regular language is regular, so yes.

1.3 Q3

The language described by the DFA is $b(a^+b^3)^*$. It is FO describable, with the (conjunction of the) following sentences:-

1. First letter is 'b': $\exists x \forall y (x \leq y \wedge Q_b(x))$
2. If the length of the word is more than 1, then it ends with a "bbb":

   $$(\exists x, y . S(x, y)) \implies \exists x . ((\forall t . t \leq x) \wedge Q_b(x) \wedge \exists y . (S(y, x) \wedge Q_b(y) \wedge \exists z . (S(z, y) \wedge Q_b(z))))$$

3. For every "bbb", there is a non-empty block of 'a's before it, preceding which there is a 'b':

   $$\forall x, y, z . ((S(x, y) \wedge S(y, z) \wedge Q_b(x) \wedge Q_b(y) \wedge Q_b(z)) \implies ($$
   $$(\exists t . S(t, x) \wedge Q_a(t)) \bigwedge$$
   $$(\exists u . u < x \wedge Q_b(u) \wedge (\forall v . (u < v < x) \implies Q_a(v))))$$

4. We don't have 4 consecutive 'b's:

   $$\neg \exists x, y, z, w . S(x, y) \wedge S(y, z) \wedge S(z, w) \wedge Q_b(x) \wedge Q_b(y) \wedge Q_b(z) \wedge Q_b(w)$$

5. We don't have "aba":

   $$\neg \exists x, y, z . S(x, y) \wedge S(y, z) \wedge Q_a(x) \wedge Q_b(y) \wedge Q_a(z)$$

6. If a 'b' is preceded by an 'a' and succeeded by a 'b', then it is a triplet,

   $$\forall x, y, z . (Q_a(x) \wedge S(x, y) \wedge Q_b(y) \wedge S(y, z) \wedge Q_b(z) \implies \exists t . (S(z, t) \wedge Q_b(t)))$$