CS754 Assignment 4 Report

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Introduction

Welcome to our report on CS754 Assignment 4. We have tried to make this report comprehensive and self-contained. We hope reading this would give you a proper flowing description of our work, methods used and the results obtained.

Also note that we installed the `Image Processing Toolbox` in MATLAB for this assignment. Thus the grader is urged to install it if she wishes to run the code on her on her machine. Also note that some of our code may take a while to run because of the intensive nature of the computations involved.

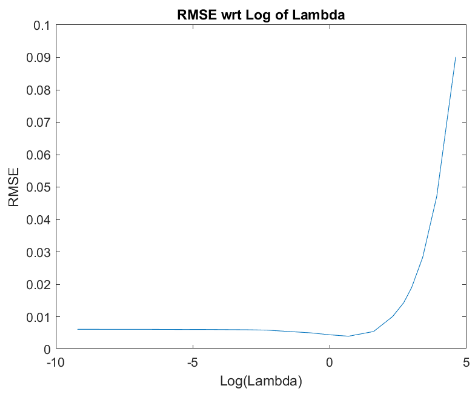
Hope you enjoy reading the report. Here we go!

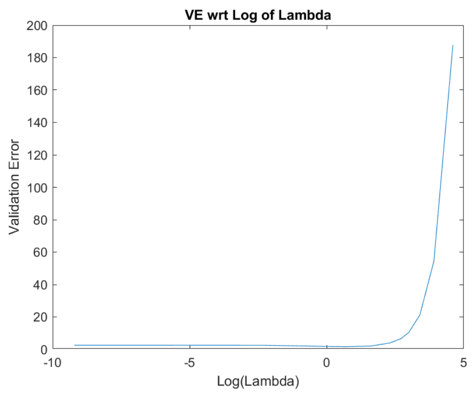
1 Problem 1

In this problem, we try out the cross validation technique and see if we can train and find an optimal value of lambda and then test that value for images that are not in our trained set. This is a very popular technique in machine learning and gives us a measure of robustness of our estimate

1.1 Plots and Graphs of RMSE and VE

Lambda	RMSE	Validation Error
0.0001	0.0061	2.5425
0.0005	0.0061	2.5421
0.0010	0.0061	2.5414
0.0050	0.0061	2.5573
0.0100	0.0061	2.5517
0.0500	0.0060	2.5378
0.1000	0.0059	2.5237
0.5000	0.0050	2.0632
1.0000	0.0044	1.7882
2.0000	0.0040	1.6477
5.0000	0.0054	2.0351
10.0000	0.0101	3.9400
15.0000	0.0143	6.5897
20.0000	0.0190	10.4407
30.0000	0.0283	21.2658
50.0000	0.0470	54.3791
100.0000	0.0901	187.8113





As you can see in the above graphs, the trend of both RMSE and Validation Error matches strongly. At $\lambda = 2$, both the RMSE and VE are at their minimum value. Thus, we can say that our estimates of lambda on the trained set are robust. This is because they match the optimal behaviour also on the new unseen images.

1.2 Coincident V and R

The whole method of cross validation is made to simulate/test out our method in real life like scenarios, where the compressive system would encounter new images, but of the same type/distribution. Since we already train over the R set, there is no point of the V set if it is the same as R.

The Validation error would just be a scaled version of the RMSE. We would not know if we have **underfitted** or **overfitted** to our training set.

In case we are performing cross-validation, it is extremely important to keep the R and V sets of the same distribution for any accurate results, but also disjoint, so as to actually test out the performance.

1.3 Proxying Ability

1.4 Theoretical θ vs Cross Validation θ

2 Problem 2

Let $D = \{d_1, d_2, ..., d_n\}$ be our dictionary, and let $I \in \mathcal{S}$ be an image in our dataset. Then by the theory of dictionary learning, $I = \sum_{i=1}^{n} \alpha_i d_i$ for some sequence of scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$. In this problem, we are given various transformations T which are applied on the images of $\mathcal{S}$, and we have to propose a modified dictionary D_T such that our transformed images are still expressible as linear combination of "atoms" in D_T.
(a) From the theory taught in class and common image processing literature, we know that derivative filters on an image are obtained through **convolutions** of the image with some known convolution filter, ie:- $I_{\text{filter}} = I * G$, where G is the filter matrix. Now, it's not hard to see that a matrix convolution is a linear mapping, ie:- the **derivative filter 'f' is itself a linear transform** on the image.Then

$$f(I) = f(\sum_{i=1}^{n} \alpha_i d_i) = \sum_{i=1}^{n} f(\alpha_i d_i) = \sum_{i=1}^{n} \alpha_i f(d_i)$$

where all the equalities follow by the properties of linear transforms.
Thus our transformed dictionary is $D_T := \{f(d_1), f(d_2), \ldots, f(d_n)\}$, where f represents our derivative filter.
(b) Once again, note that rotation is a linear transform. In particular, the rotation of an image (matrix) I anticlockwise by an angle θ can be given as below

$$I_{\text{rotated}}(x, y) = I(x\cos\theta + y\sin\theta, y\cos\theta - x\sin\theta)$$

or equivalently as

$$I_{\text{rotated}}([x \;\; y]^T) = I(R_\theta^{-1}[x \;\; y]^T)$$

where the rotation matrix R_θ represents anticlockwise rotation by an angle θ

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Thus, once again, we have a linear transform in our hands and thus the new atoms of our transformed dictionary will just be the linear transform applied on the old dictionary itself, ie:- rotated versions of the old atoms. We will have two such dictionaries (for two distinct angles of rotation). Note that we must take the union of both dictionaries as an image, say rotated by β won't be expressible as a linear combination of dictionary atoms rotated by the angle α.
Thus our transformed dictionary is $D_T := \{d_{1\alpha}, d_{2\alpha}, \dots, d_{n\alpha}, d_{1\beta}, d_{2\beta}, \dots, d_{n\beta}\}$, where $d_{k\theta}$ represents the k^{th} atom rotated by θ.
(c) Assuming the relation holds pixel-wise, we have

$$I_{\text{new}}(x,y) = \alpha(I_{\text{old}}(x,y))^2 + \beta I_{\text{old}}(x,y) + \gamma$$

But we have

$$I_{\text{old}}(x,y) = \sum_{i=1}^{n} \alpha_i d_i(x,y)$$

$$\implies I_{\text{new}}(x,y) = \alpha(\sum_{i=1}^{n} \alpha_i d_i(x,y))^2 + \beta \sum_{i=1}^{n} \alpha_i d_i(x,y) + \gamma$$

$$\implies I_{\text{new}}(x,y) = \alpha(\sum_{i=1}^{n} \alpha_i^2 d_i^2(x,y) + 2 \sum_{1\le i<j\le n} \alpha_i\alpha_j d_i(x,y) d_j(x,y)) + \beta \sum_{i=1}^{n} \alpha_i d_i(x,y) + \gamma$$

Thus our new dictionary is clear:

- First class of items: $\{d_1^{\circ 2}, d_2^{\circ 2}, ..., d_n^{\circ 2}\}$, where $^{\circ 2}$ denotes **elementwise squaring**.
- Second class of items: $\{d_i \odot d_j\}_{1\le i<j\le n}$, where $\odot$ denotes **elementwise multiplication**.
- Third item: $J_{d\times d}$, where $d_i \in \mathbb{R}^{d\times d}$, where J is the matrix of all ones, ie:- $J =$ `ones(d)`, in MATLAB notation.

Thus our transformed dictionary is $D_T := \{d_1^{\circ 2}, d_2^{\circ 2}, ..., d_n^{\circ 2}, d_1 \odot d_2, \dots, d_{n-1} \odot d_n, J_{d\times d}\}$.
(d) Blur kernel, like derivative filters, are also convolutions with kernel matrices and hence are linear transforms.
Thus our transformed dictionary is $D_T := \{f(d_1), f(d_2), \dots, f(d_n)\}$, where f represents our blur kernel transformation.
(e) Similar to the idea used in part (b), we have to take an union of all the dictionaries, each applied with a single blur kernel. Thus let $\mathcal{B} := \{b_1, b_2, ..., b_l\}$ be the set of our blur *transforms*. Then our l dictionaries are $\{b_1(d_1), b_1(d_2), \dots, b_1(d_n)\}$, $\{b_2(d_1), b_2(d_2), \dots, b_2(d_n)\}$, ..., $\{b_l(d_1), b_l(d_2), \dots, b_l(d_n)\}$, and
Thus our final dictionary is $D_T := \{b_1(d_1), \dots, b_1(d_n), \dots, b_l(d_1), b_l(d_2), \dots, b_l(d_n)\}$.
(f) Note that the meaning of the notation for this part is slightly different from all other parts of this problem in the fact that our images and dictionary atoms are now assumed to the vectorized forms of the images they represented. Now, note that the Radon transform matrix can be denoted as R_θ, which is a square matrix with entries dependent on θ, which when multiplied with our image vectors gives us their Radon transform. Once again, as above, since this is a linear transform of the vectors, our new dictionary will be the Radon transform of the dictionary atoms. **Thus our new dictionary $D_T := \{Rd_1, Rd_2, \dots, Rd_n\}$, where R is the Radon transform matrix corresponding to the angle θ.**
(g) Note that translating an image by a certain offset (x, y) basically means that we **pad the image, on the left and downwards, with x columns and y rows of zeros respectively**. Now, since we simply added an extra layer of padding of zeros, the linear combination relations still hold as the zero rows and columns obviously satisfy any linear relation and the old pixels also satisfy the relation too. Thus if we denote translation by the first offset to be t_1 and the second offset to be t_2, then, as we have seen many times before our **new dictionary will be D_T $:= \{t_1(d_1), \dots, t_1(d_n), t_2(d_1), t_2(d_2), \dots, t_2(d_n)\}$.**

3 Problem 3

3.1 Part 1

We use the **Eckart-Young-Mirsky theorem** to find the **best rank r approximation to the given matrix** $A \in \mathbb{R}^{m \times n}$, ie:- A_r, which, as we have seen in the lecture, states that the best (in terms of the Frobenius norm of the difference in between those two matrices) r-rank approximation is given by $A_r = U\Sigma_r V^*$, where $A = U\Sigma V^*$ is the **Singular Value Decomposition** of A, and Σ_r contains the r maximum singular values in Σ, ie:- we retain the r largest diagonal values of Σ and set the rest to zero.
Thus the minimizer of the objective function $J(A_r) = \|A - A_r\|_F^2$; $\text{rank}(A_r) = r$ is

$$\boldsymbol{A_r = U\Sigma_r V^*}$$

where $A = U\Sigma V^*$ is the SVD decomposition of A and Σ_r contains the r maximum singular values in Σ.
Applications of low rank approximations:

- **Robust PCA**
  - Note that in a surveillance video, the background remains stationary while our object of interest moves. Thus the background of the image can be treated as it's low rank component, which can then be identified using the algorithm outlined above.
  - In a similar vein as the idea above, one can remove occlusion by **completing** a low rank matrix to effectively "cut-out" the occluding object in the image.

3.2 Part 2

We algebraically simplify the objective function $J(R) = \|A - RB\|_F^2$ first to find out the optimum orthonormal R matrix. For that, note the following facts from linear algebra:

- $\|A\|_F^2 = \text{tr}(AA^T)$
- $\text{tr}(AB) = \text{tr}(BA)$, and thus any rearrangement of a matrix multiplication will still yield the same trace.
- $\text{tr}(A) = \text{tr}(A^T)$ for any square matrix A

Then

$$\begin{aligned}
\|A - RB\|_F^2 &= \text{tr}((A - RB)(A - RB)^T) = \text{tr}((A - RB)(A^T - B^T R^T)) \\
&= \text{tr}(AA^T - RBA^T - AB^T R^T + RBB^T R^T) \\
&= \text{tr}(AA^T) - \text{tr}(RBA^T) - \text{tr}((RBA^T)^T) + \text{tr}(RBB^T R^T) \\
&= \text{constant} - \text{tr}(RBA^T) - \text{tr}(RBA^T) + \text{tr}(R^T RBB^T) \\
&= \text{constant} - 2 \cdot \text{tr}(RBA^T) + \text{tr}(BB^T) \\
&= \text{constant} - 2 \cdot \text{tr}(RBA^T)
\end{aligned}$$

Thus minimizing $\|A - RB\|_F^2$ is equivalent to maximizing $\text{tr}(RBA^T)$.
Now, let BA^T be equal to C, and let $C = U\Sigma V^*$ be the **Singular Value Decomposition** of C. Then

$$\text{tr}(RBA^T) = \text{tr}(RC) = \text{tr}(RU\Sigma V^*) = \text{tr}(V^* RU\Sigma)$$

Now, note that since R, U and V are all orthonormal, $M = V^* RU$ is orthonormal too, and thus

$$\text{tr}(V^* RU\Sigma) = \text{tr}(M\Sigma) = \sum_{i=1}^{n} m_{ii}\sigma_i \leq \sum_{i=1}^{n} |m_{ii}|\sigma_i$$

with the last inequality following by the triangle inequality (and note that $|\sigma_i| = \sigma_i$ since singular values are non-negative).
However, since M is orthonormal, ie:- all of it's column's norms are 1, we have that $|m_{ii}| \leq 1$, and thus the maxima of $\text{tr}(V^*RU\Sigma)$ is achieved when all $|m_{ii}| = 1$. But that can happen only when $M = I$, and thus

$$V^*RU = I$$
$$\implies \boldsymbol{R = VU^*}$$

Thus the best orthonormal transform that maps B to A is given by VU^*, where U and V are obtained from the SVD of $BA^T = U\Sigma V^*$.
This problem is also widely encountered in image processing literature. We ourselves came across one such example in class, namely, in 3D Tomography under unknown angles, wherein we try to determine what rotation best maps one image to another, and there is where we note that orthonormal matrices basically represent unscaled rotation in higher dimensions, from which the problem of minimizing $\|A - RB\|_F^2$ under an orthonormal R constraint arises.

4 Problem 4

5 Problem 5

Note that we have to find the f vector which is most likely to have arisen from the y vector we know under the equation $y \sim \text{Poisson}(I_0 e^{-Rf})$, where all the operations are done element-wise on the vectors. Note that R is the **known** Radon matrix.
Stating this mathematically, we want

$$f = \arg\max_{f \in \mathbb{R}^m} P(f|y)$$

Now, note that $P(f|y) \propto P(y|f)P(f)$ by Baye's theorem, where $P(f)$ is a **prior** on f, ie:- a probability distribution on $\mathbb{R}^m$ which tells us how likely a certain point in $\mathbb{R}^m$ is to represent the natural tissue density vector.
But before specifying the functional form of the prior, we'll simplify $P(y|f)$ a bit. Also note that maximizing a certain non-negative function (such as a probability distribution) is equivalent to minimizing it's negative logarithm.
Note that since y varies with f as a Poisson relation

$$P(y|f) = \frac{e^{I_0 e^{-Rf}} (I_0 e^{-Rf})^y}{y!}$$

$$\implies -\log P(y|f) = I_0 e^{-Rf} - y \log(I_0 e^{-Rf}) + \log(y!)$$
$$\implies -\log P(y|f) = \|I_0 e^{-Rf}\|_1 + y^T Rf$$

where we drop all terms not having f in them because we're to define an objective function dependent on f. Note also that we transpose y at the end to make the matrices multipliable. Also note that to convert the vector $I_0 e^{-Rf}$ to a scalar, we take its l_1 norm (note also that the l_1 norm promotes sparsity, which is a desirable quality to have for our objective function), because $y^T Rf$ is also a scalar, and we need to add them both.
Now coming to the prior, note that f denotes natural tissue density. Now, from our study of natural images and other naturally arising phenomena, we know that **their gradients are sparse**. This is only to be expected since consecutive entries for the most part don't change rapidly.
Thus, if we define the **gradient of a vector as follows**

$$\nabla f := \sum_{i=1}^{m-1} |f_i - f_{i+1}|$$

Then

$$P(f) \propto e^{-\nabla f}$$

is a suitable prior. Taking it's negative logarithm we get

$$-\log P(f) = \nabla f$$

, and thus

$$-\log P(f|y) = -\log P(y|f) - \log P(f)$$
$$-\log P(f|y) = \|I_0 e^{-Rf}\|_1 + y^T Rf + \mu \nabla f$$

where μ is a suitable hyperparameter denoting the importance of how much the gradient sparsity is to be tuned.
Thus

$$\boldsymbol{J(f) := \|I_0 e^{-Rf}\|_1 + y^T Rf + \mu \nabla f}$$

Finally, note that for additive white Gaussian noise, we can simply minimize the mean square error (unlike Poisson noise, Gaussian noise is independent of the pixel value it distorts, and thus minimizing the mean square error should suffice). Note that the mean of a Poisson RV is equal to it's parameter, and thus y should be compared to $I_0 e^{-Rf}$, and thus

$$\boldsymbol{J_{\textbf{Gaussian}}(f) := \|y - I_0 e^{-Rf}\|_2^2}$$

Combining these two together

$$\boldsymbol{J_{\textbf{complete}}(f) := \|I_0 e^{-Rf}\|_1 + y^T Rf + \mu \nabla f + \lambda \|y - I_0 e^{-Rf}\|_2^2}$$

where λ is another hyperparameter to stress the importance of the Gaussian error function.