

CS754 Assignment 1 Report

Arpon Basu
Shashwat Garg

Spring 2022

Contents

1	Problem 1	2
1.1	Trend of Error Bound with s	2
2	Problem 2	2
2.1	Lower Bound	2
2.2	Upper Bound	3
3	Problem 3	3
4	Problem 4	3
5	Problem 5	4
6	Problem 6	4

Introduction

Welcome to our report on CS754 Assignment 1. We have tried to make this report comprehensive and self-contained. We hope reading this would give you a proper flowing description of our work, methods used and the results obtained. Feel free to keep our code scripts alongside to know the exact implementation of our tasks.

We have referred to some sites on the web for finding the MATLAB implementations (generic documentation pages) and the same has been added in the references section.

In many places, to better give context to the place from which the questions could have arisen, some theoretical discussions have been engaged in.

Hope you enjoy reading the report. Here we go!

1 Problem 1

We have been given the following problem-

Let $\theta^* : \min \|\theta\|_1$ such that $\|y - \Phi\Psi\theta\|_2 \leq \varepsilon$, where $x = \Psi\theta$ and $y = \Phi x + \eta$. ε is an upper bound on the magnitude of the noise vector η .

Also, Theorem 3 states-

If Φ obeys the restricted isometry property with isometry constant $\delta_{2s} < \sqrt{2} - 1$, then we have $\|\theta - \theta^*\|_2 \leq C_1 s^{-1/2} \|\theta - \theta_s\|_1 + C_2 \varepsilon$ where C_1 and C_2 are functions of only δ_{2s} and where $\forall i \in \mathcal{S}, \theta_{s_i} = \theta_i; \forall i \notin \mathcal{S}, \theta_{s_i} = 0$.

1.1 Trend of Error Bound with s

This is not a discrepancy. In reality, the error bound becomes worse as the value of s increases. The point is, we are only focusing on the effect of $s^{-1/2}$ and $\|\theta - \theta_s\|_1$. We must also see the change in C_1 and C_2 . These constants increase as the value of δ_{2s} changes. Thus, as the value of s increases, we observe that the bound on δ_{2s} also increases which leads to an increase in the value of C_1 and C_2 . Thus, we cannot claim that the error bound improves as the sparsity measure, s increases in value.

2 Problem 2

We note that this problem has two parts: One for demonstrating that the upper bound of $\mu(\Phi, \Psi)$ is \sqrt{n} , and one for demonstrating that the lower bound of $\mu(\Phi, \Psi)$ is 1. We shall deal with both parts separately.

2.1 Lower Bound

Proof. Since Ψ is an orthonormal matrix, its column vectors $\Psi_1, \Psi_2, \dots, \Psi_n$ form an orthonormal basis for \mathbb{K}^n , and consequently we can express g as $\sum_{k=1}^n \alpha_k \Psi_k$. Also, $\|g\|_2 = \sqrt{\sum_{j=1}^n \alpha_j^2}$, following which we get $g_{\text{normalized}} = \sum_{k=1}^n \frac{\alpha_k}{\sqrt{\sum_{j=1}^n \alpha_j^2}} \Psi_k$. Thus $\mu(g, \Psi) =$

$\sqrt{n} \cdot \max_{i \in [n]} |\mathbf{g}_{\text{normalized}}^T \Psi_i| = \sqrt{n} \cdot \max_{i \in [n]} \frac{|\alpha_i|}{\sqrt{\sum_{j=1}^n \alpha_j^2}}$. WLOG assuming that $|\alpha_1| = \max_{i \in [n]} |\alpha_i|$, we get that

$$\mu(\mathbf{g}, \Psi) \geq \sqrt{n} \cdot \max_{i \in [n]} \frac{|\alpha_i|}{\sqrt{\sum_{j=1}^n |\alpha_j|^2}} = \sqrt{n} \frac{|\alpha_1|}{\sqrt{n\alpha_1^2}} = 1$$

as desired. Note that equality is achieved iff $\alpha_1 = \alpha_2 = \dots = \alpha_n$. \square

2.2 Upper Bound

Proof. Directly borrowing from the previous proof, $\mu(\mathbf{g}, \Psi) = \sqrt{n} \cdot \max_{i \in [n]} |\mathbf{g}_{\text{normalized}}^T \Psi_i|$. Now, by the Cauchy Schwartz inequality, for any two vectors \mathbf{u}, \mathbf{v} , we have $|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \|\mathbf{u}\|_2 \|\mathbf{v}\|_2$, where $|\langle \cdot, \cdot \rangle|$ is the usual Euclidean inner product. Applying it to the coherence relation yields $|\mathbf{g}_{\text{normalized}}^T \Psi_i| \leq \|\mathbf{g}_{\text{normalized}}\|_2 \|\Psi_i\|_2 = 1$, and consequently we obtain that $\mu(\mathbf{g}, \Psi) \leq \sqrt{n}$, as desired. Note that equality is achieved iff \mathbf{g} is parallel to some column vector Ψ_i of Ψ . \square

3 Problem 3

4 Problem 4

We provide a short and sweet proof for this problem below.

Proof. We set up some notation first: Let

$$\begin{aligned} \mathbf{x}^* &:= \arg \min_{\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2 \leq e} \|\mathbf{x}\|_1 \\ l_1 &:= \min_{\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2 \leq e} \|\mathbf{x}\|_1 \\ f(t) &:= \arg \min_{\|\mathbf{x}\|_1 \leq t} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2 \end{aligned}$$

Then note that if $t < l_1$, $f(t) > e$: Why? Because if $f(t) \leq e$, then one would have that there exist \mathbf{x} with L1-norm lesser than l_1 which nevertheless make the L2 norm of $(\mathbf{y} - \mathbf{A}\mathbf{x}) \leq e$, which contradicts the minimality of \mathbf{x}^* .

Also note that $f(l_1) \leq e$, since we have a witness \mathbf{x}^* for which the L1-norm is equal to l_1 and $\|\mathbf{y} - \mathbf{A}\mathbf{x}^*\|_2 \leq e$ due to the premise of the problem P1. Thus it's clear that all minimizers of $\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2$ under the conditions $\|\mathbf{x}\|_1 \leq l_1$ actually lie only on the “sphere” $\|\mathbf{x}\|_1 = l_1$ (because for points strictly to the interior of this “sphere”, $\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2$ assumes values larger than e). Now, if it so happens that there exists another \mathbf{z} (such that $\|\mathbf{z}\|_1 = l_1$) such that $\|\mathbf{y} - \mathbf{A}\mathbf{z}\|_2 \leq e$, then we note that the *uniqueness* of \mathbf{x}^* is violated, as \mathbf{z} also becomes a minimizer for P1 (along with \mathbf{x}^*) since it satisfies the premise of P1 (which is $\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2 \leq e$).

Thus, we have that for $t = \min_{\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2 \leq e} \|\mathbf{x}\|_1$, both the problems P1 and Q1 have the **same unique minimizer**.

Hence proved. \square

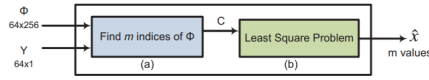


Fig. 2. Basic diagram for OMP reconstruction

Figure 1: Basic Structure of OMP

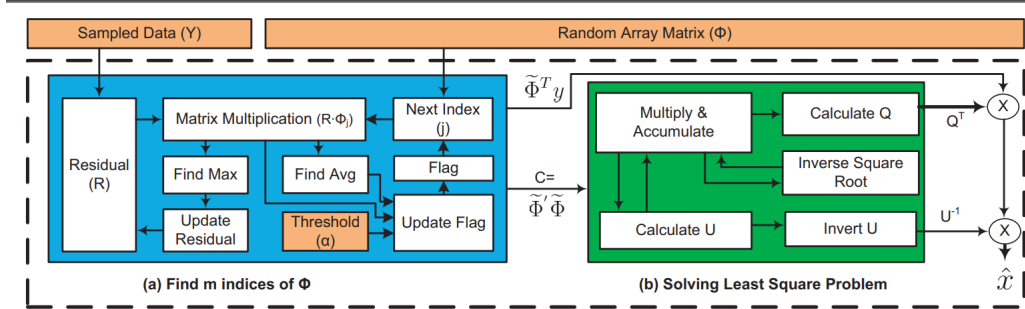
Fig. 3. Detailed architecture diagram of Improved OMP reconstruction algorithm. (a) This block iterates 8 times to solve the optimization problem eliminating columns of Φ those are poorly correlated to Y . Operates at 85 MHz. (b) Finds inverse of a 8×8 matrix after finding 8 columns of Φ . Operates at 69 MHz.

Figure 2: Proposed Hardware Architecture

5 Problem 5

(a) The title of the paper we reviewed was **Low-Complexity FPGA Implementation of Compressive Sensing Reconstruction**. It was published in **2013** by **IEEE** in the **International Conference on Computing, Networking and Communications, Multimedia Computing and Communications Symposium**.

(b) The main aim of the paper is to optimize the hardware time required for executing reconstruction algorithms like **OMP**. To that end, it develops a novel hardware architecture which promises to deliver a execution time $3.4 \times$ faster than what conventional hardware takes to execute OMP. The conceptualization and execution of the paper is best explained through the two images below.

(c) This paper optimizes the execution of the **reconstruction technique OMP** which is very widely used in compressed sensing scenario. This paper per se itself doesn't deploy compressed sensing towards any goal but rather seeks to improve what already exists.

6 Problem 6