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Solution for problem 511

We claim that the **minima is 94** which is attained when

$$(x_1, x_2, \cdots, x_{40}) = (\underbrace{1, 1, \cdots 1, 1}_{22}, \underbrace{2, 2 \cdots 2, 2}_{18})$$

If the minima occurs for some other tuplet where r twos have become one and k ones have increased , like

$$(\underbrace{1, 1, \cdots 1, 1}_{k}, \underbrace{2, 2 \cdots 2, 2}_{r}) \longrightarrow (1 + x_1, 1 + x_2, \cdots 1 + x_k, \underbrace{1, 1 \cdots 1, 1}_{r})$$

where $x_i > 0$ and $\sum_{i=1}^{k} x_i = r$ then:

Original sum (S_1)

$$S_1 = k \cdot 1^2 + r \cdot 2^2 = 4r + k$$

Changed Sum(S_2)

$$S_2 = \sum_{i=1}^{k}(1 + x_i)^2 + r \cdot 1^2 = k + 2\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} x_i^2 + r$$

$$\begin{aligned}
\therefore S_2 - S_1 &= k + 3r + (\sum_{i=1}^{k} x_i^2) - 4r - k \\
&= (\sum_{i=1}^{k} x_i^2) - r \\
&= (\sum_{i=1}^{k} x_i^2) - (\sum_{i=1}^{k} x_i) \quad \text{(replacing } r \text{ by } \sum_{i=1}^{k} x_i) \\
&\geq 0
\end{aligned} \tag{1}$$

$$\boxed{\therefore 94 \text{ is indeed the minima.}}$$

We claim that **maxima is 400** which is attained when

$$(x_1, x_2, \cdots x_{40}) = (\underbrace{1, 1, \cdots 1, 1}_{39}, 19)$$

If any other tuplet achieves maxima like

$$(1 + x_1, 1 + x_2, \cdots 1 + x_k, 19 - r)$$

where $\sum_{i=1}^{k} x_i = r$ and $r \leq 18$ then:

Original sum(S_1)

$$S_1 = k \cdot 1^2 + 19^2 = k + 361$$

Changed Sum(S_2)

$$\begin{aligned}
S_2 &= \sum_{i=1}^{k}(1 + x_i)^2 + (19 - r)^2 \\
&= k + \sum_{i=1}^{k} x_i^2 + 2\sum_{i=1}^{k} x_i + 361 + r^2 - 38r \\
&= 361 + k + \sum_{i=1}^{k} x_i^2 + 2r + r^2 - 38r \quad (\text{replacing} \sum_{i=1}^{k} \text{ by } r) \\
&= 361 + k + \sum_{i=1}^{k} x_i^2 + r^2 - 36r
\end{aligned} \tag{2}$$

$$\begin{aligned}
\therefore S_2 - S_1 &= 361 + k + \sum_{i=1}^{k} x_i^2 + r^2 - 36r - 361 - k \\
&= \sum_{i=1}^{k} x_i^2 + r^2 - 36r \\
&= \sum_{i=1}^{k} x_i^2 + (\sum_{i=1}^{k} x_i)^2 - 36r \\
&< 2(\sum_{i=1}^{k} x_i)^2 - 36r \\
&= 2r^2 - 36r \\
&= 2r(r - 18) \\
&\leq 0
\end{aligned} \tag{3}$$

$$\boxed{\therefore 400 \text{ is indeed the maxima.}}$$