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Solution for problem O435

We observe that if $a = \frac{x}{y+z}$, $b = \frac{y}{z+x}$, $c = \frac{z}{x+y}$ for some $x, y, z \in R_{\geq 0}$, then the condition $ab + bc + ca + 2abc = 1$ is satisfied.we also note that the mapping $(a, b, c) \mapsto (x, y, z)$ is bijective.
Thus

$$\sum_{cyclic} \frac{1}{8a^2+1} = \sum_{cyclic} \frac{1}{8(\frac{x}{y+z})^2+1} = \sum_{cyclic} \frac{(y+z)^2}{8x^2+(y+z)^2}$$

Since the above expression is homogeneous in (x, y, z) , we put $x+y+z = 1$ and consequently $x, y, z \in (0, 1)$ to get

$$E(x,y,z) = \sum_{cyclic} \frac{(1-x)^2}{8x^2+(1-x)^2} = \sum_{cyclic} \frac{x^2-2x+1}{9x^2-2x+1}$$

We thus ahve to minimize E under $g : x + y + z - 1 = 0$. Thus by Lagrange multipliers,

$$\mathcal{L}(x,y,z,\lambda) = E(x,y,z) - \lambda g(x,y,z)$$

$$\nabla_{x,y,z,\lambda}\mathcal{L} = 0$$
$$\therefore \lambda = \frac{16x(x-1)}{(9x^2-2x+1)^2} = \frac{16y(y-1)}{(9y^2-2y+1)^2} = \frac{16z(z-1)}{(9z^2-2x+1)^2}$$

let

$$f(r) = \frac{16r(r-1)}{(9r^2-2r+1)^2}$$
$$\therefore f(x) = f(y) = f(z) = \lambda$$

Now, $f^{'}(x)$ changes sign exactly one in $(0, 1)$. Thus any line $y = \lambda$ can intersect $y = f(x)$ (for $x \in (0, 1)$) atmost twice. Hence (x, y, z) can contain atmost two distinct numbers.
Thus the unordered triplet (x, y, z) staisfying $f(x) = f(y) = f(z)$ is of the form $(\alpha, \alpha, 1 - 2\alpha)$.

$$\therefore f(\alpha) = f(1-2\alpha)$$

$$\Rightarrow \frac{16\alpha(\alpha-1)}{(9\alpha^2-2\alpha+1)^2} = \frac{16(1-2\alpha)(-2\alpha)}{(9(1-2\alpha)^2-2(1-2\alpha)+1)^2} \tag{1}$$

Solving equation 1 with Wolfram Alpha we get $\alpha = \frac{1}{3}, 0.474119$ (Note that $\alpha \in (0, 1)$). Actually evaluating $E(x, y, z)$ for $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $(\alpha_0, \alpha_0, 1 - 2\alpha_0) = 8.7033 > 1$

Thus $E(x, y, z)$ attains its global minima on the plane $x + y + z = 1$ at $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ at that point its value is 1.

$$\therefore E(x,y,z) \geq 1$$

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$\Rightarrow \dfrac{1}{8a^2+1} + \dfrac{1}{8b^2+1} + \dfrac{1}{8c^2+1} \geq 1 \qquad$ with equality $a = b = c = \dfrac{1}{2}$