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Solution for problem O397

$$(x^3-1)(y^3-1) = 3(x^2+y^2+2)$$
$$\Leftrightarrow (x^3y^3 - x^3 - y^3 + 1 - 3x^2y^2 + 6) = 0$$
$$\Leftrightarrow (y^3-1)x^3 - 3y^2x^2 - (y^3+5) = 0$$

Now, if we consider y is an integer constant, then we are basically searching for integer roots of this cubic polynomial in xAn integer root is also a rational root . Let $\frac{p}{q}$ be a root of this equation such that $(p,q)=1, q \neq 0$. Hence by rational root theorem

$$p \mid (y^3+5), q \mid (y^3-1)$$

.
Now $\frac{p}{q}$ has to be an integer $\Rightarrow q \mid p$
A divisor of

$$(y^3-1)$$

divides a divisor of

$$(y^3+5)$$

$\Rightarrow$ there quotient divides $((y^3-1),(y^3+5))$.
Note that

$$((y^3-1),(y^3+5)) \mid ((y^3+5)-(y^3-1)) \mid 6$$
$$((y^3-1),(y^3+5)) \mid 6$$
$$\Rightarrow x = \frac{p}{q} = \pm 1, \pm 2 \pm 3$$

Checking for these values of x we get that only
$x = 1 \Rightarrow y^2 = -2$ (Impossible)
$x = -1 \Rightarrow 2y^3 + 3y^2 + 4 = 0 \Rightarrow y = -2$ (No integral root by Rational Root Theorem(RRT)
$x = 2 \Rightarrow 7y^3 - 12y^2 - 13 = 0$ (No integral root by RRT)
$x = -2 \Rightarrow 3y^3 + 4y^2 - 1 = 0 \Rightarrow y = -1$
$x = 3 \Rightarrow 26y^3 - 27y^2 - 32 = 0$ (No integral root by RRT)
$x = -3 \Rightarrow 28y^3 + 27y^2 - 22 = 0$ (No integral root by RRT)

Thus only roots are $(-1,-2)$ and $(-2,-1)$