Math Problems

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May 2021

1 Vasooli

Suppose we have n people in a room, some of whom have borrowed and/or lent money to each other. At the end of the year, all of them want to clear their dues and/or want their money back.
However, they're on very bad personal terms with each other, and are unable to complete transactions with each other in a civilized manner. So, they approach a court.
The court announces the following procedure for clearing dues:-

1. All people have to first file claims with the court. For example, if A has borrowed 1,000 $ from B, then B files a claim of 1,000 $ vis-a-vis A, while A files a claim of **-**1,000 $ vis-a-vis B.
2. Then, everybody is asked to announce their interest rates. From the above example, say A announces (an interest rate of) 8 %, B announces 5 %, and so on.
3. Then an account sheet is made as follows: $1 + 5\% = 1.05$; $1.05 \times (-1000\$)$ is added to the account sheet of A, while $1.08 \times (+1000\$)$ is added to the account sheet of B. **Note that the amount deducted from A's account sheet is dictated by the interest rate of B, and vice versa**.
4. The above procedure is carried out for all the claims filed in court.
5. Finally, to prevent exploitation, the court divides the total amount on a person's account sheet by (1 + That person's interest rate). Thus, for example, if A finally has 21,600 $ written on her account sheet, then she is **actually paid only** $\frac{21600}{1+0.08} = 20,000$ **$ by the court.**
6. We also fix the convention that everybody gets paid by the court. Thus, even if a person, say X, is **asked to pay**, say 3,000 $, by the court, we still say that X **was paid - 3,000 $ by the court.**

Prove that it is **impossible**, no matter what the borrowed/lent amounts and the interest rates are, that at the end of the above arbitration, everybody would've been paid the same **non-zero** amount by the court, ie:- either people will get paid unequal sums of money, or everybody would get paid exactly zero.

2 Children in the Park

Say we have n children in a school. They form groups with each other, of various sizes, in such a way that no child is left alone, ie:- every group has size at-least 2. Then all of these children go to a park, where they have to sit around circular tables. Note that both the order and sense of of the seating arrangements matter, ie:- 1-2-3-4-1, 1-3-2-4-1, 1-4-3-2-1, are to be considered distinct seating arrangements.
Find the number of ways this whole thing can be orchestrated, ie:- breaking up into groups and then their seating arrangement around circles. Your answer will obviously be a function of n, say $f(n)$.
Example:- If we have 4 children, say A, B, C, D, then the breaking down into groups can only be 2 + 2, or all 4 together, as shown below:
$[((A,B),(C,D)),((A,C),(B,D)),((A,D),(B,C)),(A,B,C,D)]$. In the 2 + 2 case, seating arrangement, even considering clockwise and anti-clockwise to be separate, can still be done in a unique way. In the case where everybody is together, seating can be done in 6 ways (fix anybody and permute the rest). Thus $f(4)$ **is equal to** $3+6=9$.

3 Hints

3.1 Vasooli

Express the total money owed by the people to each other in a $n \times n$ matrix $A = [a_{ij}]$ in such a way that a_{ij} represents the money owed by the j^{th} person to the i^{th} person. Clearly A is a skew symmetric matrix, and representing the interest rates as a column vector of length n, it's easy to see that the problem statement basically says that a skew symmetric matrix can't have a non-zero real eigenvalue.

3.2 Children in the Park

The number of ways has a direct bijection with the number of derangements of set of size n. The answer is thus $D_n = \lfloor \frac{n!}{e} + \frac{1}{2} \rfloor$.