Junior problems

J439. Solve in real numbers the system of equations:

$$\begin{cases} 2x^2 - 3xy + 2y^2 = 1 \\ y^2 - 3yz + 4z^2 = 2 \\ z^2 + 3zx - x^2 = 3 \end{cases}$$

Proposed by Adrian Andreescu, University of Texas at Austin

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia

Summing up first two equations and then subtracting the third one yields to

$$\begin{aligned} 3x^2 - 3xy + 3y^2 - 3yz + 3z^2 - 3zx = 0 &\Leftrightarrow x^2 + y^2 + z^2 - xy - yz - zx = 0 \\ &\Leftrightarrow (x-y)^2 + (y-z)^2 + (z-x)^2 = 0 \\ &\Leftrightarrow x = y = z. \end{aligned}$$

Hence we deduce that $x^2 = 1$ and the only solutions are $x = y = z = \pm 1$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Kevin Soto Palacios, Huarmey, Perú; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Alok Kumar, New Delhi, India; Arkady Alt, San Jose, CA, USA; Arpon Basu, AECS-4, Mumbai, India; Danae Papageorgiou, Model High School Evangelical School of Smyrna, Greece; Jason Zhang, Bowling Green High School, KY, USA; Ioannis D. Sfikas, Athens, Greece; Marin Chirciu, Colegiul Național "Zinca Golescu", Pitești, Romania; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Albert Stadler, Herrliberg, Switzerland; Polyahedra, Polk State College, FL, USA; Naïm Mégarbané, UPMC, Paris, France; Muhammad Alhafi, Syrian Mathematical Olympiad Team, Aleppo, Syria; Joehyun Kim, Fort Lee High School, NJ, USA; Nikos Kalapodis, Patras, Greece; Angela Han, The Taft School, Watertown, CT, USA; George Theodoropoulos, 2nd High school of Agrinio, Greece; Anas Kudsi, Syrian Mathematical Olympiad Team, Syria; Titu Zvonaru, Comănești, Romania.

J440. Let a, b, c, d be distinct nonnegative real numbers. Prove that

$$\frac{a^2}{(b-c)^2}+\frac{b^2}{(c-d)^2}+\frac{c^2}{(d-a)^2}+\frac{d^2}{(a-b)^2}>2.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Henry Ricardo, Westchester Area Math Circle

We have $\sum_{cyclic}\frac{a^2}{(b-c)^2}\geq\sum_{cyclic}\frac{a^2}{b^2+c^2}=S$, and we will show that $S>2$.

Consider $T=\sum_{cyclic}\frac{b^2}{b^2+c^2}$ and $U=\sum_{cyclic}\frac{c^2}{b^2+c^2}$. We note that $T+U=4$.

Now the AM-GM inequality gives us

$$S+T = \sum_{cyclic}\frac{a^2+b^2}{b^2+c^2} > 4\sqrt[4]{\prod_{cyclic}\frac{a^2+b^2}{b^2+c^2}} = 4$$

and

$$\begin{aligned} S+U &= \sum_{cyclic}\frac{a^2+c^2}{b^2+c^2} = \frac{a^2+c^2}{b^2+c^2}+\frac{a^2+c^2}{a^2+d^2}+\frac{b^2+d^2}{c^2+d^2}+\frac{b^2+d^2}{a^2+b^2} \\ &> \frac{4(a^2+c^2)}{a^2+b^2+c^2+d^2}+\frac{4(b^2+d^2)}{a^2+b^2+c^2+d^2} = 4. \end{aligned}$$

Therefore, $T+U+2S>8$, or $S>2$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Polyahedra, Polk State College, USA; Naïm Mégarbané, UPMC, Paris, France; Muhammad Alhafi, Syrian Mathematical Olympiad Team, Aleppo, Syria; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Ioannis D. Sfikas, Athens, Greece; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Albert Stadler, Herrliberg, Switzerland; Konstantinos Metaxas, Athens, Greece; Anas Kudsi, Syrian Mathematical Olympiad Team, Syria.

J441. Prove that for any positive real numbers a, b, c the following inequality holds

$$\frac{(a+b+c)^3}{3abc} + 1 \geq \left(\frac{a^2+b^2+c^2}{ab+bc+ca}\right)^2 + (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy
The inequality is

$$\frac{\sum_{cyclic} a^3 + 3\sum_{cyclic}(a^3b+ab^3) + 6abc}{3abc} + 1 \geq \frac{(\sum_{cyclic} a^2)^2}{(\sum_{cyclic} ab)^2} + 3 + \frac{\sum_{cyclic}(a^3b+ab^3)}{abc}$$

that is

$$\frac{\sum_{cyclic} a^3}{3abc} \geq \frac{(\sum_{cyclic} a^2)^2}{(\sum_{cyclic} ab)^2}$$

This in turn becomes

$$\sum_{cyclic} a^3 \left(\sum_{cyclic} ab\right)^2 \geq 3abc\left(\sum_{cyclic} a^2\right)^2$$

By using the inequality $ab+bc+ca \geq \sqrt{3abc(a+b+c)}$ it suffices

$$\sum_{cyclic} a^3 \left(3abc \sum_{cyclic} a\right) \geq 3abc\left(\sum_{cyclic} a^2\right) \iff \sum_{cyclic} a^3 \left(\sum_{cyclic} a\right) \geq \left(\sum_{cyclic} a^2\right)$$

and this becomes

$$\sum_{cyclic}(a^3b+ab^3) \geq 2\sum_{cyclic} a^2b^2$$

simply AGM.

Also solved by Daniel Lasaosa, Pamplona, Spain; Polyahedra, Polk State College, USA; Naïm Mégarbané, UPMC, Paris, France; Joehyun Kim, Fort Lee High School, NJ, USA; Nikos Kalapodis, Patras, Greece; Angela Han, The Taft School, Watertown, CT, USA; Konstantinos Metaxas, Athens, Greece; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Arkady Alt, San Jose, CA, USA; Ioannis D. Sfikas, Athens, Greece; Marin Chirciu, Colegiul Național "Zinca Golescu", Pitești, Romania; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nguyen Ngoc Tu, Ha Giang, Vietnam; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Albert Stadler, Herrliberg, Switzerland; Anas Kudsi, Syrian Mathematical Olympiad Team, Syria; Titu Zvonaru, Comănești, Romania.

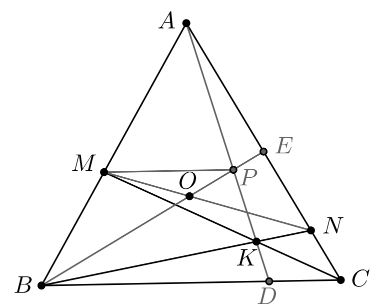
J442. Let ABC be an equilateral triangle with center O. A line passing through O intersects sides AB and AC at M and N, respectively. Segments BN and CM intersect at K and segments AK and BO intersect at P. Prove that $MB = MP$.

Proposed by Anton Vassilyev, Kazakhstan

Solution by Polyahedra, Polk State College, USA

Suppose that AK intersects BC at D and BO intersects CA at E. By Ceva's theorem,

$$\frac{CD}{DB} \cdot \frac{BM}{MA} = \frac{NC}{AN} = \frac{AN - 2EN}{AN} = 1 - \frac{2EN}{AN}.$$



Applying Menelaus' theorem to line MO in $\triangle ABE$, we get

$$\frac{MB}{AM} = \frac{BO}{OE} \cdot \frac{EN}{AN} = \frac{2EN}{AN} = 1 - \frac{CD}{DB} \cdot \frac{MB}{AM}.$$

Therefore $\frac{AM}{MB} = 1 + \frac{CD}{DB} = \frac{CB}{DB}$. Applying Menelaus' theorem to line PE in $\triangle ADC$, we have

$$\frac{AP}{PD} = \frac{CB}{DB} \cdot \frac{CE}{EA} = \frac{CB}{DB} = \frac{AM}{MB},$$

so $MP \parallel BC$. Hence $\angle MPB = \angle CBP = \angle PBM$, completing the proof.

Also solved by Daniel Lasaosa, Pamplona, Spain; Muhammad Alhafi, Syrian Mathematical Olympiad Team, Aleppo, Syria; Joehyun Kim, Fort Lee High School, NJ, USA; Albert Stadler, Herrliberg, Switzerland; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Paul Revenant, Lycée du Parc, Lyon, France; Xingze Xu, Hangzhou Foreign Languages School A-level Centre, China; Titu Zvonaru, Comănești, Romania.

J443. Find all pairs (m; n) of integers such that both equations

$$x^2 + mx - n = 0,$$

$$x^2 + nx - m = 0$$

have integer roots.

Proposed by Alessandro Ventullo, Milan, Italy

Solution by Daniel Lasaosa, Pamplona, Spain

Note that $m^2 + 4n$ and $n^2 + 4m$ must be perfect squares of the same parity as m^2 and n^2, respectively, since they are the discriminants of the quadratic equations. If wlog by the symmetry in the problem we have $n = 0$, then m^2 is clearly a perfect square, and $4m$ must be a perfect square, or $m = k^2$ must be a perfect square. In this case, the first equation has solutions $0, -m$, and the second equation has solutions $k, -k$, all clearly integers. No other solution may exist where either m or n is zero.

If $|m| = |n|$ and n is positive, then

$$|m|^2 < m^2 + 4n = |m|^2 + 4|m| < (|m| + 2)^2 ,$$

contradiction. If $|m| = |n|$ and n is negative, then

$$m^2 + 4n = m^2 - 4|m| = (|m| - 2)^2 - 4$$

must be a perfect square, and the only two perfect squares which differ by 4 are 0^2 and 2^2, or $|m| = |n| = 4$. Since n is negative, $n = -4$, and since $4^2 + 4 \cdot 4 = 32$ is not a perfect square, m must also be negative. Solution $m = n = -4$ exists in this case, but none other when $|m| = |n|$.

If $|m| \neq |n|$, Assume wlog by the symmetry in the problem that $|m| \geq |n| + 1$, and if n is positive, then

$$|m|^2 < m^2 + 4n \leq m^2 + 4|m| - 4 < (|m| + 2)^2 ,$$

contradiction. Therefore, n must be negative, and

$$|m|^2 > m^2 + 4n \geq m^2 - 4|m| + 4 = (|m| - 2)^2 ,$$

with equality iff $|m| = 1 + |n| = 1 - n$. If $m = u$ is positive, then $n = 1 - u$, and both equations have respective solutions $-1, 1 - u$ and $-1, u$, clearly integers. If $m = -u$ is negative, then $n = 1 - u$, the discriminant of the second equation is $u^2 - 6u + 1 = (u - 3)^2 - 8$, and since 1^2 and 3^2 are the only two perfect squares which differ by 8, we have $u = 6$, $m = -6$ and $n = -5$.

After restoring generality and grouping solutions, it follows that all possible pairs are

$$(m, n) = \left(0, k^2\right), \qquad (m, n) = \left(k^2, 0\right), \qquad (m, n) = (-4, -4),$$

$$(m, n) = (-6, -5), \qquad (m, n) = (-5, -6), \qquad (m, n) = (u, 1 - u),$$

where k may take any non-negative integer value, u may take any integer value, and solutions of the first kind with $k = 1$ are also described as solutions of the last kind with $u \in \{0, 1\}$.

Also solved by Polyahedra, Polk State College, USA; George Theodoropoulos, 2nd High school of Agrinio, Greece; Ioannis D. Sfikas, Athens, Greece; Albert Stadler, Herrliberg, Switzerland.

J444. Let a, b, c, d be nonnegative real numbers such that $a + b + c + d = 4$. Prove that

$$a^3b + b^3c + c^3d + d^3a + 5abcd \le 27.$$

Proposed by Marius Stănean, Zalău, România

Solution by Polyahedra, Polk State College, USA

We may assume that $a = \max\{a, b, c, d\}$. Let $f(x, y, z, w) = x^3y + y^3z + z^3w + w^3x + 5xyzw$. Then for $0 \le t \le 4$,

$$27 - f(t, 4 - t, 0, 0) = 27 - t^3(4 - t) = (t - 3)^2(t^2 + 2t + 3) \ge 0,$$

so

$$27 \ge f(a + c, b + d, 0, 0)$$

On the other hand,

$$\begin{aligned} & f(a + c, b + d, 0, 0) = (a + c)^3(b + d) \ge b(a^3 + 3a^2c) + d(a^3 + 3a^2c + ac^2) \\ = \; & f(a, b, c, d) + (a^2 - b^2)bc + 2(a - d)abc + (a^2 - d^2)ad + 3(a - b)acd + (a - c)c^2d \\ \ge \; & f(a, b, c, d). \end{aligned}$$

and the conclusion follows.

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Moubinool Omarjee, Lycée Henri IV, Paris, France; Albert Stadler, Herrliberg, Switzerland.

Senior problems

S439. Let ABC be a triangle. Let points D and E be on segment BC and line AC, respectively, such that $\triangle ABC \cong \triangle DEC$. Let M be the midpoint of BC. Let P be a point such that $\angle BPM = \angle CBE$ and $\angle MPC = \angle BED$ and A, P lie on the same side of BC. Let Q be the intersection of lines AB and PC. Prove that the lines AC, BP, QD are either concurrent or all parallel.

Proposed by Grant Yu, East Setauket NY, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Applying the Sine Law to triangles BMP and CMP, we find

$$\frac{PB}{\sin \angle PMB} = \frac{BM}{\sin \angle BPM}, \qquad \frac{PC}{\sin \angle PMC} = \frac{CM}{\sin \angle CPM},$$

or since $\angle PMB + \angle PMC = 180^\circ$, $BM = MC$, $\angle BPM = \angle CBE = \angle DBE$ and $\angle CPM = \angle BED$, we have

$$\frac{PB}{PC} = \frac{\sin \angle CPM}{\sin \angle BPM} = \frac{\sin \angle BED}{\sin \angle DBE} = \frac{BD}{DE}.$$

Note next that $\angle CDE = \angle A$ because ABC, DEC are similar. Then,

$$\angle BPC = \angle BPM + \angle MPC = \angle DBE + \angle BED = 180^\circ - \angle BDE = \angle CDE = \angle A,$$

or A, B, C, P are concyclic.

If AC, BP are parallel, then $APBC$ is an isosceles trapezoid with $AQ = QC$, $BQ = QP$, $PC = AB$ and $AP = BC$, or

$$\frac{PB}{AB} = \frac{BC - CD}{DE} = \frac{BC}{DE} - \frac{AC}{AB}, \qquad \frac{BD}{CD} = \frac{BC}{CD} - 1 = \frac{PB}{AC} = \frac{BQ}{QA},$$

and by Thales' theorem, $QD \parallel AC$, or the proposed result holds in this case.

If AC, BP concur at a point X, note that $\frac{PX}{AX} = \frac{CX}{BX}$ because $APBC$ is cyclic. Note threfore that

$$\frac{CD}{DB} \cdot \frac{BP}{PX} \cdot \frac{XA}{AC} = \frac{CD}{DB} \cdot \frac{BD \cdot PC}{DE} \cdot \frac{BX}{CX} \cdot \frac{1}{AC} = \frac{BX}{AB} \cdot \frac{PC}{CX} = \frac{BX \sin \angle BXC}{AB \sin \angle A},$$

where we have used that triangles CDE and CAB are similar, and we have next applied the Sine Law to triangle CPX. But both numerator and denominator in this last expression are equal to the distance from B to AC, or the expression equals 1, and by Ceva's theorem, lines AB, PC, DX concur. Since Q is the intersection of AB, PC, then D, Q, X are collinear, or DQ concurs at X with AD and BP, or the proposed results holds also in this case.

Also solved by Albert Stadler, Herrliberg, Switzerland

S440. Prove that for any positive real numbers a, b, c the following inequality holds:

$$\frac{a^3}{bc}+\frac{b^4}{ca}+\frac{c^4}{ab} \geq \frac{3\left(a^3+b^3+c^3\right)}{a^2+b^2+c^2}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy

The inequality can be rewritten as

$$\sum_{\text{cyc}} a^4 \sum_{\text{cyc}} a^2 \geq 3abc \sum_{\text{cyc}} a^3$$

Now

$$a^4+b^4+c^4 \geq (a^3+b^3+c^3)^{4/3}/3^{1/3}$$

Thus it suffices to show that

$$\left(\sum_{\text{cyc}} a^3\right)^{4/3} \sum_{\text{cyc}} a^2 \geq 3^{4/3} abc \sum_{\text{cyc}} a^3 \iff \left(\sum_{\text{cyc}} a^3\right)^{1/3} \sum_{\text{cyc}} a^2 \geq 3^{4/3} abc$$

which follows from AGM.

Also solved by Daniel Lasaosa, Pamplona, Spain; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Naïm Mégarbané, UPMC, Paris, France; Muhammad Alhafi, Syrian Mathematical Olympiad Team, Aleppo, Syria; Joehyun Kim, Fort Lee High School, NJ, USA; Nikos Kalapodis, Patras, Greece; Angela Han, The Taft School, Watertown, CT, USA; Konstantinos Metaxas, Athens, Greece; Dionysios Adamopoulos, 3rd High School, Pyrgos, Greece; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Arkady Alt, San Jose, CA, USA; Duy Quan Tran, University of Medicine and Pharmacy, Ho Chi Minh, Vietnam; Florin Rotaru, Focsani, Romania; Jamshidxon Qodirov, SamSU academic lyceum, Samarkand, Uzbekistan; Ioannis D. Sfikas, Athens, Greece; Marin Chirciu, Colegiul Național "Zinca Golescu", Pitești, Romania; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nguyen Ngoc Tu, Ha Giang, Vietnam; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Paul Revenant, Lycée du Parc, Lyon, France; Albert Stadler, Herrliberg, Switzerland; Anas Kudsi, Syrian Mathematical Olympiad Team, Syria; Titu Zvonaru, Comănești, Romania.

S441. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that

$$\frac{ab}{4-a^2} + \frac{bc}{4-b^2} + \frac{ca}{4-c^2} \le 1$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Nikos Kalapodis, Patras, Greece

Using the given condition the inequality can be written as

$$\frac{ab}{1+b^2+c^2} + \frac{bc}{1+c^2+a^2} + \frac{ca}{1+a^2+b^2} \le 1$$

By Cauchy-Schwarz inequality we have

$$(1+b^2+c^2)(a^2+1+1) \ge (a+b+c)^2$$

or

$$\frac{1}{1+b^2+c^2} \le \frac{a^2+2}{(a+b+c)^2}$$

or

$$\frac{ab}{1+b^2+c^2} \le \frac{a^3b+2ab}{(a+b+c)^2}$$

Analogously we obtain that $\frac{bc}{1+c^2+a^2} \le \frac{b^3c+2bc}{(a+b+c)^2}$ and $\frac{ca}{1+a^2+b^2} \le \frac{c^3a+2ca}{(a+b+c)^2}$.

It follows that

$$\frac{ab}{1+b^2+c^2} + \frac{bc}{1+c^2+a^2} + \frac{ca}{1+a^2+b^2} \le \frac{a^3b+b^3c+c^3a+2(ab+bc+ca)}{(a+b+c)^2}$$

So, it remains to prove that $(a+b+c)^2 \ge a^3b+b^3c+c^3a+2(ab+bc+ca)$ or $a^2+b^2+c^2 \ge a^3b+b^3c+c^3a$.

But this follows by the given condition and V. Cirtoaje's inequality $(a^2+b^2+c^2)^2 \ge 3(a^3b+b^3c+c^3a)$. Equality holds iff $a=b=c=1$.

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Albert Stadler, Herrliberg, Switzerland; Ioannis D. Sfikas, Athens, Greece.

S442. Solve in integers the system of equations:

$$\begin{cases} x^3 - y^2 - 7z^2 = 2018 \\ 7x^2 + y^2 + z^3 = 1312. \end{cases}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Note first that $x^3 = 2018 + y^2 + 7z^2 > 1728 = 12^3$, or $x \geq 13$. Note further that

$$7x^2 + y^2 + 7z^2 - 1362 = (x-13)\left(x^2 + 20x + 260\right) = -(z-5)\left(z^2 - 2z - 10\right),$$

or since $x^2+20x+260 = (x+10)^2+160$ is always positive, we have $(z-5)\left(z^2-2z-10\right) \leq 0$. Now, $z^2-2z-10 = (z-1)^2 - 11$ is positive for $z \geq 5$, while if $-2 \leq z \leq 4$, we have $z^2 - 2z - 10 < 0$. Thus either $z = 5$ or $z \leq -3$. Clearly, if $z = 5$ and $x = 13$, we have $y^2 = 1362 - 7 \cdot 13^2 - 7 \cdot 5^2 = 4$, yielding solutions $(x, y, z) = (13, -2, 5)$ and $(13, 2, 5)$.

Assume that $x \geq 14$ and consequently $z \leq -3$. It follows that $d = x - z \geq 17$. Then, adding both equations and denoting $s = x + z$, we have

$$13320 = 4\left(x^3 + z^3 + 7x^2 - 7z^2\right) = s(3d^2 + s^2 + 7d).$$

Now, since the RHS is positive and $3d^2 + 7d + s^2 > 0$, then s must be positive. Moreover, $3d^2 + 7d \geq 3 \cdot 17^2 + 7 \cdot 17 = 986$, or $s \leq \frac{13320}{986} < \frac{13804}{986} = 14$, and $s \leq 13$. At the same time, s must divide $13320 = 2^3 \cdot 3^2 \cdot 5 \cdot 37$, or $s \in \{1, 2, 3, 4, 5, 6, 8, 9, 10, 12\}$. Consequently, $3d^2 + 7d$ takes values

$$\frac{13320}{s} - s^2 \in \{13319, 6656, 4431, 3314, 2639, 2184, 1601, 1399, 1232, 966\}.$$

The discriminants of the corresponding quadratic equations in d end in 3 or 7 except for $s \in \{2, 3, 8, 12\}$, or integer solutions for d may only occur in this cases. However, in these cases the discriminants are 79921, 53221, 19261 and 11641, which are found not to be perfect squares.

It follows that the only possible solutions are

$$(x, y, z) = (13, 2, 5), \qquad (x, y, z) = (13, -2, 5).$$

Also solved by Ioannis D. Sfikas, Athens, Greece; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania.

S443. Let ABC be a triangle, and let r_a, r_b, r_c be its exradii. Prove that

$$r_a \cos\frac{A}{2} + r_b \cos\frac{B}{2} + r_c \cos\frac{C}{2} \le \frac{3}{2}s.$$

Proposed by Dragoljub Miloševič, Gornji Milanovac, Serbia

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain

We will prove the desired inequality in the equivalent form

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \le \frac{3}{2},$$

using the fact that $\tan\frac{A}{2} = \frac{r_a}{s}$, that is, $r_a \cos\frac{A}{2} = s \sin\frac{A}{2}$ and observing that $f(x) = \sin\frac{x}{2}$ is a concave function in the interval $(0, \pi)$. The analytic criterion for concavity of a function is that its second derivative is negative. Indeed, $f'(x) = \frac{1}{2}\cos\frac{x}{2}$ and $f''(x) = -\frac{1}{4}\sin\frac{x}{2} < 0$ for $0 < x < \pi$.

Thus, by Jensen's inequality,

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \le 3 \cdot \sin\frac{\frac{A}{2} + \frac{B}{2} + \frac{C}{2}}{3} = 3\sin\frac{\pi}{6} = \frac{3}{2}$$

with equality if and only if $A = B = C$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Muhammad Alhafi, Syrian Mathematical Olympiad Team, Aleppo, Syria; Joehyun Kim, Fort Lee High School, NJ, USA; Nikos Kalapodis, Patras, Greece; Angela Han, The Taft School, Watertown, CT, USA; Konstantinos Metaxas, Athens, Greece; Anas Kudsi, Syrian Mathematical Olympiad Team, Syria; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Arkady Alt, San Jose, CA, USA; Ioannis D. Sfikas, Athens, Greece; Marin Chirciu, Colegiul Național "Zinca Golescu", Pitești, Romania; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Scott H. Brown, Auburn University Montgomery, Montgomery, AL, USA; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania.

S444. Let $x_1, \dots, x_n$ be positive real numbers. Prove that

$$\sum_{k=1}^{n} \frac{x_k}{x_k + \sqrt{{x_1}^2 + \cdots + {x_n}^2}} \leq \frac{n}{1+\sqrt{n}}.$$

Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Solution by Henry Ricardo, Westchester Area Math Circle

We note that $f(x) = x/(x+C)$ is concave for $C > 0$ and $x \in (0, \infty)$: $f''(x) = -2C/(x+C)^3 < 0$. Letting $x = x_k$ and $C = \sqrt{x_1^2 + \cdots + x_n^2}$, we apply Jensen's inequality to see that

$$\begin{aligned}
\sum_{k=1}^{n} \frac{x_k}{x_k + \sqrt{x_1^2 + \cdots + x_n^2}} &\leq n \cdot \frac{\sum_{k=1}^{n} x_k/n}{(\sum_{k=1}^{n} x_k/n) + \sqrt{x_1^2 + \cdots + x_n^2}} \\
&= \frac{n\sum_{k=1}^{n} x_k}{\sum_{k=1}^{n} x_k + n\sqrt{x_1^2 + \cdots + x_n^2}} \\
&\leq \frac{n\sum_{k=1}^{n} x_k}{\sum_{k=1}^{n} x_k + \sqrt{n}\sum_{k=1}^{n} x_k} \\
&= \frac{n}{1+\sqrt{n}},
\end{aligned}$$

where we used the Cauchy-Schwarz inequality to deduce that $n\sqrt{x_1^2 + \cdots + x_n^2} \geq \sqrt{n}\sum_{k=1}^{n} x_k$.

Also solved by Daniel Lasaosa, Pamplona, Spain; Joehyun Kim, Fort Lee High School, NJ, USA; Nikos Kalapodis, Patras, Greece; Angela Han, The Taft School, Watertown, CT, USA; Anas Kudsi, Syrian Mathematical Olympiad Team, Syria; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Arkady Alt, San Jose, CA, USA; Duy Quan Tran, University of Medicine and Pharmacy, Ho Chi Minh, Vietnam; Haosen Chen, Zhejiang, China; Ioannis D. Sfikas, Athens, Greece; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Albert Stadler, Herrliberg, Switzerland; Paul Revenant, Lycée du Parc, Lyon, France; Titu Zvonaru, Comănești, Romania.

U439. Evaluate

$$\int_{\frac{1}{2}}^{2} \frac{x^2+2x+3}{x^4+x^2+1} dx.$$

Proposed by Alessandro Ventullo, Milan, Italy

Solution by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

Since $\frac{x^2+2x+3}{x^4+x^2+1} = \frac{-2x+5}{2(x^2-x+1)} + \frac{2x+1}{2(x^2+x+1)}$, and

$$\begin{aligned}
\frac{-2x+5}{2(x^2-x+1)} &= \frac{-1}{2}\cdot\frac{2x-5}{x^2-x+1} \\
&= \frac{-1}{2}\cdot\frac{2x-1}{x^2-x+1} + \frac{2}{x^2-x+1} \\
&= \frac{-1}{2}\cdot\frac{2x-1}{x^2-x+1} + \frac{8}{3}\cdot\frac{1}{\left(\frac{2x-1}{\sqrt{3}}\right)^2+1} \\
&= \frac{-1}{2}\cdot\frac{2x-1}{x^2-x+1} + \frac{4}{\sqrt{3}}\cdot\frac{\frac{2}{\sqrt{3}}}{\left(\frac{2x-1}{\sqrt{3}}\right)^2+1}
\end{aligned}$$

then the proposed integral is

$$\begin{aligned}
I &= \int_{1/2}^{2} \frac{-1}{2}\cdot\frac{2x-1}{x^2-x+1}\,dx + \int_{1/2}^{2}\frac{4}{\sqrt{3}}\cdot\frac{\frac{2}{\sqrt{3}}}{\left(\frac{2x-1}{\sqrt{3}}\right)^2+1}\,dx + \int_{1/2}^{2}\frac{2x+1}{2(x^2+x+1)}\,dx \\
&= \left.\frac{4\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)}{\sqrt{3}} - \frac{1}{2}\log\left(x^2-x+1\right) + \frac{1}{2}\log\left(x^2+x+1\right)\right]_{1/2}^{2} \\
&= \frac{4\pi}{3\sqrt{3}} - \log(2) + \log(2) \\
&= \frac{4\pi\sqrt{3}}{9}.
\end{aligned}$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Naïm Mégarbané, UPMC, Paris, France; Joehyun Kim, Fort Lee High School, NJ, USA; Konstantinos Metaxas, Athens, Greece; G. C. Greubel, Newport News, VA, USA; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Henry Ricardo, Westchester Area Math Circle; Ioannis D. Sfikas, Athens, Greece; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Narayanan P, Vivekananda College, Chennai, India; Paul Revenant, Lycée du Parc, Lyon, France; Albert Stadler, Herrliberg, Switzerland; Thiago Landim de Souza Leão, Federal University of Pernambuco, Brazil; Xingze Xu, Hangzhou Foreign Languages School A-level Centre, China; Titu Zvonaru, Comănești, Romania.

U440. Let $a, b, c, t \geq 1$. Prove that

$$\frac{1}{ta^3+1}+\frac{1}{tb^3+1}+\frac{1}{tc^3+1} \geq \frac{3}{tabc+1}.$$

Proposed by An Zhenping, Xianyang Normal University, China

Solution by Arkady Alt, San Jose, CA, USA

Let

$$f(x)=\frac{1}{te^x+1}, x \geq 0.$$

Note that

$$\left(\frac{1}{te^x+1}\right)' = -\frac{e^x t}{(e^x t+1)^2} = -\frac{e^x t+1-1}{(e^x t+1)^2} = -\frac{1}{te^x+1}+\frac{1}{(e^x t+1)^2}$$

$$\left(\frac{1}{te^x+1}\right)'' = \frac{1}{te^x+1}-\frac{1}{(e^x t+1)^2}-\frac{2e^x t}{(e^x t+1)^3} = \frac{e^x t\,(te^x-1)}{(te^x+1)^3}$$

Hence $f''(x) = \dfrac{e^x t\,(te^x-1)}{(te^x+1)^3} \geq 0$ for $x \geq 0, t \geq 1$ then $f(x)$ is concave up on $[0,\infty)$ and, therefore, for any $x, y, z \geq 0$ holds Jensen's Inequality

$$\frac{f(x)+f(y)+f(z)}{3} \geq f\left(\frac{x+y+z}{3}\right) \iff \frac{1}{te^x+1}+\frac{1}{te^y+1}+\frac{1}{te^z+1} \geq 3\frac{1}{te^{(x+y+z)/3}+1}.$$

By replacing (x, y, z) in the latter inequality with $(3\ln a, 3\ln b, 3\ln c)$ we obtain

$$\frac{1}{ta^3+1}+\frac{1}{tb^3+1}+\frac{1}{tc^3+1} \geq 3\cdot\frac{1}{te^{\ln abc}+1} = \frac{3}{tabc+1}.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Naïm Mégarbané, UPMC, Paris, France; Nikos Kalapodis, Patras, Greece; Konstantinos Metaxas, Athens, Greece; Ioannis D. Sfikas, Athens, Greece; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Henry Ricardo, Westchester Area Math Circle; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania.

U441. Let x, y, z be nonnegative real numbers such that $x + y + z = 1$, and let $1 \le \lambda \le \sqrt{3}$. Determine the minimum and maximum of

$$f(x, y, z) = \lambda(xy + yz + zx) + \sqrt{x^2 + y^2 + z^2}$$

in terms of λ.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy

Let's define

$$x + y + z = 3u, \quad xy + yz + zx = 3v^2$$

We need to find the minimum and maximum of

$$3\lambda v^2 + \sqrt{9u^2 - 6v^2} = 3\lambda v^2 + \sqrt{1 - 6v^2} \doteq g(v^2), \quad 0 \le v^2 \le 1/9$$

The upper bound on v^2 follows by

$$3v^2 = xy + yz + zx \le (x + y + z)^2/3 = 1/3$$

Moreover the AGM yields $v \le u$

$$g'(v^2) = 3\lambda - \frac{3}{\sqrt{1 - 6v^2}} \ge 0 \iff v^2 \le \frac{1}{6} - \frac{1}{6\lambda^2} \doteq v_M^2$$

and $v_M^2 \le 1/9$ if and only if $\lambda \le \sqrt{3}$ while $v^2 \ge 0$ if and only if $\lambda \ge 1$. The maximum of $g(v^2)$ occurs for $v^2 = v_M^2$ and

$$g(v_M^2) = 3\lambda\left(\frac{1}{6} - \frac{1}{6\lambda^2}\right) + \sqrt{1 - 1 + \frac{1}{\lambda^2}} = \frac{\lambda}{2} + \frac{1}{2\lambda}$$

The minimum of $g(v^2)$ occurs at $v^2 = 0$ or $v^2 = 1/9$. If $v^2 = 0$ we have $g(0) = 1$ while

$$g(\frac{1}{9}) = \frac{\lambda}{3} + \frac{1}{\sqrt{3}}$$

Thus the minimum of $g(v^2)$ is 1 if $(3 - \sqrt{3}) \le \lambda \le \sqrt{3}$ while it is equal to $\dfrac{\lambda}{3} + \dfrac{1}{\sqrt{3}}$ if $1 \le \lambda \le (3 - \sqrt{3})$,

Also solved by Daniel Lasaosa, Pamplona, Spain; Joehyun Kim, Fort Lee High School, NJ, USA; Ioannis D. Sfikas, Athens, Greece; Ashley Case, Ashley Case, College at Brockport, SUNY, USA; Albert Stadler, Herrliberg, Switzerland; Thiago Landim de Souza Leão, Federal University of Pernambuco, Brazil.

U442. Let $\alpha \in (0,1)$, let $(p_k)_{k\geq 1}$ be the sequence of primes and let $q_n = \prod_{k\leq n} p_k$. Evaluate

$$\lim_{n\to\infty} \frac{\sum_{p|q_n} (\log p)^\alpha}{\omega(q_n)^{1-\alpha}(\log q_n)^\alpha}.$$

($\omega(n)$ denotes the number of distinct primes of a natural number n).

Proposed by Alessandro Ventullo, Milan, Italy

Solution by the author

et $a_p = 1$ and $b_p = (\log p)^\alpha$. By Hölder's inequality, we have

$$\begin{aligned}\sum_{p|q}(\log p)^\alpha &\leq \left(\sum_{p|q} 1^{\frac{1}{1-\alpha}}\right)^{1-\alpha}\left(\sum_{p|q}((\log p)^\alpha)^{\frac{1}{\alpha}}\right)^{\alpha}\\ &= (\omega(q))^{1-\alpha}\left(\log\prod_{p|q} p\right)^\alpha \leq (\omega(q))^{1-\alpha}(\log q)^\alpha.\end{aligned}$$

Since $p_n \geq n$ for all $n \in \mathbb{N}^*$, then $\log p_n \geq \log n$ and

$$\frac{\sum_{k=1}^n (\log k)^\alpha}{n(\log n)^\alpha} \leq \frac{\sum_{p|q_n}(\log p)^\alpha}{n(\log n)^\alpha} \leq \frac{(\omega(q_n))^{1-\alpha}(\log q_n)^\alpha}{n(\log n)^\alpha}. \tag{1}$$

Let us prove that $\lim\limits_{n\to\infty} \frac{\sum_{k=1}^n(\log k)^\alpha}{n(\log n)^\alpha} = 1$. Let $a_n = \sum\limits_{k=1}^n (\log k)^\alpha$ and $b_n = n(\log n)^\alpha$. We have

$$\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty}\frac{\log(n+1)^\alpha}{(n+1)(\log(n+1))^\alpha - n(\log n)^\alpha} = 1,$$

so by the Stolz-Cesaro Theorem,

$$\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty}\frac{a_n}{b_n} = 1.$$

Now, let us prove that $\lim\limits_{n\to\infty}\frac{(\omega(q_n))^{1-\alpha}(\log q_n)^\alpha}{n(\log n)^\alpha} = 1$. We have

$$\lim_{n\to\infty}\frac{(\omega(q_n))^{1-\alpha}(\log q_n)^\alpha}{n(\log n)^\alpha} = \lim_{n\to\infty}\frac{n^{1-\alpha}(\log q_n)^\alpha}{n(\log n)^\alpha} = \lim_{n\to\infty}\left(\frac{\sum_{p\leq p_n}\log p}{n\log n}\right)^\alpha.$$

By the Prime Number Theorem, $\sum_{p\leq p_n}\log p \sim p_n$ and $n\log n \sim p_n$, so

$$\lim_{n\to\infty}\frac{(\omega(q_n))^{1-\alpha}(\log q_n)^\alpha}{n(\log n)^\alpha} = 1.$$

Using these two limits in (1), by the Squeeze Theorem, we get

$$\lim_{n\to\infty}\frac{\sum_{p|q_n}(\log p)^\alpha}{n(\log n)^\alpha} = 1.$$

Also solved by Albert Stadler, Herrliberg, Switzerland.

U443. Find

$$\lim_{n\to\infty}\int_0^{\pi}\frac{\sin x}{1+\cos^2 nx}dx.$$

Proposed by Robert Bosch, USA

Solution by Daniel Lasaosa, Pamplona, Spain

We may write

$$\int_0^{\pi}\frac{\sin x}{1+\cos^2 nx}dx=\sum_{k=1}^{n}\int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}}\frac{\sin x}{1+\cos^2 nx}dx=\sum_{k=1}^{n}I_k.$$

Note now that, in the k-th integral in the sum,

$$\frac{s_k}{1+\cos^2 nx}=\frac{\min\{\sin x\}}{1+\cos^2 nx}<\frac{\sin x}{1+\cos^2 nx}<\frac{\max\{\sin x\}}{1+\cos^2 nx}=\frac{S_k}{1+\cos^2 nx},$$

where the minimum and maximum are taken for $\frac{(k-1)\pi}{n}\le x\le\frac{k\pi}{n}$. Since all terms are positive, we have

$$s_kJ=s_k\int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}}\frac{dx}{1+\cos^2 nx}<I_k<S_k\int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}}\frac{dx}{1+\cos^2 nx}=S_kJ,$$

where we may define

$$J=\int_{\frac{(k-1)\pi}{n}}^{\frac{k\pi}{n}}\frac{dx}{1+\cos^2 nx},$$

which is independent of k because of the periodicity of the $\cos^2 nx$ function with period $\frac{\pi}{n}$. In fact, with the variable change $y=nx-(k-1)\pi$ for each k, we have

$$J=\frac{1}{n}\int_0^{\pi}\frac{dy}{1+\cos^2(y+(k-1)\pi)}=\frac{2}{n}\int_0^{\frac{\pi}{2}}\frac{dy}{1+\cos^2 y}=\frac{2}{n}\int_0^{\infty}\frac{dz}{2+z^2}=$$

$$=\frac{\sqrt{2}}{n}\arctan\left(\frac{z}{\sqrt{2}}\right)\Bigg|_0^{\infty}=\frac{\pi}{\sqrt{2}n},$$

where we have performed the variable change $z=\tan y$. It then follows that

$$\frac{1}{\sqrt{2}}\sum_{k=1}^{n}\frac{\pi s_k}{n}<\int_0^{\pi}\frac{\sin x}{1+\cos^2 nx}dx<\frac{1}{\sqrt{2}}\sum_{k=1}^{n}\frac{\pi S_k}{n}.$$

Now, the sums in both upper and lower bounds are the Riemann sums approximating the integral of $\sin x$ in the interval $[0,\pi]$, one using the maximum of the function in an infinitesimal interval, the other using the minimum of the function in the same interval, both using a partition of the interval in subintervals of length $\frac{\pi}{n}$. Or when $n\to\infty$, both bounds have the same limit, which is therefore equal to the integral, and

$$\lim_{n\to\infty}\int_0^{\pi}\frac{\sin x}{1+\cos^2 nx}dx=\frac{1}{\sqrt{2}}\int_0^{\pi}\sin x dx=-\frac{\cos x}{\sqrt{2}}\Bigg|_0^{\pi}=\sqrt{2}.$$

Also solved by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Ioannis D. Sfikas, Athens, Greece; Moubinool Omarjee, Lycée Henri IV, Paris, France; Shubhajit Roy, Mumbai, India; Albert Stadler, Herrliberg, Switzerland.

U444. Let $p > 2$ be a prime and let $f(x) \in \mathbb{Q}[x]$ be a polynomial such that $\deg(f) < p-1$ and $x^{p-1}+x^{p-2}+\cdots+1$ divides $f(x)f(x^2)\cdots f(x^{p-1})-1$. Prove that there exists a polynomial $g(x) \in \mathbb{Q}[x]$ and a positive integer i such that $i < p$, $\deg(g) < p-1$, and $x^{p-1}+x^{p-2}+\cdots+1 \mid g(x^i)f(x)-g(x)$.

Proposed by Sreejata Kishor Bhattacharya, Chennai Mathematical Institute, India

Solution by the author

Let ξ be a primitive pth root of unity and let $K = \mathbb{Q}(\xi)$. Consider the Galois group $G = \mathrm{Gal}(K|\mathbb{Q}) = \{\sigma_1, \sigma_2, \cdots, \sigma_{p-1}\}$ where σ_i is the automorphism which sends ξ to ξ^i. Let $f(x) = a_0 + a_1 x + \cdots + a_{p-2}x^{p-2}$ and $\alpha = f(\xi) = a_0 + a_1\xi + \cdots + a_{p-2}\xi^{p-2}$. We see that $\sigma_i(f(\xi)) = f(\xi^i)$, so that

$$f(\xi)f(\xi^2)\cdots f(\xi^{p-1}) = \prod_{i=1}^{n-1} \sigma_i(\alpha) = \mathrm{Nm}_{K|\mathbb{Q}}(\alpha).$$

The given condition now implies $\mathrm{Nm}_{K|\mathbb{Q}}(\alpha) = 1$. Since $\mathrm{Gal}(K|\mathbb{Q})$ is cyclic, by Hilbert's theorem 90 there exists a $\beta \in K$ and a $0 < i < p$ such that $\alpha = \dfrac{\beta}{\sigma_i(\beta)}$. Let $\beta = b_0 + b_1\xi + \cdots + b_{p-2}\xi^{p-2}$. Take $g(x) = b_0 + b_1 x + \cdots + b_{p-2}x^{p-2}$. We see that $g(\xi^i)f(\xi) - g(\xi) = 0$. Hence $g(x^i)f(x) - g(x)$ and $x^{p-1} + \cdots + 1$ share a common root. Since $x^{p-1} + \cdots + 1$ is irreducible in $\mathbb{Q}[x]$, this implies that $x^{p-1} + x^{p-2} + \cdots + 1 \mid g(x^i)f(x) - g(x)$. Note that since $x^{p-1} + x^{p-2} + \ldots + x + 1$ is the minimal polynomial of ξ (it is the cyclotomic polynomial for the p-th root of unity), then this must divide every polynomial having ξ as a root.

Also solved by Albert Stadler, Herrliberg, Switzerland.

O439. Find all triples (x, y, z) of integers such that

$$(x-y)^2 + (y-z)^2 + (z-x)^2 = 2018.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Solution by Daniel Lasaosa, Pamplona, Spain

Since we may exchange any two of x, y, z without altering the problem, we may assume wlog that $x \geq y \geq z$, or positive integers u and v which take values $x - y$ and $y - z$ exist, such that

$$2018 = u^2 + v^2 + (u+v)^2, \qquad u^2 + uv + v^2 = 1009, \qquad (2u+v)^2 = 4036 - 3v^2.$$

Since the problem is symmetric in u, v, we may assume wlog that $v \geq u$, or $3v^2 \geq 1009$, and $v \geq 19$, while at the same time $4036 - 3v^2$ is a perfect square larger than v^2, or $v^2 < 1009$ for $v \leq 31$. Note further that when the last digit of v is $1, 4, 6, 9$, the last digit of $4036 - 3v^2$ is respectively $3, 8, 8, 3$, not possible for a perfect square. We thus need to check $v \in \{20, 22, 23, 25, 27, 28, 30, 31\}$, out of which only $v = 27$ produces a perfect square $4036 - 3 \cdot 27^2 = 43$. It follows that $u = \frac{43-27}{2} = 8$. Now, given any value of $z \leq y \leq x$, we have $y = z + 8$ and $x = z + 35$. Indeed, we would then have

$$(x-y)^2 + (y-z)^2 + (z-x)^2 = 27^2 + 8^2 + 35^2 = 2018,$$

and in any other solution, (x, y, z) is a permutation of $(k + 35, k + 8, k)$, where k is any integer.

Also solved by Paul Revenant, Lycée du Parc, Lyon, France; Paraskevi-Andrianna Maroutsou, Charters Sixth Form, Sunningdale, England, UK; Arpon Basu, AECS-4, Mumbai, India; Ioannis D. Sfikas, Athens, Greece; Moubinool Omarjee, Lycée Henri IV, Paris, France; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania.

O440. Prove that in any triangle ABC the following inequality holds

$$\left(\frac{a}{b+c}\right)^2+\left(\frac{b}{c+a}\right)^2+\left(\frac{c}{a+b}\right)^2+\frac{r}{2R}\geq 1.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

First solution by Arkady Alt, San Jose, CA, USA

Applying inequality $3\left(x^2+y^2+z^2\right)\geq(x+y+z)^2$ to $(x,y,z)=\left(\frac{a}{b+c},\frac{b}{c+a},\frac{c}{a+b}\right)$ we obtain

$$\left(\frac{a}{b+c}\right)^2+\left(\frac{b}{c+a}\right)^2+\left(\frac{c}{a+b}\right)^2\geq\frac{1}{3}\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)^2.$$

Also, since by Cauchy Inequality

$$\sum_{cyc}\frac{a}{b+c}=\sum_{cyc}\frac{a^2}{ab+ca}\geq\frac{(a+b+c)^2}{2(ab+bc+ca)}$$

and $(a+b+c)^2\geq 3(ab+bc+ca)\Rightarrow\sum_{cyc}\frac{a}{b+c}\geq\frac{3}{2}$.

Hence,

$$\left(\frac{a}{b+c}\right)^2+\left(\frac{b}{c+a}\right)^2+\left(\frac{c}{a+b}\right)^2\geq\frac{1}{3}\cdot\left(\frac{3}{2}\right)^2=\frac{3}{4}.$$

Remark:

As can be seen from the proof the inequality of the problem holds for any positive real a,b,c (not only for sidelengths of a triangle).

Second solution by Arkady Alt, San Jose, CA, USA

Since

$$\frac{a}{b+c}=\frac{\sin A}{\sin B+\sin C}=\frac{\sin\frac{A}{2}\cos\frac{A}{2}}{\sin\frac{B+C}{2}\cos\frac{B-C}{2}}=\frac{\sin\frac{A}{2}}{\cos\frac{B-C}{2}}$$

and $\cos A+\cos B+\cos C=1+\frac{r}{R}$ we obtain

$$\sum_{cyc}\left(\frac{a}{b+c}\right)^2=\sum_{cyc}\frac{\sin^2\frac{A}{2}}{\cos^2\frac{B-C}{2}}\geq$$

$$\sum_{cyc}\sin^2\frac{A}{2}=\frac{1}{2}\sum_{cyc}(1-\cos A)=\frac{1}{2}\left(3-\left(1+\frac{r}{R}\right)\right)=1-\frac{r}{2R}.$$

Therefore

$$\sum_{cyc}\left(\frac{a}{b+c}\right)^2+\frac{r}{2R}\geq 1-\frac{r}{2R}+\frac{r}{2R}=1.$$

Also solved by Daniel Lasaosa, Pamplona, Spain; Joehyun Kim, Fort Lee High School, NJ, USA; Nikos Kalapodis, Patras, Greece; Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy; Marin Chirciu, Colegiul Național "Zinca Golescu", Pitești, Romania; Albert Stadler, Herrliberg, Switzerland; Titu Zvonaru, Comănești, Romania.

O441. Let „abcbe positive real numbers. Find the minimum of the expression:

$$P = \frac{1}{\sqrt{2(a^4+b^4)+4ab}} + \frac{1}{\sqrt{2(b^4+c^4)+4bc}} + \frac{1}{\sqrt{2(c^4+a^4)+4ac}} + \frac{a+b+c}{3}$$

Proposed by Hoang Le Nhat Tung, Hanoi, Vietnam

Solution by the author

We will prove that inequality:

$$\frac{1}{a^2+ab+b^2} + \frac{1}{b^2+bc+c^2} + \frac{1}{c^2+ca+a^2} \geq \frac{9}{(a+b+c)^2} \qquad (1)$$

$$(1) \Leftrightarrow (a^2+b^2+c^2+ab+bc+ca)\left(\frac{1}{a^2+ab+b^2} + \frac{1}{b^2+bc+c^2} + \frac{1}{c^2+ca+a^2}\right) \geq \frac{9(a^2+b^2+c^2+ab+bc+ca)}{(a+b+c)^2}$$

$$\Leftrightarrow \frac{(a^2+ab+b^2)+c(a+b+c)}{a^2+ab+b^2} + \frac{(b^2+bc+c^2)+a(a+b+c)}{b^2+bc+c^2} + \frac{(c^2+ca+a^2)+b(a+b+c)}{c^2+ca+a^2} \geq \frac{9(a+b+c)^2-9(ab+bc+ca)}{(a+b+c)^2}$$

$$\Leftrightarrow 1 + \frac{c(a+b+c)}{a^2+ab+b^2} + 1 + \frac{a(a+b+c)}{b^2+bc+c^2} + 1 \frac{b(a+b+c)}{c^2+ca+a^2} \geq 9 - \frac{9(ab+bc+ca)}{(a+b+c)^2}$$

$$\Leftrightarrow (a+b+c)\left(\frac{c}{a^2+ab+b^2} + \frac{a}{b^2+bc+c^2} + \frac{b}{c^2+ca+a^2}\right) + \frac{9(ab+bc+ca)}{(a+b+c)^2} \geq 6 \qquad (2)$$

On the other hand, by Cauchy- Schwarz:

$$\begin{aligned}
\frac{c}{a^2+ab+b^2} + \frac{a}{b^2+bc+c^2} + \frac{b}{c^2+ca+a^2} = & \\
&= \frac{c^2}{ca^2+cab+cb^2} + \frac{a^2}{ab^2+abc+ac^2} + \frac{b^2}{bc^2+bca+ba^2} \geq \\
&\geq \frac{(c+a+b)^2}{(ca^2+cab+cb^2)+(ab^2+abc+ac^2)+(bc^2+bca+ba^2)} = \\
&= \frac{(a+b+c)^2}{ab(a+b+c)+bc(a+b+c)+ca(a+b+c)} = \\
&= \frac{(a+b+c)^2}{(a+b+c)(ab+bc+ca)} = \frac{a+b+c}{ab+bc+ca}
\end{aligned}$$

$$\Rightarrow \frac{c}{a^2+ab+b^2} + \frac{a}{b^2+bc+c^2} + \frac{b}{c^2+ca+a^2} \geq \frac{a+b+c}{ab+bc+ca}$$

Therefore

$$\Rightarrow (a+b+c)\left(\frac{c}{a^2+ab+b^2} + \frac{a}{b^2+bc+c^2} + \frac{b}{c^2+ca+a^2}\right) + \frac{9(ab+bc+ca)}{(a+b+c)^2} \geq (a+b+c)\frac{a+b+c}{ab+bc+ca} + \frac{9(ab+bc+ca)}{(a+b+c)^2} \geq$$

$$\geq 2\sqrt{\frac{(a+b+c)^2}{ab+bc+ca}\frac{9(ab+bc+ca)}{(a+b+c)^2}} = 2\sqrt{9} = 6$$

$$\Rightarrow (2) \Rightarrow (1)$$

By Bunyakovsky:

$$(\sqrt{2(a^4+b^4)}+2ab)^2 \leq (1^2+1^2)\left[(\sqrt{2\left(a^4+b^4\right)})^2 + (2ab)^2\right] \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{\sqrt{2(a^4+b^4)}+4ab} \geq \frac{1}{2(a^2+ab+b^2)}$$

$$\Rightarrow \frac{1}{\sqrt{2(a^4+b^4)}+4ab}+\frac{1}{\sqrt{2(b^4+c^4)}+4bc}+\frac{1}{\sqrt{2(c^4+a^4)}+4ca} \geq \frac{1}{2}\left(\frac{1}{a^2+ab+b^2}+\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2}\right)$$

$$\Rightarrow \frac{1}{\sqrt{2(a^4+b^4)}+4ab}+\frac{1}{\sqrt{2(b^4+c^4)}+4bc}+\frac{1}{\sqrt{2(c^4+a^4)}+4ca}+\frac{a+b+c}{3} \geq \frac{9}{2(a+b+c)^2}+\frac{a+b+c}{3}$$

Let $a+b+c=t>0$. By AM-GM:

$$\Rightarrow P \geq \frac{9}{2t^2}+\frac{t}{3} \geq 3\sqrt[3]{\frac{9}{2t^2}\cdot\frac{t}{6}\cdot\frac{t}{6}} \Rightarrow P \geq \frac{3}{2} \Rightarrow P_{min}=\frac{3}{2}.$$

Equality occurs if:

$$\begin{cases} \dfrac{1}{a^2+ab+b^2}=\dfrac{1}{b^2+bc+c^2}=\dfrac{1}{c^2+ca+a^2} \\ \dfrac{9(ab+bc+ca)}{(a+b+c)^2}=\dfrac{(a+b+c)^2}{ab+bc+ca} \\ \sqrt{2(a^4+b^4)}=2ab; \sqrt{2(b^4+c^4)}=2bc; \sqrt{2(c^4+a^4)}=2ca \\ \dfrac{9}{2t^2}=\dfrac{t}{6} \\ t=a+b+c>0 \end{cases} \Leftrightarrow \begin{cases} a=b=c \\ a+b+c=3 \end{cases} \Leftrightarrow a=b=c=1.$$

Also solved by Joehyun Kim, Fort Lee High School, NJ, USA; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Ioannis D. Sfikas, Athens, Greece.

O442. Let a, b, c be real numbers such that $a+b+c=3$. Prove that

$$7(a^4+b^4+c^4)+27 \geq (a+b)^4+(b+c)^4+(c+a)^4$$

Proposed by Marius Stănean, Zalău, România

Solution by Paolo Perfetti, Università degli studi di Tor Vergata Roma, Rome, Italy
Let's suppose $a, b, c \geq 0$. The inequality is

$$\sum_{\text{cyc}} f(a) \doteq \sum_{\text{cyc}} 7a^4-(3-a)^4+9 \geq 0$$

$$f''(a)=9(8a^2+8a-12) \geq 0 \iff a \leq \frac{-1-\sqrt{7}}{2} \ \wedge \ a \geq \frac{\sqrt{7}-1}{2}$$

Since $f''(a)$ changes sign one time, the minimum of $f(a)+f(b)+f(c)$ is attained when at least two variables are equal so we set $b=c$ and search the minimum of

$$f(3-2b)+2f(b)=108(b-1)^2(b-2)^2 \geq 0$$

The inequality is true a fortiori if one or more variables are negative since the LHS is even in each variable.

Also solved by Joehyun Kim, Fort Lee High School, NJ, USA; Konstantinos Metaxas, Athens, Greece; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Arkady Alt, San Jose, CA, USA; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Albert Stadler, Herrliberg, Switzerland; Ioannis D. Sfikas, Athens, Greece; Dionysios Adamopoulos, 3rd High School, Pyrgos, Greece.

O443. Let $f(n)$ be the number of permutations of the set $\{1, 2, \ldots, n\}$ such that no pair of consecutive integers appears in that order; that is, 2 does not follow 1, 3 does not follow 2, and so on.

(i) Prove that $f(n) = (n-1)f(n-1) + (n-2)f(n-2)$.
(ii) For any real number α, denote by $[\alpha]$ the nearest integer to α. Prove that

$$f(n) = \frac{1}{n}\left[\frac{(n+1)!}{e}\right].$$

Proposed by Rishub Thaper, Hunterdon Central Regional High School, Flemington, NJ, USA

Solution by Joel Schlosberg, Bayside, NY, USA

(i)

Given such a permutation of the integers 1 through n, removing n yields a permutation of the integers 1 through $n-1$ with the same property, unless n is between a pair of increasing consecutive integers. But if n is between k and $k+1$, then removing $k+1$ and decreasing each of the remaining integers greater than $k+1$ by 1 yields a permutation of the integers 1 through $n-2$ with the same property. Conversely, given a permutation of the integers 1 through $n-1$ with the given property, $n-1$ permutations of the integers 1 through n can be obtained by placing n either before all the integers or after any integer $1, \ldots, n-2$. And given a permutation of the integers 1 through $n-2$ with the given property, $n-2$ permutations of the integers 1 through n can be obtained, 1 for each $k = 1, \ldots, n-2$, as follows: increase each number $k+1, \ldots, n$ by 1, then insert $n, k+1$ after k. By this process, each permutation with the given property of $1, \ldots, n$ is either among $n-1$ associated with a permutation with the given property of $1, \ldots, n-1$, or among $n-2$ associated with a permutation of the given property of $1, \ldots, n-2$. Thus

$$f(n) = (n-1)f(n-1) + (n-2)f(n-2)$$

.

(ii)

$$\frac{(n+1)!}{e} = (n+1)!\sum_{k=0}^{\infty}\frac{(-1)^k}{k!} = \sum_{k=2}^{n+1}(-1)^k\frac{(n+1)!}{k!} + (n+1)!\sum_{k=n+2}^{\infty}\frac{(-1)^k}{k!}$$

where $\sum_{k=2}^{n+1}(-1)^k\frac{(n+1)!}{k!}$ is an integer and

$$\left|(n+1)!\sum_{k=n+2}^{\infty}\frac{(-1)^k}{k!}\right| < (n+1)!\sum_{k=n+2}^{\infty}\frac{1}{k!} < (n+1)!\sum_{k=n+2}^{\infty}\frac{1}{(n+1)!(n+2)^{k-n-1}} = \frac{1}{n+1} < 1,$$

so

$$\left[\frac{(n+1)!}{e}\right] = \sum_{k=2}^{n+1}(-1)^k\frac{(n+1)!}{k!}.$$

The values $f(2) = 1$ (permutation $(2,1)$) and $f(3) = 3$ (permutations $(1,3,2),(2,1,3),(3,2,1)$) match the formula for $f(n)$; assume by induction that it holds for $n-1$ and $n-2$ for $n \geq 3$. Then

$$\begin{aligned}
f(n) &= (n-1)f(n-1) + (n-2)f(n-2) \\
&= \left[\frac{n!}{e}\right] + \left[\frac{(n-1)!}{e}\right] \\
&= \sum_{k=2}^{n} (-1)^k \frac{n!}{k!} + \sum_{k=2}^{n-1} (-1)^k \frac{(n-1)!}{k!} \\
&= (-1)^n + \sum_{k=2}^{n-1} (-1)^k \frac{n! + (n-1)!}{k!} \\
&= \frac{1}{n}\left((n+1)(-1)^n + (-1)^{n+1} + \sum_{k=2}^{n-1} (-1)^k \frac{(n+1)!}{k!}\right) \\
&= \frac{1}{n} \sum_{k=2}^{n+1} (-1)^k \frac{(n+1)!}{k!} = \frac{1}{n}\left[\frac{(n+1)!}{e}\right],
\end{aligned}$$

completing the induction.

Also solved by Dionysios Adamopoulos, 3rd High School, Pyrgos, Greece; Albert Stadler, Herrliberg, Switzerland; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

O444. Let T be Toricelli point of a triangle ABC. Prove that

$$\frac{1}{BC^2}+\frac{1}{CA^2}+\frac{1}{AB^2}\geq\frac{9}{(AT+BT+CT)^2}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Solution by Arkady Alt, San Jose, CA, USA

Let $x:=TA, y:=TB, z:=TC$. Since

$$BC^2=TB^2+TC^2-2TB\cdot TC\cos 120^\circ=y^2+z^2+yz$$

and similarly $CA^2=z^2+x^2+zx, AB^2=x^2+y^2+xy$ then inequality of the problem becomes

$$\frac{1}{x^2+xy+y^2}+\frac{1}{y^2+yz+z^2}+\frac{1}{z^2+zx+x^2}\geq\frac{9}{(x+y+z)^2}. \tag{1}$$

Assuming $x+y+z=1$ (due homogeneity of (1)) and denoting $p:=xy+yz+zx, q:=xyz$ we obtain

$$\begin{aligned}
&\left(x^2+xy+y^2\right)\left(y^2+yz+z^2\right)\left(z^2+zx+x^2\right)=\\
&= \sum\left(x^4y^2+x^2y^4\right)+3x^2y^2z^2+xyz\sum x^3+\sum x^3y^3+2xyz\sum xy\,(x+y)=\\
&= \left(x^2+xy+y^2\right)\left(y^2+yz+z^2\right)\left(z^2+zx+x^2\right)\\
&= \sum x^3y^3+\left(x^2y^2+x^2z^2+y^2z^2\right)\left(x^2+y^2+z^2\right)+\\
&+ 2xyz\,(xy+xz+yz)\,(x+y+z)-6x^2y^2z^2+xyz\left(x^3+y^3+z^3\right)=
\end{aligned}$$

$$p^3+3q^2-3pq+(1-2p)\left(p^2-2q\right)+2pq-6q^2+q\,(1+3q-3p)=p^2-p^3-q$$

$$\begin{aligned}
&\sum\left(x^2+xy+y^2\right)\left(y^2+yz+z^2\right)=\\
&= \sum\left(x^2y^2+y^2z^2+x^2z^2+x^2yz+xy^2z+xyz^2+xy^3+y^3z+y^4\right)=\\
&= 3\left(x^2y^2+y^2z^2+x^2z^2\right)+xyz\sum(x+y+z)+\sum x^4+\sum xy\left(x^2+y^2\right)=\\
&= 3\left(x^2y^2+y^2z^2+x^2z^2\right)+2xyz\,(x+y+z)+\\
&+ \left(x^4+y^4+z^4\right)+(xy+xz+yz)\left(x^2+y^2+z^2\right)=\\
&= 3\left(p^2-2q\right)+2q+1+4q-4p+2p^2+p\,(1-2p)=3p^2-3p+1
\end{aligned}$$

and then can rewrite inequality (1) as

$$\frac{3p^2-3p+1}{p^2-p^3-q}\geq 9.$$

Note that $3p = (xy + yz + zx) \le (x + y + z)^2 = 1,\ q \ge \dfrac{4p - 1}{9}$
(Schure inequality $\sum_{cyc} x (x - y) (x - z) \ge 0$ in p,q-notation and normalized by $x + y + z = 1$)

and also

$$q = xyz (x + y + z) \le \frac{(xy + yz + zx)^2}{3} = \frac{p^2}{3}.$$

Since $\dfrac{3p^2 - 3p + 1}{p^2 - p^3 - q}$ increases in $q \le \dfrac{p^2}{3}$
($3p^2 - 3p + 1 > 0$ and $p^2 - p^3 - q \ge p^2 - p^3 - \dfrac{p^2}{3} = \dfrac{p^2 (2 - 3p)}{3} > 0$)

then

$$\frac{3p^2 - 3p + 1}{p^2 - p^3 - q} - 9 \ge \frac{3p^2 - 3p + 1}{p^2 - p^3 - \left(\frac{4p - 1}{9}\right)} - 9 =$$

$$\frac{9p (1 - 3p)^2}{1 - 4p + 9p^2 - 9p^3} \ge 0$$

because

$$1 - 4p + 9p^2 - 9p^3 = (1 - 2p)^2 + p^2 (5 - 9p) \ge (1 - 2p)^2 + p^2 \left(5 - 9 \cdot \frac{1}{3}\right) = (1 - 2p)^2 + 2p^2 > 0$$

Also solved by Joehyun Kim, Fort Lee High School, NJ, USA; Albert Stadler, Herrliberg, Switzerland; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania.