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Solution for problem 513

The characteristic equation of the recurrence given in (1) is

$$x^3 - 3x^2 + 3x - 1 = 0$$
$$\Rightarrow (x-1)^3 = 0$$

Thus by the theory of homogeneous linear recurrences we get that

$$a_n = k_1 \cdot 1^n + k_2 \cdot n \cdot 1^n + k_3 \cdot n^2 \cdot 1^n$$

$$= k_1 + k_2 \cdot n + k_3 \cdot n^2 \qquad \forall n \geq 0$$

Now by condition (2) we have

$$2(k_1 + k_2 + k_3) = (k_1) + (k_1 + 2k_2 + 4k_3) - 2 \Rightarrow k_3 = 1$$

$$\therefore a_n = n^2 + (k_2)n + k_1$$

Now $k_1 = a_0 \in \mathbb{Z}$ and $a_1 = 1 + k_1 + k_2 \in \mathbb{Z} \Rightarrow k_2 \in \mathbb{Z}$ that is both k_1, k_2 are integers.
If $k_2^2 - 4k_1 \neq 0$ then

$x^2 = a_n = n^2 + k_2 n + k_1$
$\Rightarrow 4x^2 = 4n^2 + 4k_2 n + 4k_1 = (2n)^2 + 2 \cdot 2n \cdot k_2^2 + (4k_1 - k_2^2)$
$\Rightarrow (2x)^2 = (2n + k_2)^2 + (4k_1 - k_2^2)$
$\Rightarrow |(2x)^2 - (2n + k_2)^2| = |(k_2^2 - 4k_1)|$
This equation can have atmost $\tau(|(k_2^2 - 4k_1)|)$ solutions for x, where $\tau(n)$ is number of positive divisors of n.
But this contradicts condition (3) according to which $x^2 = a_n$ can have arbitrarily large number of solutions.
$\therefore (k_2^2 - 4k_1) = 0$
$\Rightarrow k_2 = 2r; k_1 = r^2$ for some $r \in \mathbb{Z}$
$\therefore a_n = n^2 + 2rn + r^2 = (n + r)^2$ is a perfect square for all n.

$\therefore$ **Hence proved.**