Dense divisors of a null matrix

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Statement:
Prove that $\forall\ A \in \mathbb{K}^{n\times n}\ \exists\ X \in \mathbb{K}^{n\times n}$ such that $AX = \mathbf{O}$ and X has atmost nr zeroes in it, where $r = \text{rank}(A)$. By a dense matrix, we mean a matrix having more number of non zero elements than zeros.
A statement differing from the above only in the order of multiplication of the two matrices still holds, i.e, $\forall\ A \in \mathbb{K}^{n\times n}\ \exists\ Y \in \mathbb{K}^{n\times n}$ such that $YA = \mathbf{O}$ and Y has at most nr zeroes in it.

Proof:
Note that if A is an invertible matrix, then $X = Y = \mathbf{O}$ does the job for us. Also, if $A = \mathbf{O}$, then any matrix $X \in \mathbb{K}^{n\times n}$ suffices, and once again the theorem is trivial (note that $\text{rank}(\mathbf{O}) = 0$).
If not, then for any $n \times n$ (non invertible) matrix A with rank $r \geq 1$, we have $(n-r)$ "basic" solutions of the equation $A\mathbf{x} = \mathbf{O}$, $\mathbf{x} \in \mathbb{K}^{n\times 1}$, ie:- column vectors whose entries are 1 at the position of any one of the $(n-r)$ non-pivotal variable(s) and zero at the rest of the non-pivotal variables, while the pivotal variables are determined by back substitution. Thus, by taking a linear combination (with all coefficients non-zero) of all the the basic solutions of the matrix, one can generate a new column vector which is a solution of $A\mathbf{x} = \mathbf{O}$ and all it's non-pivotal entries are non-zero, i.e, it has at most r zero entries in it. Stack that column vector together side by side, with itself, n times to get a $n \times n$ matrix X such that $AX = \mathbf{O}$ and X has at most $n\cdot r$ zeroes in it.
And as for the second part of the statement, we know $\exists\ X' \in \mathbb{K}^{n\times n}$ such that $A^T X' = \mathbf{O}$ and X' has at most nr zeroes in it. Since $\text{rank}(A^T) = \text{rank}(A)$, and $(A^T X')^T = (X')^T A = \mathbf{O}$, we get that $Y = (X')^T$ satisfies all the desired properties. Hence proved.

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