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Solution for Problem J377

We know that $sin^2\frac{A}{2}=\frac{1-cosA}{2}$.Hence the statement to be proved is $\frac{m_a}{R}\geq(1-cosA)$.
Now,we know that $m_a=\frac{\sqrt{2b^2+2c^2-a^2}}{2}$.
Using the cosine law to substitute for a^2,we get that $m_a=\frac{\sqrt{b^2+c^2+2bccosA}}{2}$.
Also,using the extended sine law,we substitute $\frac{a}{2sinA}$ for R.
Thus the statement to be proved reduces to:-
$\frac{\sqrt{b^2+c^2+2bccosA}}{\frac{a}{sinA}}\geq 1-cosA \Leftrightarrow (b^2+c^2+2bccosA)(sin^2A)\geq a^2(1+cos^2A-2cosA)\Leftrightarrow b^2sin^2A+c^2sin^2A+2bccosAsin^2A\geq (b^2+c^2-2bccosA)(1+cos^2A-2cosA)=b^2+c^2-2bccosA+b^2cos^2A+c^2cos^2A-2bccos^3A-2b^2cosA-2c^2cosA+4bccos^2A$
Shifting everything to the RHS of the inequality,simplifying the resulting expression and factoring out $cosA$,we get:-
$0\geq(-2)(cosA)(1-cosA)(b-c)^2$.
But,since $A\in(0,\frac{\pi}{2}]$, $cosA\geq 0$. Also,since $(1-cosA)$ and $(b-c)^2$ are always non-negative,it's obvious that the inequality $0\geq(-2)(cosA)(1-cosA)(b-c)^2$ is true because -2 is negative.
For the upper bound,we know that $cos^2\frac{A}{2}=\frac{1+cosA}{2}$.So replacing $1-cosA$ with $1+cosA$ and simplifying as above,we get:-
$0\leq 2(cosA)(1+cosA)(b-c)^2$.Again,since all of the terms $cosA$,$(1+cosA)$ and $(b-c)^2$ are non-negative,and since 2 is positive,we see that this inequality is also true.
Note that in both the inequalities,equality occurs when $A=\frac{\pi}{2}$,when $cosA=0$.