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Solution for problem J442

We use barrycentric coordinates. let

$$A : (1, 0, 0)$$

$$B : (0, 1, 0)$$

$$C : (0, 0, 1)$$

Where $\triangle ABC$ is our reference triangle.

$$\therefore \quad O : (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$$

where O is the centre of triangle.
Let M be $(1-t, t, 0)$ where $t \in \mathbb{R}$. We note that $M \in \overleftrightarrow{AB}$
$\because M, O, N$ are collinear ,

$$\begin{vmatrix} 1-t & t & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ X_N & Y_N & Z_N \end{vmatrix} = 0$$

$\because N \in \overleftrightarrow{AC}, Y_N = 0$ solving for X_N, Z_N we get $N : (2t-1 : 0 : t)$

$\therefore \overleftrightarrow{BN}$ is represented by

$$\begin{vmatrix} 0 & 1 & 0 \\ (2t-1) & 0 & t \\ x & y & z \end{vmatrix} = 0$$
$$\Rightarrow tx + (1-2t)z = 0$$

$\overleftrightarrow{CM}$ is represented by

$$\begin{vmatrix} 0 & 0 & 1 \\ (1-t) & t & 0 \\ x & y & z \end{vmatrix} = 0$$
$$\Rightarrow tx + (1-t)y = 0$$

$$\begin{aligned} \therefore k &= \overleftrightarrow{BN} \cap \overleftrightarrow{CM} \\ &= \big((1-t)(2t-1) : t(2t-1) : t(1-t)\big) \end{aligned} \tag{1}$$

$$\therefore \overleftrightarrow{AK} : \begin{vmatrix} 1 & 0 & 0 \\ (1-t)(2t-1) & t(2t-1) & t(1-t) \\ x & y & z \end{vmatrix} = 0$$
$$\Rightarrow (1-t)y + (1-2t)z = 0$$

Also

$$\overleftrightarrow{BO} : \begin{vmatrix} 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ x & y & z \end{vmatrix} = 0$$
$$\Rightarrow (x - z) = 0$$

$$\begin{aligned} \therefore P &= \overleftrightarrow{AK} \cap \overleftrightarrow{BO} \\ &= \big((1-t), (2t-1), (1-t)\big) \end{aligned} \tag{2}$$

$\therefore$ the displacement vectors are

$$\overrightarrow{MB} = \big((1-t), (t-1), 0\big)$$

$$\overrightarrow{MP} = \big(0, (1-t), (t-1)\big)$$

$$\begin{aligned} \therefore |\overrightarrow{MB}|^2 &= -a^2yz - b^2zx - c^2xy = c^2(1-t)^2 \\ \therefore |\overrightarrow{MP}|^2 &= -a^2(1-t)(t-1) = a^2(1-t)^2 \end{aligned} \tag{3}$$

$\because \triangle ABC$ is equilateral $\therefore a = c \Rightarrow MB = MP$

Hence Proved
Note: Our proof works exactly even if only $BC = A$ as the centroid