Arpon Basu, School: AECS-4 , Mumbai-400094
Solution for problem O397

We have to prove that $\sqrt{\frac{9a+b}{9b+a}} + \sqrt{\frac{9b+c}{9c+b}} + \sqrt{\frac{9c+a}{9a+c}} \geq 3 \ \forall a, b, c \in R^+$
Putting $b = ax, c = by$ where $x, y \in R^+$ we get that

$$\begin{aligned} &\sqrt{\frac{9a+b}{9b+a}} + \sqrt{\frac{9b+c}{9c+b}} + \sqrt{\frac{9c+a}{9a+c}} \\ &= \sqrt{\frac{9a+ax}{9ax+a}} + \sqrt{\frac{9ax+axy}{9axy+ax}} + \sqrt{\frac{9axy+a}{9a+axy}} \\ &= \sqrt{\frac{x+9}{9x+1}} + \sqrt{\frac{y+9}{9y+1}} + \sqrt{\frac{9xy+1}{xy+9}} \end{aligned}$$

We define $f(x, y) = \sqrt{\frac{x+9}{9x+1}} + \sqrt{\frac{y+9}{9y+1}} + \sqrt{\frac{9xy+1}{xy+9}}$

$$\therefore \frac{\delta f}{\delta x} = 40\left(\frac{y}{\sqrt{(9xy+1)(xy+9)^3}}\right) - \left(\frac{1}{\sqrt{(x+9)(9x+1)^3}}\right)$$

Note that we get the expression for $\frac{\delta f}{\delta x}$ if we plug in y in place of x.

$$\therefore \frac{\delta f}{\delta x} = 0 \Rightarrow y = rx$$

for some $r \in R^+$

$$\Rightarrow r = \pm 1 \Rightarrow x = \pm y$$

$$\because x, y > 0 \Rightarrow x = y$$

Considering y as constant, we find $minf(x, y)$ for $x \in R^+$

$$\therefore \frac{\delta f}{\delta x}|_{x=y} = 0 \Rightarrow \frac{x}{\sqrt{(x+9)(9x+1)^3}} = \frac{1}{\sqrt{(x+9)(9x+1)^3}}$$

$$\Rightarrow x^2(x+9)(9x+1)^3 - (9x^2+1)(x^2+9)^3 = 0$$

Using Wolfram Alpha, one can check that the only positive real roots of above equation are 1 and α where $\alpha \approx 0.09$. Also note that $f(1, 1) = 3$ while $f(\alpha, \alpha) > 3 = f(1, 1)$ Similarly , considering x as a constant, we get $minf(x, y) = f(1, 1)$ for $y \in R^+$. Checking any other value of $f(x, y)$ say $f(2, 3) \approx 3.330$ varifies that $f(1, 1)$ is indeed the minima of $f(x, y)$

$$\therefore \sqrt{\frac{9a+b}{9b+a}} + \sqrt{\frac{9b+c}{9c+b}} + \sqrt{\frac{9c+a}{9a+c}} \geq 3$$

and equality occurs iff $x = y = 1$, that is $b = a, c = b \therefore a = b = c$