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Solution for Problem J376

$\frac{1}{5-4\cos\alpha}+\frac{1}{5-4\cos\beta}+\frac{1}{5-4\cos\gamma}=\frac{1}{1+4(1-\cos\alpha)}+\frac{1}{1+4(1-\cos\beta)}+\frac{1}{1+4(1-\cos\gamma)}=\frac{3+8\Sigma(1-\cos\alpha)+16\Sigma(1-\cos\alpha)(1-\cos\beta)}{1+4\Sigma(1-\cos\alpha)+16\Sigma(1-\cos\alpha)(1-\cos\beta)+64\Pi(1-\cos\alpha)}$
Thus the statement to be proved is $\frac{3+8\Sigma(1-\cos\alpha)+16\Sigma(1-\cos\alpha)(1-\cos\beta)}{1+4\Sigma(1-\cos\alpha)+16\Sigma(1-\cos\alpha)(1-\cos\beta)+64\Pi(1-\cos\alpha)}\geq 1$.
Multiplying both sides by the denominator and simplifying we get:-
$2+4\Sigma(1-\cos\alpha)\geq 64\Pi(1-\cos\alpha)\Rightarrow 1+2\Sigma(1-\cos\alpha)\geq 32\Pi(1-\cos\alpha)$
Now,using the inequality $\Sigma\cos\alpha\leq\frac{3}{2}$ if α, β and γ are angles of a triangle,we get that:-
$\Sigma(1-\cos\alpha)=3-\Sigma\cos\alpha\geq 3-\frac{3}{2}=\frac{3}{2}\Rightarrow 1+2\Sigma(1-\cos\alpha)\geq 4$.Thus if we can prove that $4\geq 32\Pi(1-\cos\alpha)\Rightarrow\frac{1}{8}\geq\Pi(1-\cos\alpha)$ then we're done.
We note that $(1-\cos\alpha)=2\sin^2\frac{\alpha}{2}\Rightarrow\Pi(1-\cos\alpha)=8(\Pi\sin(\frac{\alpha}{2}))^2$ implies that $\frac{1}{8}\geq\Pi(1-\cos\alpha)\Rightarrow\frac{1}{64}\geq(\Pi\sin(\frac{\alpha}{2}))^2\Rightarrow\frac{1}{8}\geq\Pi\sin(\frac{\alpha}{2})$.
Finally,using the fact that $\Pi\sin(\frac{\alpha}{2})=\frac{r}{4R}$,where r and R denote the inradius and circumradius of the triangle respectively,we get that the statement to be proved is $\frac{r}{4R}\leq\frac{1}{8}\Rightarrow 2r\leq R$,which is obviously true.
Note that in both the inequalities $\Sigma\cos\alpha\leq\frac{3}{2}$ and $R\geq 2r$ used for proving our result,equality holds iff $\alpha=\beta=\gamma=\frac{\pi}{3}$,and hence:-
$\frac{1}{5-4\cos\alpha}+\frac{1}{5-4\cos\beta}+\frac{1}{5-4\cos\gamma}\geq 1$,*if* α, β *and* γ *are angles of a triangle.Equality holds iff* $\alpha=\beta=\gamma=\frac{\pi}{3}$.