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Solution for problem O392

We know that $r_a = \frac{\triangle}{(s-a)}$, $r_b = \frac{\triangle}{(s-b)}$ and $r_c = \frac{\triangle}{(s-c)}$ where s is semi perimeter.

$$\therefore \frac{1}{r_a} = \frac{(s-a)}{\triangle} \Rightarrow \frac{1}{r_a h_a} = \frac{(s-a)}{\triangle h_a}$$

$$\therefore \sum \frac{1}{r_a h_a} = \sum \frac{(s-a)}{\triangle h_a} = \frac{1}{\triangle}(\sum \frac{(s-a)}{h_a})$$

Now

$$\frac{(s-a)}{h_a} = \frac{(\frac{b+c-a}{2})}{\frac{2\triangle}{a}} = \frac{a(b+c-a)}{4\triangle}$$

$$\sum \frac{1}{r_a h_a} = \frac{1}{4\triangle^2}\sum[(a)(b+c-a)]$$

$$= \frac{1}{4\triangle^2}[2(ab+bc+ca) - (a^2+b^2+c^2)]$$

$$= \frac{1}{4\triangle^2}[a^2+b^2+c^2-(a-b)^2-(b-c)^2-(c-a)^2]$$

Now,by Hadwiger Finsler inequality , we know that

$$a^2+b^2+c^2-(a-b)^2-(b-c)^2-(c-a)^2 \geq 4\sqrt{3}\triangle$$

$\therefore \sum \frac{1}{r_a h_a} \geq \frac{4\sqrt{3}\triangle}{4\triangle^2} = \frac{\sqrt{3}}{\triangle}$.(Proved)

Now

$$\frac{1}{3r^2} = \frac{1}{3(\frac{\triangle}{s})^2} = \frac{1}{3}(\frac{s}{\triangle})^2$$

We have to prove that

$$\frac{1}{3r^2} \geq \sum \frac{1}{r_a h_a}$$

$$\iff \frac{s^2}{3} \geq \frac{1}{4}\sum[a(b+c-a)]$$

$$\iff 4s^2 \geq 3\sum[a(b+c-a)]$$

$$\iff (a+b+c)^2 \geq 3[2(ab+bc+ca)-(a^2+b^2+c^2)]$$

After algebraic manipulation we get

$$(a^2+b^2+c^2) \geq (ab+bc+ca)$$

But it is obviously true. Equalities occur, in both inequalities iff

$$a=b=c$$

that is iff $\triangle$ABC is equilateral.