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Solution for problem U397

We know that $T_n = \frac{(n)(n+1)}{2}$

$$(8T_n - 3)(8T_{n+1} - 3) = (4n^2 + 4n - 3)(4n^2 + 12n + 5)$$
$$= (n + \frac{3}{2})(n - \frac{1}{2})(n + \frac{5}{2})(n + \frac{1}{2})$$
$$= \frac{(2n+3)(2n-1)(2n+5)(2n+1)}{16}$$
$$\therefore 16\frac{1}{(8T_n - 3)(8T_{n+1} - 3)} = \frac{16}{(2n+3)(2n-1)(2n+5)(2n+1)}$$
$$= \frac{4}{(2n+5)(2n+1)}\frac{4}{(2n+3)(2n-1)}$$
$$= \Big(\frac{1}{(2n+1)} - \frac{1}{(2n+5)}\Big)\Big(\frac{1}{(2n-1)} - \frac{1}{(2n+3)}\Big)$$

After algebraic manipulation we obtain

$$16\frac{1}{(8T_n - 3)(8T_{n+1} - 3)} = \frac{1}{3}\Big(\frac{1}{2n-1} - \frac{1}{2n+5}\Big) + \Big(\frac{1}{2n+3} - \frac{1}{2n+1}\Big)$$

$$\therefore \sum_{n\geq 1} \frac{16}{(8T_n - 3)(8T_{n+1} - 3)} = \frac{1}{3}\sum_{n\geq 1}\Big(\frac{1}{2n-1}\Big) - \frac{1}{3}\sum_{n\geq 1}\Big(\frac{1}{2n+5}\Big) + \sum_{n\geq 1}\Big(\frac{1}{2n+3}\Big) - \sum_{n\geq 1}\Big(\frac{1}{2n+1}\Big)$$
$$= \frac{1}{3}[(\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7}\dots) - (\frac{1}{7} + \dots)] + [(\frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11}\dots) - (\frac{1}{3} + \frac{1}{5} + \frac{1}{7}\dots)]$$
$$= \frac{1}{3}[\frac{1}{1} + \frac{1}{3} + \frac{1}{5}] + [\frac{-1}{3}]$$
$$= \frac{1}{3}[\frac{1}{3} + \frac{1}{5}]$$
$$= \frac{1}{3}\frac{8}{15} = \frac{8}{45}$$
$$\therefore \sum_{n\geq 1} \frac{16}{(8T_n - 3)(8T_{n+1} - 3)} = \frac{8}{45}$$
$$\Rightarrow \sum_{n\geq 1} \frac{1}{(8T_n - 3)(8T_{n+1} - 3)} = \frac{1}{90}$$