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Solution for problem U401

We know that $p(x)=\frac{1}{x^2}$ for $x\in\{1,2,\ldots,n+1\}$ we define $g(x)=x^2p(x)-1$. As degree of $g(x)$ is $(n+2)$ so it has $(n+2)$ roots.

$$g(x)=x^2p(x)-1=x^2\frac{1}{x^2}-1=0,\forall x\in\{1,2,\ldots,n+1\}$$

$\therefore (n+1)$ roots of $g(x)$ are $\{1,2,\ldots,n+1\}$.

$$\therefore g(x)=(x-1)(x-2)\ldots(x-(n+1))(x-\alpha)$$

$$\Rightarrow g(0)=(-1)^{n+1}(n+1)!(-\alpha)$$
$$\Rightarrow 0^2p(0)-1=(-1)^{n+2}(n+1)!(\alpha)$$
$$\Rightarrow (-1)^{n+2}(n+1)!(\alpha)=-1$$
$$\alpha=\frac{(-1)^{n+1}}{(n+1)!}$$

$$\therefore g(n+2)=\big((n+2)-1\big)\big((n+2)-2\big)\ldots\big((n+2)-(n+1)\big)\Big((n+2)-\frac{(-1)^{n+1}}{(n+1)!}\Big)$$

$$=(n+1)!(\frac{(n+2)!-(-1)^{n+1}}{(n+1)!})=(n+2)!+(-1)^{n+2}$$
$$\therefore (n+2)^2p(n+2)-1=(n+2)!+(-1)^{n+2}$$

$$\therefore p(n+2)=\frac{(n+2)!+(-1)^{n+2}+1}{(n+2)^2}$$

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