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Solution for problem O402

We have to prove that

$$\sin^2 2A + \sin^2 2B + \sin^2 2C \geq 2\sqrt{3}\sin 2A \sin 2B \sin 2C$$
$$\Rightarrow 4\sum_{sym} \sin^2 A \cos^2 A \geq 16\sqrt{3}\prod_{sym} \sin A \cos A$$
$$\Rightarrow \sum_{sym} \sin^2 A \cos^2 A \geq 4\sqrt{3}\prod_{sym} \sin A \cos A$$

let $x = \sin A \cos A$, $y = \sin B \cos B$ & $z = \sin C \cos C$ then we need to prove that

$$x^2 + y^2 + z^2 \geq 4\sqrt{3}xyz$$

If any of $A, B, C \in [\frac{\pi}{2}, \pi)$ then $xyz \leq 0$ while $x^2 + y^2 + z^2 \geq 0$ so inequality is obviously true.
Thus we may assume $x, y, z > 0$ It suffice to prove that

$$3\sqrt[3]{x^2y^2z^2} \geq 4\sqrt{3}xyz$$
$$\Leftrightarrow 3\sqrt[3]{(xyz)^{\frac{2}{3}}} \geq 4\sqrt{3}xyz$$
$$\Leftrightarrow 3 \geq 4\sqrt{3}(xyz)^{\frac{1}{3}}$$
$$\Leftrightarrow (\frac{\sqrt{3}}{4})^3 \geq xyz$$
$$\Leftrightarrow (\frac{3\sqrt{3}}{64}) \geq xyz$$

Now

$$xyz = \prod_{sym} \sin A \cos A = \prod_{sym} \sin A \prod_{sym} \cos A$$

Using the inequalities

$$\prod_{sym} \sin A \leq \frac{3\sqrt{3}}{8}, \prod_{sym} \cos A \leq \frac{1}{8}$$

we get

$$xyz \leq \frac{3\sqrt{3}}{8}\frac{1}{8} = \frac{3\sqrt{3}}{64}$$

QED
Equality occurs iff

$$x = y = z \Rightarrow A = B = C = \frac{\pi}{3}$$