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Solution for problem O436

Let $a = x + y$,$b = y + z$,$c = z + x$ where $x, y, z \in R_{\geq 0}$. Then a, b, c follow the triangle inequality .
Also

$$s = \frac{a+b+c}{2} = \frac{(x+y)+(y+z)+(z+x)}{2} = x+y+z$$

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} = \sqrt{\frac{xy}{(y+z)(z+x)}}$$

$$\sum_{cyclic} \frac{a^2}{\sin(\frac{A}{2})} = \sum_{cyclic} \frac{(x+y)^2}{\sqrt{\frac{xy}{(y+z)(z+x)}}}$$

$$= \sqrt{\frac{(x+y)(y+z)(z+x)}{xyz}} \sum_{cyclic} \sqrt{x(y+z)^3}$$

We now observe that the inequality is homogeneous, so we put $x+y+z = 1$
We get

$$\sqrt{\frac{(1-x)(1-y)(1-z)}{xyz}} \sum_{cyclic} \sqrt{x(1-x)^3} \geq 0 \qquad \text{(To be proved)}$$

Applyying AM-GM inequality

$$\sum_{cyclic} \sqrt{x(1-x)^3} \geq 3\sqrt[3]{\prod_{cyclic} \sqrt{x(1-x)^3}}$$

$$\prod_{cyclic} \sqrt{\frac{1-x}{x}} \sum_{cyclic} \sqrt{x(1-x)^3} \geq 3 \prod_{cyclic} \frac{(1-x)}{\sqrt[3]{x}}$$

Thus if we can prove that

$$3 \prod_{cyclic} \frac{(1-x)}{\sqrt[3]{x}} \geq \frac{8}{3} \qquad \text{then we are done}$$

$$\Rightarrow \prod_{cyclic} \frac{(1-x)}{\sqrt[3]{x}} \geq \frac{8}{9}$$

We define

$$\mathcal{L}(x, y, z, \lambda) = \underbrace{\prod_{cyclic} \frac{(1-x)}{\sqrt[3]{x}}}_{\text{function to be optimized}} - \lambda \underbrace{(x+y+z-1)}_{\text{constraints}} \quad \text{for applying Lagrange multiplier}$$

$\Rightarrow \nabla \mathcal{L} = 0$

$\therefore \lambda = -x^{-\frac{4}{3}} \dfrac{(1-y)(1-z)}{\sqrt[3]{yz}}$ (cyclic equation for y, z hold)

$\therefore -x^{-\frac{4}{3}} \dfrac{(1-y)(1-z)}{\sqrt[3]{yz}} = -y^{-\frac{4}{3}} \dfrac{(1-z)(1-x)}{\sqrt[3]{zx}} = -z^{-\frac{4}{3}} \dfrac{(1-x)(1-y)}{\sqrt[3]{xy}}$

$\Rightarrow \dfrac{1-x}{x} = \dfrac{1-y}{y} = \dfrac{1-z}{z}$

$\Rightarrow x = y = z$

$\Rightarrow x = y = z = \dfrac{1}{3}$

Evaluating

$$\prod_{cyclic} \frac{(1-x)}{\sqrt[3]{x}}$$

for $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, we get the value $\frac{8}{9}$.
Putting any other value (x, y, z) subject to $x + y + z = 1$ makes

$$\prod_{cyclic} \frac{(1-x)}{\sqrt[3]{x}}$$

greater than $\frac{8}{9}$, implying $\frac{8}{9}$ is the global minima on $x + y + z = 1$.
Thus $\prod \frac{(1-x)}{\sqrt[3]{x}} \geq \frac{8}{9}$ hence our original inequality is proved.