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Solution for Problem U374

By actually computing the first few values of a_n,we observe that $a_n < n + \frac{n}{n+1}$.
Thus if we can prove that

$$(\frac{((n+1)^2+1)^2}{(n+1)^4+(n+1)^2+1})^{\frac{1}{n+1}} < 1 + \frac{1}{n+1}$$

then,by applying strong induction we can conclude that $\lfloor a_n \rfloor = n$ (as then $n+1 < a_{n+1} < n+2$).
Now

$$(\frac{((n+1)^2+1)^2}{(n+1)^4+(n+1)^2+1})^{\frac{1}{n+1}} < 1 + \frac{1}{n+1}$$

$$\Leftrightarrow (\frac{((n+1)^2+1)^2}{(n+1)^4+(n+1)^2+1}) < (1 + \frac{1}{n+1})^{n+1}$$

Lemma 1 $(\frac{((n+1)^2+1)^2}{(n+1)^4+(n+1)^2+1}) < 2$

Proof

$(\frac{((n+1)^2+1)^2}{(n+1)^4+(n+1)^2+1}) < 2 \Leftrightarrow (n+1)^4 + 1 > 0$

Thus

$$2 < (\frac{((n+1)^2+1)^2}{(n+1)^4+(n+1)^2+1}) < (1 + \frac{1}{n+1})^{n+1} < e$$

$$\Rightarrow \lfloor a_n \rfloor = n$$

Thus

$$a_n = n + r$$

where $r \in R$ and $0 \leq r < 1$
Hence

$$\lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} 1 + \frac{r}{n} = 1 + \lim_{n\to\infty} \frac{r}{n} < 1 + \lim_{n\to\infty} \frac{1}{n}$$

Since $0 \leq r < 1$

$$\lim_{n\to\infty} \frac{0}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$$

By Squeeze theorem (Heine's theorem) we get that

$$\lim_{n\to\infty} \frac{r}{n} = 0$$

Hence,

$$\lim_{n\to\infty} \frac{a_n}{n} = 1 + 0 = 1$$