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Solution for Problem J375

$x^{1/3}+y^{1/3}=\frac{1}{2}+(x+y+\frac{1}{4})^{1/2}$
$\Rightarrow x^{1/3}+y^{1/3}=\frac{1}{2}+\frac{(4x+4y+1)^{1/2}}{2}$
Now,let $x^{1/3}=a$ and $y^{1/3}=b$.Thus,
$2(a+b)-1=(1+4a^3+4b^3)^{\frac{1}{2}}$
$\Rightarrow 4(a+b)^2+1-4(a+b)=1+4a^3+4b^3$
$\Rightarrow (a+b)^2-(a+b)=a^3+b^3=(a+b)(a^2-ab+b^2)$.
$\Rightarrow (a+b)(a+b-1)=(a+b)(a^2-ab+b^2)$.
$(a+b)\neq 0$ for if $a+b=0$ then LHS is 0 while RHS is 1
So factoring out the $a+b$ and canceling we get:-
$\Rightarrow a+b-1=a^2-ab+b^2$.
$\Rightarrow a^2-(b+1)a+b^2-b+1=0$. Solving this equation by treating it as a quadratic in a,we get,
$\Rightarrow a=\frac{b+1\pm\sqrt{-3b^2+6b-3}}{2}$
$\Rightarrow a=\frac{b+1\pm\sqrt{-3(b-1)^2}}{2}$
Since $-3(b-1)^2\leq 0$,and since a has to be a real number,this forces
$-3(b-1)^2=0\Rightarrow b=1\Rightarrow a=1$
$\Rightarrow x=a^3=1$,$y=b^3=1$
Thus $(1,1)$ is the *only real solutions* of the given equation.