Improved Lower Bounds for all Odd-Query LDCs

Arpon Basu*, Jun-Ting Hsieh+, Pravesh K. Kothari*, Andrew D. Lin*

*Princeton University, +MIT

Locally Decodable Codes (LDCs)

- A code $E : \{\pm 1\}^k \to \{\pm 1\}^n$ is a q-**L**ocally **D**ecodable **C**ode if any message bit x_i can be decoded by reading $\leqslant q$ codeword bits of the encoding $E(x)$.
- We study the optimal tradeoff between the *blocklength* n and the message length k and query complexity q.

Bounds in Terms of q

- For $q = 2$, Hadamard codes achieve $k = \log n$, which is optimal.
- For $q \geqslant 3$, Efremenko-Yekhanin codes achieve $k \geqslant 2^{(\log\log n)^{2-o(1)}}$.
- The previously best known lower bounds were the following:

$$k \lesssim \begin{cases} \log n & q = 2 \text{ [GKST06]} \\ n^{1-2/q} & q \geqslant 4 \text{ even [KW04] or } q = 3 \text{ [AGKM23]} \\ n^{1-2/(q+1)} & q \geqslant 5 \text{ odd [KW04]} \end{cases}$$

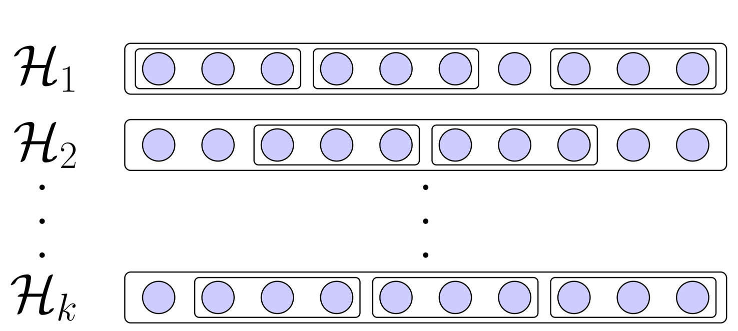
- Prior to [AGKM23], all best known bounds for odd q were directly from treating the q-LDC as an even $(q+1)$-LDC.
- In our work, we improve the bounds for odd q.

Main Theorem: $q \geqslant 3$-LDCs

For all $q \geqslant 3$-LDCs, we have $k \lesssim n^{1-2/q}$.

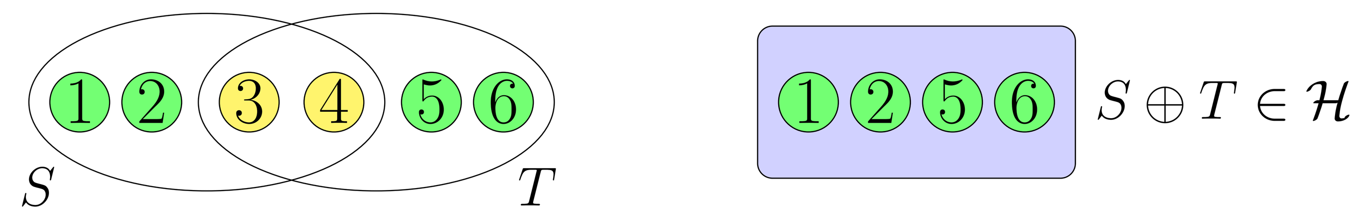
Normal LDCs: Query Set Model

- A q-uniform hypergraph matching on $[n]$ is a collection of pairwise disjoint size-q subsets of $[n]$.
- If E is a *normal* q-LDC, then for any $i \in [k]$, there exists a q-uniform hypergraph matching $\mathcal{H}_i$ on $[n]$ with $\Omega(n)$ hyperedges such that for all $C \in \mathcal{H}_i$, we have $x_i = E(x)_C := \prod_{j \in C} E(x)_j$. We write $\mathcal{H} = \cup_{i \in [k]} \mathcal{H}_i$.
- Using a standard reduction, we can assume our LDC is normal.



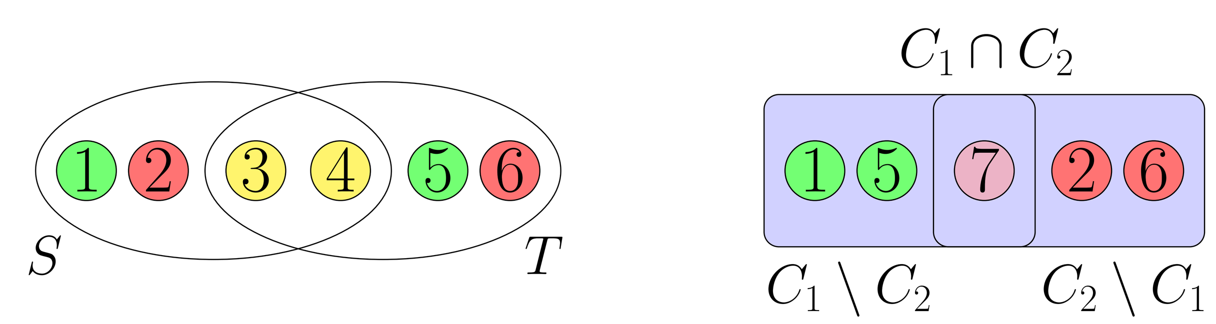
Kikuchi Graphs: Even q

We define the *Kikuchi graph* $K(\mathcal{H})$ at level ℓ to be the graph with vertex set $\binom{[n]}{\ell}$ and edges between any $S, T \in \binom{[n]}{\ell}$ such that $S \oplus T \in \mathcal{H}$.



Kikuchi Graphs: Odd q

For any $t < q$, we define the *Kikuchi graph* $K_t(\mathcal{H})$ at level ℓ to be the graph with vertex set $\binom{[n]}{\ell}$ and edges between any $S, T \in \binom{[n]}{\ell}$ such that $S \oplus T = C_1 \oplus C_2$ for some $C_1, C_2 \in \mathcal{H}$ such that $|C_1 \cap C_2| = t$.



Regularity implies Bounds

- The average degree of a Kikuchi graph is $\Omega(1)$ when $\ell \geqslant n^{1-2/q}$.
- If the Kikuchi graph is *approximately regular*, i.e. most vertices have degree $\lesssim$ the average degree, then $k \lesssim \ell$.
- If the average degree is $\Omega(1)$ and q is even, we delete an $o(1)$ fraction of vertices to ensure the resulting subgraph is approximately regular.
- For odd q, the same holds w.h.p. if the $\mathcal{H}_i$s are random hypergraphs.

Reducing to the Random Case [AGKM23]

- If we have some $\{u, v\} \subseteq [n]$ such that $\omega(\log n)$ hyperedges contain both u and v, the degree of S containing u or v in $K_1(\mathcal{H})$ is too big.
- When $q = 3$, we can take C, C' which both contain a "heavy pair" $\{u, v\}$ and look at the size 2 sets $C \oplus C'$.
- We then use *exponential lower bounds* for 2-LDCs to prove the desired bound on k if many heavy pairs exist.
- Therefore, we reduce the problem to the case where all pairs are contained in $O(\log n)$ hyperedges like in the random case.
- When $q = 5$ and heavy pairs or triples appear, the above process produces sets $C \oplus C'$ of size 6 or 4.
- There are no exponential lower bounds for 4- or 6-LDCs.

Finding a Weaker Sufficient Condition

- Since we cannot reduce to the random case for $q \geqslant 5$, we look for a weaker condition for Kikuchi regularity that allows for heavy tuples.
- Let d_r be the *co-degree* of r-tuples, i.e. the maximum number of hyperedges that contain any r-tuple in $[n]$.
- When heavy tuples appear, we work with some $K_t(\mathcal{H})$ for $t > 1$.
- While $d_1 = \Theta(k)$ always holds, d_t can take a wide range of values.
- To ensure $d_{\text{avg}}(K_t(\mathcal{H})) \geqslant \Omega(1)$, we need **$d_t$ to be sufficiently large** and **each $C \in \mathcal{H}$ must contain some t-tuple that appears in $\Omega(d_t)$ other $C' \in \mathcal{H}$.**

Relative Regularity Conditions

- As before, we need to exclude heavy tuples: i.e. we require *upper bounds* on each of $d_1, \ldots, d_{t-1}, d_{t+1}, \ldots, d_q$.
- These upper bounds must ensure $K_t(\mathcal{H})$ is regular even though d_t is not asymptotically fixed.
- Instead of strict upper bounds, the bounds on d_r will depend on d_t.
- By using polynomial concentration techniques, we can show that **there exists a function f_t such that if $d_r/d_t \leqslant f_t(r)$ for all $r \neq t$, then $K_t(\mathcal{H})$ is approximately regular.**
- The **blue** and **red** conditions together are the **approximate strong regularity** conditions for ensuring regularity of $K_t(\mathcal{H})$.

Picking t and pruning $\mathcal{H}$

- For the **conditions on the co-degrees**, we show we can always find some t such that d_t is large and all other d_r are upper bounded by $d_t \cdot f_t(r)$. **We then work with $K_t(\mathcal{H})$.**
- For the **condition on the hypergraph**, we drop hyperedges that do not contain a t-tuple which appears $\Omega(d_t)$ times, as this keeps d_t invariant and can only decrease the other d_r.

Conclusions and Further Directions

- We improved the bound $k \lesssim n^{1-2/(q+1)}$ to $k \lesssim n^{1-2/q}$ for all odd $q \geqslant 5$, matching the best known bounds for even q and $q = 3$.
- Can the Kikuchi method be improved to obtain better bounds, perhaps even $k \lesssim n^{o(1)}$?
- What other problems can we approach using the Kikuchi method?