# PULLING BACK A GALOIS CORRESPONDENCE

ARPON RAKSIT

original: October 25, 2014
updated: January 19, 2015

**1. Definition.** For $G$ a topological group, we denote by $\mathrm{Sub}_G$ the set of open normal subgroups of $G$, which is a poset under inclusion.

**2. Proposition.** Let $G$ be a profinite group. Suppose we have a morphism of topological groups $\phi\colon A \longrightarrow G$ such that for every $N \in \mathrm{Sub}_G$ the induced map $\phi_N\colon A/\phi^{-1}(N) \longrightarrow G/N$ is an isomorphism. Then:

(a) the preimage map $\mathrm{Sub}_G \longrightarrow \mathrm{Sub}_A$, defined by $N \mapsto \phi^{-1}(N)$, is injective;

(b) $\phi^{-1}(N_1N_2) = \phi^{-1}(N_1)\phi^{-1}(N_2)$ for $N_1, N_2 \in \mathrm{Sub}_G$.

**Proof.** Let $N_1, N_2 \in \mathrm{Sub}_G$. To prove (a) it suffices to show that $N_1 \subseteq N_2$ if $\phi^{-1}(N_1) \subseteq \phi^{-1}(N_2)$, so assume the latter. Then

$$\phi^{-1}(N_1 \cap N_2) = \phi^{-1}(N_1) \cap \phi^{-1}(N_2) = \phi^{-1}(N_2). \tag{3}$$

Now consider the commutative diagram

$$\begin{array}{ccccc}
A & \longrightarrow & A/\phi^{-1}(N_1 \cap N_2) & \longrightarrow & A/\phi^{-1}(N_2) \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi_{N_1 \cap N_2}} & & \downarrow{\scriptstyle \phi_{N_2}} \\
G & \longrightarrow & G/(N_1 \cap N_2) & \longrightarrow & G/N_2
\end{array}$$

with the horizontal maps the projections. In the right-hand square, the vertical maps are isomorphisms by hypothesis, and the top horizontal map is an isomorphism by (3); thus the bottom horizontal map is an isomorphism, which implies $N_1 \subseteq N_2$.

We now prove (b). Certainly $\phi^{-1}(N_1N_2) \supseteq \phi^{-1}(N_1)\phi^{-1}(N_2)$. And it's fairly easy to see we have the sequence of identifications

$$\begin{aligned}
\frac{\phi^{-1}(N_1)\phi^{-1}(N_2)}{\phi^{-1}(N_1)} &\simeq \frac{\phi^{-1}(N_2)}{\phi^{-1}(N_1) \cap \phi^{-1}(N_2)} = \frac{\phi^{-1}(N_2)}{\phi^{-1}(N_1 \cap N_2)} \\
&= \phi^{-1}_{N_1 \cap N_2}\left(\frac{N_1}{N_1 \cap N_2}\right) \simeq \phi^{-1}_{N_1}\left(\frac{N_1N_2}{N_1}\right) = \frac{\phi^{-1}(N_1N_2)}{\phi^{-1}(N_1)}.
\end{aligned}$$

But since $\phi^{-1}(N_1)$ has finite index in $A$, it follows that

$$[\phi^{-1}(N_1N_2) : \phi^{-1}(N_1)\phi^{-1}(N_2)] = \left[\frac{\phi^{-1}(N_1N_2)}{\phi^{-1}(N_1)} : \frac{\phi^{-1}(N_1)\phi^{-1}(N_2)}{\phi^{-1}(N_1)}\right] = 1,$$

proving the desired claim. □

**4. Example.** If we take $G$ to be a Galois group in (2), then the proposition says that when we have a suitable morphism $A \longrightarrow G$, the Galois theory described by $G$ is in fact controlled by $A$. This is what I meant by "pulling back a Galois correspondence" in the title. Let's state this in more detail in the motivating example.

Let $K$ be a non-archimedean local field. Let $K^{\mathrm{ab}}$ be a maximal abelian extension of $K$. The Galois group $G := \mathrm{Gal}(K^{\mathrm{ab}}/K)$ is a profinite group, and Galois theory tells us that the poset $\mathrm{Sub}_G$ is (contravariantly) equivalent to the poset $\mathrm{Ab}_K$ of finite abelian extensions of $K$, i.e. the set of finite subextensions of $K^{\mathrm{ab}}$ ordered by inclusion. The "reciprocity" statement in local class field theory asserts:

- existence of a morphism $\phi_K \colon K^\times \longrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K)$ satisfying the hypothesis of (2);
- if $N := \mathrm{Stab}_L \in \mathrm{Sub}_G$ is the subgroup corresponding to a finite abelian extension $K \hookrightarrow L$, then $\phi_K^{-1}(N)$ is the *norm group* $N_{L/K}(L^\times) \subseteq K^\times$.

Thus, putting (2) and Galois theory together gives us that the poset $\mathrm{Ab}_K$ is (contravariantly) equivalent to the poset of norm groups in $K^\times$ by the correspondence $L \longmapsto N_{L/K}(L^\times)$. Then there is an "existence" theorem in local class field theory stating that the norm groups are precisely the open subgroups of finite index in $K^\times$.