FINITE PRESENTATION (IN COMMUTATIVE ALGEBRA)

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1. Locality

In this section we prove two basic statements about finite presentation, that ”finitely presented implies always finitely presented”[1] and that it is a local property. The proofs in the setting of modules and in the setting of algebras are completely analogous, so we will abstract very slightly and address both at once.

1.1. Notation. (a) All rings and algebras are commutative with unit.

(b) For a ring A, let $\mathcal{C}_A$ denote either the category of A-modules or the category of A-algebras. All arrows indicate morphisms in $\mathcal{C}_A$.

(c) Let $\amalg$ denote the coproduct in $\mathcal{C}_A$, i.e., direct sum $\oplus$ for A-modules or tensor product $\otimes_A$ for A-algebras.

(d) For $C \in \mathrm{obj}(\mathcal{C}_A)$ and $n \in \mathbb{N}$ let $C^{\amalg n}$ denote the n-fold coproduct $C \amalg \cdots \amalg C$ and C^n denote the n-fold product $C \times \cdots \times C$.

(e) Let $F_A \in \mathrm{obj}(\mathcal{C}_A)$ denote the free object on one element, i.e., $F_A \simeq A$ for A-modules and $F_A \simeq A[t]$ for A-algebras. Thus for any $n \in \mathbb{N}$ there is an isomorphism $\hom_{\mathcal{C}_A}(F_A^{\amalg n}, C) \simeq C^n$ natural in $C \in \mathrm{obj}(\mathcal{C}_A)$.

And now fix a ring A for the remainder.

1.2. Definition. An object $C \in \mathrm{obj}(\mathcal{C}_A)$ is:

(a) *finitely generated* if for some $n \in \mathbb{N}$ there is a surjection $F_A^{\amalg n} \longrightarrow M$;

(b) *finitely presented* if for some $n \in \mathbb{N}$ there is a surjection $F_A^{\amalg n} \longrightarrow M$ with finitely generated kernel.

1.3. Lemma. Let $f\colon C \longrightarrow D$ and $g\colon D \longrightarrow E$ be morphisms in $\mathcal{C}_A$. Define $h := g \circ f$. Then if f is surjective, the canonical morphism $\ker(h) \longrightarrow \ker(g)$ induced by f is surjective. In particular, if $\ker(h)$ is finitely generated then so is $\ker(g)$.

Proof. Evident. □

1.4. Proposition. Let $C \in \mathrm{obj}(\mathcal{C}_A)$ be finitely presented. If $\alpha\colon D \longrightarrow C$ is a surjection with D finitely generated, then $\ker(\alpha)$ is finitely generated.

Proof. By (1.3) we may assume $D \simeq F_A^{\amalg m}$. By definition we have a surjection $\beta\colon F_A^{\amalg n} \longrightarrow C$ with $\ker(\beta)$ finitely generated. Consider the diagram

$$\begin{tikzcd}
& F_A^{\amalg m} \arrow[dr, "\alpha"] & \\
F_A^{\amalg m} \amalg F_A^{\amalg n} \arrow[ur, "\mathrm{id}\amalg\xi"] \arrow[rr, "\gamma"] \arrow[dr, "\eta\amalg\mathrm{id}"'] & & C, \\
& F_A^{\amalg n} \arrow[ur, "\beta"'] &
\end{tikzcd}$$

where $\xi\colon F_A^{\amalg n} \longrightarrow F_A^{\amalg m}$ and $\eta\colon F_A^{\amalg m} \longrightarrow F_A^{\amalg n}$ are defined such that $\alpha \circ \xi = \beta$ and $\beta \circ \eta = \alpha$—which exist since free objects are projective in $\mathcal{C}_A$—so that defining $\gamma := \alpha \amalg \beta$ makes the diagram commute. Now, if $t_1, \ldots, t_m$ are the canonical

[1]This catchy phrasing for the proposition is taken from Vakil's notes.

generators (in the appropriate sense according to $\mathcal{C}_A$) of $F_A^{\amalg m}$, then clearly $\ker(\gamma)$ is generated by $\ker(\beta)$ and $\{t_i - \eta(t_i)\}_{1\le i\le m}$. Thus $\ker(\beta)$ being finitely generated implies $\ker(\gamma)$ is finitely generated, which implies $\ker(\alpha)$ is finitely generated by (1.3). ☐

1.5. Proposition. Let $C \in \mathrm{obj}(\mathcal{C}_A)$.

(a) If C is finitely generated (resp. presented), then for any $f \in A$ the localisation C_f is finitely generated (resp. presented) in $\mathcal{C}_{A_f}$.

(b) If $f_1, \dots, f_n \in A$ generate the unit ideal and C_{f_i} is finitely generated (resp. presented) in $\mathcal{C}_{A_{f_i}}$ for each i, then C is finitely generated (resp. presented).

Proof. We first address finite generation. Observe $(F_A)_f \simeq F_{A_f}$. Thus, since localisation is exact, hence commutes with finite products, a surjection $F_A^{\amalg n} \longrightarrow C$ localises to a surjection $F_{A_f}^{\amalg n} \longrightarrow C_f$. This proves (a). Now suppose for each i that C_{f_i} is generated (in the appropriate sense) by $\{a_{i,j}/f_i^{r_{i,j}}\}_{1\le j\le m_i}$. Take $R \ge \max\{r_{i,j}\}$. Then for all i we have that $f_i^R a$ is generated by $\{a_{i,j}\}$ for any $a \in A$. Since $f_1, \dots, f_n$ generate the unit ideal, so do $f_1^R, \dots, f_n^R$ and hence there exist $c_i \in A$ such that $\sum c_i f_i^R = 1$. It is then clear that $\{c_i\} \cup \{a_{i,j}\}$ generates C. This proves (b).

Now finite presentation. Since localisation is exact, hence commutes with kernels, (a) for finite presentation is immediate from (a) for finite generation. Now assume the hypothesis of (b). Since we have proved (b) for finite generation there is a surjection $\alpha\colon F_A^{\amalg n} \longrightarrow C$. Then for each i,

$$\alpha_{f_i}\colon F_{A_{f_i}}^{\amalg n} \longrightarrow C_{f_i}$$

is a surjection, so by (1.4) we know $\ker(\alpha_{f_i}) \simeq \ker(\alpha)_{f_i}$ is finitely generated. By (b) for finite generation this implies $\ker(\alpha)$ is finitely generated as desired. ☐

1.6. Remark. (1.5) is what one needs to prove when showing that a morphism of schemes being of locally of finite type or locally of finite presentation is local on the source.

2. Duality

2.1. Let M and N be A-modules. Let $S \subset A$ be a multiplicative set. There is a canonical map of A_S-modules

$$(\mathrm{hom}_A(M, N))_S \longrightarrow \mathrm{hom}_{A_s}(M_S, N_S), \tag{2.2}$$

sending $\phi/s \in (\mathrm{hom}_A(M,N))_S$ to the map $x/t \longmapsto \phi(x)/(st)$ in $\mathrm{hom}_{A_s}(M_S, N_S)$.

2.3. Proposition. The map (2.2) is injective if M is finitely generated and an isomorphism if M is finitely presented.

Proof. Assume M is finitely generated, say by $x_1, \dots, x_n \in M$. Let $\phi/s \in (\mathrm{hom}_A(M,N))_S$ and suppose $x/t \longmapsto \phi(x)/(st)$ is the zero map. This means for each i there exists $t_i \in S$ such that $t_i\phi(x_i) = 0$. Let $t := \prod_{i=1}^n t_i$ so that $t\phi(x_i) = 0$ for all i. Since $x_1, \dots, x_n$ are generators, this implies $t\phi(x) = 0$ for all $x \in M$, whence $\phi/s = 0$. This proves injectivity.

Now assume M is moreover finitely presented. Let $x := (x_1, \dots, x_n)$ and let $r_1(x), \dots, r_m(x)$ denote the relations on $x_1, \dots, x_n$. Let $\psi \in \mathrm{hom}_{A_s}(M_S, N_S)$. Suppose $\psi(x_i) = y_i/s_i$ with $y_i \in N$. By taking $s := \prod_{i=1}^n s_i$, we can in fact assume $s_1 = \dots = s_n = s$. Writing $y = (y_1, \dots, y_n)$, we must have

$$\psi(r_j(x)) = r_j(y)/s = 0 \implies t_j r_j(y) = 0$$

for some $t_j \in S$. Let $t := \prod_{j=1}^m t_j$ so that $tr_j(y) = r_j(ty) = 0$ for each j. Then we have a well-defined map $\phi \in \hom_A(M, N)$ given by $\phi(x_i) = ty_i$. And since

$$\phi(x_i)/(st) = ty_i/(st) = y_i/s = \psi(x_i),$$

and $x_1, \ldots, x_n$ generate M_S over A_S, the map (2.2) sends $\phi/(st)$ to ψ. This proves surjectivity. □

References