EXACTNESS OF FUNCTORS

ARPON RAKSIT

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1. Products and coproducts

1.1. Definition. Let $\mathcal{A}$ a category and $\{A_i\}_{i \in I} \subseteq \mathrm{obj}(\mathcal{A})$. A *biproduct* of the A_i is an object $\bigoplus A_i \in \mathrm{obj}(\mathcal{A})$ equipped with morphisms

$$A_i \longrightarrow \bigoplus A_i \quad \text{and} \quad \bigoplus A_i \longrightarrow A_i \quad \text{for } i \in I,$$

making $\bigoplus A_i$ simultaneously a product $\prod A_i$ and coproduct $\coprod A_i$. By the usual abuse of notation, to say we have such data attached to an object $B \in \mathrm{obj}(\mathcal{A})$, we will simply write $B \simeq \bigoplus A_i$.

1.2. Lemma. Let $\mathcal{A}$ an additive category and $A, B, C \in \mathrm{obj}(\mathcal{A})$. If C is a product or coproduct of A and B, then there exist morphisms

$$i\colon A \longrightarrow C, \quad j\colon B \longrightarrow C, \quad p\colon C \longrightarrow A, \quad q\colon C \longrightarrow B$$

satisfying the following identities:

$$pi = \mathrm{id}_A, \quad qj = \mathrm{id}_B, \quad pj = 0, \quad qi = 0, \quad ip + jq = \mathrm{id}_C.$$

In this case, C equipped with i, j, p, q is in fact a biproduct of A and B.

Proof. Suppose we have a coproduct $A \xrightarrow{i} C \xleftarrow{j} B$. The identity and zero morphisms induce $p\colon C \longrightarrow A$ and $q\colon C \longrightarrow B$ with

$$pi = \mathrm{id}_A, \quad qj = \mathrm{id}_B, \quad pj = 0, \quad qi = 0.$$

Consider the morphism $ip + jq\colon C \longrightarrow C$. From the identities above and the universal property of the coproduct we get that $(ip + jq)i = i$ and $(ip + jq)j = j$, implying $ip + jq = \mathrm{id}_C$.

Now suppose we are given morphisms $f\colon D \longrightarrow A$ and $g\colon D \longrightarrow B$. By the above, we have a morphism $if + jg\colon D \longrightarrow C$ with $p(if + jg) = f$ and $q(if + jg) = g$. Conversely, for any morphism $h\colon D \longrightarrow C$ with $ph = f$ and $qh = g$ we must have

$$h = (ip + jq)h = i(ph) + j(qh) = if + jg.$$

So the induced morphism h is indeed unique, implying $A \xleftarrow{p} C \xrightarrow{q} B$ is a product, and hence $C \simeq A \oplus B$.

If we start with the product instead of the coproduct, we just dualise the above argument (which I learned from [Tri10]). □

1.3. Notation. In light of (1.2), in an additive category we will use finite products, coproducts, and biproducts interchangeably, and denote all of these with $\oplus$.

1.4. Lemma. Let $\mathcal{A}$ and $\mathcal{B}$ additive categories and $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor. Then F preserves finite products.

Proof. Let $A, B \in \mathrm{obj}(\mathcal{A})$. Let $A \oplus B$ their biproduct, and the morphisms i, j, p, and q as in (1.2). Since F is an additive functor we get the same identities for $F(i)$, $F(j)$, $F(p)$, and $F(q)$. But then the argument from (1.2) implies that $F(A \oplus B) \simeq F(A) \oplus F(B)$. □

1.5. Lemma. Let $\mathcal{A}$ an additive category and $A, B \in \mathrm{obj}(\mathcal{A})$. Let $f, g\colon A \longrightarrow B$ two morphisms. Then $f + g\colon A \longrightarrow B$ can be factored as

$$A \xrightarrow{f \times g} B \oplus B \xrightarrow{\mathrm{id}_B \amalg \mathrm{id}_B} B,$$

where $f \times g$ is the morphism induced to the product by f and g, and $\mathrm{id}_B \amalg \mathrm{id}_B$ is the morphism induced from the coproduct by id_B and id_B.

Proof. In the notation of (1.2), we have $f \times g = if + jg$ and $\mathrm{id}_B \amalg \mathrm{id}_B = p + q$, so

$$(\mathrm{id}_B \amalg \mathrm{id}_B)(f \times g) = (p+q)(if + jg) = f + g. \qquad \square$$

2. Left and right exact functors

2.1. Definition. Let $\mathcal{A}$ and $\mathcal{B}$ categories. We say a functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ is:

(a) *left exact* if F preserves finite limits;

(b) *right exact* if F preserves finite colimits;

(c) *exact* if F is both left and right exact.

2.2. Example. Left (resp. right) adjoints will be right (resp. left) exact since they preserve *all* limits (resp. colimits). This will seem nicer when we actually characterise exactness of functors in terms of exactness of sequences below.

2.3. Lemma. Let $\mathcal{A}$ and $\mathcal{B}$ additive categories and $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ a functor. If F is left or right exact then F is additive.

Proof. Let $A, B \in \mathrm{obj}(\mathcal{A})$ and $f, g\colon A \longrightarrow B$ two morphisms. Since if F preserves finite limits or colimits then F preserves biproducts, we have the commutative diagram

$$\begin{tikzcd}
F(A) \arrow[r, "F(f\times g)"] \arrow[dr, "F(f)\times F(g)"'] & F(B \oplus B) \arrow[r, "F(\mathrm{id}_B \amalg \mathrm{id}_B)"] & F(B), \\
& F(B) \oplus F(B) \arrow[u, "\wr"] \arrow[ur, "\mathrm{id}_{F(B)} \amalg \mathrm{id}_{F(B)}"'] &
\end{tikzcd}$$

notation as in (1.5). Then we're done by (1.5). $\square$

2.4. Lemma. Let $\mathcal{A}$ and $\mathcal{B}$ abelian categories and $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ an additive functor. The following are equivalent.

(a) For every exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C$ in $\mathcal{A}$ the induced sequence $0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$ in $\mathcal{B}$ is exact.

(b) For every exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ in $\mathcal{A}$ the induced sequence $0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$ in $\mathcal{B}$ is exact.

(c) F preserves kernels.

(d) F is left exact.

Proof. $(1 \Rightarrow 2)$ Tautology.

$(2 \Rightarrow 3)$ Let $\phi\colon A \longrightarrow B$ a morphism in $\mathcal{A}$. Then we have short exact sequences

$$0 \longrightarrow \ker(\phi) \longrightarrow A \longrightarrow \mathrm{im}(\phi) \longrightarrow 0, \quad 0 \longrightarrow \mathrm{im}(\phi) \longrightarrow B \longrightarrow \mathrm{coker}(\phi) \longrightarrow 0,$$

which by assumption give rise to exact sequences

$$\begin{gathered} 0 \longrightarrow F(\ker(\phi)) \longrightarrow F(A) \longrightarrow F(\mathrm{im}(\phi)), \\ 0 \longrightarrow F(\mathrm{im}(\phi)) \longrightarrow F(B) \longrightarrow F(\mathrm{coker}(\phi)). \end{gathered}$$

Since $F(\phi)$ factors as $F(A) \longrightarrow F(\mathrm{im}(\phi)) \longrightarrow F(B)$, the exactness above implies

$$\ker(F(\phi)) \simeq \ker(F(A) \longrightarrow F(\mathrm{im}(\phi)) \simeq F(\ker(\phi)).$$

$(3 \Rightarrow 4)$ By (1.4), F preserves finite products. And equalisers can be expressed as kernels in additive categories. Thus this is immediate from the fact that all (finite) limits can be expressed in terms of (finite) products and equalisers.

(4 $\Rightarrow$ 1) To say $0 \longrightarrow A \longrightarrow B \longrightarrow C$ is exact is just to say that $A \simeq \ker(B \longrightarrow C)$. Thus this follows from the fact that kernels are finite limits. $\square$

Of course we can dualise (2.4) to obtain the analogous charactersiations of right exact functors.

References

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