DEFINING THE COTANGENT COMPLEX

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Opening nonsense

I've recently learned a bit about the general framework of model categories. It's sort of amazing, in the sense that seemingly a lot of the basic homotopy theory of spaces which I've seen can be axiomatised in a totally formal and categorical way. This includes something like Whitehead's theorem (and its converse)—that a map of CW complexes is a homotopy equivalence if and only if it's a weak homotopy equivalence—which I never would have thought of as a categorical fact.

Anyway, this axiomatisation and generalisation is beautiful and all, but of course one hopes that it leads to new homotopy theory, in addition to providing a nice framework for existing homotopy theory to live in. Obviously it did. My goal in writing these notes was to learn about one of the earliest examples of this, the development of a (co)homology theory of commutative algebras by André [And74] and Quillen [Qui68].

Notation. — Denote by Set the category of sets.
— All rings (and algebras) are commutative.
— For a ring R denote by $\mathrm{Mod}(R)$ the category of R-modules and $\mathrm{Alg}(R)$ the category of R-algebras.
— For a category $\mathcal{C}$ denote by $\mathrm{s}\mathcal{C}$ the category of simplicial objects in $\mathcal{C}$.

1. Homology and abelianisation

In hopes of generalising the notion of homology via the theory of model categories, we first ask the following.

1.1. Question. *How can we view singular homology in a homotopyish way?*

1.2. Perhaps the most immediate answer is the following. Suppose X is a topological space. We can choose a CW approximation, that is, a CW complex Y and a weak equivalence $Y \to X$. Let $\mathrm{Ab}(Y)$ denote the free topological abelian group on Y. Then one version of the Dold-Thom theorem [McC69] states that

$$\pi_*(\mathrm{Ab}(Y)) \simeq \mathrm{H}_*(Y) \simeq \mathrm{H}_*(X).$$

Thus we should view $\mathrm{Ab}(Y)$ as the object which gives, via its homotopy, singular homology. Note that to apply Dold-Thom we had to choose a CW approximation to X before applying Ab. From the point of view of model categories, this is precisely choosing a cofibrant replacement (in the standard model structure on topological spaces). So finally we should view singular homology as the derived functor of Ab.

1.3. Another answer to the question goes through simplicial sets rather than topological spaces. Suppose $K \in \mathrm{sSet}$, perhaps thinking $K = \mathrm{Sing}(X)$ for some topological

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space X. Let $\mathrm{Ab}(K) := \mathbb{Z}K \in \mathrm{sMod}(\mathbb{Z})$ the free simplicial abelian group on K. Then the Dold-Kan correspondence[1] tells us that

$$\pi_*(\mathrm{Ab}(K)) \simeq \mathrm{H}_*(K) \simeq \mathrm{H}_*(X).$$

So again we view $\mathrm{Ab}(K)$ as giving homology via its homotopy. Here we did not have to choose a cofibrant replacement, but this is simply because (in the standard model structure) all $K \in \mathrm{sSet}$ are cofibrant!

These two points of view on singular homology motivate the following general philosophy, due to Quillen.

1.4. Definitions. (1) An *abelian group object* in a category $\mathcal{C}$ is an object $A \in \mathcal{C}$ such that the functor $\hom_{\mathcal{C}}(-, A)\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ factors through the forgetful functor $\mathrm{Mod}(\mathbb{Z}) \to \mathrm{Set}$, i.e., $\hom_{\mathcal{C}}(B, A)$ is naturally an abelian group.

(2) A *morphism of abelian group objects* $A, A' \in \mathcal{C}$ is a morphism $A \to A'$ in $\mathcal{C}$ for which the induced map $\hom_{\mathcal{C}}(B, A) \to \hom_{\mathcal{C}}(B, A')$ is a group homomorphism for $B \in \mathcal{C}$.

(3) The resulting *subcategory of abelian group objects* in $\mathcal{C}$ is denoted by $\mathcal{C}_{\mathrm{ab}}$.

1.5. One can check that if $\mathcal{C}$ has a terminal object $*$ and binary products, then $A \in \mathcal{C}$ is an abelian group object if and only if there exist morphisms

— $\mu\colon A \times A \to A$ (multiplication),
— $\epsilon\colon * \to A$ (identity),
— $\iota\colon A \to A$ (inverse),

such that the usual diagrams encoding abelian group structure commute, and a morphism of abelian group objects is one which respects multiplication.

1.6. Let $\mathcal{C}$ be a category with a terminal object and binary products. It is immediate from (1.5) that $(\mathrm{s}\mathcal{C})_{\mathrm{ab}} \simeq \mathrm{s}(\mathcal{C}_{\mathrm{ab}})$ That is, the two subcategories of $\mathrm{s}\mathcal{C}$ consisting of:

— abelian group objects
— objects which are degree-wise abelian group objects and whose face and degeneracy maps are morphisms of abelian group objects

can be identified in the natural manner. We won't distinguish between the two, and will denote them by $\mathrm{s}\mathcal{C}_{\mathrm{ab}}$.

Suppose moreover that the inclusion $\mathcal{C}_{\mathrm{ab}} \to \mathcal{C}$ has a left adjoint $\mathrm{Ab}\colon \mathcal{C} \to \mathcal{C}_{\mathrm{ab}}$. It is easy to see then that the inclusion $\mathrm{s}\mathcal{C}_{\mathrm{ab}} \to \mathrm{s}\mathcal{C}$ has left adjoint $\mathrm{s}\mathcal{C} \to \mathrm{s}\mathcal{C}_{\mathrm{ab}}$ given by applying Ab degree-wise.

1.7. Definition. Let $\mathcal{C}$ be a model category. Suppose there is a model structure on $\mathcal{C}_{\mathrm{ab}}$ such that the inclusion $\mathcal{C}_{\mathrm{ab}} \to \mathcal{C}$ is a right Quillen functor, with left adjoint $\mathrm{Ab}\colon \mathcal{C} \to \mathcal{C}_{\mathrm{ab}}$, called *abelianisation*. Then *homology* is the total left derived functor $\mathbb{L}\,\mathrm{Ab}$, i.e., the homology of $X \in \mathcal{C}$ is the object $\mathbb{L}\,\mathrm{Ab}(X) \in \mathcal{C}_{\mathrm{ab}}$.

1.8. Examples. This philosophy is of course a generalisation of (1.2) and (1.3). If $\mathcal{C} = \mathrm{Top}$ then $\mathcal{C}_{\mathrm{ab}}$ is the subcategory of topological abelian groups and if $\mathcal{C} = \mathrm{sSet}$ then $\mathcal{C}_{\mathrm{ab}}$ is the subcategory of simplicial abelian groups. In both cases the natural model structure on $\mathcal{C}_{\mathrm{ab}}$ is defined precisely such that $\mathcal{C}_{\mathrm{ab}} \to \mathcal{C}$ is right Quillen, with left adjoints $\mathrm{Ab}\colon \mathrm{Top} \to \mathrm{Top}_{\mathrm{ab}}$ and $\mathrm{Ab}\colon \mathrm{sSet} \to \mathrm{sSet}_{\mathrm{ab}} \simeq \mathrm{sMod}(\mathbb{Z})$ the free topological and simplicial abelian group functors, respectively.

[1] Many thanks to Albrecht Dold in this section!

2. Kähler differentials

Of course we must then ask: to what other categories $\mathcal{C}$ can we fruitfully apply this notion of homology to? As stated earlier, the goal here is to describe what happens when we consider the category of commutative rings or algebras. So first we need to understand what abelianisation is in this context.

2.1. Notation. We fix a ring R.

2.2. Proposition. *The only abelian group object in* $\mathrm{Alg}(R)$ *is the zero ring.*

Proof. The terminal object in $\mathrm{Alg}(R)$ is the zero ring 0, and the only $A \in \mathrm{Alg}(R)$ for which there exists a morphism $0 \to A$ is $A = 0$ itself. $\square$

2.3. So we've already run into a subtlety in this category. We'll resolve this issue by working in a slightly different category. Fix an R-algebra A. Let

$$\mathcal{C} := \mathrm{Alg}(R)_{/A}$$

be the overcategory of R-algebras over A. I.e., an object of $\mathcal{C}$ is a sequence of ring morphisms $R \to B \to A$ (which we of course abusively refer to as B) whose composition is the structure morphism $R \to A$.

Before continuing the discussion on abelianisation, we review derivations and Kähler differentials.

2.4. Definitions. Let B be an R-algebra and M a B-module.

(1) An *R-derivation* $d\colon B \to M$ is a morphism of R-modules satisfying the Leibniz rule, $d(xy) = x\,d(y) + y\,d(x)$ for $x, y \in B$.
(2) The set $\mathrm{Der}_R(B, M)$ of R-derivations has a natural B-module (in particular abelian group) structure.
(3) The B-module $\Omega_{B/R}$ of *Kähler differentials* is defined by the universal property

$$\hom_{\mathrm{Mod}(B)}(\Omega_{B/R}, M) \simeq \mathrm{Der}_R(B, M).$$

In fact, if we let $\mu\colon B \otimes_R B \to B$ be the product map and $I := \ker(\mu)$, then there is an R-derivation $d\colon B \to I/I^2$ sending $x \mapsto 1 \otimes x - x \otimes 1$ such that the map

$$\hom_{\mathrm{Mod}(B)}(I/I^2, M) \to \mathrm{Der}_R(B, M), \quad \phi \mapsto \phi \circ d$$

is an isomorphism, so $\Omega_{B/R} \simeq I/I^2$. We omit the verification of this construction.

2.5. We define a functor

$$\mathrm{Mod}(A) \to \mathcal{C}, \quad M \mapsto A \oplus M,$$

where the ring structure on $A \oplus M$ is given by $(a, x)(b, y) = (ab, ay + bx)$. Then the structure morphism $R \to A$ and projection $A \oplus M \to A$ determine $R \to A \oplus M \to A$ as an object in $\mathcal{C}$. If we have a morphism $\phi\colon M \to N$ in $\mathrm{Mod}(A)$ then $\mathrm{id}_A \oplus \phi\colon A \oplus M \to A \oplus N$ evidently gives a morphism in $\mathcal{C}$.

2.6. Proposition. *The functor* $\mathrm{Mod}(A) \to \mathcal{C}$ *defined in (2.5) factors through the inclusion* $\mathcal{C}_{\mathrm{ab}} \to \mathcal{C}$*, and furthermore induces an equivalence of categories*

$$\mathrm{Mod}(A) \simeq \mathcal{C}_{\mathrm{ab}}.$$

Proof. The first claim is that $A \oplus M$ is an abelian group object for any $M \in \mathrm{Mod}(A)$. Let $B \in \mathcal{C}$. By definition of $\mathcal{C}$ and $A \oplus M$, giving a morphism $B \to A \oplus M$ is equivalent to giving an R-linear map $d\colon B \to M$ satisfying $d(xy) = x\,d(y) + y\,d(x)$ for $x, y \in B$.

(Here we have viewed M as a module over R and B via restriction of scalars.) I.e., we have a natural isomorphism

$$\hom_{\mathcal{C}}(B, A \oplus M) \simeq \mathrm{Der}_R(B, M), \tag{2.7}$$

which is an abelian group, giving our claim.

We next show that if $B \in \mathcal{C}_{\mathrm{ab}}$ is an abelian group object then it is isomorphic to $A \oplus M$ for some $M \in \mathrm{Mod}(A)$. Let $\eta\colon B \to A$ be the map over A. Note the terminal object of $\mathcal{C}$ is A and the product is given by fibred product $B \times_A B$. To have an abelian group object we in particular need morphisms $\mu\colon B \times_A B \to B$ and $\epsilon\colon A \to B$. Since ϵ is a morphism over A, it must be a section of η. Thus if we set $M := \ker(\eta)$ we have a splitting $B \simeq A \oplus M$ as A-modules. Next, ϵ being a "two-sided identity" means

$$\mu \circ (\epsilon\eta, \mathrm{id}_B) = \mathrm{id}_B = \mu \circ (\mathrm{id}_B, \epsilon\eta),$$

and hence $\mu(x, 0) = x = \mu(0, x)$ for $x \in M$. This implies $xy = \mu(x, 0)\mu(0, y) = \mu(0, 0) = 0$ for $x, y \in M$, and hence for $a, b \in A$ we have

$$(\epsilon(a) + x)(\epsilon(b) + y) = \epsilon(ab) + \epsilon(a)y + \epsilon(b)x$$

It follows that in fact $B \simeq A \oplus M$ as objects of $\mathcal{C}$.

Finally we must show the functor is fully faithful. Faithfulness is clear. A morphism $\psi\colon A \oplus M \to A \oplus N$ over A must be of the form $\mathrm{id}_A \oplus \phi$. And since

$$(1, x)(1, y) = (1, x + y), \quad (a, 0)(1, x) = (1, ax)$$

for $a \in A$ and $x, y \in M$ or $x, y \in N$, if ψ is a morphism of abelian group objects, then ϕ is A-linear. □

2.8. Proposition. *The assignment $B \mapsto \Omega_{B/R} \otimes_B A$ defines a functor $\mathcal{C} \to \mathrm{Mod}(A)$ which is left adjoint to inclusion $\mathrm{Mod}(A) \simeq \mathcal{C}_{\mathrm{ab}} \to \mathcal{C}$.*

Proof. Functoriality is clear from the definition (2.4.3) of $\Omega_{B/R}$ and the Yoneda lemma, as one can pull back derivations. Then adjointness follows from (2.7):

$$\begin{aligned} \hom_{\mathcal{C}}(B, A \oplus M) &\simeq \mathrm{Der}_R(B, M) \\ &\simeq \hom_{\mathrm{Mod}(B)}(\Omega_{B/R}, M) \\ &\simeq \hom_{\mathrm{Mod}(A)}(\Omega_{B/R} \otimes_B A, M). \end{aligned}$$ □

2.9. We now have a convenient (and pretty intruiging, actually) notion of abelianisation for the category $\mathcal{C} = \mathrm{Alg}(R)_{/A}$. As per the philosophy of §1, we should now take the derived functor. But wait! We don't even have a model structure on $\mathcal{C}$! And why would we expect to? Indeed, we should extend the above discussion (2.5)–(2.8) to *simplicial* objects via (1.6) to get an adjunction

$$\Omega_{-/R} \otimes_{-} A \;:\; \mathrm{sAlg}(R)_{/A} \rightleftarrows \mathrm{sMod}(A) \;:\; A \oplus -.$$

(Recall that the functors extend to the simplicial categories by degree-wise application.) It seems much more natural to give model structures to these categories, and in fact this is what we will do in the following section.

3. Promoting model structures

Why does it seem more natural to give model structures to simplicial categories? Well, we certainly have a familiar model structure on simplicial sets, so hopefully we can leverage this to our advantages, via the forgetful functors from the categories at hand to sSet. This is precisely what we do here, following [GJ99], though in a bit more generality, as stated in [GS07].

3.1. Definition. Let $\mathcal{C}$ be a category and $\mathcal{F}$ a class of morphisms in $\mathcal{C}$. We say $X \in \mathcal{C}$ is *small* for $\mathcal{F}$ if (assuming $\mathcal{C}$ has enough colimits), for any sequence of morphisms $Y_1 \to Y_2 \to Y_3 \to \cdots$ in $\mathcal{F}$, the canonical morphism

$$\operatorname{colim} \hom_{\mathcal{C}}(X, Y_n) \to \hom_{\mathcal{C}}(X, \operatorname{colim} Y_n),$$

is an isomorphism, or equivalently, if any morphism $X \to \operatorname{colim} Y_n$ factors through some inclusion $Y_m \to \operatorname{colim} Y_n$.

3.2. Example. By the equivalent restatement in (3.1), it is clear that any $X \in$ sSet which has finitely many nondegenerate simplices is small for $\mathcal{F}$ the class of *all* morphisms in sSet.

3.3. Definition. Let $\mathcal{C}$ be a model category. Let $\mathcal{F}$ be the class of cofibrations in $\mathcal{C}$ and $\mathcal{G}$ the class of acyclic cofibrations. We say $\mathcal{C}$ is *cofibrantly generated* if there exist sets of morphisms $\mathcal{I}$ and $\mathcal{J}$ in $\mathcal{C}$ such that:

(1) for any morphism $X \to Y$ in $\mathcal{I}$, X is small for $\mathcal{F}$;
(2) a morphism $X \to Y$ in $\mathcal{C}$ is an acyclic fibration if and only if it has the right lifting property against $\mathcal{I}$;
(3) for any morphism $X \to Y$ in $\mathcal{J}$, X is small for $\mathcal{G}$;
(4) a morphism $X \to Y$ in $\mathcal{C}$ is a fibration if and only if it has the right lifting property against $\mathcal{J}$;

In particular, $\mathcal{I}$ and $\mathcal{J}$ generate (in the relevant sense) $\mathcal{F}$ and $\mathcal{G}$, respectively.

3.4. Example. By (3.2), the standard model structure on sSet is cofibrantly generated with

$$\mathcal{I} := \{\partial\Delta^n \hookrightarrow \Delta^n \mid n \geq 0\}, \quad \mathcal{J} := \{\Lambda^n_k \hookrightarrow \Delta^n \mid n \geq 1, 0 \leq k \leq n\},$$

the sphere and horn inclusions, respectively.

3.5. Let $\mathcal{C}$ be a model category and $\mathcal{D}$ be any category. Assume we have a functor $G\colon \mathcal{D} \to \mathcal{C}$ with a left adjoint $F\colon \mathcal{C} \to \mathcal{D}$. Define a morphism in f in $\mathcal{D}$ to be a:

(1) weak equivalence if $G(f)$ is a weak equivalence in $\mathcal{C}$;
(2) fibration if $G(f)$ is a fibration in $\mathcal{C}$;
(3) cofibration if it has the left lifting property against all acyclic fibrations (as just defined).

Thinking of $\mathcal{C}$ as sSet and $\mathcal{D}$ as our simplicial algebraic category, with G the forgetful functor, our hope is to prove that this defines a model structure on $\mathcal{D}$. The precise result, due to Quillen, is the following.

3.6. Theorem. *In the situation of (3.5). Assume:*

(1) $\mathcal{D}$ is complete and cocomplete;
(2) $\mathcal{C}$ is cofibrantly generated, with generating sets $\mathcal{I}$ and $\mathcal{J}$ of cofibrations and acyclic cofibrations, respectively;
(3) G preserves sequential colimits;
(4) a cofibration in $\mathcal{D}$ with the left lifting property against all fibrations is acyclic.

Then the given definitions indeed give a model structure on $\mathcal{D}$, and hence F and G form a Quillen adjunction.

For convenience, we isolate two steps of the proof.

3.7. Lemma. *A morphism $f\colon X \to Y$ in $\mathcal{D}$ can be factored as*

$$X \xrightarrow{i} Z \xrightarrow{q} Y,$$

where q is an acyclic fibration and i has the left lifting property against all acyclic fibrations.

Proof. Define $Z_0 := X$ and $q_0 := f$. Inductively define $i_n \colon Z_{n-1} \to Z_n$ and $q_n \colon Z_n \to Y$ for $n \geq 1$ as follows. Consider all diagrams D of the form

$$\begin{tikzcd} F(U) \arrow[r, "\alpha"] \arrow[d, "F(j)"] & Z_{n-1} \arrow[d, "q_{n-1}"] \\ F(V) \arrow[r, "\beta"] & Y \end{tikzcd}$$

where $j \in \mathcal{I}$. Then we can form the pushout

$$\begin{tikzcd} \coprod_D F(U) \arrow[r, "(\alpha)"] \arrow[d, "(F(j))"] & Z_{n-1} \arrow[d, "i_n"] \arrow[dr, "q_{n-1}"] & \\ \coprod_D F(V) \arrow[r, "(\beta)"] & Z_n \arrow[r, "q_n"] & Y. \end{tikzcd}$$

Define $Z := \operatorname{colim} Z_n$, $i \colon X = Z_0 \to Z$ the canonical morphism, and $q \colon Z \to Y$ induced by the q_n. Then we have a factoring

$$X \xrightarrow{i} Z \xrightarrow{q} Y$$

of f. Each $j \in \mathcal{I}$ has the left lifting property against acyclic fibrations in $\mathcal{C}$, implying via the adjunction and the definition of acyclic fibrations in $\mathcal{D}$ that $F(j)$ has the same property in $\mathcal{D}$. Since the left lifting property is stable under coproducts, pushouts, and sequential colimits, i also has it.

So we just need to show q is an acyclic fibration, i.e., that $G(q)$ is an acyclic fibration. By assumption (3) we have $G(Z) \simeq \operatorname{colim} G(Z_n)$, so by definition of $\mathcal{I}$ it suffices to show any lifting problem

$$\begin{tikzcd} U \arrow[r, "\alpha"] \arrow[d, "j"] & \operatorname{colim} G(Z_n) \arrow[d, "G(q)"] \\ V \arrow[r, "\beta"] & G(Y) \end{tikzcd}$$

with $j \in \mathcal{I}$ can be solved. Now, U must be small for cofibrations in $\mathcal{C}$, and again by the adjunction and lifting properties we see that $G(i_n)$ is a cofibration for all $n \geq 1$. Thus α in fact factors through some $G(Z_m) \to \operatorname{colim} G(Z_n)$. But then applying the adjunction to

$$\begin{tikzcd} U \arrow[r, "\alpha"] \arrow[d, "j"] & G(Z_m) \arrow[d, "G(q_m)"] \\ V \arrow[r, "\beta"] & G(Y) \end{tikzcd}$$

and considering the definition of Z_{m+1}, we see that we can lift β to a morphism $V \to G(Z_{m+1}) \to \operatorname{colim} G(Z_n)$, solving the original lifting problem. $\square$

3.8. Lemma. *A morphism $f \colon X \to Y$ in $\mathcal{D}$ can be factored as*

$$X \xrightarrow{i} Z \xrightarrow{q} Y,$$

where q is a fibration and i has the left lifting property against all fibrations.

Proof. Replace $\mathcal{I}$ with $\mathcal{J}$ in the proof of (3.7). $\square$

Proof of (3.6). We have assumed $\mathcal{D}$ is complete and cocomplete, giving the first model category axiom. The second and third (two-out-of-three and retract) axioms are immediate from the definitions and these holding in $\mathcal{C}$.

The fourth axiom is the lifting axiom. One half is simply our definition of a cofibration, so let's prove the other half. Suppose $f\colon X \to Y$ is an acyclic cofibration in $\mathcal{D}$. Let $q \circ i\colon X \to Z \to Y$ the factorisation of f given by (3.8). Then assumption (4) gives that i is a weak equivalence. By two-out-of-three then so is q. But then we know a lift exists in

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & Z \arrow[d, "q"] \\ Y \arrow[r, "\mathrm{id}_Y"] & Y, \end{tikzcd}$$

exhibiting f as a retract of i. Since lifting properties are stable under retracts, we win.

The fifth axiom is the factorisation axiom, which is immediate from (3.7) and (3.8), along with the definition of cofibrations in $\mathcal{D}$ and assumption (4). $\square$

3.9. Now consider the example $\mathcal{C} = \mathrm{sSet}$ and $\mathcal{D} = \mathrm{sMod}(R)$ or $\mathcal{D} = \mathrm{sAlg}(R)$.[2] We have forgetful functors $\mathcal{D} \to \mathcal{C}$ with left adjoints given by free functors $\mathcal{C} \to \mathcal{D}$, which commute with filtered colimits. Then to define a model structure on $\mathcal{D}$ via (3.6), the only condition which needs checking is (4). We omit this verification for the sake of concision, but one can see [GJ99] for a proof.

3.10. We state one more result without proof to end this discussion of defining model structures. Suppose $\mathcal{C}$ is a model category and $X \in \mathcal{C}$. If we define a morphism $f\colon X \to Y$ in the overcategory $\mathcal{C}_{/X}$ to be a weak equivalence, fibration, or cofibration if it is such in $\mathcal{C}$, this defines a model structure on $\mathcal{C}_{/X}$ [Sch13].

4. The cotangent complex

With (3.9) and (3.10) we've managed to construct natural model structures on the categories $\mathrm{sAlg}(R)_{/A}$ and $\mathrm{sMod}(A)$ we were discussing back in §2.[3] (Recall R is a ring and A an R-algebra.)

4.1. Notation. We recall that in the current setting we have an adjunction

$$\mathrm{Ab} \;:\; \mathrm{sAlg}(R)_{/A} \rightleftarrows \mathrm{sMod}(A) \;:\; \mathrm{In},$$

where In is the inclusion functor $M \mapsto A \oplus M$ and Ab is the abelianisation (or Kähler differentials) functor $B \mapsto \Omega_{B/R} \otimes_B A$—and both of these are defined by applying the definitions (2.5) and (2.4.3) degree-wise.

The next step is to derive Ab as in (1.7), to get our "homology object". But to be able to do this, we first need to check that the adjunction is in fact a Quillen adjunction.

4.2. Lemma. *The functor* In *is right Quillen, and hence* Ab *is left Quillen.*

[2]Or $\mathcal{D}$ any number of other simplicial algebraic categories, which don't happen to be relevant to our current discussion.

[3]I've glossed over a technical detail here. To get the model structure on $\mathrm{sAlg}(R)_{/A}$ via (3.10) we should first observe that there is a natural identification $\mathrm{sAlg}(R)_{/A} \simeq (\mathrm{sAlg}(R))_{/c(A)}$, where $c(A)$ is the constant simplicial object as defined below in (4.4).

Proof. Let $\phi\colon M \to N$ be any morphism in $\mathrm{sMod}(A)$ and $\mathrm{id}_A \oplus \phi\colon A \oplus M \to A \oplus N$ the morphism in $\mathrm{sAlg}(R)_{/A}$ given by applying In. Consider any lifting problems

$$\begin{tikzcd} K \arrow[r, "b"] \arrow[d] & M \arrow[d, "\phi"] \\ L \arrow[r, "d"] & N \end{tikzcd} \qquad \begin{tikzcd} K \arrow[r, "{(a,b)}"] \arrow[d] & A \oplus M \arrow[d, "\mathrm{id}_A \oplus \phi"] \\ L \arrow[r, "{(c,d)}"] & A \oplus N \end{tikzcd}$$

in sSet. If there is a lift $e\colon L \to M$ in the first problem, then clearly $(c, e)\colon L \to A \oplus M$ is a lift in the second problem. Since fibrations and acyclic fibrations in $\mathrm{sMod}(A)$ and $\mathrm{sAlg}(R)_{/A}$ are defined by lifting properties in sSet, this shows that In is right Quillen. □

Finally it makes sense to make the following definition.

4.3. Definition. The *cotangent complex functor* is the total left derived functor $\mathbb{L}\,\mathrm{Ab}\colon\ \mathrm{ho}(\mathrm{sAlg}(R)_{/A}) \to \mathrm{ho}(\mathrm{sMod}(A))$.

But hold on, why do we care about that? At the start, we wanted a homology theory for the algebra $R \to A$. Somehow we've ended up with a functor on the *simplicial* R-algebras *over* A. Well ok, don't worry, we just have a bit more work to do.

4.4. Recall for any category $\mathcal{C}$ we have an embedding $c\colon \mathcal{C} \to \mathrm{s}\mathcal{C}$ by taking the constant simplicial objects. That is, for $X \in \mathcal{C}$ we can define $c(X) \in \mathrm{s}\mathcal{C}$ by $c(X)_n := X$ for $n \geq 0$ with all face and degeneracy maps just the identity id_X.

Now we get to the definitions we've really been waiting for.

4.5. Definitions. (1) The *cotangent complex* of the R-algebra A is

$$\mathrm{L}_{A/R} := \mathbb{L}\,\mathrm{Ab}(c(A)),$$

i.e., the cotangent complex functor evaluated on the constant simplicial object determined by A. Note $\mathrm{L}_{A/R} \in \mathrm{sMod}(A)$, but we will identify it via the Dold-Kan correspondence with the associated chain complex of A-modules.

(2) As (the value of) a total left derived functor, $\mathrm{L}_{A/R}$ is well-defined up to weak equivalence, and hence we can define the *André-Quillen homology and cohomology* of $R \to A$ with coefficients in an A-module M by

$$\mathrm{D}_n(A, R; M) := \mathrm{H}_n(\mathrm{L}_{A/R} \otimes_A M),$$
$$\mathrm{D}^n(A, R; M) := \mathrm{H}^n(\hom_{\mathrm{Mod}(A)}(\mathrm{L}_{A/R}, M)).$$

Note in particular that homology with coefficients in A is given by $\pi_*(\mathrm{L}_{A/R})$.

5. Simplicial resolutions

To end our formal discussion, we remark briefly about the passage to simplicial objects needed to define the cotangent complex.

Consider the theory of derived functors in abelian categories. Suppose $F\colon \mathcal{A} \to \mathcal{B}$ is a right exact functor between abelian categories. To compute the total left derived functor of F on $X \in \mathcal{A}$, we apply F to a projective resolution of X, which we can then take homology of. In the standard model structure on categories of chain complexes, a projective resolution of X is precisely a cofibrant replacement of the chain complex with X concentrated in degree 0. Finally via the Dold-Kan correspondence, this is equivalent to picking a cofibrant replacement of $c(X) \in \mathrm{s}\mathcal{A}$.

To define the cotangent complex, we wanted to derive a functor from a non-abelian category. The above tells us that instead of taking projective resolutions, the correct analogue is to take a *simplicial resolution*, that is, a cofibrant replacement of the constant simplicial object. Indeed, this is exactly how we defined $\mathrm{L}_{A/R}$.

Concluding nonsense

At the moment, I don't have the time or space to learn or write about what this homology theory of algebras really means or why it's useful. As one might expect of a left derived functor, it turns out that André-Quillen homology finishes the long exact sequence ending at a certain right exact sequence which one has for Kähler differentials. In addition, the cotangent complex is supposed to carry a lot of information about the "deformation theory" of algebras, and when suitably generalised, of schemes and other algebro-geometric objects. But now we've really reached the point where I don't know what I'm talking about.

Hopefully I can learn about these things properly in the near future. In any case, I think just the work we did to define the cotangent complex is a pretty awesome and convincing example of model categories—in particular the theory of non-abelian derived functors and simplicial resolutions—being an excellent tool for introducing homotopical ideas into new settings.

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