VECTOR FIELDS AND THE J-HOMOMORPHISM

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1. The problem

To start at the very beginning, we state the basic definitions.

1.1. Notation. Throughout we have integers $n \geq 2$ and $0 \leq k \leq n$.

1.2. Definitions. Let M be a differentiable manifold. Let $\pi\colon T \longrightarrow M$ be the tangent bundle on M.

1. A *vector field* on M is a continuous section $v\colon M \longrightarrow T$ of π.
2. A set of vector fields $\{v_1, \dots, v_k\}$ on M is *linearly independent* if for each $p \in M$ the vectors $v_1(p), \dots, v_k(p)$ are linearly independent in the tangent space T_p. In particular a single vector field v forms a linearly independent set if and only if it is nowhere vanishing.

Now the actual story: we are taught to love spheres from our very first days in the land of topology. But perhaps it is the following result—or perhaps really its title—which first truly beguiles us.

1.3. Theorem (Hairy ball)**.** The sphere S^{n-1} admits a nowhere vanishing vector field if and only if n is even.

Of course, so enticed, we cannot just leave it there. We must ask the following.

1.4. Question. Then how many vector fields does S^{n-1} admit? Or more precisely, what is the maximum size of a set of linearly independent vector fields on S^{n-1}?

This is one of those questions that occupied people for a while. Its answer was one of the first applications of generalised cohomology theory, and involves some really nice ideas, some of which are hopefully conveyed in this exposition.

2. Lower bound

We first briefly review the positive side of the problem, that is, how to actually construct some vector fields and achieve a lower bound. We do this for completeness, and because it clarifies the sort of numerology we see in the upper bound. One can see, e.g., [HM12, Mil10] for more details.

2.1. First of all, we have our very nice embedding $S^{n-1} \hookrightarrow \mathbb{R}^n$, which gives the tangent spaces of S^{n-1} a very concrete description. In particular, a vector field on S^{n-1} is just a map $v\colon S^{n-1} \longrightarrow \mathbb{R}^n$ such that $v(x) \perp x$ (in $\mathbb{R}^n$) for all $x \in S^{n-1}$. Note that by Gram-Schmidt, giving k linearly independent vector fields $v_1, \dots, v_k\colon S^{n-1} \longrightarrow \mathbb{R}^n$ is equivalent to giving k pointwise orthonormal maps $v_1, \dots, v_k\colon S^{n-1} \longrightarrow S^{n-1}$.

2.2. Notation. • Write $n = 2^a b$ with b odd, and write $a = 4c + d$ with $0 \le d \le 3$. We define $\rho(n) := 2^d + 8c$.[1] Note in particular $\rho(n) = 1$ if n is odd.

• Define $e_k := |\{0 < j \le k : j \equiv 0, 1, 2, 4 \pmod 8\}|$. It's easy to see that

$$\rho(n) - 1 = \max\{l \ge 0 : 2^{e_l} \mid n\} = \max\{l \ge 0 : e_l \le a\}. \tag{2.3}$$

2.4. The *Clifford algebras* Cl_l for $l \ge 0$ are the free associative $\mathbb{R}$-algebras with generators $q_1, \ldots, q_l$ subject to the relations $q_i^2 = -1$ and $q_i q_j = -q_j q_i$ for $i \ne j$. For small values of l these are as follows[2], where $A(d)$ denotes the algebra of d-by-d matrices in A.

l	0	1	2	3	4	5	6	7	8
Cl_l	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}^{\oplus 2}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)^{\oplus 2}$	$\mathbb{R}(16)$

For larger values of l we have a periodicity $\mathrm{Cl}_{l+8} \simeq \mathrm{Cl}_l \otimes_{\mathbb{R}} \mathrm{Cl}_8$.

What does this have to do with vector fields? Well, suppose V is an $\mathbb{R}$-vector space of dimension n with a representation $\mathrm{Cl}_l \otimes_{\mathbb{R}} V \longrightarrow V$. Let $G_l \subset \mathrm{Cl}_l$ be the multiplicative group genereated by $\{\pm q_i\}$; it's easy to see $|G_l| = 2^{l+1}$. We can construct a G_l-invariant inner product on V (e.g., by averaging any inner product over G_l), and then we see that (under this inner product) we get l orthonormal vector fields on the unit sphere $S(V) \simeq S^{n-1}$ via $x \longmapsto q_i x$ for $1 \le i \le l$.

Thus we are interested in knowing the minimal dimension of a representation of Cl_l. By the periodicity above, one can show without much difficulty that this dimension is precisely 2^{e_l}, where e_l is as defined in [2.2]. Thus there is a representation of Cl_l on $\mathbb{R}^n$ whenever $2^{e_l} \mid n$, by writing $\mathbb{R}^n \simeq \mathbb{R}^{2^{e_l}} \times \cdots \times \mathbb{R}^{2^{e_l}}$ and acting diagonally via the minimal representation. This gives the following lower bound on our question [1.4].

2.5. Theorem. There is a set of $\rho(n) - 1$ linearly independent vector fields on S^{n-1}.

Proof. Let $l := \rho(n) - 1$. By [2.3], $2^{e_l} \mid n$, so the discussion in [2.4] gives the claim. $\square$

3. Upper bound: a reduction

Getting an upper bound is where the real difficulty lies. Well, we know one upper bound: S^{n-1} certainly can't admit n linearly independent vector fields, since $\dim S^{n-1} = n - 1$. And to say S^{n-1} admits $n - 1$ linearly independent vector fields is to say S^{n-1} is parallelisable, which famously is true if and only if $n \in \{2, 4, 8\}$. An optimal upper bound would at least tell us this parallelisability result. So let's think about it for a second and reduce the question to one more attackable by algebra.

3.1. Definition. For $l \in \mathbb{N}$, the *Stiefel manifold* $V_{l,n}$ is the space

$$\{(v_1, \ldots, v_l) : v_i \in S^{n-1}, \langle v_i, v_j \rangle = \delta_{i,j}\}$$

of orthonormal l-frames on $\mathbb{R}^n$.

[1] The $\rho(n)$ are called the *Radon-Hurwitz numbers*.

[2] One should note however that the descriptions in this table are given by completely non-canonical isomorphisms.

3.2. Lemma. Let $\pi_k \colon V_{k+1,n} \longrightarrow S^{n-1}$ be the projection $(v_1, \ldots, v_{k+1}) \longmapsto v_1$. Then S^{n-1} admits a set of k linearly independent vector fields if and only if there is a section $S^{n-1} \longrightarrow V_{k+1,n}$ of π_k.

Proof. This is immediate from the discussion in [2.1]. □

So we've reduced our problem to the existence of some map. Already one can imagine using algebra to get at the problem now. E.g., we can ask for what k this map can exist in singular homology or cohomology. This was the strategy of Steenrod and Whitehead [SW51], who achieve an upper bound $k \leq 2^a$, in the notation of [2.2]. Of course this result doesn't tell us that S^{15} is not parallelisable, and leaves a large gap from the lower bound [2.5]. This gap was finally closed by Adams, who ingeniously employed K-theory instead to show the lower bound [2.5] is in fact optimal.

3.3. Theorem ([Ada62])**.** There does not exist a set of $\rho(n)$ linearly independent vector fields on S^{n-1}.

Our goal for the remainder is to explain the ideas behind a proof of this theorem, which of course completely answers our question [1.4]. Following [Mil10], the argument we give is not in the original form presented in [Ada62], but rather one utilising Adams's later work on bounding the image of the J-homomorphism [Ada65].

However, we will only work in complex K-theory. As a result we will achieve a very slightly worse upper bound than promised by [3.3], but this way we get to avoid a few subtleties that arise in studying real K-theory. At least to the author, it seems the argument in complex K-theory retains the main ideas and yet is much easier to absorb. At the end we will explain why, after sorting out the subtleties, translating the argument into real K-theory gives the full result.

4. Connection to the J-homomorphism

We first review our basic notation and the general setup of Adams's study of the J-homomorphism.

4.1. Notation. 1. X will always denote a connected finite CW-complex.

2. Denote[3] real and complex K-theory by $K_\Lambda(X)$ with $\Lambda = \mathbb{R}$ or $\Lambda = \mathbb{C}$, respectively, and reduced K-theory by $\widetilde{K}_\Lambda(X)$.
3. We have maps:
   - $\iota\colon K_\mathbb{C}(X) \longrightarrow K_\mathbb{R}(X)$ by forgetting complex structure;
   - $\kappa\colon K_\mathbb{R}(X) \longrightarrow K_\mathbb{C}(X)$ by complexification, i.e., tensoring with $X \times \mathbb{C}$.

   Note that since $\mathbb{C} \simeq \mathbb{R} \oplus \mathbb{R}$ the map $\iota \circ \kappa$ is just multiplication by 2.
4. If $\xi \longrightarrow X$ is a real or complex vector bundle, we abusively denote its class in $K_\Lambda(X)$ by ξ as well.
5. Let ϵ_Λ for $\Lambda = \mathbb{R}$ or $\Lambda = \mathbb{C}$ denote the trivial real or complex line bundle (over a space understood from context), respectively.

4.2. Remark. Since we always work over a connected base, any vector bundle $\xi \longrightarrow X$ has a well-defined dimension over a specified Λ, and this extends to a ring morphism $\dim\colon K_\Lambda(X) \longrightarrow \mathbb{Z}$. Recall $\widetilde{K}_\Lambda(X) = \ker(\dim)$.

[3] We use Adams's notation for K-theory rather than the what is the standard notation these days, since it seems more convenient for our purposes.

4.3. Definitions. 1. A *spherical fibration* over X is a fibre bundle $S \longrightarrow X$ whose fibre has the homotopy type of a sphere.

2. Let $S \longrightarrow X$ and $S' \longrightarrow X$ be two spaces over X. We say S and S' are *fibre homotopy equivalent* if there are maps $f\colon S \longrightarrow S'$ and $g\colon S' \longrightarrow S$ over X and homotopies $gf \simeq \mathrm{id}_S$ and $fg \simeq \mathrm{id}_{S'}$ over X.[4]
3. Denote by $SF(X)$ the Grothendieck group of the monoid of fibre homotopy equivalence classes of spherical fibrations over X with fibre-wise smash product.
4. The *(real) J-homomorphism* is the map $J_{\mathbb{R}}\colon K_{\mathbb{R}}(X) \longrightarrow SF(X)$ induced by $\xi \longmapsto \xi - 0$ for bundles $\xi \longrightarrow X$, i.e., removing the zero section. Or if we equip ξ with a metric, then the *unit sphere bundle* $S(\xi) := \{v \in \xi : |v| = 1\}$ is evidently fibre homotopy equivalent to $\xi - 0$, so we have $J(\xi) = S(\xi)$.
5. We get a *complex J-homomorphism* $J_{\mathbb{C}} := J_{\mathbb{R}} \circ \iota\colon K_{\mathbb{C}}(X) \longrightarrow K_{\mathbb{R}}(X) \longrightarrow SF(X)$.
6. The *image of the J-homomorphism* is denoted $J_\Lambda(X)$ with $\Lambda = \mathbb{R}$ or $\Lambda = \mathbb{C}$.

We now translate our problem into a question about the J-homomorphism.

4.4. Lemma (Dold-Lashof)**.** Let $Y \longrightarrow X$ and $Y' \longrightarrow X$ be fibre bundles. Assume Y and Y' have the homotopy type of CW-complexes and X is connected. Then a map $Y \longrightarrow Y'$ over X inducing a homotopy equivalence on all fibres is in fact a fibre homotopy equivalence.

Proof. Omitted. □

4.5. Notation. Let γ denote the tautological real line bundle over $\mathbb{RP}^k$.

4.6. Lemma. Assume $\pi_k\colon V_{k+1,n} \longrightarrow S^{n-1}$ has a section. Then there is a fibre homotopy equivalence $S(\epsilon_{\mathbb{R}}^{\oplus n}) \longrightarrow S(\gamma^{\oplus n})$ over $\mathbb{RP}^k$.

Proof. First let $Y := (S^k \times S^{n-1})/(\mathbb{Z}/2)$, with $\mathbb{Z}/2$ acting diagonally via the antipodal maps, and observe there is a homeomorphism $Y \simeq S(\gamma^{\oplus n})$ over $\mathbb{RP}^k$, where the map $Y \longrightarrow \mathbb{RP}^k$ is induced by projection

$$S^k \times S^{n-1} \longrightarrow S^k \longrightarrow \mathbb{RP}^k, \quad (v, x) \longmapsto \ell_v.$$

Indeed, viewing $S(\gamma^{\oplus n}) \subset \gamma^{\oplus n} \subset \mathbb{RP}^k \times (\mathbb{R}^{k+1})^n$, the homeomorphism is induced by

$$S^k \times S^{n-1} \longrightarrow \mathbb{RP}^k \times (\mathbb{R}^{k+1})^n, \quad (v, x) \longmapsto (\ell_v, x_1 v, \ldots, x_n v),$$

where $(x_1, \ldots, x_n) := x \in S^{n-1} \subset \mathbb{R}^n$. Then since $S(\epsilon_{\mathbb{R}}^{\oplus n}) \simeq \mathbb{RP}^k \times S^{n-1}$, we are left to give a fibre homotopy equivalence $\mathbb{RP}^k \times S^{n-1} \longrightarrow Y$.

If $s\colon S^{n-1} \longrightarrow V_{k+1,n}$ is a section of π_k then we get a map $f\colon \mathbb{RP}^k \times S^{n-1} \longrightarrow Y$ over $\mathbb{RP}^k$ induced by the map

$$S^k \times S^{n-1} \longrightarrow S^k \times S^{n-1}, \quad (v, x) \longmapsto (v, s(x)v),$$

where $s(x) \in V_{k+1,n}$ acts on S^k by identifying an orthonormal frame with an n-by-$(k+1)$ matrix. Note that since s is a section, if $e_1 \in S^k$ is the first canonical basis vector, then $s(x)e_1 = x$.

Since f preserves fibres, over each $\ell \in \mathbb{RP}^k$ it induces a map $f_\ell\colon S^{n-1} \longrightarrow S^{n-1}$. On any connected open set $U \subseteq \mathbb{RP}^k$ over which $\gamma^{\oplus n}$ is trivial, the assignment $\ell \longmapsto f_\ell$ gives a continuous map $U \longrightarrow \mathrm{map}(S^{n-1}, S^{n-1})$, whose

[4] As usual, "over X" just means preserving fibres.

image lies in a single connected component. Since $\mathbb{RP}^k$ is connected and can be covered by such U, it follows that $\deg(f_\ell)$ is constant. But above we noted that f induces the identity over ℓ_{e_1}, so in fact f_ℓ must be a homotopy equivalence for all $\ell \in \mathbb{RP}^k$. We are then done by [4.4]. $\square$

4.7. Notation. Define $\lambda := \gamma - 1 \in \widetilde{K}_{\mathbb{R}}(\mathbb{RP}^k)$ and $\nu := \kappa(\lambda) = \kappa(\gamma) - 1 \in \widetilde{K}_{\mathbb{C}}(\mathbb{RP}^k)$.

4.8. Lemma. Suppose $n = 2m$. If S^{n-1} admits a set of k linearly independent vector fields, then $J_{\mathbb{C}}(m\nu) = J_{\mathbb{R}}(n\lambda) = 0$.

Proof. Given the hypothesis, it is immediate from [3.2] and [4.6] that $J_{\mathbb{R}}(n\lambda) = 0$. And by definition we have $J_{\mathbb{C}}(m\nu) = J_{\mathbb{R}}(\iota(\kappa(m\lambda))) = J_{\mathbb{R}}(2m\lambda) = J_{\mathbb{R}}(n\lambda)$. $\square$

So solving our problem now reduces to understanding $K_\Lambda(\mathbb{RP}^k)$ and $J_\Lambda(\mathbb{RP}^k)$, and as indicated earlier we will work with $\Lambda = \mathbb{C}$. We first address the former.

5. K-theory of projective space

5.1. Notation. Let $f_k := \lfloor k/2 \rfloor$.

5.2. Theorem. $\widetilde{K}_{\mathbb{C}}(\mathbb{RP}^k)$ is generated by ν, which satisfies the relations $\nu^2 = -2\nu$ and $\nu^{f_k+1} = 0$. This determines a group isomorphism $\widetilde{K}_{\mathbb{C}}(\mathbb{RP}^{k-1}) \simeq \mathbb{Z}/2^{f_k}$. Finally, the Adams operations are given by

$$\psi^l(\nu) = \begin{cases} 0 & \text{if } l \text{ even} \\ \nu & \text{if } l \text{ odd.} \end{cases}$$

Proof. The case $k = 1$ is trivial, so assume $k > 1$. Since real line bundles have structure group $\mathrm{O}(1) \simeq \{\pm 1\}$, we automatically have $\gamma^2 = 1 \in K_{\mathbb{R}}(\mathbb{RP}^k)$. It follows that $\kappa(\gamma)^2 = 1 \in K_{\mathbb{C}}(\mathbb{RP}^k)$, and hence that $\nu^2 = -2\nu$.

Next we prove $\nu^{f_k+1} = 0$. Since the tautological line bundle on $\mathbb{RP}^{k+1}$ pulls back to the tautological line bundle on $\mathbb{RP}^k$ via the inclusion $\mathbb{RP}^k \longrightarrow \mathbb{RP}^{k+1}$, by naturality we may assume $k = 2f_k + 1$ is odd. Let $\pi\colon \mathbb{RP}^k \longrightarrow \mathbb{CP}^{f_k}$ be the canonical projection, and let $\xi \longrightarrow \mathbb{CP}^{f_k}$ be the tautological (complex) line bundle. It is easy to see directly that $\pi^*\xi \simeq \kappa(\gamma)$. Thus that $\nu^{f_k+1} = 0$ follows from the fact[5] that $\widetilde{K}_{\mathbb{C}}(\mathbb{CP}^{f_k}) \simeq \mathbb{Z}[t]/t^{f_k+1}$ where $t := \xi - 1$.

We now show $\widetilde{K}_{\mathbb{C}}(\mathbb{RP}^k) \simeq \mathbb{Z}/2^{f_k}$, using the Atiyah-Hirzebruch spectral sequence

$$E_2^{p,q} := H^p(\mathbb{RP}^k, K_{\mathbb{C}}^q(*)) \implies K_{\mathbb{C}}^{p+q}(\mathbb{RP}^k).$$

Recall Bott periodicity and the cohomology of projective space:

$$K_{\mathbb{C}}^q(*) \simeq \begin{cases} \mathbb{Z} & \text{if } q \text{ even} \\ 0 & \text{if } q \text{ odd,} \end{cases} \qquad H^p(\mathbb{RP}^k;\mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } p = 0 \\ \mathbb{Z}/2 & \text{if } 0 < p \leq k,\ p \text{ even} \\ \mathbb{Z} & \text{if } p = k,\ k \text{ odd} \\ 0 & \text{otherwise.} \end{cases}$$

Thus the E_2 page looks as follows, where A depends on the parity of k as indicated above.

[5] One can see [Ati67] or [Ada62] for (two different) proofs.

-1	0	1	2	3	4	$\cdots$	k	$k+1$
4	$\mathbb{Z}$		$\mathbb{Z}/2$		$\mathbb{Z}/2$	$\cdots$	A	
3								
2	$\mathbb{Z}$		$\mathbb{Z}/2$		$\mathbb{Z}/2$	$\cdots$	A	
1								
0	$\mathbb{Z}$		$\mathbb{Z}/2$		$\mathbb{Z}/2$	$\cdots$	A	
-1								
-2	$\mathbb{Z}$		$\mathbb{Z}/2$		$\mathbb{Z}/2$	$\cdots$	A	
-3								
-4	$\mathbb{Z}$		$\mathbb{Z}/2$		$\mathbb{Z}/2$	$\cdots$	A	

We claim the spectral sequence is trivial, i.e., all the differentials on every page vanish. By naturality of the spectral sequence in a point inclusion $* \longrightarrow \mathbb{RP}^k$ we know all the differentials on the column $p = 0$ vanish. All differentials change the parity of the total degree $p + q$, so a differential from a $\mathbb{Z}/2$ can only possibly map to 0 or $A \simeq \mathbb{Z}$ (in the case k odd) and hence must vanish. And obviously the differentials on the column $p = k$ vanish. So we conclude

$$\mathrm{Gr}_p\, K_{\mathbb{C}}(\mathbb{RP}^k) \simeq E_\infty^{p,-p} \simeq E_2^{p,-p} \simeq \begin{cases} \mathbb{Z} & \text{if } p = 0 \\ \mathbb{Z}/2 & \text{if } 1 \le p \le f_k \\ 0 & \text{otherwise.} \end{cases}$$

We next claim that $\mathrm{Gr}_p\, K_{\mathbb{C}}(\mathbb{RP}^k) \simeq \mathbb{Z}/2$ is generated by the class of ν^p for $1 \le p \le f_k$. It suffices to show this for $p = 1$, because the spectral sequence is multiplicative, and has multiplication induced by the cup product in singular cohomology on E_2, and if $x \in H^2(\mathbb{RP}^k; \mathbb{Z})$ is a generator then we know $x^p \neq 0$ for $1 \le p \le f_k$. And for $p = 1$ it suffices to treat the case $k = 2$, since the spectral sequence is natural and the inclusion $\mathbb{RP}^2 \longrightarrow \mathbb{RP}^k$ both:

- pulls back the tautological bundle on $\mathbb{RP}^k$ to the tautological bundle on $\mathbb{RP}^2$;
- induces an isomorphism $H^2(\mathbb{RP}^k; \mathbb{Z}) \simeq H^2(\mathbb{RP}^2; \mathbb{Z}) \simeq \mathbb{Z}/2$.

But in the case $k = 2$, we have $\mathrm{Gr}_1\, K_{\mathbb{C}}(\mathbb{RP}^2) \simeq \widetilde{K}_{\mathbb{C}}(\mathbb{RP}^2)$, so to say that ν is a generator is just to say $\nu \neq 0$, or equivalently $\kappa(\gamma) \neq 1$. One can prove this with the Stiefel-Whitney class: it suffices to show $1 \neq \iota(\kappa(\gamma)) = 2\gamma \in K_{\mathbb{R}}(\mathbb{RP}^2)$, which is witnessed by $w(\gamma \oplus \gamma) = w(\gamma)^2 = 1 + x^2 \neq 1$, where $x \in H^1(\mathbb{RP}^2; \mathbb{Z}/2)$ is a generator.

Now we can determine the extensions needed to compute $K_{\mathbb{C}}(\mathbb{RP}^k)$ from the associated graded $\mathrm{Gr}_*\, K_{\mathbb{C}}(\mathbb{RP}^k)$. Let F_p be the p-th filtered piece of $K_{\mathbb{C}}(\mathbb{RP}^k)$, so that $F_p/F_{p+1} \simeq \mathrm{Gr}_p\, K_{\mathbb{C}}(\mathbb{RP}^k) \simeq \mathbb{Z}/2$ for $1 \le p \le f_k$ and $F_1 \simeq \widetilde{K}_{\mathbb{C}}(\mathbb{RP}^k)$. We inductively show that $F_p \simeq \mathbb{Z}/2^{f_k - p + 1}$ with generator ν^{f_k}; the base case $p = f_k$ is done already. To induct, we have the extension problem

$$0 \longrightarrow \mathbb{Z}/2^{f_k - p} \longrightarrow F_p \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

We just need to show ν^p has order 2^{f_k-p+1}. But we have the identity $\nu^{p+1} = -2\nu^p$, and we inductively know ν^p has order 2^{f_k-p}. Since we know F_p is a 2-group this implies the claim.

To finish the proof we just need to verify the Adams operations, but this is evident from the identity $\kappa(\gamma)^2 = 1$ shown above, since $\psi^l(\nu) = \psi^l(\kappa(\gamma)) - 1 = \kappa(\gamma)^l - 1$. □

5.3. Remark. Given the information from [5.2], the multiplicative structure on $\widetilde{K}_{\mathbb{C}}(\mathbb{RP}^k)$ can be clarified by observing that we can define an injective morphism of (non-unital) rings $\alpha\colon \widetilde{K}_{\mathbb{C}}(\mathbb{RP}^{k-1}) \longrightarrow \mathbb{Z}/2^{f_k+1}$ via $\nu \longmapsto -2$.

6. The Thom isomorphism

Before moving on to studying $J_{\mathbb{C}}(\mathbb{RP}^{k-1})$ we briefly review the basic theory of the Thom isomorphism.

6.1. Definition. Let $\xi \longrightarrow X$ be a vector bundle of real dimension r. The *Thom space* $\mathrm{Th}(\xi)$ of ξ is obtained by fibre-wise one-point compactifying ξ and then identifying all the points at infinity. More precisely we have $\mathrm{Th}(\xi) \simeq \mathbb{P}(\xi \oplus \epsilon_{\mathbb{R}})/\mathbb{P}(\xi)$, or if we equip ξ with a metric then we have $\mathrm{Th}(\xi) \simeq D(\xi)/S(\xi)$ the unit disk bundle quotiented by the unit sphere bundle.

6.2. Remark. For any $p \in X$ the fibre $\mathbb{R}^r \simeq \xi_p \hookrightarrow \xi$ determines a "fibre" $S^r \simeq \mathrm{Th}(\xi_p) \hookrightarrow \mathrm{Th}(\xi)$.

6.3. Let $\xi \longrightarrow X$ be an oriented vector bundle of real dimension r. Let E be a multiplicative cohomology theory. Then we have

$$\widetilde{E}^*(\mathrm{Th}(\xi)) \simeq E^*(D(\xi), S(\xi)) \simeq E^*(\xi, \xi - 0),$$

and in this way $\widetilde{E}^*(\mathrm{Th}(\xi))$ is a module over $E^*(X) \simeq E^*(\xi)$.

A *Thom class* in E of ξ is an element $u \in \widetilde{E}^r(\mathrm{Th}(\xi))$ such that for any fibre $i\colon S^r \simeq \mathrm{Th}(\xi_p) \longrightarrow \mathrm{Th}(\xi)$, where the identification $S^r \simeq \mathrm{Th}(\xi_p)$ is determined by the orientation of ξ, the restriction

$$\widetilde{E}^r(\mathrm{Th}(\xi)) \xrightarrow{i^*} \widetilde{E}^r(S^r) \xrightarrow{\sim} E^0(*)$$

sends u to the canonical unit $1 \in E^0(*)$.

The *Thom isomorphism theorem* states that if ξ has a Thom class u then the map

$$\phi\colon E^*(X) \longrightarrow \widetilde{E}^{*+r}(\mathrm{Th}(\xi)), \quad x \longmapsto x \cdot u$$

is an isomorphism of $E^*(X)$-modules.

6.4. Proposition. There exist Thom classes u_ξ in $K_{\mathbb{C}}$ for all complex vector bundles $\xi \longrightarrow X$. These can be chosen to satisfy the following pleasant properties.

1. Naturality: for any pullback square

$$\begin{tikzcd} f^*\xi \arrow[r, "g"] \arrow[d] & \xi \arrow[d] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

   of complex vector bundles, $u_{f^*\xi} = g^*(u_\xi)$.

2. Multiplicativity: let $\xi \longrightarrow X$ and $\eta \longrightarrow Y$ be complex vector bundles. Consider the product bundle $\xi \times \eta \longrightarrow X \times Y$, and let $\pi_\xi \colon \xi \times \eta \longrightarrow \xi$ and $\pi_\eta \colon \xi \times \eta \longrightarrow \eta$ be the projections. Then
$$u_{\xi \times \eta} = \pi_\xi^*(u_\xi) \cdot \pi_\eta^*(u_\eta).$$
By naturality this implies the same multiplicativity when $X = Y$ and we replace $\xi \times \eta$ with $\xi \oplus \eta$.

Proof. This is achieved by the "difference bundle" construction of [ABS64], but this topic is omitted here, unfortunately. □

6.5. Example. Let $\xi \longrightarrow \mathbb{CP}^\infty$ be the tautological line bundle. We claim that in fact $\mathrm{Th}(\xi) \simeq \mathbb{CP}^\infty$. Indeed since $S(\xi) \simeq S^\infty$ is contractible, we have $\mathrm{Th}(\xi) \simeq D(\xi)/S(\xi) \simeq D(\xi) \simeq \mathbb{CP}^\infty$.[6] It is easy to see then that the $K^*_{\mathbb{C}}(\mathbb{CP}^\infty)$-module structure on $\mathrm{K}^*_{\mathbb{C}}(\mathrm{Th}(\xi))$ is just given by multiplication in $K^*_{\mathbb{C}}(\mathbb{CP}^\infty)$, and hence the Thom class $u_\xi \in \widetilde{K}^2_{\mathbb{C}}(\mathrm{Th}(\xi)) \simeq \widetilde{K}_{\mathbb{C}}(\mathbb{CP}^\infty)$ must be a generator $\pm(\xi - 1)$.

7. Characteristic classes

We now begin our quest to understand $J_{\mathbb{C}}(\mathbb{RP}^k)$. This will fall out of Adams's more general results on the J-homomorphism, namely, the construction of a certain quotient $J_\Lambda(X) \twoheadrightarrow J'_\Lambda(X)$, a "lower bound" on $J_\Lambda(X)$. We begin with a general discussion of characteristic classes, which are used to define the group $J'_\Lambda(X)$.

7.1. Let E and F be multiplicative cohomology theories. Consider the following data.

1. Let $\mathcal{V}$ be some class of vector bundles $\xi \longrightarrow X$ equipped with natural[7] Thom classes u_ξ in E and t_ξ in F.
2. Let $T \colon E \longrightarrow F$ be a natural transformation of cohomology theories.

Then for any $\xi \longrightarrow X$ in $\mathcal{V}$ we can form a *characteristic class*
$$\mathrm{cl}(T, \xi) := \psi_\xi^{-1} T \phi_\xi(1) = T(u_\xi)/t_\xi \in F^*(X),$$
where ϕ_ξ, ψ_ξ denote the Thom isomorphisms in E, F respectively. Note by naturality of our Thom classes, $\mathrm{cl}(T, \xi)$ is natural in bundle maps $f^*\xi \longrightarrow \xi$.

7.2. Example. If we take $E = F = H^*(-; \mathbb{Z}/2)$ in [7.1] then we have natural orientations on all vector bundles. It turns out that if we set T to be the Steenrod square Sq^i then the resulting characteristic class $\mathrm{cl}(\mathrm{Sq}^i, -)$ is just the Stiefel-Whitney class w_i.

7.3. Definition. Take $E = F = K^*_{\mathbb{C}}$ and $T = \psi^l$ the Adams operation for $l \in \mathbb{N}$ in [7.1]. The resulting classes $\rho^l := \mathrm{cl}(\psi^l, -)$ are called the *cannibalistic classes.*[8]

7.4. Remark. By [6.4] the cannibalistic classes ρ^l are defined on all complex vector bundles, and moreover satisfy the exponential property
$$\rho^l(\xi \oplus \eta) = \rho^l(\xi)\rho^l(\eta). \tag{7.5}$$

7.6. Convention. For the remainder, all bundles are complex vector bundles, and dimension always refers to complex dimension.

[6] I learned this argument from [May12].

[7] Here "natural" is used in the same sense as [6.4].

[8] Because they live in K-theory and also eat things in K-theory, and because Adams was awesome.

7.7. Lemma. Let $\xi \longrightarrow X$ be a line bundle. Then $\rho^l(\xi) = 1 + \xi + \cdots + \xi^{l-1}$ for $l \in \mathbb{N}$.

Proof. By naturality it suffices to prove this for the universal line bundle $\xi \longrightarrow \mathbb{CP}^\infty$. By [6.5], $\mathrm{Th}(\xi) \simeq \mathbb{CP}^\infty$ with Thom class $\pm(\xi - 1) \in \widetilde{K}_{\mathbb{C}}(\mathbb{CP}^\infty)$. Then by definition of ρ^l and ψ^l we have

$$\rho^l(\xi) = \frac{\psi^l(\pm(\xi-1))}{\pm(\xi-1)} = \frac{\xi^l - 1}{\xi - 1} = 1 + \xi + \cdots + \xi^{l-1}. \qquad \square$$

7.8. Remark. By the splitting principle, [7.5] and [7.7] imply that $\dim \rho^l(\xi) = l^{\dim \xi}$ for any bundle $\xi \longrightarrow X$.

7.9. Combining [7.5] and [7.7] gives $\rho^l(\epsilon_{\mathbb{C}}^{\oplus r}) = l^r$. This tells us that, after inverting l, we can extend the definition of ρ^l from bundles to all of $K_{\mathbb{C}}(X)$. Namely, since any element of $K_{\mathbb{C}}(X)$ can be written in the form $\xi - r$ for some bundle ξ, we can define

$$\rho^l \colon K_{\mathbb{C}}(X) \longrightarrow K_{\mathbb{C}}(X)_l, \quad \xi - r \longmapsto \rho^l(\xi)/l^r,$$

where $K_{\mathbb{C}}(X)_l$ denotes the localisation at l. Since $\xi - r = \eta - s \iff \xi + s = \eta + r$, this is obviously well defined. And note we have preserved the exponential property: $\rho^l(x+y) = \rho^l(x)\rho^l(y)$ for $x, y \in K_{\mathbb{C}}(X)$.

We next show how the cannibalistic classes ρ^l allows us to bound $J_{\mathbb{C}}(X)$.

7.10. Lemma. Suppose $J_{\mathbb{C}}(\xi) = J_{\mathbb{C}}(\eta)$ for two bundles $\xi \longrightarrow X$ and $\eta \longrightarrow X$. Then there exists $y \in \widetilde{K}_{\mathbb{C}}(X)$ such that $1 + y \in K_{\mathbb{C}}(X)$ is invertible and

$$\rho^l(\eta) = \rho^l(\xi) \cdot \frac{\psi^l(1+y)}{1+y} \quad \text{for all } l \in \mathbb{N}.$$

Proof. The hypothesis is that there is a fibre homotopy equivalence $f \colon S(\xi) \longrightarrow S(\eta)$ over X. This extends to a homotopy equivalence of Thom complexes $g \colon \mathrm{Th}(\xi) \longrightarrow \mathrm{Th}(\eta)$. On each fibre, g induces a map $g_p \colon \mathrm{Th}(\xi_p) \longrightarrow \mathrm{Th}(\eta_p)$ which is just (homotopic to) the suspension of $f_p \colon S(\xi_p) \longrightarrow S(\eta_p)$; since f_p is a homotopy equivalence so is g_p.

Let $v := \phi_\xi^{-1} g^* \phi_\eta(1) = g^*(u_\eta)/u_\xi \in K_{\mathbb{C}}(X)$. Since g_p is a homotopy equivalence,

$$g_p^*(u_{\eta_p}) = \pm u_{\xi_p} \quad \text{for } p \in X.$$

It follows that $\dim v = \pm 1$. Let $\epsilon := \dim v \in K_{\mathbb{C}}(X)$, so that $\dim \epsilon v = 1$, and hence $\epsilon v = 1 + y$ for some $y \in \widetilde{K}_{\mathbb{C}}(X)$.

Let $h \colon \mathrm{Th}(\eta) \longrightarrow \mathrm{Th}(\xi)$ be a homotopy inverse to g and define $w := \phi_\eta^{-1} h^* \phi_\xi(1) \in K_{\mathbb{C}}(X)$. Since ϕ_ξ, ϕ_η are isomorphisms of $K_{\mathbb{C}}(X)$-modules and pullback respects multiplication, we have

$$vw = \phi_\xi^{-1} g^* \phi_\eta(1) \cdot \phi_\eta^{-1} h^* \phi_\xi(1) = \phi_\eta^{-1} h^* \phi_\xi(\phi_\xi^{-1} g^* \phi_\eta(1)) = 1,$$

and symmetrically $wv = 1$. So w is inverse to v, and hence ϵw is inverse to ϵv, implying $1 + y$ is invertible.

Finally let $l \in \mathbb{N}$. By naturality of the Adams operation ψ^l we have

$$((\phi_\xi^{-1} g^* \phi_\eta)(\phi_\eta^{-1} \psi^l \phi_\eta))(1) = ((\phi_\xi^{-1} \psi^l \phi_\xi)(\phi_\xi^{-1} g^* \phi_\eta))(1).$$

Then by definition of ρ^l, multplicativity of ψ_l, and the fact that ϕ_ξ is an isomorphism of $K_{\mathbb{C}}(X)$-modules, we get

$$v \cdot \rho^l(\eta) = \phi_\xi^{-1}(\psi^l(v \cdot u_\xi)) = \phi_\xi^{-1}(\psi^l(u_\xi)) \cdot \psi^l(v) = \rho^l(\xi) \cdot \psi^l(v).$$

Now, multiplying this equation by $\epsilon = \psi^l(\epsilon)$ we get

$$(1+y)\cdot\rho^l(\eta) = \rho^l(\xi)\cdot\psi^l(1+y),$$

and since $1+y$ is invertible we are done. $\square$

7.11. Definitions. 1. Let $V_{\mathbb{C}}(X) \subseteq K_{\mathbb{C}}(X)$ be the subgroup of elements x for which there exists $y \in \widetilde{K}_{\mathbb{C}}(X)$ such that $1+y$ is invertible and

$$\rho^l(x) = \frac{\psi^l(1+y)}{1+y} \in K_{\mathbb{C}}(X)_l \quad \text{for } l \in \mathbb{N}. \tag{7.12}$$

That $V_{\mathbb{C}}(X)$ is in fact a subgroup follows from the fact that each ρ^l is exponential and each ψ^l is multiplicative.

2. Define $J'_{\mathbb{C}}(X) := K_{\mathbb{C}}(X)/V_{\mathbb{C}}(X)$.

7.13. Lemma. $V_{\mathbb{C}}(X) \subseteq \widetilde{K}_{\mathbb{C}}(X)$.

Proof. Suppose $x \in V_{\mathbb{C}}(X)$, and let y is as in [7.11]. Writing $x = \xi - \eta$ for bundles ξ and η, we must have

$$\rho^l(x) = \frac{\psi^l(1+y)}{1+y} \in K_{\mathbb{C}}(X)_l \implies l^r(1+y)\rho^l(\xi) = l^r\psi^l(1+y)\rho^l(\eta) \in K_{\mathbb{C}}(X).$$

The Adams operations preserve dimension (by the splitting principle and their definition), so $\dim\psi^l(1+y) = \dim(1+y)$. It follows that $\dim\rho^l(\xi) = \dim\rho^l(\eta)$, whence $\dim\xi = \dim\eta$ by [7.8]. Therefore $x \in \widetilde{K}_{\mathbb{C}}(X)$. $\square$

7.14. Proposition. The quotient map $K_{\mathbb{C}}(X) \longrightarrow J'_{\mathbb{C}}(X)$ factors through $J_{\mathbb{C}}$. That is, $\ker(J_{\mathbb{C}}) \subseteq V_{\mathbb{C}}(X)$, so $J'_{\mathbb{C}}(X)$ is a quotient of $J_{\mathbb{C}}(X)$.

Proof. Immediate from [7.10]. $\square$

8. Finishing

We now specialise to the case $X = \mathbb{RP}^k$. Computing $J'_{\mathbb{C}}(\mathbb{RP}^k)$ will give us enough information about $J_{\mathbb{C}}(\mathbb{RP}^k)$ to finally give an upper bound to our question [1.4] on vector fields.

8.1. Lemma. Let ν be the generator of $\widetilde{K}_{\mathbb{C}}(\mathbb{RP}^k)$ as in [5.2]. For $l \in \mathbb{N}$ odd we have

$$\rho^l(t\nu) = 1 + \frac{l^t - 1}{2l^t}\nu \quad \text{for } 0 \leq t < 2^{f_k}.$$

Proof. Let $\xi := \kappa(\gamma)$. By [7.7] and the identity $\xi^2 = 1$ we have

$$\rho^l(\xi) = 1 + \xi + \cdots + \xi^{l-1} = \frac{l+1}{2} + \frac{l-1}{2}\xi = l + \frac{l-1}{2}\nu.$$

The desired identity for $t = 1$ then follows from $\nu = \xi - 1 \implies \rho^l(\nu) = \rho^l(\xi)/l$. Then $t > 1$ follows by induction using the relation $\nu^2 = -2\nu$ (and $t = 0$ is trivial). $\square$

8.2. Lemma. $V_{\mathbb{C}}(\mathbb{RP}^k) \subseteq \{0, 2^{f_k - 1}\nu\}$.

Proof. Let $x \in V_{\mathbb{C}}(\mathbb{RP}^k)$, and let $y \in \widetilde{K}_{\mathbb{C}}(\mathbb{RP}^k)$ be such that [7.12] holds. By [5.2] we have $\psi^l(1+y) = 1+y$ for l odd, so we in fact have $\rho^l(x) = 1$ for l odd.

Next, $x \in \widetilde{K}_{\mathbb{C}}(X)$ by [7.13], so by [5.2] we can write $x = t\nu$ for some $0 \leq t < 2^{f_k}$. Then from [8.1] we know that

$$\rho^l(x) = \rho^l(t\nu) = 1 + \frac{l^t - 1}{2l^t}\nu.$$

Recall the ring morphism $\alpha\colon \widetilde{K}_{\mathbb{C}}(\mathbb{RP}^k) \longrightarrow \mathbb{Z}/2^{f_k+1}$ defined in [5.3] by $\alpha(\nu) := -2$. Observe this gives a morphism of multiplicative groups $\beta\colon 1+\widetilde{K}_{\mathbb{C}}(\mathbb{RP}^k) \longrightarrow (\mathbb{Z}/2^{f_k+1})^\times$ via $\beta(1+z) := 1+\alpha(z)$.

So finally let l be odd. Note $\widetilde{K}_{\mathbb{C}}(\mathbb{RP}^k) \simeq \mathbb{Z}/2^{f_k} \implies \widetilde{K}_{\mathbb{C}}(\mathbb{RP}^k)_l \simeq \widetilde{K}_{\mathbb{C}}(\mathbb{RP}^k)$. Thus from the above we get

$$1 = \rho^l(x) \implies 1 = \beta\left(1 + \frac{l^t - 1}{2l^t}\nu\right) = 1/l^t.$$

We now use the fact that $(\mathbb{Z}/2^{f_k+1})^\times \simeq \mathbb{Z}/2 \times \mathbb{Z}/2^{f_k-1}$ has an element of order 2^{f_k-1} to see that choosing l appropriately implies $2^{f_k-1} \mid t$, as desired. $\square$

8.3. Theorem. There does not exist a set of $\rho(n)+4$ linearly independent vector fields on S^{n-1}.

Proof. If n is odd this is trivial by the hairy ball theorem [1.3]. So assume $n = 2m$, and suppose S^{n-1} admits a set of k vector fields. Combining [4.8], [7.14], and [8.2] gives that $2^{f_k-1} \mid m \implies 2^{f_k} \mid n$. Then observe that $f_k \geq e_k - 1 \geq e_{k-4}$. By [2.3] it follows that $k-4 \leq \rho(n) - 1 \implies k \leq \rho(n)+3$. $\square$

So ends our journey.

8.4. Remark. Our final upper bound in [8.3] is off by 4 from the right answer [3.3]. If we were computer scientists, that would be good enough. As remarked above, that 4 goes away if one translates our work from complex K-theory into real K-theory. Indeed one can define ρ^l using the Adams operations in $K_{\mathbb{R}}$ rather than $K_{\mathbb{C}}$, and then define analogous groups $V_{\mathbb{R}}(X)$ and $J'_{\mathbb{R}}(X)$. But there is a bit of nastiness:

- Thom classes don't exist for all real vector bundles, so the analogue of [6.4] is more subtle. However, in [ABS64] natural Thom clases are also constructed for certain Spin bundles, and this is where one must begin.
- This means our proof of [7.7] won't go through in the real case, and indeed this implies that extending ρ^l to $K_{\mathbb{R}}(X)$ is much more technical than the simple discussion of [7.9].

These subtleties are handled in [Ada65], and the reward is the correct bound. The argument is essentially the same: one uses $J_{\mathbb{R}}(n\lambda) = 0$ from [4.8], and this time computes $V_{\mathbb{R}}(\mathbb{RP}^k) \simeq 0$, so $K_{\mathbb{R}}(\mathbb{RP}^k) \simeq J_{\mathbb{R}}(\mathbb{RP}^k) \simeq J'_{\mathbb{R}}(\mathbb{RP}^k)$. The crucial difference is that $\widetilde{K}_{\mathbb{R}}(\mathbb{RP}^k) \simeq \mathbb{Z}/2^{e_k}$, so now we get $2^{e_k} \mid n$ rather than $2^{f_k} \mid n$ when proving the upper bound [8.3]. Getting precisely e_k instead of the approximation f_k means that no pesky 4 will show up.

8.5. Remark. In addition to computing $J'_{\mathbb{R}}(\mathbb{RP}^k)$, Adams computes $J'_{\mathbb{R}}$ for spheres as well, by using another characteristic class coming from the formalism of [7.1]. This gives bounds on the image of the J-homomorphism for spheres, where the study of the J-homomorphism originated.

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