THE YONEDA LEMMA

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1. Introduction

This is going to be short. I feel like every so often I realise how magical the Yoneda lemma is. One of those realisations just occurred so let me write something down. I'll record the statement and proof of the lemma here too, for completeness.

1.1. Notation. We fix throughout a (locally small) category $\mathcal{C}$. Denote by $\mathrm{PSh}(\mathcal{C})$ the category of (contravariant) functors $\mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}$.

1.2. Definition. For $X \in \mathrm{obj}(\mathcal{C})$ we define the functor $h_X : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}$ by:

- $h_X(T) := \hom_{\mathcal{C}}(T, X)$ for $T \in \mathrm{obj}(\mathcal{C})$;
- $h_X(f) : h_X(T) \longrightarrow h_X(S)$ sends $g \in \hom_{\mathcal{C}}(T, X) \longmapsto g \circ f \in \hom_{\mathcal{C}}(S, X)$ for $f \in \hom_{\mathcal{C}}(S, T)$.

Observe also that any $f \in \hom_{\mathcal{C}}(X, Y)$ defines a natural transformation $h_f : h_X \longrightarrow h_Y$ by $g \in \hom_{\mathcal{C}}(T, X) \longmapsto f \circ g \in \hom_{\mathcal{C}}(T, Y)$. Thus we have defined a functor $h : \mathcal{C} \longrightarrow \mathrm{PSh}(\mathcal{C})$.

1.3. Lemma (Yoneda)**.** Let $F : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}$ any (contravariant) functor. Let $X \in \mathrm{obj}(C)$. Then there is a natural bijection

$$\hom_{\mathrm{PSh}(\mathcal{C})}(h_X, F) \simeq F(X)$$

between the natural transformations $h_X \longrightarrow F$ and the set $F(X)$.

Proof. Let $\alpha : h_X \longrightarrow F$ a natural transformation. Then for each $T \in \mathrm{obj}(\mathcal{C})$ and $f \in \hom_{\mathcal{C}}(T, X)$, by definition of h we have the commutative diagram

$$\begin{tikzcd}
\hom_{\mathcal{C}}(X, X) \arrow[r, "g \longmapsto g \circ f"] \arrow[d, "\alpha_x"] & \hom_{\mathcal{C}}(T, X) \arrow[d, "\alpha_T"] \\
F(X) \arrow[r, "F(f)"] & F(T).
\end{tikzcd}$$

Setting $g = \mathrm{id}_X$ gives us that $\alpha_T(f) = F(f)(\alpha_X(\mathrm{id}_X))$. It follows that

$$\hom_{\mathrm{PSh}(\mathcal{C})}(h_X, F) \longrightarrow F(X), \quad \alpha \longmapsto \alpha_X(\mathrm{id}_X)$$

gives the necessary bijection. □

1.4. Corollary. The functor $h : \mathcal{C} \longrightarrow \mathrm{PSh}(\mathcal{C})$ is fully faithful. In particular, $X \simeq Y$ if and only if $h_X \simeq h_Y$ for $X, Y \in \mathrm{obj}(\mathcal{C})$.

Proof. Setting $F := h_Y$ in the Yoneda lemma above gives us precisely that the map $\hom_{\mathcal{C}}(X, Y) \longrightarrow \hom_{\mathrm{PSh}(\mathcal{C})}(h_X, h_Y)$ induced by h is a bijection. □

1.5. Remark. In light of (1.4), we call h the *Yoneda embedding*.

2. Some magic

We review the notion of a fibre product, the discussion of which in [GW10] being what prompted these notes in the first place. Throughout we let $u : X \longrightarrow S$ and $v : Y \longrightarrow S$ two morphisms in $\mathcal{C}$.

2.1. Definition. The *fibre product of* X *and* Y *over* S is an object $X \times_S Y \in \mathrm{obj}(\mathcal{C})$ equipped with *projections* $p : X \times_S Y \longrightarrow X$ and $q : X \times_S Y \longrightarrow Y$ satisfying:

(a) $u \circ p = v \circ q$, and

(b) for any $T \in \operatorname{obj}(\mathcal{C})$ and morphisms $a : T \longrightarrow X$ and $b : T \longrightarrow Y$ such that $u \circ a = v \circ b$, there exists a unique morphism $a \times b : T \longrightarrow X \times_S Y$ such that $a = p \circ (a \times b)$ and $b = q \circ (a \times b)$.

This is encapuslated by the following diagram:

$$\begin{tikzcd}
T \arrow[dr, dashed, "\exists!\, a\times b"] \arrow[drr, bend left, "b"] \arrow[ddr, bend right, "a"'] & & \\
& X \times_S Y \arrow[r, "q"] \arrow[d, "p"] & Y \arrow[d, "v"] \\
& X \arrow[r, "u"] & S.
\end{tikzcd}$$

For the usual reasons, this characterises $X \times_S Y$ up to (unique) isomorphism.

One more bit of terminology: if we have the data in (2) such that c is moreover an isomorphism—that is, T equipped with a, b is in fact a fibre product of X and Y over S—we say the commutative square

$$\begin{tikzcd}
T \arrow[r, "b"] \arrow[d, "a"] & Y \arrow[d, "v"] \\
X \arrow[r, "u"] & S
\end{tikzcd}$$

is *cartesian*, or a *pullback diagram*.

2.2. Example. It is easy to see that fibre products exist in the category Set. Indeed, they're exactly what you expect them to be (check that this construction satisfies the universal property above):

$$X \times_S Y \simeq \{(x, y) \mid x \in X, y \in Y, u(x) = v(y)\}.$$

On the other hand, it can be much more difficult to understand fibre products in other categories in this concrete set-theoretic manner (e.g., the category of schemes, where fibre products are incredibly prevalent). Indeed, as with other universal constructions, it is often much easier to work with the fibre product abstractly. But here's another way to work with it: the Yoneda embedding magically lets us reduce things to the familiar category Set! Let's see what I mean.

2.3. Lemma. An object $Z \in \operatorname{obj}(\mathcal{C})$ equipped with $p : Z \longrightarrow X$ and $q : Z \longrightarrow Y$ is a fibre product of X and Y over S if and only if the diagram (in the category Set)

$$\begin{tikzcd}
\hom_{\mathcal{C}}(T, Z) \arrow[r, "h_q"] \arrow[d, "h_p"] & \hom_{\mathcal{C}}(T, Y) \arrow[d, "h_v"] \\
\hom_{\mathcal{C}}(T, X) \arrow[r, "h_u"] & \hom_{\mathcal{C}}(T, S).
\end{tikzcd}$$

is commutative and cartesian, i.e., for all $T \in \operatorname{obj}(\mathcal{C})$ we have

$$\hom_{\mathcal{C}}(T, Z) \simeq \hom_{\mathcal{C}}(T, X) \times_{\hom_{\mathcal{C}}(T,S)} \hom_{\mathcal{C}}(T, Y).$$

Proof. The condition that the diagram commutes is equivalent to the condition that $u \circ p = v \circ q$ since the Yoneda embedding is fully faithful. And, just by staring at things for long enough, we see that saying

$$\begin{aligned}
\hom_{\mathcal{C}}(T, Z) &\simeq \hom_{\mathcal{C}}(T, X) \times_{\hom_{\mathcal{C}}(T,S)} \hom_{\mathcal{C}}(T, Y) \\
&\simeq \{(a, b) \mid a : T \longrightarrow X, b : T \longrightarrow Y, u \circ a = v \circ b\}
\end{aligned}$$

is equivalent to saying that Z satisfies the necessary universal property. $\square$

2.4. Remark. The above lemma is a great example of a tautology, albeit wrapped in formalism. The point is that saying we have a cartesian square in $\mathcal{C}$ and saying the Yoneda embedding of the square is cartesian in Set are really precisely the same statement: to give a map into $X \times_S Y$ is to give maps into X and Y which induce the same map into S.

Ok, so that's what I mean when I say the Yoneda embedding reduces us to talking about sets. Let's see it in action (briefly).

2.5. Lemma. For all morphisms $S \longrightarrow T$ the diagram

$$\begin{tikzcd} X \times_S Y \arrow[r, "p \times q"] \arrow[d, "u \circ p = v \circ q"'] & X \times_T Y \arrow[d, "u \times v"] \\ S \arrow[r, "\mathrm{id}_S \times \mathrm{id}_S"] & S \times_T S \end{tikzcd}$$

is commutative and cartesian, assuming all the fibre products written exist. That is, $X \times_S Y \simeq S \times_{S \times_T S} (X \times_T Y)$.

Proof. We are reduced to the case that $\mathcal{C} = \text{Set}$ immediately by (2.3)! In this case, as you can check, our concrete description of the fibre product makes the claim obvious! $\square$

Observe that this last proof was two sentences, each ending with an exclamation point. I seriously feel like this Yoneda trick is magic here. Sure, one can argue completely abstractly using only the universal properties of all the fibre products written down—that's how I proved this the first time I saw it—but this is just so much cleaner and intuitive, I think!

2.6. Remark. As an aside, this last lemma is actually pretty useful. It implies for instance in the category of schemes that if $S \longrightarrow T$ is separated then, by stability under base change, $X \times_S Y \longrightarrow X \times_T Y$ (note the graph morphism is a special case of this) is a closed immersion.

References

[GW10] Ulrich Görtz and Torsten Wedhorn, *Algebraic Geometry I: Schemes with Examples and Exercises*, Springer, 2010.