INVERTIBLE MODULES

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1. Notation. Let A be a commutative ring.

2. Lemma. Let M be a finitely generated A-module. If $M_{\mathfrak{p}} \simeq 0$ for some $\mathfrak{p} \in \operatorname{Spec}(A)$, then $M_f \simeq 0$ for some $f \in A - \mathfrak{p}$.

Proof. Choose generators $x_1, \ldots, x_n \in M$. If $M_{\mathfrak{p}} \simeq 0$ then there are $f_1, \ldots, f_n \in A - \mathfrak{p}$ such that $f_i x_i = 0$ for $1 \leq i \leq n$. Taking $f := \prod_{i=1}^n f_i \in A - \mathfrak{p}$, we have $fM \simeq 0 \implies M_f \simeq 0$. $\square$

3. Lemma. Let M be a finitely generated A-module. Then M is locally free of finite rank if and only if $M_{\mathfrak{p}}$ is a finite free $A_{\mathfrak{p}}$-module for each $\mathfrak{p} \in \operatorname{Spec}(A)$.

Proof. The only if direction is obvious, so assume $M_{\mathfrak{p}}$ is a finite free $A_{\mathfrak{p}}$-module for each $\mathfrak{p} \in \operatorname{Spec}(A)$. Fixing $\mathfrak{p} \in \operatorname{Spec}(A)$, it suffices to show there exists $f \in A - \mathfrak{p}$ such that M_f is a free A_f-module. We have an isomorphism $A_{\mathfrak{p}}^n \xrightarrow{\sim} M_{\mathfrak{p}}$, which without loss of generality is the localisation of a morphism $\alpha \colon A^n \longrightarrow M$. It follows from (2) that since $\alpha_{\mathfrak{p}}$ is an isomorphim, so is α_f for some $f \in A - \mathfrak{p}$. $\square$

4. Definition. We say an A-module M is *invertible* if there exists an A-module N and an isomorphism $M \otimes_A N \simeq A$.

5. Lemma. An invertible A-module M is finitely generated.

Proof. Assume we have an A-module N and an isomorphism $\alpha \colon A \xrightarrow{\sim} M \otimes_A N$. Observe the image of α must be generated by a single element of $M \otimes_A N$, so α must factor through an isomorphism $\beta \colon A \xrightarrow{\sim} M_0 \otimes_A N$ for some finitely generated submodule $M_0 \subseteq M$. Then we have an isomorphism

$$\begin{tikzcd}
M \arrow[rrr, "\sim"] \arrow[d, "\wr"'] & & & M_0 \arrow[d, "\wr"] \\
M \otimes_A A \arrow[r, "\mathrm{id}\otimes\beta"] & M \otimes_A M_0 \otimes_A N \arrow[r, "\sim"] & M \otimes_A N \otimes_A M_0 \arrow[r, "\alpha^{-1}\otimes\mathrm{id}"] & A \otimes_A M_0.
\end{tikzcd}$$

$\square$

6. Lemma. An A-module M is invertible if and only if M is locally free of rank 1.

Proof. First assume M is invertible: there is an A-module N such that $M \otimes_A N \simeq A$. Fix any $\mathfrak{p} \in \operatorname{Spec}(A)$. By (3) it suffices to show $M_{\mathfrak{p}}$ is free of rank 1 over $A_{\mathfrak{p}}$. We have

$$M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} \simeq (M \otimes N)_{\mathfrak{p}} \simeq A_{\mathfrak{p}},$$

i.e. $M_{\mathfrak{p}}$ is invertible. Thus we are reduced to the case that A is a local ring. Let $\mathfrak{m} \in \operatorname{Spec}(A)$ denote the maximal ideal and $k := A/\mathfrak{m}$ the residue field. We have

$$(M \otimes_A k) \otimes_k (N \otimes_A k) \simeq M \otimes_A N \otimes_A k \simeq k,$$

i.e. $M \otimes_A k$ is an invertible k-module. But k is a field so obviously this implies $M/\mathfrak{m}M \simeq M \otimes_A k \simeq k$. Let $x \in M$ be a lift of a generator of $M/\mathfrak{m}M$ and $\alpha \colon A \longrightarrow M$ the map defined by $1 \longmapsto x$. By construction the induced map $\alpha \otimes k \colon k \longrightarrow M \otimes_A k$ is an isomorphism. Then Nakayama's lemma implies α is an isomorphism.

Conversely, assume M is locally free of rank 1. Let $M^\vee := \hom_A(M, A)$ denote the dual of M and consider the canonical evaluation map $M \otimes_A M^\vee \longrightarrow A$. By hypothesis this morphism locally looks like[1] evaluation $A \otimes_A A^\vee \longrightarrow A$, which is clearly an isomorphism. This finishes the proof. $\square$

7. Definition. It follows from (5) that the collection $\mathrm{Pic}(A)$ of isomorphism classes of invertible modules forms a set. Under tensor product $\mathrm{Pic}(A)$ forms an abelian group called the *Picard group* of A. Note that by the proof of (6) the inverse in $\mathrm{Pic}(A)$ of (the isomorphism class) of an invertible module L is given by (the isomorphism class of) the dual $L^\vee$.

8. Lemma. Let A_* be a graded ring. Let $\otimes$ denote $\otimes_{A_0}$. Assume there is an isomorphism of A_0-modules $A_1 \otimes A_{-1} \simeq A_0$. Then for $n \geq 1$ we have

$$A_n \simeq A_1^{\otimes n} \quad \text{and} \quad A_{-n} \simeq A_{-1}^{\otimes n} \simeq (A_1^\vee)^{\otimes n}.$$

Proof. We inductively prove the first isomorphism, the second is symmetric. The case $n = 1$ is tautological. Let $n > 1$. Multiplication gives a map $A_1^{\otimes n} \longrightarrow A_n$. An inverse map is given by

$$A_n \simeq A_n \otimes A_0 \simeq A_n \otimes A_{-1} \otimes A_1 \longrightarrow A_{n-1} \otimes A_1 \simeq A_1^{\otimes n},$$

the last isomorphism coming from the inductive hypothesis. $\square$

[1] I'm also using that locally free implies finitely presented, which implies that dualising will commute with localisation.