Math 216A Homework 4

Finish reading §3 of Chapter 2, and do Exercises 3.1, 3.2, and 3.3 there.

The aim of (most of) the rest of this assignment is to work out the precise relationship between classical varieties and certain schemes. Let k be an algebraically closed field.

For a reduced k-algebra A of finite type, we define a locally ringed space $\mathrm{MaxSpec}(A)$ as follows. The underlying topological space is the set of maximal ideals of A with the usual Zariski topology (closed sets are those sets of maximal ideals which contain a given ideal), so we have the usual Nullstellensatz correspondence between closed sets and radical ideals in A. As for the sheaf, we can mimic the $\mathcal{B}$-sheaf construction from Liu's book, so $\mathrm{MaxSpec}(A)$ has the structure of a ringed space of k-algebras. This is intrinsic to the k-algebra A.

Define an *abstract algebraic set* to be a ringed space of k-algebras locally isomorphic (as a ringed space of k-algebras) to ones of the form $\mathrm{MaxSpec}(A)$ for reduced k-algebras A of finite type (we don't need to include the word 'locally' for our ringed spaces, as this is automatic after the fact, since all residue fields wind up being isomorphic to k).

Your task is to prove the following generalization of Hartshorne's Proposition 2.6 in Chapter 2 (which imposed irreducibility hypotheses).

Theorem. *There is an equivalence of categories between abstract algebraic sets over k and reduced k-schemes which are locally of finite type.*

1*. Let $i : X \to Y$ be a continuous map of topological spaces such that $U \rightsquigarrow i^{-1}(U)$ sets up a bijection between open sets in Y and open sets in X. Show that the functors $\mathcal{F} \rightsquigarrow i^{-1}(\mathcal{F})$, $\mathcal{G} \rightsquigarrow i_*(\mathcal{G})$ give up an equivalence of categories between sheaves of abelian groups (rings, sets, etc.) on X and Y (i.e., construct natural isomorphisms of functors $f : i^{-1}i_* \simeq \mathrm{id}_X$ and $g : i_*i^{-1} \simeq \mathrm{id}_Y$ so that $i_*(f) = g \circ i_*$ and $i^{-1}(g) = f \circ i^{-1}$).

2*. Let X be a reduced k-scheme which is locally of finite type. Define $T(X)$ to be the topological space consisting of the closed points in X, with $i : T(X) \to X$ the inclusion. Define $\mathcal{O}_{T(X)} = i^{-1}(\mathcal{O}_X)$. Show that Exercise 1 applies to the present setting and that $(T(X), \mathcal{O}_{T(X)})$ is an abstract algebraic set over k. If $X = \mathrm{Spec}(A)$, show that $T(X)$ is isomorphic to $\mathrm{MaxSpec}(A)$.

3*. For any topological space Y, define the set $t(Y)$ to be the set of irreducible closed sets in Y (so $t(\emptyset) = \emptyset$). Define a *closed* set in $t(Y)$ to be a set of the form $t(Z)$ for a closed set Z in Y.

(*i*) Check that this makes $t(Y)$ a topological space, and t is a covariant functor from the category of topological spaces to itself. Moreover, if $f : U \to Y$ is an open embedding of topological spaces, check that $t(f) : t(U) \to t(Y)$ is also an open embedding (and likewise for closed embeddings).

(*ii*) Check that the canonical map $i : Y \to t(Y)$ defined by $i(y) = \overline{\{y\}}$ is continuous and that Exercise 1 applies to i.

(*iii*) Let Y be an abstract algebraic set. Show that $i : Y \to t(Y)$ gives a homeomorphism of Y onto the set of closed points in $t(Y)$. Define $\mathcal{O}_{t(Y)} = i_*(\mathcal{O}_Y)$, a sheaf of *reduced* k-algebras, so $t(Y)$ is now regarded as a ringed space of k-algebras. Prove that $t(Y)$ is a reduced k-scheme, locally of finite type over k. In fact, if Y is $\mathrm{MaxSpec}(A)$, then construct an isomorphism of locally ringed spaces of k-algebras $t(Y) \simeq \mathrm{Spec}(A)$.

4*. Show that t and T are naturally *functors* between the categories of locally finite type reduced k-schemes and abstract algebraic sets over k. Construct explicit *natural* isomorphisms $F_X : X \simeq tT(X)$ and $G_Y : Y \simeq Tt(Y)$ given on underlying sets by $x \mapsto \overline{\{x\}}$ and $y \mapsto T(\{y\})$ respectively. (Exercise 1 will be helpful for getting the sheaf maps). Be sure to check that $T(F_X) = G_{T(X)}$ and $t(G_Y) = F_{t(Y)}$, so F and G are truly 'inverses'. Keep in mind that in both categories, maps are completely determined by their effect on the underlying topological spaces, so this makes some compatibility checks painless.

5*. It is a very important fact that nearly all geometric facts about reduced schemes locally of finite type over k can be formulated completely within the framework of abstract algebraic sets; in terms of commutative algebra, this comes down to the fact that k-algebras of finite type are Jacobson rings and have all residue fields at maximals equal to k. For example, check:

(*i*) If $f : X \to Y$ is a morphism of abstract algebraic sets over k, then f is a closed immersion (i.e., f is topologically a closed embedding and $\mathcal{O}_Y \to f_*(\mathcal{O}_X)$ is surjective) if and only if $t(f)$ is a closed immersion of schemes. Likewise for open immersions.

(*ii*) Let Y be any affine abstract algebraic set, so $\mathcal{O}_{t(Y)}(t(Y)) = \mathcal{O}_Y(Y)$ is a finite type reduced k-algebra. Using the universal mapping property of affine schemes, there is a natural map of k-schemes $t(Y) \to \mathrm{Spec}(\mathcal{O}_Y(Y))$. Show this is an isomorphism (this is the precise isomorphism which is always implicit when passing between affine abstract algebraic sets and k-schemes, but no one bothers to mention it).

Next, let $\mathbf{A}_k^n$, $\mathbf{P}_k^n$ denote the usual abstract algebraic sets over k ($\mathrm{MaxSpec}(k[X_1, \ldots, X_n])$ for the former, the latter defined by gluing affine spaces along open subsets in the usual way). Show that there is a unique isomorphism of k-schemes

$$t(\mathbf{P}_k^n) \simeq \mathrm{Proj}(k[T_0, \ldots, T_n])$$

which is compatible with the $n+1$ standard open immersions of $t(\mathbf{A}_k^n) \simeq \mathrm{Spec}(k[X_1, \ldots, X_n])$ into each side. More generally, let $Y \subseteq \mathbf{P}_k^n$ be a closed subset with homogenous coordinate ring $S(Y)$. Give Y its canonical structure of abstract algebraic set. Show that there is a unique isomorphism of k-schemes $t(Y) \simeq \mathrm{Proj}(S(Y))$ compatible with the natural maps from each to $\mathrm{Proj}(k[T_0, \ldots, T_n])$.

(*iv*) Let X be a reduced scheme locally of finite type over k. Show that $\dim X = \dim t(X)$ and X is irreducible if and only if $t(X)$ is irreducible. In fact, show that the functor t on topological spaces sets up a bijection between irreducible closed sets in X and $t(X)$. Finally, check that X is quasi-compact if and only if $t(X)$ is.

Much of the above carries over to an arbitrary field k, and with the reducedness hypothesis removed. In essence, working with a scheme locally of finite type over a field k is 'equivalent' to working with the underlying set of closed points and the restricted structure sheaf (whose residue fields are now merely finite extensions of k). This is theoretically clumsy. From now on we work exclusively with schemes. But keep the classical picture in mind; it is very very important.

6. Let X be a scheme, Y a closed subset (let $i : Y \to X$ denote the inclusion).

(i) For open U in X, define $\mathcal{I}_Y(U)$ to be the ideal of all $f \in \mathcal{O}_X(U)$ which vanish on $Y \cap U$ (i.e., $f(y) = 0$ in $\kappa(y)$ for all $y \in Y \cap U$). Prove that $\mathcal{I}_Y$ is a sheaf of ideals in $\mathcal{O}_X$ on X and if we define $\mathcal{O}_Y = i^{-1}(\mathcal{O}_X/\mathcal{I}_Y)$, then $(Y, \mathcal{O}_Y)$ is a *reduced scheme*. In particular, every closed *subset* of a scheme X can be realized as the underlying space of a *closed subscheme* of X (the correct analogue in analytic geometry is deep).

(ii) Check that $i : Y \to X$ and $i^\# : \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I}_Y \simeq i_*(\mathcal{O}_Y)$ is a morphism of schemes $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ and that this is 'universal' in the sense that for any *reduced* scheme Z,

$$\mathrm{Hom}(Z, Y) \to \{f \in \mathrm{Hom}(Z, X) \mid f(Z) \subseteq Y \text{ as sets}\}$$

is a bijection. We say that $(Y, \mathcal{O}_Y)$ is the *reduced induced structure* on Y. If $Y = X$, this is just X_{red}.

This shows that closed subsets of X can be identified with certain 'radical' ideal sheaves. We'll later describe precisely which ideal sheaves arise in this way.

7. Let $\mathbf{C}(\!(T)\!)$ denote the field of formal Laurent series over $\mathbf{C}$, which is to say the fraction field of the complete discrete valuation ring $\mathbf{C}[\![T]\!]$. Since this complete dvr has a coefficient field which is algebraically closed of characteristic 0, it follows from the theory of local fields (e.g., see Serre's book *Local Fields*) that the finite extensions of $\mathbf{C}(\!(T)\!)$ are precisely fields of the form $\mathbf{C}(\!(T^{1/n})\!)$. Prove that $\mathbf{C}(\!(T)\!)$ has infinite transcendence degree over $\mathbf{C}$. Choose a transcendence basis $\{Y_j\}$. Prove that the (separable) algebraic extension $\mathbf{C}(\!(T)\!)/\mathbf{C}(\{Y_j\})$ has infinite degree. Show that $\mathbf{C}(\!(T)\!) \otimes_{\mathbf{C}} \mathbf{C}(\!(T)\!)$ is not a noetherian ring. But prove this beast *is* a domain (an earlier commutative algebra homework will be useful here).

Conclude that $\mathbf{C}[\![T]\!] \otimes_{\mathbf{C}} \mathbf{C}[\![T]\!]$ is not a noetherian ring, so the natural map

$$\mathbf{C}[\![T]\!] \otimes_{\mathbf{C}} \mathbf{C}[\![T]\!] \to \mathbf{C}[\![U, V]\!]$$

is not an isomorphism. (Can you rigorously prove in a more direct manner that this is not an isomorphism?)