Math 216A Homework 2

Read §2 of Chapter 2 as well as the important Remark 2.6 in §2.2 of Qing Liu's book "Algebraic Geometry and Arithmetic Curves", and the application of that Remark in Prop. 3.1 of §2.3.1 of Liu's book to give a more elegant construction of the structure sheaf on $\mathrm{Spec}(A)$ than is done in Hartshorne's book.

See the Proj handout (or §2.3.3 of Liu's book) for a nicer "gluing" approach to Hartshorne's construction of $\mathrm{Proj}(S)$. In HW4 you will develop a more elegant version of Prop. 2.6 in Hartshorne's Chapter 2, dropping the irreducibility hypotheses there.

Also work on the exercises below from §1 of Chapter 2 (each person will write up the solution to one asterisked problem to share with the others).

Remark. If you already have experience with sheaves of abelian groups (the only type Hartshorne considers) – often called "abelian sheaves" – then think about how the ideas and exercises in §1 would adapt to the consideration of sheaves of sets (a viewpoint that is important in étale cohomology and beyond).

Ch 2: 1.2*, 1.5, 1.6*, 1.7, 1.8, 1.9, 1.10, 1.11, 1.12, 1.15 1.16* (need Zorn's Lemma for (b)), 1.18*, 1.22* (give a universal mapping property of the gluing using the categories of sheaves of abelian groups on X and the U_i's; also works for sheaves of sets, rings, etc.).

For 1.18, note that the map is not just a bijection of Hom sets, but is compatible with the additive group structure on the Hom sets as well. Also, be sure to check suitable functoriality for a composite $g \circ f$, using the functorial properties of f^{-1} and f_* with respect to composites as you should precisely formulate. Finally, if $\varphi : f^{-1}\mathcal{G} \to \mathcal{F}$ corresponds to $\psi : \mathcal{G} \to f_*(\mathcal{F})$, check that for all $P \in X$, the diagram

$$\begin{array}{ccc} (f^{-1}\mathcal{G})_P & \overset{\varphi_P}{\to} & \mathcal{F}_P \\ \simeq\downarrow & & \uparrow \\ \mathcal{G}_{f(P)} & \overset{\psi_{f(P)}}{\to} & (f_*\mathcal{F})_{f(P)} \end{array}$$

commutes (the left column was defined in class)!

The upshot of this is that Exercise 1.18 might be a bit time-consuming, but it is absolutely essential in life to verify that the isomorphisms you prove have various functorial properties, for otherwise these isomorphisms would not be as useful (later experience will convince you of this). In practice, one tries to check as much of this as possible, so one doesn't need to recheck it later. But it is hard without experience to predict what functorialities will be needed later, so just check as much as you can think of, and if something is needed later which you have not verified before, then check it!

Mathematics is not about checking commutative diagrams, but only practice with checking these things will enable you to handle more complicated diagrams that will arise later on, where it might not be so trivial to check the necessary compatibilities. Consistency among possibly quite different-looking constructions can be extremely important; develop good habits early on to understand such things.