MATH 216A HOMEWORK 8

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§1 Reducing valuative criteria to DVRs

1.1 Let $(\mathscr{O}, \mathfrak{m})$ be a noetherian local domain with fraction field K.

1.1.1 Lemma — Let $\{x_1, \dots, x_n\}$ be a set of generators for $\mathfrak{m}$. Then there exists $i \in \{1, \dots, n\}$ such that if we define $\mathscr{O}' := \mathscr{O}[x_1/x_i, \dots, x_n/x_i] \subseteq K$ then the principal ideal (x_i) of $\mathscr{O}'$ is not the unit ideal.

Proof — Choose a valuation ring $(A, \mathfrak{m}_A)$ of K dominating $(\mathscr{O}, \mathfrak{m})$ (Matsumura, Theorem 10.2), and let $v \colon K^\times \longrightarrow \Gamma$ be the associated valuation. Recall this means

$$A = \{x \in K^\times : v(x) \geq 0\} \cup \{0\}, \quad \mathfrak{m}_A = \{x \in K^\times : v(x) > 0\} \cup \{0\},$$

so since A dominates $\mathscr{O}$ we know $v(x_1), \dots, v(x_n) > 0$. We choose $i \in \{1, \dots, n\}$ such that $v(x_i)$ is minimal among $v(x_1), \dots, v(x_n)$. Then clearly $\mathscr{O}' \subseteq A$. And so the intersection $\mathfrak{m}_A \cap \mathscr{O}'$ is a prime ideal, in particular is not the unit ideal. Since $x_1 \in \mathfrak{m}_A$, the claim follows. □

1.1.2 Lemma — There is a noetherian local domain $(\mathscr{O}'', \mathfrak{m}'') \subseteq K$ of dimension 1 dominating $(\mathscr{O}, \mathfrak{m})$.

Proof — Let $\{x_1, \dots, x_n\}$ be a set of generators for $\mathfrak{m}$. Choose $i \in \{1, \dots, n\}$ and define $\mathscr{O}'$ as in (1.1.1). Then $(x_i) \subseteq \mathscr{O}'$ is not the unit ideal so we may choose a minimal prime ideal $\mathfrak{p}$ containing it. Define $(\mathscr{O}'', \mathfrak{m}'') := (\mathscr{O}'_\mathfrak{p}, \mathfrak{p}\mathscr{O}'_\mathfrak{p})$. This is evidently a local domain, and is noetherian since it is a localization of $\mathscr{O}'$, which is noetherian by construction since $\mathscr{O}$ is. It is dimension 1 by the hauptidealsatz. Finally, $\mathfrak{m}'' \cap \mathscr{O}$ is a prime ideal in $\mathscr{O}$ evidently containing $x_1, \dots, x_n$, and hence must be $\mathfrak{m}$, so indeed this dominates $\mathscr{O}$. □

1.1.3 Proposition — Let L be a finitely generated field extension of K. Then there exists a discrete valuation ring $(R, \mathfrak{m}_R)$ of L dominating $(\mathscr{O}, \mathfrak{m})$.

Proof — First observe that we may replace $(K, \mathscr{O}, \mathfrak{m})$ with any $(K', \mathscr{O}', \mathfrak{m}')$ where K' is an intermediate extension of L/K and $(\mathscr{O}', \mathfrak{m}')$ is a noetherian local domain in K' dominating $(\mathscr{O}, \mathfrak{m})$. This gives us two reductions:

- If L is an extension of a transcendental intermediate field $K' = K(t)$, then we may take $\mathscr{O}' = \mathscr{O}[t]$ and $\mathfrak{m}'$ to be any maximal ideal containing $\mathfrak{m}\mathscr{O}'$. This reduces us to the case that L is a finite extension of K.
- By (1.1.2) we are reduced us to the case that $\mathscr{O}$ is dimension 1.

Now let $\widetilde{\mathscr{O}}$ be the integral closure of $\mathscr{O}$ in L. By Krull-Akizuki (Matsumura, Corollary to Theorem 11.7), $\widetilde{\mathscr{O}}$ is a Dedekind domain. Thus if we take $\widetilde{\mathfrak{m}}$ to be any maximal ideal of

$\widetilde{\mathscr{O}}$ containing $\mathfrak{m}\widetilde{\mathscr{O}}$, the localization $(R, \mathfrak{m}_R) := (\widetilde{\mathscr{O}}_{\widetilde{\mathfrak{m}}}, \widetilde{\mathfrak{m}}\widetilde{\mathscr{O}}_{\widetilde{\mathfrak{m}}})$ will be a DVR. The intersection $\mathfrak{m}_R \cap \mathscr{O}$ is a prime ideal containing $\mathfrak{m}$ and hence must be $\mathfrak{m}$, so R indeed dominates $\mathscr{O}$. □

1.2 We now explain how in the noetherian setting we may use (1.1.3) to reduce the checking of valuative criteria to the case of discrete valuation rings.

1.2.1 PROPOSITION — Let X be a locally noetherian scheme. Let $f\colon X \longrightarrow Y$ be a map locally of finite type. Then f is separated if and only if f satisfies the valuative criterion for separatedness for discrete valuation rings.

PROOF — The "only if" direction still follows from the usual valuative criterion, so we just have to show the "if" direction. Here we just have to slightly modify the proof of Hartshorne, Theorem II.4.3:

- Firstly note that we only need X locally noetherian for the diagonal

$$\Delta\colon X \longrightarrow X \times_Y X$$

  to be quasicompact.

- Secondly, note that since f is locally of finite type, so is the base change

$$f'\colon X \times_Y X \longrightarrow X;$$

  and since X is locally noetherian this implies $X \times_Y X$ is locally noetherian. Thus, given $\xi_1 \in \Delta(X)$ and a specialization $\xi_0 \in X \times_Y X$, (1.1.3) now allows us to choose a *discrete* valuation ring R of $K := \kappa(\xi_1)$ dominating the *noetherian* local ring $\mathscr{O}$ of ξ_0 in the reduced induced scheme structure on $\overline{\{\xi_1\}} \subseteq X \times_Y X$. □

1.2.2 LEMMA — Let Y be a noetherian scheme. Let $f\colon X \longrightarrow Y$ be a separated map of finite type. Then the following are equivalent:

(*a*) f is proper;

(*b*) for any finite type map $g\colon Y' \longrightarrow Y$, the base change $f'\colon X' := X \times_Y Y' \longrightarrow Y'$ is closed;

(*c*) for $n \geq 1$, the base change $f'\colon \mathbf{A}^n_X \simeq X \times_Y \mathbf{A}^n_Y \longrightarrow \mathbf{A}^n_Y$ is closed.

PROOF — Note that since we're assuming f is separated and of finite type, (*a*) is equivalent to f being universally closed. Observe also that all three conditions are local on Y, so we may assume Y is an affine scheme $\operatorname{Spec} A$ with A a noetherian ring.

The implications (*a*) ⇒ (*b*) and (*b*) ⇒ (*c*) are tautological.

(*c*) ⇒ (*b*): Since (*b*) is local also on Y', we may assume Y' is an affine scheme $\operatorname{Spec} B$ of finite type over $Y = \operatorname{Spec} A$. In this case we may choose a closed immersion $Y' \hookrightarrow \mathbf{A}^n_Y$, from which we get a commutative diagram

$$\begin{tikzcd} X' \arrow[r, hook] \arrow[d] & \mathbf{A}^n_X \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r, hook] & \mathbf{A}^n_Y \arrow[r] & Y. \end{tikzcd}$$

By hypothesis the middle vertical map is closed. Since the upper left and lower left maps are closed immersions, the left vertical map f' is also closed, as desired.

(b) $\Rightarrow$ (a): By Chow's lemma we may find a surjective projective map $p\colon X' \longrightarrow X$ such that $p' := fp\colon X' \longrightarrow Y$ is quasiprojective. The latter means we may factor p' as a composite of an immersion $j\colon X' \hookrightarrow X''$ and a projective map $p''\colon X'' \longrightarrow Y$. To show f is proper it suffices to show p' is proper (since X is the image of X' in p, and the image of a proper map is proper (Hartshorne, Exercise II.4.4)). It would therefore suffice to show j is closed, hence in fact a closed immersion. To see this, we factor j as the composite

$$X' \xrightarrow{\Gamma_j} X'' \times_Y X' \xrightarrow{(\mathrm{id}_{X''}, p)} X'' \times_Y X \xrightarrow{f''} X'',$$

where Γ_j is the graph of j and f'' is the base change of f along $p''\colon X'' \longrightarrow X$. Since j is immersion, hence separated, Γ_j is closed; since p is projective so is $(\mathrm{id}_{X''}, p)$, which is hence closed; and since p'' is projective, in particular finite type, the base change f'' is closed by hypothesis. Thus the composite j is closed, as desired. □

1.2.3 Proposition — Let $f\colon X \longrightarrow Y$ be a finite type map of noetherian schemes. Then f is proper if and only if f satisfies the valuative criterion for properness for discrete valuation rings.

Proof — As in (1.2.1) the "only if" direction is automatic from the usual valuative criterion and we want to show the "if direction", so assume f satisfies the valuative criterion for discrete valuation rings.

By (1.2.1) we know that f is separated, so we need only show that f is universally closed. By (1.2.2) it suffices to check for finite type maps $g\colon Y' \longrightarrow Y$ that the base change $f'\colon X' := X \times_Y Y' \longrightarrow Y'$ is closed; but if g is of finite type then Y' is also locally noetherian, and we know the valuative criterion is stable under base change, so the map $f'\colon X' \longrightarrow Y'$ satisfies the same hypotheses as f. This reduces to checking that f is closed.

We may now follow the proof of the usual valuative criterion (say the one in Brian's notes). We first reduce to checking that $f(X)$ is stable under specialization. We then have to show for any $x \in X$ and specialization $y_0 \in Y$ of $y := f(x)$ that there is an $x_0 \in X$ with $f(x_0) = y_0$. Let $\mathcal{O}$ be the local ring at y_0 in the reduced induced subscheme structure on $\overline{\{y\}}$. By noetherianess of Y, this is a local noetherian domain with fraction field $\kappa(y)$. Since f is of finite type, $\kappa(x)$ is a finitely generated field extension of $\kappa(y)$, and hence we may use (1.1.3) to find a discrete valuation ring A with fraction field $\kappa(x)$ and a map $\operatorname{Spec} A \longrightarrow Y$ sending the generic point to y_0 and the closed point to y. But by the valuative criterion the canonical map $\operatorname{Spec} \kappa(x) \longrightarrow X$ lifts to a map $\operatorname{Spec} A \longrightarrow X$, and the image of the closed point in this map will be the desired x_0 lying over y_0. □