THE REGULAR REPRESENTATION

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1. Finite groups

Let G be a finite group. Let k be an algebraically closed field such that $\mathrm{char}(k)$ does not divide G. All vector spaces and representations in this section are over k.

1.1. Definition. Consider the vector space $\mathrm{Fun}(G)$ of functions from $\phi\colon G \to k$. This has the structure of a $G \times G$-representation by letting $(g, h) \in G \times G$ act on $\phi\colon G \to k$ via the formula

$$((g, h) \cdot \phi)(x) = \phi(g^{-1}xh).$$

This is perhaps what should be called the *regular representation* associated to G. Well, note that it's a representation of $G \times G$, but it then naturally induces G-representation structures on $\mathrm{Fun}(G)$:

(1.1.1) restricting in the map $(\mathrm{id}_G, 1)\colon G \to G \times G$ sending $g \mapsto (g, 1)$ gives us the *left regular representation*, where $(g \cdot \phi)(x) = \phi(g^{-1}x)$;

(1.1.2) restricting in the map $(1, \mathrm{id}_G)\colon G \to G \times G$ sending $g \mapsto (1, g)$ gives us the *right regular representation*, where $(g \cdot \phi)(x) = \phi(xg)$.

Unless otherwise stated we consider $\mathrm{Fun}(G)$ as a G-representation using the left regular representation structure.

1.2. Lemma. Let V be a representation of G. Then there is a natural isomorphism (of vector spaces)

$$\mathrm{Hom}_G(V, \mathrm{Fun}(G)) \xrightarrow{\sim} V^\vee,$$

given by sending $T\colon V \to \mathrm{Fun}(G)$ to the linear functional $v \mapsto T(v)(1)$.

Proof. Trivial. □

1.3. Corollary. Let $\{V_i\}_{i \in I}$ be a set of representatives for the isomorphism classes of irreducible representations of G. Then there is an isomorphism of G-representations

$$\mathrm{Fun}(G) \simeq \bigoplus_{i \in I} V_i^\vee \otimes V_i$$

(where here $V_i^\vee$ is viewed just as a vector space, i.e. a trivial G-representation). In particular, taking dimensions of each side, we get

$$|G| = \sum_{i \in I} \dim(V_i)^2.$$

Proof. For any representation V we have by semisimplicity and Schur's lemma that

$$V \simeq \bigoplus_{i \in I} \mathrm{Hom}_G(V_i, V) \otimes V_i.$$

The claim then follows from plugging in $V = \mathrm{Fun}(G)$ and applying (1.2). □

2. Compact Lie groups

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