MATH 216A HOMEWORK 9

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§1 Geometric connectedness

1.1 Let $\pi\colon Y \longrightarrow X$ be a map of topological spaces. Let $\mathscr{C}_X, \mathscr{C}_Y$ denote the sets of connected components in X, Y. Note that since images of connected spaces are connected, π induces a map of sets $\pi_*\colon \mathscr{C}_Y \longrightarrow \mathscr{C}_X$.

1.1.1 Lemma — Suppose that for any connected component C of X the preimage $\pi^{-1}(C)$ is connected. Then $\pi_*\colon \mathscr{C}_Y \longrightarrow \mathscr{C}_X$ is a bijection.

Proof — In this case, taking preimages defines a two-sided inverse $\pi^{-1}\colon \mathscr{C}_X \longrightarrow \mathscr{C}_Y$ to π_*. □

1.1.2 Lemma — Suppose that:

(*a*) π is open or closed;

(*b*) the fibers of π are connected (in particular nonempty, so π is surjective).

Then $\pi_*\colon \mathscr{C}_Y \longrightarrow \mathscr{C}_X$ is a bijection.

Proof — By (1.1.1) it suffices to show that the preimage of any connected component C of X is connected. Thus we may reduce to the case that X is connected, where we want to show Y is connected.

Suppose not: then we may write $Y = A \amalg B$ with A, B nonempty and clopen subspaces of Y. Since π is surjective we have $X = \pi(A) \cup \pi(B)$. Since X is connected, and π is open or closed, $\pi(A)$ and $\pi(B)$ cannot be disjoint; thus we may choose $x \in \pi(A) \cap \pi(B)$. That means the fiber Y_x meets both A and B, and hence $Y_x = (A \cap Y_x) \amalg (B \cap Y_x)$ is disconnected. This proves the claim. □

1.2 Proposition — Let X be a scheme over an algebraically closed field k. Let K/k be an extension field, and let $X_K := X \times_k K$ be the base-change. Then the projection $\pi\colon X_K \longrightarrow X$ induces a bijection on connected components; in particular, X_K is connected if and only if X is connected.

Proof — By exercise 2(iv) the map $\pi\colon X_K \longrightarrow X$ is open. So by (1.1.2) it suffices to show the fibers of π are connected.

Pick any point $x \in X$, and consider the canonical map $\mathrm{Spec}(\kappa(x)) \longrightarrow X$. The fiber of X_K over x is given by

$$X_K \times_X \mathrm{Spec}(\kappa(x)) \simeq \mathrm{Spec}(K) \times_{\mathrm{Spec}(k)} X \times_X \mathrm{Spec}(\kappa(x)) \simeq \mathrm{Spec}(K \otimes_k \kappa(x)).$$

In homework 1 we proved that, since k is algebraically closed, $K \otimes_k \kappa(x)$ is a domain; hence its spectrum is connected. □

1.3 Proposition — Let X be a scheme over a field k. The following are equivalent:

(a) The base-change $X_K := X \times_k K$ is connected for every extension K/k.

(b) The base-change $X_{\overline{k}} := X \times_k \overline{k}$ is connected, where $\overline{k}$ is any algebraic closure of k.

Proof — (a) $\Rightarrow$ (b): Tautological.

(b) $\Rightarrow$ (a): Let K/k be any extension. Let $\overline{k}$ and $\overline{K}$ be algebraic closures of k and K. The canonical map

$$X_{\overline{K}} := X \times_k \overline{K} \simeq X_K \times_K \overline{K} \longrightarrow X_K$$

is surjective, so it suffices to show $X_{\overline{K}}$ is connected. We may embed $\overline{k}$ as a subextension of $\overline{K}/k$, and then we're done by (1.2). □

1.4 Proposition — Let X be a scheme over a field k. Suppose X has a k-rational point. Let K/k be an extension field. If X is connected, then the base change $X_K := X \times_k K$ is connected.

Proof — By (1.3) we may reduce to the case $K = \overline{k}$. Suppose $X_K = X_{\overline{k}}$ is not connected, so we may write $X_K = A \amalg B$ for A, B nonempty clopen subspaces. Let $\pi \colon X_K \longrightarrow X$ be the projection. Let $x \colon \operatorname{Spec} k \longrightarrow X$ be a k-rational point, and observe that the fiber $(X_K)_x$ is

$$X_K \times_X \operatorname{Spec}(k) \simeq \operatorname{Spec}(K) \times_{\operatorname{Spec}(k)} X \times_X \operatorname{Spec}(k) \simeq \operatorname{Spec}(K),$$

hence is a single point $x' \in X_K$. Without loss of generality suppose $x' \in A$, so $x' \notin B$, and hence (by the above computation of the fiber) $\pi(B)$ does not contain x and hence is a proper nonempty subspace of X. But now, as $\operatorname{Spec}(K) = \operatorname{Spec}(\overline{k}) \longrightarrow \operatorname{Spec}(k)$ is an integral map, so is its base change $\pi \colon X_K \longrightarrow X$, so π is closed. And by exercise 2(iv), π is open. Thus B being clopen implies $\pi(B)$ is clopen, and hence X is also not connected. □

1.5 Proposition — Let X be a scheme locally of finite type over a field k. Suppose $X \times_k L$ is connected for all finite separable extensions L/k. Then $X \times_k K$ is connected for all extensions K/k.

Proof — By (1.3) it suffices to show $X \times_k \overline{k}$ is connected. Since X is locally of finite type over k, it has a closed point x, and moreover $\kappa(x)/k$ is a finite extension. Then $X \times_k \kappa(x)$ is a scheme over $\kappa(x)$ with a $\kappa(x)$-rational point, and we may embed $\kappa(x)$ in $\overline{k}$, so by (1.4) it suffices to show $X \times_k \kappa(x)$ is connected. Let L/k be the maximal separable subextension of $\kappa(x)$. Then $\kappa(x)/L$ is purely inseparable, so $\operatorname{Spec}(\kappa(x)) \longrightarrow \operatorname{Spec}(L)$ is a universal homeomorphism, and thus $X \times_k \kappa(x)$ is homeomorphic to $X \times_k L$, which is connected by hypothesis, finishing the proof. □