MATH 216A HOMEWORK 7

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§1 Group actions on schemes

Part (i)

Let G be a (discrete) group.

1.1 Let BG denote the category with one object whose automorphism group is G (and with no other objects or morphisms). Let $\mathscr{C}$ be a category, and suppose given a left-action of G on an object $C \in \mathscr{C}$, i.e. a group homomorphism $G \longrightarrow \mathrm{Aut}(C)$. (All of the following goes through for right-actions as well, by replacing G with G^{op}.) Observe that this data is equivalent to a functor $\alpha\colon \mathrm{B}G \longrightarrow \mathscr{C}$ taking the unique object of BG to C. We let $\alpha_g\colon C \longrightarrow C$ denote the action of $g \in G$ on C.

1.1.1 Definition — In this generality, the *object of invariants* of this action, denoted C^G, is defined to be the limit of this functor α (if it exists), and the *quotient object* of this action, denoted C_G, is defined to be the colimit of this functor α (if it exists).

Unwrapping the definitions of limit and colimit, C^G is the universal object of $\mathscr{C}$ equipped with a map $\iota\colon C^G \longrightarrow C$ satisfying $\alpha_g \iota = \iota$ for all $g \in G$, and C_G is the universal object of $\mathscr{C}$ equipped with a map $\pi\colon C \longrightarrow C_G$ satisfying $\pi\alpha_g = \pi$ for all $g \in G$.

1.2 Let $(Z, \mathscr{O}_Z)$ be a ringed space with a right-action of G (in the category of ringed spaces). That is, we have for each $g \in G$ an automorphism $\alpha_g\colon Z \xrightarrow{\sim} Z$ and an isomorphism $\phi_g\colon \mathscr{O}_Z \xrightarrow{\sim} (\alpha_g)_*\mathscr{O}_Z$, and for pairs $g, h \in G$ we have $\alpha_{gh} = \alpha_h\alpha_g$ and a commutative diagram

1.2.1
$$\begin{tikzcd} \mathscr{O}_Z \arrow[r, "\phi_h"] \arrow[d, "\phi_{gh}"'] & (\alpha_h)_*\mathscr{O}_Z \arrow[d, "(\alpha_h)_*(\phi_g)"] \\ (\alpha_{gh})_*\mathscr{O}_Z \arrow[r, "\sim"] & (\alpha_h)_*(\alpha_g)_*\mathscr{O}_Z. \end{tikzcd}$$

1.2.2 Construction — Now let Z/G be a the quotient space, $\pi\colon Z \longrightarrow Z/G$ the quotient map. By definition we have that $\pi\alpha_g = \pi$ for each $g \in G$, and hence pushing forward the maps ϕ_g in π gives us automorphisms

$$\psi_g := \pi_*(\phi_g)\colon \pi_*(\mathscr{O}_Z) \xrightarrow{\sim} \pi_*(\alpha_g)_*\mathscr{O}_Z \simeq \pi_*\mathscr{O}_Z;$$

and (1.2.1) implies that $\psi_{gh} = \psi_g\psi_h$ for $g, h \in G$, so these automorphisms determine a left-action of G on $\pi_*(\mathscr{O}_Z)$ in the category of sheaves of rings on Z/G.

Define $\mathscr{O}_{Z/G} := (\pi_*\mathscr{O}_Z)^G$ to be the invariants of this action (in the category of sheaves of rings on Z/G (which admits all limits), as defined in (1.1.1)). There is by definition a canonical map $\iota\colon \mathscr{O}_{Z/G} \longrightarrow \pi_*\mathscr{O}_Z$. Together with the quotient map this gives us a map of ringed spaces $(\pi,\iota)\colon (Z,\mathscr{O}_Z) \longrightarrow (Z/G,\mathscr{O}_{Z/G})$.

1.2.3 Lemma — The map $\iota\colon \mathscr{O}_{Z/G} \longrightarrow \pi_*\mathscr{O}_Z$ can be identified with the inclusion of the presheaf of G-invariant sections of $\pi_*\mathscr{O}_Z$.

Proof — This follows from the facts that the forgetful functors from

- the category of sheaves of sets on a space to the category of presheaves of sets on that space,
- and the category of rings to the category of sets

both preserve limits. □

1.2.4 Proposition — The map $(\pi,\iota)\colon (Z,\mathscr{O}_Z) \longrightarrow (Z/G,\mathscr{O}_{Z/G})$ exhibits $(Z/G,\mathscr{O}_{Z/G})$ as the quotient object $(Z,\mathscr{O}_Z)_G$ in the category of ringed spaces.

Proof — Let $(Z',\mathscr{O}_{Z'})$ be any ringed space. A map of ringed spaces $(Z/G,\mathscr{O}_{Z/G}) \longrightarrow (Z',\mathscr{O}_{Z'})$ is given by a map of spaces $\rho\colon Z/G \longrightarrow Z'$ and a map of sheaves of rings $\theta\colon \mathscr{O}_{Z'} \longrightarrow \rho_*\mathscr{O}_{Z/G}$.

The quotient space Z/G is a quotient object in the category of topological spaces, so giving ρ is equivalent to giving the map $\widetilde{\rho} := \rho\pi\colon Z \longrightarrow Z'$ satisfying $\widetilde{\rho}\alpha_g = \widetilde{\rho}$ for $g \in G$.

Since ρ_* preserves limits (it is a right adjoint) and $\mathscr{O}_{Z/G}$ is defined to be the invariants $(\pi_*\mathscr{O}_Z)^G$, we have that $\rho_*(\mathscr{O}_{Z/G})$ is the invariants $(\rho_*\pi_*\mathscr{O}_Z)^G$ of the action given by the automorphisms $\rho_*(\gamma_g)$. Thus giving the map θ is equivalent to giving the map $\widetilde{\theta} := \rho_*(\iota)\theta$ satisfying $\rho_*(\gamma_g)\widetilde{\theta} = \widetilde{\theta}$. But now γ_g was defined to be $\pi_*(\phi_g)$, so we can rewrite this condition as follows:

$$\widetilde{\theta} = \rho_*\pi_*(\phi_g) = \widetilde{\rho}_*(\phi_g) = \widetilde{\rho}_*(\alpha_g)_*(\phi_g).$$

The above demonstrates that giving the map of ringed spaces $(\rho,\theta)\colon (Z/G,\mathscr{O}_{Z/G}) \longrightarrow (Z',\mathscr{O}_{Z'})$ is equivalent to giving the map of ringed spaces $(\widetilde{\rho},\widetilde{\theta})\colon (Z,\mathscr{O}_Z) \longrightarrow (Z',\mathscr{O}_{Z'})$ satisfying $(\widetilde{\rho},\widetilde{\theta})(\alpha_g,\phi_g) = (\widetilde{\rho},\widetilde{\theta})$ for all $g \in G$. This proves the claim. □

1.2.5 Lemma — Let $U \subseteq Z$ be a G-stable open subset, i.e. an open subset with $\alpha_g(U) = U$ for all $g \in G$, and let $\mathscr{O}_U := \mathscr{O}_Z|_U$. Then the action of G on $(Z,\mathscr{O}_Z)$ restricts to one on $(U,\mathscr{O}_U)$. And by definition of the quotient topology, $U/G = \pi(U) \subseteq Z/G$ is open. In fact, $(U/G,\mathscr{O}_{Z/G}|_U)$ is the quotient of $(U,\mathscr{O}_U)$ by G.

Proof — It follows easily from (1.2.3) that our construction above behaves well with restricting to open subsets. □

Part (ii)

For the remainder we (crucially) assume G is finite.

1.3 Suppose given a ring A with a left-action of G. This is equivalent to a right-action of G on the affine scheme $X := \operatorname{Spec}(A)$. Let A^G denote the ring of invariants and $Y := \operatorname{Spec}(A^G)$. Let $\pi\colon X \longrightarrow Y$ be the map of schemes induced by the inclusion $A^G \hookrightarrow A$.

1.3.1 LEMMA — The inclusion $A^G \hookrightarrow A$ is an integral extension of rings.

PROOF — Any $a \in A$ is a root of the monic polynomial $\prod_{g \in G}(t - ga) \in A^G[t]$. $\square$

1.3.2 LEMMA — On underlying topological spaces, the map $\pi\colon X \longrightarrow Y$ is G-invariant, hence factors through the quotient map $X \longrightarrow X/G$. Moreover, the resulting map $X/G \longrightarrow Y$ is a homeomorphism.

PROOF — It follows from (1.3.1) that π is surjective and closed, so it suffices to show that π is G-invariant and that G acts transitively on the fibers of π.

We first show π is G-invariant. This amounts to showing that for any $\mathfrak{p} \in \operatorname{Spec}(A)$ and $g \in G$ we have $\mathfrak{p} \cap A^G = (g\mathfrak{p}) \cap A^G$. This follows from the fact that $ga \in A^G \iff a \in A^G$ for any $a \in A$ and $g \in G$, which is straightforward to check.

We now show that G acts transitively on the fibers of π. This amounts to showing that if $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} A$ satisfy $\mathfrak{p} \cap A^G = \mathfrak{q} \cap A^G$, then $\mathfrak{p} = g\mathfrak{q}$ for some $g \in G$. We first claim that in this situation we must have $\mathfrak{p} \subseteq g\mathfrak{q}$ for some $g \in G$. For any $a \in \mathfrak{p}$,

$$\prod_{g \in G}(ga) \in \mathfrak{p} \cap A^G = \mathfrak{q} \cap A^G,$$

so since $\mathfrak{q}$ is prime we must have $ga \in \mathfrak{q}$ for some $g \in G$. Thus $\mathfrak{p} \subseteq \bigcup_{g \in G} g\mathfrak{q}$, and now by prime avoidance we get $\mathfrak{p} \subseteq g\mathfrak{q}$ for some $g \in G$, as claimed.

Symmetrically, we may find $h \in G$ such that $\mathfrak{q} \subseteq h\mathfrak{p}$. We then have $\mathfrak{p} \subseteq g\mathfrak{q} \subseteq gh\mathfrak{p}$. Iterating this statement, we get

$$\mathfrak{p} \subseteq gh\mathfrak{p} \subseteq (gh)^2\mathfrak{p} \subseteq \cdots \subseteq (gh)^n\mathfrak{p} = \mathfrak{p}$$

where n is the order of gh in G. We see that all these containments must be equalities, and then deduce the containment $\mathfrak{p} \subseteq g\mathfrak{q}$ must be an equality, finishing the proof. $\square$

1.3.3 LEMMA — Let $A^G \longrightarrow B$ be a flat ring map. Let $B' := A \otimes_{A^G} B$. Then with G acting on B' via its action on A, the map $B \longrightarrow B'$ identifies B with the G-invariants $(B')^G$.

PROOF — Forming invariants $(-)^G$ is a finite limit and flat base change commutes with finite limits. $\square$

1.3.4 PROPOSITION — The map $\pi\colon X \longrightarrow Y$ exhibits Y as the quotient ringed space X/G.

PROOF — We showed in (1.3.2) that the underlying space of Y can be identified with the quotient space X/G. From our construction (1.2) it now suffices to show that the map $\mathcal{O}_Y \longrightarrow \pi_*\mathcal{O}_X$ identifies $\mathcal{O}_Y$ with the sheaf of invariants $(\pi_*\mathcal{O}_X)^G$. We just need to check this on sections over the base of distinguished affines Y_f for $f \in A^G$. Specifically, we need to check that the map

$$(A^G)_f \simeq \Gamma(Y_f, \mathcal{O}_Y) \longrightarrow \Gamma(X_f, \mathcal{O}_X) \simeq A_f$$

is the inclusion of the G-invariants. This is an application of (1.3.3), taking B to be $(A^G)_f$. $\square$

Part (iii)

Now let X be any scheme.

1.4 Lemma — Suppose X has the property that any finite subset has an open affine neighborhood. Then any open subscheme of X has this property.

Proof — Let $U \subseteq X$ be an open subscheme, and $E \subseteq U$ a finite subset. Replacing X with an open affine neighborhood of E in X (and U with its intersection of this neighborhood), we may assume X is an affine scheme $\mathrm{Spec}(A)$. Then $X \setminus U = V(I)$ for some ideal I of A and $E = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. It suffices to find a distinguished affine $D(f)$ contained in U and containing E, i.e. to find $f \in A$ such that $f \in I$ and $f \notin \mathfrak{p}_1, \ldots, \mathfrak{p}_n$. This can be arranged by prime avoidance, since we know $I \not\subseteq \mathfrak{p}_1, \ldots, \mathfrak{p}_n$, as $E \cap V(I) = \emptyset$. □

1.5 Suppose given a right G-action on our scheme X. Let X/G denote the quotient ringed space.

1.5.1 Lemma — Let $x \in X$. Suppose the G-orbit E of X has an open affine neighborhood. Then x (and hence E) has a G-stable open affine neighborhood $U \subseteq X$.

1.5.2 Proposition — The following are equivalent:

(*a*) The ringed space X/G is a scheme and the quotient map $\pi\colon X \longrightarrow X/G$ is affine.

(*b*) The orbit of any point $x \in X$ has an open affine neighborhood in X.

Part (iv)

We now shift to the relative situation.

1.6 Let S be a scheme and X be an S-scheme. Suppose given a right G-action on X (in the category of S-schemes).

1.6.1 Assumption — Assume all finite subsets of X lie in an open affine of X. Then by (1.5.2), X/G is a scheme an $\pi\colon X \longrightarrow X/G$ is affine. Note that since the G-action on X is via morphisms over S, the structure map $X \longrightarrow S$ is G-invariant, and hence factors through the quotient π to give a map $X/G \longrightarrow S$. Thus we may canonically view X/G as an S-scheme and π as a map over S.

1.6.2 Lemma — Suppose X is finite type over S. Then the quotient $\pi\colon X \longrightarrow X/G$ is a finite map.

Proof — By (1.6.1) we know π is affine, so it's quasicompact. Thus X being finite type over S implies that π is finite type (Hartshorne exercise II.3.13(f)). Since finite is equivalent to finite type and integral, it now suffices to show π is integral.

Moreover π is integral by (1.3.1). □

Part (v)

We continue in the setup of (1.6).

1.7 Let S' be another S-scheme, and define $X' := X \times_S S'$.

1.7.1 Construction — By functoriality of the fiber product, the action of G on X via S-morphisms determines an action of G on X' via S'-morphisms. Namely, the automorphism $g\colon X \longrightarrow X$ over S gives us an automorphism $(g, \mathrm{id}_{S'})\colon X \times_S S' \longrightarrow X \times_S S'$ over S'.

Observe that since any G-orbit of X lies in an open affine of X and lies over a single point of S, the same holds for X' over S', and hence X'/G is a scheme over S' with affine quotient map $\pi'\colon X' \longrightarrow X'/G$.

Composing the projection $X' \longrightarrow X$ with the quotient $X \longrightarrow X/G$, we get a G-invariant map of S-schemes $X' \longrightarrow X/G$, which factors uniquely as a map of S'-schemes $X'/G \longrightarrow X/G$. By the universal property of base change, this determines a canonical map $\rho\colon X'/G \longrightarrow X/G \times_S S'$.

I.7.2 Proposition — Suppose S' is flat over S. Then the map $\rho\colon X'/G \longrightarrow X/G \times_S S'$ is an isomorphism.

Proof — Being an isomorphism is local on the target, so we can reduce to the case where X, S, S' (and hence X') are affine, as follows.

Let $p \in X/G \times_S S'$. Let $x \in X$ be a represntative of the projection of p to X/G; let s' be the projection of p to S'; let $s \in S$ be the image of p in the base. Let $V \subseteq S$ be an open affine neighborhood of s. Since $\sigma\colon X \longrightarrow S$ is G-invariant, $\sigma^{-1}(V)$ is a G-stable open neighborhood of x; applying our assumption (I.6.1) and (I.5.1, I.4), we may find a G-stable open affine neighborhood $U \subseteq \sigma^{-1}(V)$ of x. Then let $V' \subseteq S'$ be any open affine neighborhood of s' mapping to V in S. Set $U' := U \times_V V'$.

By definition of ρ we have $\rho^{-1}(U/G \times_V V') = \pi'(U')$. And by definition of the G-action on $X' = X \times_S S'$, the open subscheme U' is G-stable, so by (I.2.5) we have $\rho^{-1}(U/G \times_V V') \simeq U'/G$. Thus the restriction of the map ρ to the open $U/G \times_V V'$ of the target is given by analagous map $U'/G \longrightarrow U/G \times_V V'$ with X, X', S, S' replaced by U, U', V, V' all affine, giving the desired reduction.

Finally, the case where everything is affine was proved in (I.3.3). □

I.7.3 Proposition — Suppose $S = \mathrm{Spec}(k)$ for k a field, and X is finite type over k (and satisfying (I.6.1), so e.g. X is quasi-projective over k). Let K/k be an algebraically closed extension. Then there is a natural bijection $X(K)/G \simeq (X/G)(K)$.

Proof — The quotient map $\pi\colon X \longrightarrow X/G$ induces a map $X(K) \longrightarrow (X/G)(K)$, and since π is G-invariant this factors through a map $X(K)/G \longrightarrow (X/G)(K)$. We claim this is bijective. By (I.7.2) we may base-change from k to K and hence assume $k = K$ is algebraically closed. Note also that by hypothesis X is finite type, and by part (iv) this implies X/G is finite type.

In particular, now K-points of X and X/G are equivalently closed points of the schemes. Since $\pi\colon X \longrightarrow X/G$ is a surjective map of finite type schemes over a field, it will also be surjective on closed points, so $X(K)/G \longrightarrow (X/G)(K)$ is surjective. And to show injectivity we just need that G acts transitively on the fibers of $X(K) \longrightarrow (X/G)(K)$, but now this follows immediately from the fact that G acts transitively on the fibers of $\pi\colon X \longrightarrow X/G$. □