Math 216A Homework 2
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§1 Hartshorne exercise II.2.22

Let X be a topological space and $\{U_i\}$ an open cover of X. Let $U_{i,j} := U_i \cap U_j$ (considering (i,j) as an *ordered* pair). Let

$$f_i \colon U_i \hookrightarrow X, \quad f_{i,j} \colon U_{i,j} \hookrightarrow X, \quad f^i_{i,j} \colon U_{i,j} \hookrightarrow U_i, \quad f^j_{i,j} \colon U_{i,j} \hookrightarrow U_i$$

denote the inclusions.

1.1 *Lemma* — Let $\mathscr{F}$ be a sheaf on X. Let $\mathscr{F}_i := (f_i)^*\mathscr{F}$ and $\mathscr{F}_{i,j} := (f_{i,j})^*\mathscr{F} = (f^i_{i,j})^*\mathscr{F}_i = (f^j_{i,j})^*\mathscr{F}_j$. Then we have a canonical equalizer sequence of sheaves

$$\mathscr{F} \longrightarrow \prod_i (f_i)_*\mathscr{F}_i \rightrightarrows \prod_{i,j} (f_{i,j})_*\mathscr{F}_{i,j},$$

where here:

- the left map is given on the i-th factor of the target by the unit map $\mathscr{F} \longrightarrow (f_i)_*\mathscr{F}_i = (f_i)_*(f_i)^*\mathscr{F}$ of the adjunction $(f_i)^* \dashv (f_i)_*$;
- the upper right map is given on the (i,j)-th factor of the target by projection onto the i-th factor of the source followed by $(f_i)_*$ applied to the unit map $\mathscr{F}_i \longrightarrow (f^i_{i,j})_*\mathscr{F}_{i,j} = (f^i_{i,j})_*(f^i_{i,j})^*\mathscr{F}_i$ of the adjunction $(f^i_{i,j})^* \dashv (f^i_{i,j})_*$;
- the lower right map is given on the (i,j)-th factor of the target by projection onto the j-th factor of the source followed by $(f_j)_*$ applied to the unit map $\mathscr{F}_j \longrightarrow (f^j_{i,j})_*\mathscr{F}_{i,j} = (f^j_{i,j})_*(f^j_{i,j})^*\mathscr{F}_i$ of the adjunction $(f^j_{i,j})^* \dashv (f^j_{i,j})_*$.

Proof — Limits of sheaves are computed as limits of presheaves, so it suffices to check that the above gives an equalizer sequence after evaluating on any open set $V \subseteq X$. Well, this is precisely the sequence

$$\mathscr{F}(V) \longrightarrow \prod_i \mathscr{F}(V \cap U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(V \cap U_i \cap U_j)$$

that the sheaf condition (applied to the cover $V \cap U_i$ of V) tells us is an equalizer. $\square$

1.2 *Proposition* — Suppose given for each i a sheaf $\mathscr{F}_i$ on U_i, and for each (i,j) an isomorphism $\phi_{i,j} \colon (f^i_{i,j})^*\mathscr{F}_i \xrightarrow{\sim} (f^j_{i,j})^*\mathscr{F}_j$, such that $\phi_{i,i}$ is the identity for each i and for any (i,j,k) we have $\phi_{i,k} = \phi_{j,k} \circ \phi_{i,j}$ when restricted to $U_{i,j,k} := U_i \cap U_j \cap U_k$. Then there exists a unique (up to unique isomorphism) sheaf $\mathscr{F}$ on X equipped with isomorphisms $\psi_i \colon (f_i)^*\mathscr{F} \xrightarrow{\sim} \mathscr{F}_i$ such that $\phi_{i,j} = \psi_j \circ \psi_i^{-1}$ (restricted to $U_{i,j}$).

Proof — Inspired by (1.1), we will consider two maps

$$\prod_i (f_i)_*\mathscr{F}_i \rightrightarrows \prod_{i,j}(f_{i,j})_*\mathscr{F}_{i,j},$$

defined as follows:

- the upper map is just as in (1.1), given on the (i,j)-th factor of the target by projection onto the i-th factor of the source followed by $(f_i)_*$ applied to the unit map $\mathscr{F}_i \longrightarrow (f^i_{i,j})_*\mathscr{F}_{i,j} = (f^i_{i,j})_*(f^i_{i,j})^*\mathscr{F}_i$ of the adjunction $(f^i_{i,j})^* \dashv (f^i_{i,j})_*$;
- the lower map is less tautological, given on the (i,j)-th factor of the target by projection onto the j-th factor of the source followed by $(f_j)_*$ applied to the map $\mathscr{F}_j \longrightarrow (f^j_{i,j})_*\mathscr{F}_{i,j} = (f^j_{i,j})_*(f^j_{i,j})^*\mathscr{F}_i$ which is adjoint to $\phi_{j,i}$ in the adjunction $(f^j_{i,j})^* \dashv (f^j_{i,j})_*$.

To prove existence, we will show that the equalizer of these maps supplies us with the desired $\mathscr{F}$ and ψ_i. To prove uniqueness, we will show that any such $\mathscr{F}$ and ψ_i supply us with an equalizer of these two maps[1].

Let's begin with existence. So let $\mathscr{F}$ be the equalizer of the above two maps. This gives us a canonical map $\alpha\colon \mathscr{F} \longrightarrow \prod_i (f_i)_*\mathscr{F}_i$, and we define $\psi_i\colon (f_i)^*\mathscr{F} \longrightarrow \mathscr{F}_i$ to be adjoint to the map $\alpha_i\colon \mathscr{F} \longrightarrow (f_i)_*\mathscr{F}_i$ in the i-th factor. That α equalizes the two maps above tells us precisely that for each (i,j) we have $\psi_j = \phi_{i,j} \circ \psi_i$ on $U_{i,j}$. Thus we need only show for each i that ψ_i is an isomorphism.

Since f_i is an open inclusion, it's clear that $(f_i)^*$ preserves limits (including products and equalizers), and hence we may pull back our equalizer sequence to U_i:

$$(f_i)^*\mathscr{F} \longrightarrow \prod_j (f_i)^*(f_j)_*\mathscr{F}_j \rightrightarrows \prod_{j,k}(f_i)^*(f_{j,k})_*\mathscr{F}_{j,k}.$$

Then since $(f_i)^*(f_j)_* = (f^i_{i,j})_*(f_i)^*$ and similarly for $(f_i)^*(f_{j,k})_*$, we may reduce to the case that $U_i = X$ (and $f_i = \mathrm{id}_X$), and $\psi_i = \alpha_i$ is just a map $\mathscr{F} \longrightarrow \mathscr{F}_i$.

We define an inverse map $\beta_i\colon \mathscr{F}_i \longrightarrow \mathscr{F}$. By definition of equalizer, this means we must define a map $\beta'_i\colon \mathscr{F}_i \longrightarrow \prod_j (f_j)_*\mathscr{F}_j$ which equalizes the two maps to $\prod_{j,k}(f_{j,k})_*\mathscr{F}_{j,k}$. We achieve this by defining the j-th factor $\beta'_{i,j}$ of β'_i to be the map $\mathscr{F}_i \longrightarrow (f_j)_*\mathscr{F}_j$ adjoint to the isomorphism $\phi_{j,i}\colon (f_j)^*\mathscr{F}_i \xrightarrow{\sim} \mathscr{F}_i$; that β'_i equalizes the two maps follows from the cocycle condition $\phi_{i,k} = \phi_{j,k} \circ \phi_{i,j}$. That $\alpha_i \circ \beta_i = \mathrm{id}_{\mathscr{F}_i}$ is immediate from the condition that $\phi_{i,i}$ is the identity. To check $\beta_i \circ \alpha_i = \mathrm{id}_{\mathscr{F}}$ is equivalent to checking $\alpha_j = \beta'_{i,j} \circ \alpha_i\colon \mathscr{F} \longrightarrow (f_j)_*\mathscr{F}_j$. Chasing through the definitions (note $\beta'_{i,j}$ is defined identically to the (i,j)-th factor of the lower map of the equalizer), this is immediate from α equalizing the two maps. This finishes the proof that α_i is an isomorphism, and hence finishes the construction.

We now prove uniqueness. Suppose given a sheaf $\mathscr{F}$ on X with isomorphisms ψ_i as in the statement of the proposition. Observe that this gives us a commutative diagram (or rather two overlaid commutative diagrams)

$$\begin{array}{ccc}
\prod_i (f_i)_*(f_i)^*\mathscr{F} & \rightrightarrows & \prod_{i,j}(f_{i,j})_*(f_{i,j})^*\mathscr{F} \\
\Big\downarrow{\scriptstyle ((f_i)_*(\psi_i))} & & \Big\downarrow{\scriptstyle ((f_i)_*(f^i_{i,j})^*(\psi_i))} \\
\prod_i (f_i)_*\mathscr{F}_i & \rightrightarrows & \prod_{i,j}(f_{i,j})_*\mathscr{F}_{i,j},
\end{array}$$

[1] This also provides a universal property for mapping into the glued sheaf.

where the first row is the pair of maps appearing in (1.1) and the second row is the pair we've been considering in this proof. By (1.1) $\mathscr{F}$ is an equalizer of the top row. Since the vertical maps are isomorphisms, it must be an equalizer of the bottom row as well. $\square$