MATH 216A HOMEWORK 5

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§1 Normalization

Let X be an integral scheme.

1.1 Definition — We say an X-scheme $\pi\colon \widetilde{X} \longrightarrow X$ exhibits $\widetilde{X}$ as a *normalization* of X if:

(*a*) $\widetilde{X}$ is normal and integral, and π is dominant;

(*b*) for any normal and integral scheme Y and dominant map of schemes $\rho\colon Y \longrightarrow X$, there exists a unique map $\widetilde{\rho}\colon Y \longrightarrow \widetilde{X}$ over X.

1.2 Remark — By the usual universal property/final object/Yoneda business, a normalization of X is unique up to unique isomorphism (as an X-scheme). So from here on out we are entitled to call something *the* normalization rather than *a* normalization.

1.3 Remark — Recall (as stated on the homework for exercise 3.7) that a map $\pi\colon T \longrightarrow S$ of irreducible schemes is dominant if and only if it takes the generic point of Y to the generic point of S. This in particular implies that such a π is irreducible if and only if for any single nonempty open $U \subseteq S$ and open $V \subseteq \pi^{-1}(U)$ the restriction $\pi|_V\colon V \longrightarrow U$ is dominant.

1.4 Lemma — Let $U \subseteq X$ be a nonempty open subscheme (note U is still integral). Suppose given a normalization $\pi\colon \widetilde{X} \longrightarrow X$ of X. Let $\widetilde{U} := \pi^{-1}(U)$. Then the restriction $\pi|_{\widetilde{U}}\colon \widetilde{U} \longrightarrow U$ exhibits $\widetilde{U}$ as the normalization of U.

Proof — Since $\widetilde{U}$ is an open subscheme of $\widetilde{X}$, it's clear that $\widetilde{U}$ is normal and integral. And since U is nonempty it is immediate from (1.3) that the restriction $\pi|_{\widetilde{U}}$ remains dominant.

Now suppose given a dominant map of schemes $\rho\colon Y \longrightarrow U$ with Y normal and integral. The composite $Y \longrightarrow U \hookrightarrow X$ factors uniquely through $\pi\colon \widetilde{X} \longrightarrow X$, so it's clear that ρ factors uniquely through $\pi|_{\widetilde{U}}\colon \widetilde{U} \longrightarrow Y$. □

1.5 Proposition — Let $\{U_i\}_{i\in I}$ be an cover of X by nonempty opens. Suppose given normalizations $\pi_i\colon \widetilde{U}_i \longrightarrow U_i$ of U_i for each $i \in I$. Then these glue to a map $\pi\colon \widetilde{X} \longrightarrow X$ exhibiting $\widetilde{X}$ as the normalization of X.

Proof — To construct $\pi\colon \widetilde{X} \longrightarrow X$, we need to exhibit isomorphisms

$$\phi_{i,j}\colon \pi_i^{-1}(U_i \cap U_j) \xrightarrow{\sim} \pi_j^{-1}(U_i \cap U_j)$$

that are coherent in the usual sense. But this data comes to us immediately from the fact that normalizations restrict well to open subschemes (1.4) together with the uniqueness of normalization up to unique isomorphism (1.2).

By construction $\widetilde{X}$ has an open cover by normal integral schemes, hence itself is a normal integral scheme. And π is dominant by (1.3).

Suppose given a dominant map of schemes $\rho\colon Y \longrightarrow X$ with Y normal and integral. For $i \in I$, let $Y_i := \rho^{-1}(U_i)$ and let $\rho_i := \rho|_{Y_i}\colon Y_i \longrightarrow U_i$ denote the restriction. Then each Y_i is an open subscheme of Y, hence normal and integral, and by (1.3) ρ_i is dominant, so we obtain unique maps $\widetilde{\rho}_i\colon Y_i \longrightarrow \widetilde{U}_i$ over U_i. As noted in the construction of $\widetilde{X}$, by (1.4) the intersection $\widetilde{U}_i \cap \widetilde{U}_j$ in $\widetilde{X}$ is the normalization of $U_i \cap U_j$. Thus the two maps

$$\widetilde{\rho}_i|_{Y_i \cap Y_j}, \widetilde{\rho}_j|_{Y_i \cap Y_j}\colon Y_i \cap Y_j \longrightarrow \widetilde{U}_i \cap \widetilde{U}_j,$$

both lifting $\rho|_{Y_i \cap Y_j}\colon Y_i \cap U_j \longrightarrow U_i \cap U_j$, must be equal. Hence we may glue the lifts $\widetilde{\rho}_i$ to obtain a lift $\widetilde{\rho}\colon Y \longrightarrow \widetilde{X}$ of ρ. The lift is unique since the lifts $\widetilde{\rho}_i$ were unique. Thus $\widetilde{X}$ is indeed the normalization of X. □

1.6 Now suppose X is an affine scheme $\operatorname{Spec} A$, still integral so A is a domain. Let K be the field of fractions of A.

1.6.1 Lemma — The following are equivalent:

(*a*) A is integrally closed in K.

(*b*) X is normal, i.e. for all prime ideals $\mathfrak{p}$ in A, $A_{\mathfrak{p}}$ is integrally closed in K.

(*c*) For all maximal ideals $\mathfrak{m}$ in A, $A_{\mathfrak{m}}$ is integrally closed in K.

Proof — Some commutative algebra, omitted here. □

1.6.2 Proposition — Let $\widetilde{A}$ be the integral closure of A in K. Let $\widetilde{X} := \operatorname{Spec} \widetilde{A}$. Then the map $\pi\colon \widetilde{X} \longrightarrow X$ induced by the inclusion $\pi^\sharp\colon A \hookrightarrow \widetilde{A}$ exhibits $\widetilde{X}$ as the normalization of X.

Proof — Obviously $\widetilde{X}$ is integral, and by (1.6.1) it's normal. And π is dominant since $\pi^\sharp$ is injective (alternatively this follows from (1.3)).

Suppose given a dominant map of schemes $\rho\colon Y \longrightarrow X$ with Y normal and integral. Recall that, since X is affine, this is determined uniqely by the map on global sections $\rho^\sharp\colon A \longrightarrow \Gamma(Y, \mathscr{O}_Y)$. Since $\widetilde{X}$ is also affine, it suffices to show that there is a unique extension $\widetilde{\rho}^\sharp\colon \widetilde{A} \longrightarrow \Gamma(Y, \mathscr{O}_Y)$. Let L denote the function field of Y, which is canonically isomorphic to the fraction field of $\Gamma(U, \mathscr{O}_Y)$ for any nonempty affine open $U \subseteq Y$ by [Hartshorne, Exercise II.3.6].

Consider the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\rho^\sharp"] \arrow[d] & \Gamma(Y, \mathscr{O}_Y) \arrow[d] \\
\widetilde{A} \arrow[d] & \Gamma(U, \mathscr{O}_Y) \arrow[d] \\
K \arrow[r, "\rho^\sharp"] & L,
\end{tikzcd}$$

where here $U \subseteq Y$ is any nonempty affine open, the vertical maps are the canonical inclusions and restrictions, and the bottom map is the map on function fields induced by $\rho^\sharp$ (which makes sense by (1.3)). Now, since Y is normal, so is U, and hence by (1.6.1) $\Gamma(U, \mathscr{O}_Y)$ is integrally closed in L. By definition of $\widetilde{A}$, it follows that there is a map

$\widetilde{\rho}_U^\sharp \colon \widetilde{A} \longrightarrow \Gamma(U, \mathscr{O}_Y)$ which fills in the middle row of the above diagram, and since the map $\Gamma(U, \mathscr{O}_Y) \longrightarrow L$ is injective, this $\widetilde{\rho}_U^\sharp$ is unique.

Finally, note that our U above was arbitrary, and in this situation the sheaf condition tells us that $\Gamma(Y, \mathscr{O}_Y)$ is the intersection $\bigcap_U \Gamma(U, \mathscr{O}_Y)$ in L. Thus from the above we actually get the desired unique extension $\widetilde{\rho}^\sharp \colon \widetilde{A} \longrightarrow \Gamma(Y, \mathscr{O}_Y)$. □

1.7 Remark — Let k be a field. It's a fact that if A is a finite type k-algebra and a domain then the integral closure $\widetilde{A}$ of A in its fraction field is finite over A.[1] Since finiteness of maps of schemes is local on the target, it then follows from (1.5) and (1.6) that if X is a finite type scheme over k and integral then the normalization $\pi \colon \widetilde{X} \longrightarrow X$ is a finite map.

[1] E.g. this is stated with a reference as [Hartshorne, Theorem I.3.9A].