Math 216A Homework 5

"... The study of category theory for its own sake (surely one of the most sterile of all intellectual pursuits) also dates from this time. Grothendieck himself can't necessarily be blamed for this, since his own use of categories was very successful in solving problems."

Miles Reid, in *Undergraduate Algebraic Geometry.*

No reading in Hartshorne this week; instead, work on the following exercises. (Others from §3 will be in next week's homework.)

Ch 2: 3.4*, 3.5 (require f to also be of finite type; this is essential for the notion of quasi-finiteness to possess certain good properties, as we will see later), 3.6 (assume $U \neq \emptyset$, and note that if X is just irreducible and locally noetherian, then $\mathcal{O}_\xi$ is a local artin ring), 3.7* (if X is also irreducible, observe that f is dominant if and only if f takes the generic point of X to the generic point of Y), 3.8* (assume $U \neq \emptyset$, and see p. 64 Remark in Matsumura), 3.11(b)*, 3.12 (for (b), also show there is a unique scheme isomorphism $\mathrm{Proj}(S/I') \simeq \mathrm{Proj}(S/I)$ compatible with the closed immersions of each into X), 3.17, 3.18, 3.19*, 3.20 (omit (f) and assume only that X is locally of finite type over k, and note that in (a), necessarily $\dim X < \infty$).

Extra 1: Use Zorn's Lemma to prove the existence of minimal primes in any non-zero ring. Now let X be a scheme. Show that for $x, y \in X$, if $\overline{\{x\}} = \overline{\{y\}}$, then $x = y$. Next, prove that there exist maximal irreducible sets in X and that these are necessarily closed and every irreducible set lives inside one. These subsets are called the *irreducible components* of X (note that this all depends only on the underlying topological space of X, though you need to use the scheme structure to prove things). Show that the irreducible components of X are in natural bijection with the set of $x \in X$ such that $\mathcal{O}_{X,x}$ is a 0-dimensional ring (with the corresponding irreducible component being $\overline{\{x\}}$). For $x \in X$, describe irreducible closed sets in X containing x in terms of $\mathcal{O}_{X,x}$ (the topological embedding $\mathrm{Spec}(\mathcal{O}_{X,x}) \hookrightarrow X$ may be useful here).

This is most useful when the underlying space of X is locally noetherian, for then the set $\{X_i\}$ of irreducible components is locally finite (in the sense that for all $x \in X$, there is an open U around x which meets only finitely many X_i, with the non-empty $X_i \cap U$ being the irreducible components of U). In particular, a locally noetherian scheme is a locally connected space. Amuse yourself by proving that if A is a product of infinitely many copies of a field, then $\mathrm{Spec}(A)$ is *not* locally connected!

Extra 2: Let X and Y be integral schemes locally of finite type over an affine $\mathrm{Spec}(R)$, and $K(X)$, $K(Y)$ their function fields (which are canonically R-algebras via $\mathrm{Spec}(K(X)) \to X \to \mathrm{Spec}(R)$ and likewise for Y). A *rational* map from X to Y is a map $U \to Y$ for a non-empty open U in X. We say that two *dominant* rational maps $f_i : U_i \to Y$ are *equivalent* if there exists a non-empty open $U \subseteq U_1 \cap U_2 \neq \emptyset$ so that the maps $U \to Y$ induced by the f_i's are the same. Check this is an equivalence relation.

If f is a dominant rational R-map from X to Y, note that it induces an R-algebra map $K(Y) \to K(X)$. Show that this map of fields depends only on the equivalence class of f and that in this way we get a bijection between equivalence classes of dominant rational R-maps from X to Y and R-algebra maps $K(Y) \to K(X)$. Explain why we require R to be noetherian (hint: look back at HW1, Exercise 1). If you are ambitious, read EGA IV_1, §1.4 (especially Prop. 1.4.6) and relax the noetherian hypothesis. Beware that there is a relative notion of R-rational map which is *not* what I just defined above (the problem is that the property of being a *dense* open does not behave well under base change); the relative definition is somewhat complicated in general (see EGA IV_4, §20.2).

When R is an algebraically closed field k and X and Y are integral schemes locally of finite type over k, this gives a scheme-theoretic formulation of the classical method of working with function fields instead of dominant rational maps. In the old days, Weil did force in a notion of generic point, but there were horribly many of these points and it was sort of strange from a geometric viewpoint; not nearly as elegant as the definition provided by scheme theory.

Extra 3: Let S be an $\mathbf{N}$-graded ring. It is often important to know that various commutative algebra constructions on the abstract ring S yields 'graded' results. For example, read Theorem 13.7 in Matsumura (and the proof). Conclude from this that for noetherian S, $\mathrm{Proj}(S) = \emptyset$ if and only if the ideal S_+ consists of nilpotent elements. In fact, prove this without the noetherian hypothesis, and check that if I is a homogeneous ideal in a possibly non-noetherian S, then $\mathrm{rad}(I)$ is the intersection of all homogeneous primes containing I. Use this to give an algebraic description of what it means to say that for homogeneous $f_i \in S$, the opens $D_+(f_i)$ cover $\mathrm{Proj}(S)$. Thus, some properties of the scheme $\mathrm{Proj}(S)$ do encode things about the graded ring S.

Extra 4: Let S be an $\mathbf{N}$-graded ring, d a positive integer. Write out the proof that $\mathfrak{p} \rightsquigarrow \mathfrak{p} \cap S^{(d)}$ gives a bijection between $\mathrm{Proj}(S)$ and $\mathrm{Proj}(S^{(d)})$ as sets (be careful about the 0th graded pieces). Define S' to be the same graded ring as S, except we set the 0th graded piece to be $\mathbf{Z}$. Show that contraction of primes under $S' \to S$ induces a bijection from $\mathrm{Proj}(S)$ to $\mathrm{Proj}(S')$. Thus, $\mathrm{Proj}(S)$ really 'knows nothing' about S_0.

Now enhance these bijections to scheme isomorphisms and check functoriality.

Extra 5: We say that a morphism $f : Y \to X$ of locally ringed spaces is an *immersion* if f is a homeomorphism of Y onto a locally closed subset of X and there exists an open set U containing $f(Y)$ as a closed subset so that the induced morphism $f_U : Y \to U$, in addition to being a closed embedding topologically, has the property that the map of sheaves of rings $\mathcal{O}_U \to (f_U)_*(\mathcal{O}_Y)$ is surjective. Check the following useful properties of immersions (not hard, but should be checked once in your life).

(*i*) Check that the condition in the definition is independent of the choice of open U containing $f(Y)$ as a closed subset, and note that there is a unique largest such U (containing all others). In particular, if $f(Y)$ is closed in X, then f is an immersion if and only if f is a closed immersion in the usual sense.

(*ii*) Prove that a composite of immersions is an immersion and that an immersion is a monomorphism in the category of locally ringed spaces (i.e., if $Y \to X$ is an immersion, then the map of sets $\mathrm{Hom}(Z, Y) \to \mathrm{Hom}(Z, X)$ is injective for all locally ringed spaces Z). Show that for a scheme X and a point $x \in X$, the canonical map $\mathrm{Spec}(\mathcal{O}_x) \to X$ is a monomorphism in the category of schemes which is typically not an immersion.

(*iii*) Let $\mathcal{I}$ be an ideal sheaf on an open set U in X. Assume that the closed set $Y = \mathrm{supp}(\mathcal{O}_U/\mathcal{I})$ in U is not closed in any larger opens in X. Let $i : Y \to X$ denote the inclusion map. To the data $(\mathcal{I}, U)$, attach an immersion $(Y, i^{-1}(\mathcal{O}_U/\mathcal{I})) \to (X, \mathcal{O}_X)$, and show that every immersion of a locally ringed space into $(X, \mathcal{O}_X)$ is equivalent to one coming from a unique pair $(\mathcal{I}, U)$ as above. This singles out distinguished representatives for equivalence classes of closed immersions into X, for example.

(*iv*) Let $Y \to X$ be an immersion, with U an open in X in which the image of Y is closed. Let $\mathcal{I}$ be the associated ideal sheaf on U for Y. Prove the following universal property. Consider a morphism $g : Z \to X$ of locally ringed spaces with $g(Z) \subseteq U$, with $g_U : Z \to U$ the induced map. Using the adjoint $(g_U)^{-1}(\mathcal{O}_U) \to \mathcal{O}_Z$ to $\mathcal{O}_U \to (g_U)_*(\mathcal{O}_Z)$ (!!), we get a map $(g_U)^{-1}(\mathcal{I}) \subseteq (g_U)^{-1}(\mathcal{O}_U) \to \mathcal{O}_Z$ (pullback of sections of $\mathcal{I}$ to functions on Z). Then g factors through $Y \to X$ if and only if this map $(g_U)^{-1}(\mathcal{I}) \to \mathcal{O}_Z$ is 0, in which case it factors through $Y \to X$ uniquely. In other words, $Y \to X$ is universal for the property 'pullback annihilates $\mathcal{I}$'.

An equivalent formulation of the annihilation condition (check!) is that for all $z \in Z$, the map $\mathcal{O}_{X,g(z)} = \mathcal{O}_{U,g_U(z)} \to \mathcal{O}_{Z,z}$ kills the ideal $\mathcal{I}_{g_U(z)}$. Think about the case $U = X$ (the essential case) to see what is happening (in which case this gives a universal mapping property for factoring through a closed immersion in terms of killing an ideal sheaf, analogous to the property for a ring map from A to factor through A/I).