Math 216A Homework 1

This week, read section 1 of Chapter 2, and get started on the task of gradually reading sections 1–15 of Matsumura's Commutative Ring Theory (a review of the material in Atiyah–MacDonald, but with some new angles). The Matsumura reading is something you should work on throughout the fall so that in the winter youll be in position to delve further into it before such further material is needed in Hartshorne's book.

Do as much as you can in the 4 exercises below; it is some commutative algebra with a geometric flavor, highlighting notions and issues which come up a lot in practice later on.

1. Let R be any ring, and $A \simeq R[X_1, \dots, X_n]/I$ and B two R-algebras, with I a finitely generated ideal (one says A is 'of finite presentation' over R; when R is noetherian this is of course equivalent to saying A is finitely generated as an R-algebra). Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals in A and B respectively. For any $a \in A - \mathfrak{p}$, $b \in B - \mathfrak{q}$, and any R-algebra map $f : A_a \to B_b$ with $f^{-1}(\mathfrak{q}_b) = \mathfrak{p}_a$, there is a naturally induced R-algebra map $f_{\mathfrak{q}} : A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ which is a local map. Prove the following:

Theorem. *For every R-algebra map $F : A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ which is a local map, there exist $b \in B - \mathfrak{q}$ and an R-algebra map $f : A \to B_b$ for which $f^{-1}(\mathfrak{q}_b) = \mathfrak{p}$ and $f_{\mathfrak{q}} = F$.*

Hint: look for "common denominators" and use mapping properties of localization and polynomial rings to get things well-defined.

This will have a nice geometric interpretation in the context of schemes; you may want to think about the geometric meaning in the setting of affine varieties over an algebraically closed field when using maximal ideals.

If you have time to kill, try to formulate a suitable 'uniqueness' result for how f depends on F above (think about the geometry!). One consequence of the above is that if A and B are *domains* finitely generated over a field K, then a K-algebra isomorphism between the fraction fields is always induced by a K-algebra isomorphism $A_a \simeq B_b$ for suitable non-zero $a \in A$, $b \in B$ (take $\mathfrak{p} = 0$, $\mathfrak{q} = 0$ in the above Theorem).

2. Let A be a finitely generated k-algebra, where k is an algebraically closed field. Let k'/k be an extension field with k' algebraically closed. Let $A' = k' \otimes_k A$, finitely generated over k'. It often is the case that A has a property P if and only if A' has a property P for various properties P. Prove this for P the following properties: non-zero, domain, reduced, of dimension d, has a unique minimal prime, regular (i.e., all localizations at maximal ideals are regular local rings – see Theorem 5.1 in Chapter 1 for the important link to the rank of a Jacobian matrix). This sort of thing is essential in order to pass between algebraic geometry over $\overline{\mathbf{Q}}$ and algebraic geometry over $\mathbf{C}$ (where analytic tools become available).

Also prove that if A_1 and A_2 are finitely generated k-algebras, then $A_1 \otimes_k A_2$ is reduced (respectively, is a domain) if A_1 and A_2 are of this type. Give counterexamples if we drop the hypothesis that k is algebraically closed.

In this exercise, you'll need to use your commutative algebra skills (e.g., Noether normalization, localizations, etc.); if you aren't aware of the notion of a separating transcendence basis for finitely generated extensions of perfect fields then read up on this (e.g., see Volume 1 of Zariski–Samuel's "Commutative Algebra", or the self-contained first few pages of §26 of Matsumura's "Commutative Ring Theory"). Try to think geometrically, if possible.

3. Let $f \in K[T_1, \ldots, T_n]$ be a polynomial, K a field. Explain how the condition that f be irreducible over K is equivalent to the non-solvability over K of a suitable system of polynomial equations. Using the Nullstellensatz over K (!), give a formulation in terms of your polynomial constraints for what it means to say that f is irreducible over an algebraic closure of K. Conclude that if $f \in \mathbf{Z}[T_1, \ldots, T_n]$ is irreducible over an algebraic closure of $\mathbf{Q}$ (over equivalently (!), over $\mathbf{C}$), then for all but finitely many primes p, f is irreducible when viewed with coefficients over an algebraically closed field of characteristic p.

Note that $X^2 - 2 \in \mathbf{Z}[X]$ is irreducible in $\mathbf{Q}[X]$, but is reducible in $\mathbf{F}_p[X]$ for infinitely many p (and is irreducible in $\mathbf{F}_p[X]$ for infinitely many p also). Thus, the 'geometric' condition of irreducibility over an algebraic closure of $\mathbf{Q}$ is essential (of course, $X^2 - 2$ is *not* irreducible in $\overline{\mathbf{Q}}[X]$).

4. First, give a direct proof of the following special (but important) case of the Cohen Struture Theorem (Theorem 5.5A, Ch. 1) which includes the cases of interest in classical algebraic geometry:

Theorem. *Let K be a field and $(A, \mathfrak{m})$ be a complete local noetherian K-algebra. Assume there exists a subfield $k \subseteq A$ containing K such that $k \to A/\mathfrak{m}$ is an isomorphism. Show that A is regular if and only if there is an isomorphism of K-algebras $A \simeq k[\![T_1, \ldots, T_n]\!]$, for some n, and that in such cases there is even an isomorphism as k-algebras.*

(Hint: if A is regular of dimension n, exploit completeness of A to make a local k-algebra map $k[\![T_1, \ldots, T_n]\!] \to A$ that you rigorously show is surjective via "successive approximation" and the Krull Intersection Theorem. Use that A is a domain to deduce via dimension considerations that such a map must be an isomorphism.)

Suppose now that K is not a perfect field and $a \in K$ is not a pth power, with $p = \operatorname{char}(K) > 0$. Define $\mathfrak{p} = (T^p - a) \subseteq K[T]$ and $A = \widehat{K[T]_{\mathfrak{p}}}$. Show that A is a complete discrete valuation ring (in particular, it is regular) with residue field isomorphic to $K(a^{1/p})$ (as a K-algebra). However, show that there does *not* exist a subfield $k \subseteq A$ containing K for which $k \to A/\mathfrak{m}$ is an isomorphism (hint: look at $A \otimes_K K(a^{1/p})$). Explain why this is not inconsistent with the Cohen Structure Theorem.