MATH 216A HOMEWORK 3

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§1 Hartshorne exercise II.2.4

1.1 Let X be a locally ringed space and $s \in \Gamma(X, \mathscr{O}_X)$ a global section of its structure sheaf.

Notation — For $x \in X$, we will denote the *stalk* of s at x in the local ring $\mathscr{O}_x$ by s_x, and the *value* of s at x in the residue field $\kappa(x) := \mathscr{O}_x/\mathfrak{m}_x$ by $s(x)$.

Notation — We define $X_s := \{x \in X : s(x) \neq 0\}$.

1.1.1 Lemma — (*a*) X_s is an open subset of X.

(*b*) The restriction of s to X_s is invertible in $\Gamma(X_s, \mathscr{O}_X)$.

Proof — Let $x \in X_s$. Then $s(x) \neq 0$ means $s_x \notin \mathfrak{m}_x$, and since $\mathscr{O}_x$ is local this means there exists $t_x \in \mathscr{O}_x$ such that $s_x t_x = 1$. Thus we can find an open neighborhood U of x and a local section $t \in \Gamma(U, \mathscr{O}_X)$ with stalk t_x at x such that $s|_U t = 1 \in \Gamma(U, \mathscr{O}_X)$. This clearly implies $s(y) \neq 0$ for $y \in U$, i.e. $U \subseteq X_s$. This proves X_s is open.

We have also proven that we may cover X_s by open subsets U_i which have local sections $t_i \in \Gamma(U_i, \mathscr{O}_X)$ inverse to (the restrictions of) s. But since inverses are unique, these local sections evidently glue to give a well-defined section $t \in \Gamma(X_s, \mathscr{O}_X)$ inverse to (the restriction of) s. □

1.1.2 Lemma — Let $\pi\colon Y \longrightarrow X$ be a map of locally ringed spaces. Let $\phi\colon \Gamma(X, \mathscr{O}_X) \longrightarrow \Gamma(Y, \mathscr{O}_Y)$ denote the pullback of global sections in π. Then $\pi^{-1}(X_s) = Y_{\phi(s)}$.

Proof — This is immediate from π being a map of *locally* ringed spaces, which implies that $s(\pi(y)) = 0 \iff \phi(s)(y) = 0$. □

1.2 Proposition — Let X be a locally ringed space. Let A be a ring. Then the map[1]

$$\alpha\colon \mathrm{Map}_{\mathrm{LocRingSpaces}}(X, \operatorname{Spec} A) \longrightarrow \mathrm{Map}_{\mathrm{Rings}}(A, \Gamma(X, \mathscr{O}_X))$$

given by taking a map $\pi\colon X \longrightarrow \operatorname{Spec} A$ to pullback of global sections in π is a bijection.

Proof — Let's write $Y := \operatorname{Spec} A$.

We first prove α is injective. Let $\pi\colon X \longrightarrow \operatorname{Spec} A$ be a map of locally ringed spaces, and let $\phi := \alpha(\pi)\colon A \longrightarrow \Gamma(X, \mathscr{O}_X)$ be pullback of global sections. We will show that π is uniquely determined by ϕ.

Let's first consider the map on topological spaces; for $x \in X$ and $s \in A$, we see from (1.1.2) that $x \in \pi^{-1}(Y_s) \iff \phi(s)(x) \neq 0$, or equivalently $\pi(x) \in V(s) \iff \phi(s)(x) =$

[1] In case it's not clear, here LocRingSpaces denotes the category of locally ringed spaces and Rings denotes the category of (commutative, unital) rings.

0. This implies that $\pi(x)$ must be the prime ideal $\{s \in A : \phi(s)(x) = 0\}$. Hence π is determined on topological spaces by ϕ.

We now consider the map on structure sheaves $\pi^\sharp\colon \mathcal{O}_Y \longrightarrow \pi_*\mathcal{O}_X$. It suffices to show that the map is determined by ϕ over the distinguished basic opens $Y_s = D(s)$ for $s \in A$. By (I.1.2) we have $\pi^{-1}(Y_s) = X_{\phi(s)}$. Since $\pi^\sharp$ is a map of sheaves, the diagram

$$\begin{tikzcd} \Gamma(Y,\mathcal{O}_Y) \arrow[r, "\pi^\sharp"] \arrow[d] & \Gamma(X,\mathcal{O}_X) \arrow[d] \\ \Gamma(Y_s,\mathcal{O}_Y) \arrow[r, "\pi^\sharp"] & \Gamma(X_{\phi(s)},\mathcal{O}_X) \end{tikzcd}$$

must commute (where the vertical maps are restriction). But now recall the construction of $\operatorname{Spec} A$: the left map is isomorphic to the localization map $A \longrightarrow A_s$. By the universal property of localization it follows that the bottom map is determined uniquely by the top map, which is ϕ. This completes the proof that α is injective.

We now prove surjectivity, taking cue for our construction from the argument for injectivity. Suppose given a map of rings $\phi\colon A \longrightarrow \Gamma(X,\mathcal{O}_X)$. We define a map of sets $\pi\colon X \longrightarrow \operatorname{Spec} A$ by setting

$$\pi(x) := \{s \in A\colon \phi(s)(x) = 0 \in \kappa(x)\}$$

(it's easy to see this is a prime ideal in A). For $s \in A$ we have immediately from the definitions that $\pi^{-1}(D(s)) = X_{\phi(s)}$, which is open by (I.1.1)(*a*); since $\{D(s)\}_{s\in S}$ form a base of $\operatorname{Spec} A$, this implies π is in fact continuous.

Now, again writing $Y := \operatorname{Spec} A$, we must define a (local) map of sheaves $\pi^\sharp\colon \mathcal{O}_Y \longrightarrow \pi_*\mathcal{O}_X$. It suffices to do so on the distinguished base of opens $\{D(s)\}_{s\in A}$ of Y (in a consistent fashion). Globally, i.e. on $D(1) = Y$, we define $\pi^\sharp\colon A = \Gamma(Y,\mathcal{O}_Y) \longrightarrow \Gamma(X,\mathcal{O}_X)$ to be ϕ; thus, when we've finished constructing the map, we'll have $\alpha(\pi) = \phi$, and hence have proven surjectivity. For each basic open $D(s) = Y_s$ we have from the above that $\pi^{-1}(Y_s) = X_{\phi(s)}$; by (I.1.1)(*b*) the restriction of $\phi(s)$ to $X_{\phi(s)}$ is invertible. Since $\Gamma(Y,\mathcal{O}_Y) \longrightarrow \Gamma(Y_s,\mathcal{O}_Y)$ is isomorphic to the localization map $A \longrightarrow A_s$, it follows that there is a unique map ϕ_s making the diagram

$$\begin{tikzcd} \Gamma(Y,\mathcal{O}_Y) \arrow[r, "\phi"] \arrow[d] & \Gamma(X,\mathcal{O}_X) \arrow[d] \\ \Gamma(Y_s,\mathcal{O}_Y) \arrow[r, "\phi_s"] & \Gamma(X_{\phi(s)},\mathcal{O}_X) \end{tikzcd}$$

commute; we define $\pi^\sharp$ on Y_s to be ϕ_s. It is immediate from this construction that whenever $Y_t \subseteq Y_s$ the diagram

$$\begin{tikzcd} \Gamma(Y_s,\mathcal{O}_Y) \arrow[r, "\phi_s"] \arrow[d] & \Gamma(X_{\phi(s)},\mathcal{O}_X) \arrow[d] \\ \Gamma(Y_t,\mathcal{O}_Y) \arrow[r, "\phi_t"] & \Gamma(X_{\phi(t)},\mathcal{O}_X) \end{tikzcd}$$

commutes. Hence $\pi^\sharp$ is a well-defined map of sheaves.

Finally, for $x \in X$, we see from the above construction that the induced map on stalks $\pi^\sharp\colon A_{\pi(x)} \simeq \mathcal{O}_{\pi(x)} \longrightarrow \mathcal{O}_x$ is the localization of the composite $A \longrightarrow \Gamma(X,\mathcal{O}_X) \longrightarrow \mathcal{O}_x$

at the prime ideal $\pi(x)$. By definition, $\pi(x)$ is the preimage of the maximal ideal $\mathfrak{m}_x \subseteq \mathscr{O}_{\pi(x)}$, and hence this map on stalks is local. Thus $\pi^\sharp$ is a map of locally ringed spaces, finishing the proof. □

I.2.1 Remark — Suppose we take X to be an affine scheme $\operatorname{Spec} B$ in the above. Then from the construction of the proof we see that α^{-1} is given precisely by applying the functor Spec.

I.2.2 Corollary — For any locally ringed space X, there is a natural map of locally ringed spaces $\eta\colon X \longrightarrow \operatorname{Spec}\Gamma(X,\mathscr{O}_X)$, which is an isomorphism if and only if X is an affine scheme.

Proof — The map η corresponds to the identity map of rings $\Gamma(X,\mathscr{O}_X) \longrightarrow \Gamma(X,\mathscr{O}_X)$ under the bijection α^{-1} of (I.2). If η is an isomorphism then $X \simeq \operatorname{Spec}\Gamma(X,\mathscr{O}_X)$ is tautologically an affine scheme. The converse, that η is an isomorphism if X is affine, follows from (I.2.1). □