REPRESENTATIONS OF ABELIAN GROUPS

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Let k be an algebraically closed field. Let $\mathcal{A}$ be an abelian k-linear category. Let G be an abelian group and let $\mathcal{A}^G$ denote the category of G-representations in $\mathcal{A}$, i.e. the functor category $[\mathrm{B}G, \mathcal{A}]$ (which is also obviously abelian and k-linear in a natural way).

1. Proposition. Let M be a simple object in $\mathcal{A}^G$. Then there's a character $\lambda\colon G \to k^\times$ such that the action of G on M is given by multiplication by λ.

Proof. The claim is simply that the action $\gamma(g) \in \mathrm{End}_{\mathcal{A}}(M)$ of each $g \in G$ is a multiple $\lambda(g) \in k$ of the identity. Since G is abelian we have

$$\gamma(g)\gamma(h) = \gamma(gh) = \gamma(hg) = \gamma(h)\gamma(g)$$

for all $h \in H$. I.e. $\gamma(g)$ is also a morphism of representations. The claim then follows from the formulation of Schur's lemma [`abelian-category-linear-algebra`, 1.4.2] stating that M being simple implies its only endomorphisms are multiplication by scalars. □

2. Remark. It follows immediately from (1) that if M is simple in $\mathcal{A}^G$ then M is also simple when viewed as an object in $\mathcal{A}$.

3. Corollary. Assume $\mathcal{A}$ is semisimple. Then any representation $M \in \mathcal{A}^G$ has an *isotypic decomposition*

$$M \simeq \bigoplus_{\lambda \in G^\vee} M_\lambda,$$

where for $\lambda \in G^\vee$ the representation $M_\lambda \in \mathcal{A}^G$ is acted on by G via λ, i.e. for $g \in G$ the action $g\colon M_\lambda \to M_\lambda$ is given by multiplication by $\lambda(g)$.

Proof. Since $\mathcal{A}$ is semisimple, by [`representations-semisimplicity`, 2] we know that $\mathcal{A}^G$ is semisimple. Thus any $M \in \mathcal{A}^G$ is a direct sum of simple objects. The claim now follows from (1). □

4. Remark. Note that the isotypic decomposition provided by (3) is automatically unique: for $\lambda \in G^\vee$ we can recover M_λ as the kernel of the map $M \to \bigoplus_{g \in G} M$ which on the factor indexed by $g \in G$ is given by multiplication by $1 - \lambda(g)$.