LINEAR ALGEBRA IN AN ABELIAN CATEGORY

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Throughout, let $\mathcal{A}$ be an abelian category.

1. (Semi)simplicity and Schur's lemma

1.1. Definition. A nonzero object $M \in \mathcal{A}$ is called *simple* if it has no non-trivial subobjects, i.e. any monomorphism $M' \hookrightarrow M$ is either an isomorphism or the zero map $0 \hookrightarrow M$.

1.2. Definition. (1.2.1) An object $X \in \mathcal{A}$ is said to be *semisimple* if it is a direct sum of finitely many simple objects.

(1.2.2) We say the entire category $\mathcal{A}$ is *semisimple* if all objects in $\mathcal{A}$ are semisimple.

1.3. Lemma (Schur). Let $M, N \in \mathcal{A}$ be simple objects. Then any morphism $T \colon M \to N$ is either zero or an isomorphism.

Proof. Consider the subobjects $i\colon \ker(T) \hookrightarrow M$ and $j\colon \operatorname{im}(T) \hookrightarrow N$. Since M, N are simple each of i and j is either zero or an isomorphism. If i is an isomorphism or j is zero then T is zero. Otherwise i is zero and j is an isomorphism, in which case T is an isomorphism. □

1.4. Corollary. Let k be a commutative ring, and suppose $\mathcal{A}$ is equipped with the structure of a k-linear category.

(1.4.1) Let $M \in \mathcal{A}$ be a simple object. Then the k-algebra $\operatorname{End}_{\mathcal{A}}(M)$ is a division algebra.

(1.4.2) Assume k is an algebraically closed field. Let $M \in \mathcal{A}$ be a simple object such that $\operatorname{End}_{\mathcal{A}}(M)$ is finite-dimensional over k. Then every endomorphism of M is multiplication by a scalar, i.e. the canonical map

$$k \to \operatorname{End}_{\mathcal{A}}(M), \quad a \mapsto a \cdot \mathrm{id}_M$$

is an isomorphism.

Proof. (1.4.1) is a restatement of (1.3), and (1.4.2) follows from (1.4.1) once you recall that the only finite-dimensional division algebra over an algebraically closed field k is k itself. □

1.5. Proposition. $\mathcal{A}$ is semisimple if and only if all objects in $\mathcal{A}$ have finite length and every short exact sequence in $\mathcal{A}$ splits.

Proof. ($\Leftarrow$) We want to show any $X \in \mathcal{A}$ is semisimple. We induct on the length of X (finite by hypothesis). The cases $\lg(X) = 0$ (i.e. $X \simeq 0$) and $\lg(X) = 1$ (i.e. X simple) are tautological. If $\lg(X) > 1$ then we can choose a simple subobject $X' \hookrightarrow X$ and consider the resulting short exact sequence

$$0 \to X' \to X \to X'' \to 0.$$

By hypothesis this splits. Since X' is simple, this reduces the semisimplicity of X to the semisimplicity of X''. But $\lg(X'') = \lg(X) - 1$ so we're done by induction.

(⇒) Obviously if $X \in \mathcal{A}$ is semisimple then it has finite length. So suppose given a short exact sequence

$$0 \to X' \to X \to X'' \to 0.$$

By hypothesis we may write $X \simeq \bigoplus_{i \in I} X_i$ and $X' \simeq \bigoplus_{j \in J} X'_j$ where I, J are finite sets and the X_i, X'_j are simple. Then the inclusion $\phi \colon X' \to X$ is given by a matrix of maps $\phi_{i,j} \colon X'_j \to X_i$, each of which is either zero or an isomorphism by Schur's lemma (1.3). Since ϕ is injective each $\phi_{i,j}$ must in fact be an isomorphism. I.e. we may take $J \subseteq I$ and $X'_j \simeq X_j$. Then there's an obvious projection $X \to X'$ splitting the sequence. $\square$