Math 216A. Valuative criteria

Let $f : X \to Y$ be a morphism of schemes. We wish to formulate criteria of Grothendieck (suggested to him by Serre) for f to be separated or universally closed. These criteria are stated in terms of morphisms from spectra of valuation rings; we refer the reader to basic books on commutative algebra for the elementary properties of valuation rings.

1. Separatedness

We begin with the separated case. The *valuative criterion for separatedness* is the condition that for any valuation ring A with fraction field K and any fixed map $y : \operatorname{Spec} A \to Y$ inducing $y_K : \operatorname{Spec} K \to Y$ by restriction, and any map $x_K : \operatorname{Spec} K \to X$ such that $f(x_K) = y_K$ in $Y(K)$, there is *at most one* map $x : \operatorname{Spec} A \to X$ extending x_K.

Put in more compact terms, for any fixed map $y : \operatorname{Spec} A \to Y$ the map of sets

$$\operatorname{Hom}_Y(\operatorname{Spec} A, X) \to \operatorname{Hom}_Y(\operatorname{Spec} K, X)$$

is injective; here, $\operatorname{Spec} A$ and $\operatorname{Spec} K$ are considered as Y-schemes via the fixed but arbitrary choice y and X is a Y-scheme via f. One often describes this by saying "every rational point of X (over Y) extends in at most one way to an integral point of X (over Y)."

Let's formulate this is yet another way, whose proof we leave to the reader.

Lemma 1.1. *For a map of schemes $f : X \to Y$, the valuative criterion for separatedness says exactly that for any valuation ring A with fraction field K, the diagram of sets*

$$\begin{tikzcd} X(A) \arrow[r] \arrow[d] & X(A) \times_{Y(A)} X(A) \arrow[d] \\ X(K) \arrow[r] & X(K) \times_{Y(K)} X(K) \end{tikzcd}$$

is cartesian. In other words, the diagonal map $\Delta_{X/Y} : X \to X \times_Y X$ has the property that

$$\operatorname{Hom}_{X \times_Y X}(\operatorname{Spec} A, X) \to \operatorname{Hom}_{X \times_Y X}(\operatorname{Spec} K, X)$$

is bijective for every map $\operatorname{Spec} A \to X \times_Y X$ with A a valuation ring.

Note the important fact the bijectivity statement at the end of the lemma is equivalent to a surjectivity statement, because that final map of Hom-sets is *automatically* injective. Indeed, rather more generally if $h : X \to P$ is a monomorphism of schemes (such as the diagonal $\Delta_{X/Y}$ above) then for any domain A with fraction field K and any map $\operatorname{Spec} A \to P$, we claim that the map of sets

$$\operatorname{Hom}_P(\operatorname{Spec} A, X) \to \operatorname{Hom}_P(\operatorname{Spec} K, X)$$

is injective. Indeed, this is a tautology because the set $\operatorname{Hom}_P(\operatorname{Spec} A, X)$ has at most one element (because $X \to P$ is a monomorphism)!

To summarize, we may reformulate the valuative criterion of separatedness for f as a property of the diagonal map $\Delta_{X/Y} : X \to X \times_Y X$: for any valuation ring A with fraction field K, and any fixed but arbitrary map $\operatorname{Spec} A \to X \times_Y X$, the (necessarily injective) map of sets

$$\operatorname{Hom}_{X \times_Y X}(\operatorname{Spec} A, X) \to \operatorname{Hom}_{X \times_Y X}(\operatorname{Spec} K, X)$$

is surjective. Observe also that f is separated if and only if its diagonal $\Delta_{X/Y}$ is a closed immersion, and since $\Delta_{X/Y}$ is *a priori* an immersion we see that it is a closed immersion if and only if it has closed image, and thus is a closed immersion if and only if it is universally closed (as closed immersions are carried to closed immersions by base change and hence are universally closed).

Thus, $f : X \to Y$ is separated if and only if its diagonal is universally closed, and f satisfies the valuative criterion for separatedness if and only if

$$\operatorname{Hom}_{X \times_Y X}(\operatorname{Spec} A, X) \to \operatorname{Hom}_{X \times_Y X}(\operatorname{Spec} K, X)$$

is surjective for any valuation ring A with fraction field K and any fixed but arbitrary map $\operatorname{Spec} A \to X \times_Y X$. We conclude that both the separatedness of f and the valuative criterion for separatedness of f may be reformulated as intrinsic properties of the diagonal of f.

We shall see in the next section that the resulting "valuative" property being asserted for the diagonal of f is exactly the valuative criterion of universal closedness (applied to this diagonal map), and we will prove in the next section that a *quasi-compact* morphism is universally closed if and only if it satisfies the valuative criterion for universal closedness. Since quasi-compactness of the diagonal of f is the property that we have called quasi-separatedness of f, the equivalence in the next section (see Theorem 2.1) applied to the diagonal of f yields:

Theorem 1.2. *Let $f : X \to Y$ be a quasi-separated map. The map f is separated if and only if it satisfies the valuative criterion for separatedness.*

We remind the reader that when X is locally noetherian, any map $f : X \to Y$ is quasi-separated. Thus, the mild extra hypothesis on f in the theorem is rarely violated in practice.

2. Universal closedness

Let $f : X \to Y$ be a map of schemes. The *valuative criterion for universal closedness* of f is the condition that for any valuation ring A with fraction field K and any map $y : \operatorname{Spec} A \to Y$ restricting to $y_K : \operatorname{Spec} K \to Y$ and any $x_K : \operatorname{Spec} K \to X$ lifting y_K (that is, $f(x_K) = y_K$ in $Y(K)$), there exists a map $x : \operatorname{Spec} A \to X$ extending x_K and satisfying $f(x) = y$ in $Y(A)$. In other words, for a fixed but arbitrary map $y : \operatorname{Spec} A \to Y$, the map of sets

$$\operatorname{Hom}_Y(\operatorname{Spec} A, X) \to \operatorname{Hom}_Y(\operatorname{Spec} K, X)$$

is surjective. One often just says "every rational point of X extends to an integral point of X". Here is the main theorem we wish to prove:

Theorem 2.1. *Let $f : X \to Y$ be a quasi-compact map of schemes. The map f is universally closed if and only if it satisfies the valuative criterion for universal closedness.*

Remark 2.2. In practice, it is important to know that it is not necessary to check *all* valuation rings in this criterion. For example, when X and Y are locally noetherian and f is finite-type, it turns out to be sufficient to check just discrete valuation rings (and even only those that are complete with algebraically closed residue field). One can restrict even further in more specialized situations, as you will learn with experience. For example, when working with finite-type schemes over an algebraically closed field k, it turns out to be sufficient to take $A = k[\![t]\!]$ and to consider just those k-morphisms carrying the closed point of $\operatorname{Spec} A$ to a closed point of Y!

Before proving Theorem 2.1, let us record an important corollary, usually called the *valuative criterion for properness*; it follows immediately from Theorem 1.2 and Theorem 2.1, together with the fact that proper maps are finite-type maps that are separated and universally closed:

Corollary 2.3. *Let $f : X \to Y$ be a finite-type map of schemes. The map f is proper if and only if for every valuative ring A with fraction field K and every fixed but arbitrary map* $\operatorname{Spec} A \to Y$*, the map of sets*

$$\operatorname{Hom}_Y(\operatorname{Spec} A, X) \to \operatorname{Hom}_Y(\operatorname{Spec} K, X)$$

is bijective.

Remark 2.4. Don't forget that we have to *assume* f to be finite-type in this corollary! There is no known functorial criterion for the property of being finite-type (though Grothendieck discovered that there is a functorial criterion for being locally of finite presentation). The reason we insist on the finite-type condition in the definition of properness is that this is necessary in order that there be interesting theorems concerning proper maps.

We shall now prepare to prove Theorem 2.1. We begin with some simple but useful lemmas.

Lemma 2.5. *Let $i : Z \to X$ be a closed immersion of schemes, and let A be a domain with fraction field K. Let $f : \operatorname{Spec} A \to X$ be a map of schemes such that the restriction $f_K : \operatorname{Spec} K \to X$ factors (necessarily uniquely) through i. The map f then factors (uniquely) through i.*

Proof. Consider the closed subscheme $Z' \hookrightarrow \operatorname{Spec} A$ obtained by pullback of i along f. The hypotheses tells us that this closed subscheme contains the generic point of $\operatorname{Spec} A$, which is to say that the ideal $I \subseteq A$ defining Z' is contained in the prime ideal (0) defining the generic point. That is, $Z' = \operatorname{Spec} A$, and so the projection map $\operatorname{Spec} A = Z' \to Z$ provides the desired factorization, and its uniqueness is immediate since i is a monomorphism. ∎

Lemma 2.6. *If $f : X \to Y$ satisfies the valuative criterion for universal closedness, then so does the base change $f' : X' \to Y'$ for any map $Y' \to Y$, and so does $f \circ i$ for any closed immersion $i : Z \hookrightarrow X$.*

Proof. The case of base change is a simple consequence of the universal mapping property of $X' = X \times_Y Y'$, and the case of closed immersions follows from the preceding lemma. ∎

Consider the problem of showing that if a quasi-compact map $f : X \to Y$ satisfies the valuative criterion for universal closedness, then f is universally closed. We would like to show that for any map $Y' \to Y$, the base change $f' : X' \to Y'$ is a closed map; that is, $f'(Z')$ is a closed set in Y' for any closed set Z' in X'. By Lemma 2.6, the map f' satisfies the valuative criterion for universal closedness, and if we given Z' its unique structure of reduced closed subscheme of X' then another application of Lemma 2.6 (to the map f' and the closed immersion $i' : Z' \to X'$) shows that $f' \circ i'$ satisfies the valuative criterion for universal closedness. Note also that $f' \circ i'$ is a quasi-compact map (as f is quasi-compact, hence so is f', and i' is quasi-compact since it is a closed immersion).

Consequently, we may rename $f' \circ i'$ as f and thereby conclude that *to prove the "if" direction of Theorem* 2.1 it suffices to show that a quasi-compact $f : X \to Y$ satisfying the valuative criterion for universal closedness has closed image. This completes the "general nonsense", and in the next section we will give the arguments involving genuine ideas that are needed to finish the proof of Theorem 2.1

3. Completion of proof of Theorem 2.1

Let us first suppose that $f : X \to Y$ is universally closed, and we will prove that f satisfying the valuative criterion for universal closedness; this implication will not make use of the hypothesis that f is quasi-compact, and it will just barely (but in a crucial way at the end) make use of the fact that the valuative criterion involves valuation rings and not arbitrary local domains.

Let A be a valuation ring with fraction field K, and let $y : \operatorname{Spec} A \to Y$ be a map and $x_K : \operatorname{Spec} K \to X$ be a map satisfying $f(x_K) = y_K$ in $Y(K)$; we seek to construct a map $x : \operatorname{Spec} A \to X$ extending x_K such that $f(x) = y$ in $Y(A)$. The valuation ring A will be fixed throughout the argument in this part.

As an initial step, note that $\operatorname{Hom}_Y(\operatorname{Spec} A, X) = \operatorname{Hom}_{\operatorname{Spec} A}(\operatorname{Spec} A, X \times_Y \operatorname{Spec} A)$, and similarly with $\operatorname{Spec} K$ replacing $\operatorname{Spec} A$. Thus, we may replace $f : X \to Y$ with the (closed!) base-change $X \times_Y \operatorname{Spec} A \to A$ along y to reduce to the special case where $Y = \operatorname{Spec} A$ and where we wish to prove that a section

$$x_K : \operatorname{Spec} K \to X_K$$

of the generic fiber of $f : X \to \operatorname{Spec} A$ extends to a section $x : \operatorname{Spec} A \to X$ of f. Let Z be the reduced scheme structure on the closure of the image-point of x_K in X, so by closedness of f we know that the image of Z in $Y = \operatorname{Spec} A$ is closed and (by its definition in terms of x_K) contains the generic point of $\operatorname{Spec} A$. Since $\operatorname{Spec} A$ is irreducible, the only closed set in $\operatorname{Spec} A$ containing the generic point is the whole space, so there exists $z \in Z$ mapping to the closed point of $\operatorname{Spec} A$.

Now consider the local domain $\mathcal{O}_{Z,z}$ at the point z on the integral scheme Z with function field equal to the residue field K at the image-point of the section x_K of the generic fiber $X_K \to \operatorname{Spec} K$. We thereby identify K with the fraction field of the local domain $\mathcal{O}_{Z,z}$, and since f carries z to the closed point of $\operatorname{Spec} A$ we obtain a local map $A \to \mathcal{O}_{Z,z}$ that respects the identification of fraction fields of these domains with K (since the computation of the residue field at the image-point of x_K was via the structure map $X \to \operatorname{Spec} A$).

In other words, $\mathcal{O}_{Z,z}$ is a local domain in the fraction field K of A that dominates A. Since A is a *valuation ring*, it is maximal with respect to domination in its fraction field, so the map $A \to \mathcal{O}_{Z,z}$ induced by f is an isomorphism. In other words, the composite map

$$\operatorname{Spec} \mathcal{O}_{Z,z} \to Z \hookrightarrow X \to \operatorname{Spec} A$$

is an isomorphism, so composing the inverse of this isomorphism with the natural map $\operatorname{Spec} \mathcal{O}_{Z,z} \to X$ gives a map $x : \operatorname{Spec} A \to X$ that is a section of f and extends x_K. This shows that f satisfies the valuation criterion for universal closedness.

It remains to treat the most interesting aspect, namely the fact that the valuative criterion for universal closedness actually implies the property of being universally closed, at least for quasi-compact maps $f : X \to Y$. As we saw at the end of §2, for this implication it is enough to prove in general the weaker claim that a quasi-compact map $f : X \to Y$ satisfying the valuative criterion for universal closedness has image $f(X)$ that is closed in Y.

Let us first state a useful "specialization" criterion for closedness of the image of a quasi-compact map:

Lemma 3.1. *Let $f : X \to Y$ be a quasi-compact map of schemes. If $f(X)$ is stable under specialization (that is, for each $y \in f(X)$, the closure $\overline{\{y\}}$ is contained in $f(X)$), then $f(X)$ is closed.*

Proof. If $\{U_i\}$ is an open covering of Y, with $f_i : f^{-1}(U_i) \to U_i$ the restriction of f over U_i, then $f(X) \cap U_i$ is the image of f_i, and so $f(X)$ is closed in Y if and only if f_i has closed image in U_i for all i. Since each f_i has image in U_i that is stable under specialization in U_i (because formation of closure is compatible with intersecting with an open subset), and each f_i is quasi-compact, it is therefore enough to prove the result for each of the f_i's; that is, we can work locally on Y. Thus, we can assume $Y = \operatorname{Spec} R$ is affine, so X is quasi-compact. If $\{V_j\}$ is a finite open affine cover of X then we can replace X with $\coprod V_j$ to reduce to the case where X is also affine, say $X = \operatorname{Spec} B$.

We may also replace R with its quotient by the kernel of $R \to B$ (as this replaces Y with a closed subset containing $f(X)$) to reduce to the case when $R \to B$ is injective. In this case we will prove that the subset $f(X) \subseteq Y$ contains all generic points of Y and thus by stability under specialization is equal to Y (as every point of a scheme is a specialization of a generic point, since every prime of a ring contains a minimal prime).

Let η be a minimal prime of R, so by localizing we have an injection $R_\eta \to B_\eta$. Hence, B_η is a nonzero ring, so B contains a prime $\mathfrak{q}$ whose contraction to R is contained in η and thus (by minimality of η) is equal to η. This shows that $f(X)$ contains all generic points of Y. ■

Returning to our setup, we wish to prove that if $f : X \to Y$ is quasi-compact and satisfies the valuative criterion for universal closedness, then $f(X) \subseteq Y$ is stable under specialization. Let $y = f(x)$ for a point $x \in X$, and let $y_0 \in Y$ be a specialization of y. We wish to construct a point $x_0 \in X$ such that $f(x_0) = y_0$. As we saw in the discussion of specialization and generization of points in lecture, there exists a valuation ring A and a map $h : \operatorname{Spec} A \to Y$ such that h carries the generic point to y and the closed point to y_0. Using Lemma 2.6, we may replace $f : X \to Y$ with its base change by h to reduce to the case $Y = \operatorname{Spec} A$ with a point x in the fiber X_K of f over the generic point $\operatorname{Spec} K$ of Y, and we seek to find $x_0 \in X$ over the closed point of Y (indeed, making such a point x_0 on the scheme that was $X \times_Y \operatorname{Spec} A$ before renaming yields the point we really want on the original X by projection to the first factor).

Now the miracle happens. The residue field $\kappa(x)$ is an extension of the residue field K of the generic point y of $Y = \operatorname{Spec} A$, and basic general results in the theory of valuation rings ensure that the valuation ring A of K is dominated by a valuation ring A' of the extension field $\kappa(x)$ (another way to say this is that valuations may always be extended through field extensions, though one may have to enlarge the value group in the process). The composite map

$$\xi : \operatorname{Spec} A' \to \operatorname{Spec} A \to Y$$

sends the closed point to y_0 (since $A \to A'$ is local) and has generic-point restriction $\operatorname{Spec} \kappa(x) \to Y$ that factors through f via the natural map $\operatorname{Spec} \kappa(x) \to X$. Thus, *by the valuative criterion for universal closedness*, the map $\xi : \operatorname{Spec} A' \to Y$ factors through f, and the resulting map $\operatorname{Spec} A' \to X$ must (by compatibility with ξ) carrying the closed point of $\operatorname{Spec} A'$ to a point $x_0 \in X$ that lies over the closed point y_0 of Y! This completes the verification of the specialization criterion for closedness of $f(X)$ (using f and

X as in the initial setup before we replaced the original base Y with $\operatorname{Spec} A$), and so this finishes the proof of Theorem 2.1.