KÄHLER DIFFERENTIALS

ARPON RAKSIT

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§ 1 PRELIMINARIES

1.1 **Convention —** All rings and algebras are commutative and unital.

1.2 **Notation —** If X is a topological space, we denote the (po)set of open subsets of X by $\mathrm{Op}(X)$.

1.3 **Notation —** Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be ringed spaces. Recall that a map of ringed spaces $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ consists of a continuous map $\pi\colon X \to Y$ together with a map $\alpha\colon \pi^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ of sheaves of rings on X (or equivalently $\alpha\colon \mathscr{O}_Y \to \pi_*\mathscr{O}_Y$ on Y). Thus we will denote maps of ringed spaces by pairs (π, α).

1.4 **Definitions —** (*a*) Let Mod denote the category with:

- objects given by pairs (A, M), where A is a ring and M is an A-module;
- morphisms $(A, M) \to (B, N)$ are given by triples (α, ϕ) where α is a map of rings $A \to B$ and ϕ is a map of A-modules $M \to N$.

(*b*) Let TopMod denote the category with:

- objects given by triples $(X, \mathscr{O}_X, M)$, where $(X, \mathscr{O}_X)$ is a ringed space and M is an $\mathscr{O}_X$-module;
- morphisms $(X, \mathscr{O}_X, M) \to (Y, \mathscr{O}_Y, N)$ are given by triples (π, α, ϕ) where (π, α) is a map of ringed spaces $(X, \mathscr{O}_X, M) \to (Y, \mathscr{O}_Y, N)$ and ϕ is a map of $\mathscr{O}_X$-modules $\pi^*N \to M$ (or equivalently a map of $\mathscr{O}_Y$-modules $N \to \pi_*M$).

Observe that the full subcategory of TopMod on objects $(X, \mathscr{O}_X, M)$ for which X is a point is canonically equivalent to the category $\mathrm{Mod}^{\mathrm{op}}$.

(*c*) Let Alg denote the arrow category of rings, i.e. the category with:

- objects given by maps of rings $A \to B$;
- morphisms from $A \to B$ to $A' \to B'$ given by commutative squares

$$\begin{tikzcd} B \arrow[r] & B' \\ A \arrow[u] \arrow[r] & A'. \arrow[u] \end{tikzcd}$$

(*d*) Let TopAlg denote the arrow category of ringed spaces, i.e. the category with:

- objects given by maps of ringed spaces $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$;
- morphisms from $(X, \mathscr{O}_X) \to (Y', \mathscr{O}'_Y)$ to $(X, \mathscr{O}_{X'}) \to (Y', \mathscr{O}_{Y'})$ given by com-

mutative squares

$$\begin{tikzcd}
(X, \mathcal{O}_X) \arrow[r] \arrow[d] & (X', \mathcal{O}_{X'}) \arrow[d] \\
(Y, \mathcal{O}_Y) \arrow[r] & (Y', \mathcal{O}_{Y'}).
\end{tikzcd}$$

Observe that the full subcategory of TopAlg on objects $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ for which X and Y are points is canonically equivalent to the category $\mathrm{Alg}^{\mathrm{op}}$.

§ 2 DERIVATIONS AND DIFFERENTIALS

In this section we fix a map of ringed spaces $(\pi, \alpha)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$.

2.1 **Definition —** Let M be an $\mathcal{O}_X$-module. A *Y-derivation on X in M* is a map of $\pi^{-1}\mathcal{O}_Y$-modules $d\colon \mathcal{O}_X \to M$ satisfying the Leibniz rule,

$$d(fg) = f\,d(g) + d(f)\,g \quad \text{for } f, g \in \mathcal{O}_X(U),\ U \in \mathrm{Op}(X).$$

2.2 **Remark —** Note that the Leibniz rule implies

$$d(1) = d(1 \cdot 1) = 1 \cdot d(1) + d(1) \cdot 1 = 2 \cdot d(1) \implies d(1) = 0;$$

and by $\pi^{-1}\mathcal{O}_Y$-linearity this implies more generally that $d \circ \alpha = 0$. Conversely, if $d\colon \mathcal{O}_X \to M$ is a map of sheaves of abelian groups satisfying the Leibniz rule, then the condition $d \circ \alpha = 0$ clearly implies that d is $\pi^{-1}\mathcal{O}_Y$-linear.

2.3 **Notation —** Denote the set of Y-derivations on X in M by $\mathrm{Der}(X/Y, M)$. If $d\colon \mathcal{O}_X \to M$ is a Y-derivation and $\phi\colon M \to N$ a map of $\mathcal{O}_X$-modules, then $\phi \circ d\colon \mathcal{O}_X \to N$ is a Y-derivation on X in N. Thus the assignment $M \mapsto \mathrm{Der}(X/Y, M)$ defines a functor $\mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Set}$.

2.4 **Definition —** We define the $\mathcal{O}_X$-module $\Omega_{X/Y}$ of *Kähler differentials on X over Y* to be an object corepresenting the functor $\mathrm{Der}(X/Y, -)\colon \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Set}$. That is, $\Omega_{X,Y}$ is defined by the property that there is a *universal Y-derivation d* on X in $\Omega_{Y/X}$, where "universal" means as usual that composition with d gives an isomorphism

$$\mathrm{Hom}_{\mathrm{Mod}_{\mathcal{O}_X}}(\Omega_{X/Y}, M) \simeq \mathrm{Der}(X/Y, M).$$

Of course, this property uniquely characterizes $\Omega_{X/Y}$, but it remains to prove its existence. We do so in [2.7].

2.5 **Remark —** This definition [2.4] takes in as input a map $(\pi, \alpha)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, i.e. an object of TopAlg, and outputs an $\mathcal{O}_X$-module $\Omega_{X/Y}$, i.e. an object $(X, \mathcal{O}_X, \Omega_{X/Y}) \in$ TopMod.

Suppose

$$\begin{tikzcd}
(X, \mathcal{O}_X) \arrow[r, "(\sigma,\gamma)"] \arrow[d, "(\pi,\alpha)"'] & (X', \mathcal{O}_{X'}) \arrow[d, "(\rho,\beta)"] \\
(Y, \mathcal{O}_Y) \arrow[r, "(\tau,\delta)"] & (Y', \mathcal{O}_{Y'})
\end{tikzcd}$$

is a commutative diagram, i.e. a map in TopAlg. Let M be an $\mathcal{O}_X$-module. There is a map of sets $\mathrm{Der}(X/Y, M) \to \mathrm{Der}(X'/Y', \sigma_* M)$ defined as follows: given a Y-derivation

$\mathscr{O}_X \to M$, restrict in γ to obtain a map $\sigma^{-1}\mathscr{O}_{X'} \to M$, then pass to the adjoint map $\sigma\colon \mathscr{O}_{X'} \to \sigma_* M$, which one easily checks is a Y'-derivation.

This is clearly natural in M, so we have defined a natural transformation $\mathrm{Der}(X/Y,-) \to \mathrm{Der}(X'/Y', \sigma_*(-))$. By definition of Kähler differentials [2.4], this is equivalent to a natural transformation

$$\mathrm{Hom}_{\mathrm{Mod}_{\mathscr{O}_X}}(\Omega_{X/Y},-) \to \mathrm{Hom}_{\mathrm{Mod}_{\mathscr{O}_{X'}}}(\Omega_{X'/Y'},\sigma_*(-)) \simeq \mathrm{Hom}_{\mathrm{Mod}_{\mathscr{O}_X}}(\sigma^*\Omega_{X'/Y'},-),$$

which, by the Yoneda lemma, is equivalent to a map of $\mathscr{O}_X$-modules $\phi\colon \sigma^*\Omega_{X'/Y'} \to \Omega_{X/Y}$, determining a map $(\sigma,\gamma,\phi)\colon (X,\mathscr{O}_X,\Omega_{X/Y}) \to (X',\mathscr{O}_{X'},\Omega_{X'/Y'})$ in TopMod. Thus, the assignment of Kähler differentials

$$(\pi,\alpha)\colon (X,\mathscr{O}_X) \to (Y,\mathscr{O}_Y) \quad \mapsto \quad (X,\mathscr{O}_X,\Omega_{X/Y})$$

defines a functor $\Omega\colon \mathrm{TopAlg} \to \mathrm{TopMod}$.

2.6 **Remark —** Note that when restricted to the full subcategory $\mathrm{Alg}^{\mathrm{op}} \hookrightarrow \mathrm{TopAlg}$ [1.4 (*d*)], the Kähler differentials functor Ω defined in [2.5] takes values in the full subcategory $\mathrm{Mod}^{\mathrm{op}} \hookrightarrow \mathrm{TopMod}$ [1.4 (*b*)]. I.e. Ω restricts to a functor $\mathrm{Alg} \to \mathrm{Mod}$, given by the assignment $(A \to B) \mapsto (B, \Omega_{B/A})$.

In general, when we restrict to Alg, i.e. when X and Y are points, we replace X and Y in our terminology and notation with B and A. E.g. we have *A-derivations on B* in B-modules M, the set of such derivations is denoted $\mathrm{Der}(B/A, M)$, and as above, the B-module of *Kähler differentials of B over A* is denoted $\Omega_{B/A}$.

2.7 **Construction —** We now construct the object $\Omega_{X/Y}$ defined in [2.4], proving its existence. We start by doing this for the case that X and Y are points. Our notation is as discussed in [2.6]: we consider a ring map $\alpha\colon A \to B$ and construct $\Omega_{B/A}$. This is straightforward: we define $\Omega_{B/A}$ to be the B-module generated by symbols $\{df\}_{f\in B}$, subject only to the required relations:

$$d(f+g) = df + dg, \qquad d(fg) = f(dg) + (df)g, \qquad d\alpha(c) = 0,$$

for $f, g \in B$ and $c \in A$. Then the universal A-derivation $d\colon B \to \Omega_{B/A}$ is given simply by $f \mapsto df$.

We now return to the general case. Applying the previous construction locally, we obtain a pre-$\mathscr{O}_X$-module $\Omega^{\mathrm{pre}}_{X/Y}$ by assigning to $U \in \mathrm{Op}(X)$ the Kähler differentials of $\mathscr{O}_X(U)$ over $\pi^{-1}\mathscr{O}_Y(U)$; i.e. we define

$$\Omega^{\mathrm{pre}}_{X/Y}(U) := \Omega_{\mathscr{O}_X(U)/\pi^{-1}\mathscr{O}_Y(U)} \in \mathrm{Mod}_{\mathscr{O}_X(U)}.$$

Finally, we define $\Omega_{X/Y} \in \mathrm{Mod}_{\mathscr{O}_X}$ to be the sheafification of $\Omega^{\mathrm{pre}}_{X/Y}$.

§ 3 SQUARE-ZERO EXTENSIONS

3.1 **Construction —** Let $(X, \mathscr{O}_X)$ be a ringed space and M an $\mathscr{O}_X$-module. We construct an $\mathscr{O}_X$-algebra $A \oplus M$, called the *square-zero extension of $\mathscr{O}_X$ by M*. As the notation suggests, the underlying $\mathscr{O}_X$-module is the direct sum $\mathscr{O}_X \oplus M$. And as the name suggests, the ring structure is defined by requiring:

- that M (or rather its image in the canonical inclusion $M \to \mathscr{O}_X \oplus M$) be a square-zero ideal in $\mathscr{O}_X \oplus M$;

- that the canonical inclusion $\iota_{X,M}\colon \mathcal{O}_X \to \mathcal{O}_X \oplus M$ be a ring map (so that we indeed get an $\mathcal{O}_X$-algebra).

These conditions lead to the following formula for multiplication:

$$(f, m)(g, n) = (fg, fn + gm) \quad \text{for } (f, m), (g, n) \in (\mathcal{O}_X \oplus M)(U),\ U \in \mathrm{Op}(X),$$

noting that $(\mathcal{O}_X \oplus M)(U) \simeq \mathcal{O}_X(U) \oplus M(U)$.

3.2 **Remark —** This construction [3.1] takes in as input a triple $(X, \mathcal{O}_X, M) \in \mathrm{TopMod}$ and outputs an $\mathcal{O}_X$-algebra $\iota_{X,M}\colon \mathcal{O}_X \to \mathcal{O}_X \oplus M$, which we may interpret as an object $(\mathrm{id}_X, \iota_{X,M})\colon (X, \mathcal{O}_X \oplus M) \to (X, \mathcal{O}_X)$ in TopAlg.

Moreover, if (π, α, ϕ) is a map $(X, \mathcal{O}_X, M) \to (Y, \mathcal{O}_Y, N)$ in TopMod, then the diagram

$$\begin{tikzcd}
(X, \mathcal{O}_X \oplus M) \arrow[r, "(\pi, \alpha\oplus\phi)"] \arrow[d, "(\mathrm{id}_X, \iota_{X,M})"] & (Y, \mathcal{O}_Y \oplus N) \arrow[d, "(\mathrm{id}_Y, \iota_{Y,N})"] \\
(X, \mathcal{O}_X) \arrow[r, "(\pi, \alpha)"] & (Y, \mathcal{O}_Y)
\end{tikzcd}$$

is a map $(\mathrm{id}_X, \iota_{X,M}) \to (\mathrm{id}_Y, \iota_{Y,N})$ in TopAlg. Thus the square-zero extension construction

$$(X, \mathcal{O}_X, M) \quad \mapsto \quad (\mathrm{id}_X, \iota_{X,M})\colon (X, \mathcal{O}_X \oplus M) \to (X, \mathcal{O}_X)$$

defines a functor $\oplus\colon \mathrm{TopMod} \to \mathrm{TopAlg}$.

3.3 **Remark —** Note that when restricted to the full subcategory $\mathrm{Mod}^{\mathrm{op}} \hookrightarrow \mathrm{TopMod}$ [1.4 *(b)*], the square-zero extension functor $\oplus$ defined in [3.2] takes values in the full subcategory $\mathrm{Alg}^{\mathrm{op}} \hookrightarrow \mathrm{TopAlg}$ [1.4 *(d)*]. I.e. $\oplus$ restricts to a functor $\mathrm{Mod} \to \mathrm{Alg}$, given by the assignment $(A, M) \mapsto (A \to A \oplus M)$.

3.4 **Lemma —** Let $(Z, \mathcal{O}_Z)$ be a ringed space and M an $\mathcal{O}_Z$-module. Let $(\pi, \alpha)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a map of ringed spaces. Then the following data are equivalent:

- a map from $(\mathrm{id}_Z, \iota_{Z,M})\colon (Z, \mathcal{O}_Z \oplus M) \to (Z, \mathcal{O}_Z)$ to $(\pi, \alpha)\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ in TopAlg;
- a map of ringed spaces $(\rho, \beta)\colon (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ together with a Y-derivation on X in $\rho_* M$.

Moreover, the bijection between the sets of these data is functorial in $(Z, \mathcal{O}_Z, M) \in \mathrm{Mod}^{\mathrm{op}}$ and $(\pi, \alpha) \in \mathrm{TopAlg}$.

Proof — This boils down to the following fact, which is straightforwardly verified. Let $\gamma\colon \mathcal{O}_X \to \rho_*\mathcal{O}_Z \oplus \rho_* M$ be a map of sheaves of sets. Write $\gamma = (\beta, d)$ for maps $\beta\colon \mathcal{O}_X \to \rho_*\mathcal{O}_Z$ and $d\colon \mathcal{O}_X \to \rho_* M$. Then the following are equivalent:

- γ is a map of sheaves of rings whose restriction $\gamma \circ \alpha\colon \pi^{-1}\mathcal{O}_Y \to \rho_*\mathcal{O}_Z \oplus \rho_* M$ factors through $\rho_*\mathcal{O}_Z$.
- β is a map of sheaves of rings and d is a Y-derivation. □

3.5 **Proposition —** There is an adjunction

$$\oplus\colon \mathrm{TopMod} \rightleftarrows \mathrm{TopAlg} \,:\Omega,$$

where the left adjoint $\oplus$ denotes the square-zero extension functor [3.2], and the right adjoint Ω denotes the Kähler differentials functor [2.5].

Proof — Combine [3.4] with the definition of Kähler differentials [2.4]. □

3.6 **Corollary —** By restricting the adjunction supplied by [3.5] to the full subcategories $\mathrm{Mod}^{\mathrm{op}} \hookrightarrow \mathrm{TopMod}$ and $\mathrm{Alg}^{\mathrm{op}} \hookrightarrow \mathrm{TopMod}$ [2.6, 3.3], we obtain an adjunction

$$\Omega\colon \mathrm{Alg} \rightleftarrows \mathrm{Mod} :\oplus,$$

where now (after taking opposite categories) Kähler differentials Ω is left adjoint to square-zero extension $\oplus$.

§4 PROPERTIES OF DIFFERENTIALS

4.1 **Lemma —** Let $(\pi, \alpha)\colon (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a map of ringed spaces. Let $x \in X$. Then the stalk of the Kähler differentials sheaf $\Omega_{X/Y}$ at x is given by the Kähler differentials module of the ring map of stalks $\alpha_x\colon \mathscr{O}_{Y,\pi(x)} \to \mathscr{O}_{X,x}$. That is, there is a canonical isomorphism of $\mathscr{O}_{X,x}$-modules $(\Omega_{X/Y})_x \simeq \Omega_{\mathscr{O}_{X,x}/\mathscr{O}_{Y,\pi(x)}}$.

Proof — Recall from [2.7] that $\Omega_{X/Y}$ is the sheafification of the presheaf $\Omega^{\mathrm{pre}}_{X/Y}$ defined by the assignment

$$\mathrm{Op}(X) \ni U \mapsto \Omega_{\mathscr{O}_X(U)/\pi^{-1}\mathscr{O}_Y(U)} \in \mathrm{Mod}_{\mathscr{O}_X(U)}.$$

Since sheafification preserves stalks, it follows that

$$(\Omega_{X/Y})_x \simeq (\Omega^{\mathrm{pre}}_{X/Y})_x \simeq \operatorname*{colim}_{U \ni x} \Omega^{\mathrm{pre}}_{X/Y}(U) \simeq \operatorname*{colim}_{U \ni x} \Omega_{\mathscr{O}_X(U)/\pi^{-1}\mathscr{O}_Y(U)}.$$

As this is a filtered colimit (taken over the poset of $U \in \mathrm{Op}(X)$ containg x), it also computes the colimit inside the category Mod; more precisely, we have isomorphisms

$$\begin{aligned}(\mathscr{O}_{X,x}, (\Omega_{X/Y})_x) &\simeq \operatorname*{colim}_{U \ni x} (\mathscr{O}_X(U), \Omega_{\mathscr{O}_X(U)/\pi^{-1}\mathscr{O}_Y(U)}) \\ &\simeq \operatorname*{colim}_{U \ni x} \Omega(\alpha_U\colon \pi^{-1}\mathscr{O}_Y(U) \to \mathscr{O}_X(U))\end{aligned}$$

in Mod. By [3.6], $\Omega\colon \mathrm{Alg} \to \mathrm{Mod}$ is a left adjoint and hence preserves colimits, so we obtain an isomorphism (still in Mod)

$$(\mathscr{O}_{X,x}, (\Omega_{X/Y})_x) \simeq \Omega\left(\operatorname*{colim}_{U \ni x} \alpha_U\right) \simeq \Omega(\alpha_x),$$

which is the desired isomorphism of $\mathscr{O}_{X,x}$-modules $(\Omega_{X/Y})_x \simeq \Omega_{\mathscr{O}_{X,x}/\mathscr{O}_{Y,\pi(x)}}$. □

4.2 **Lemma —** Let k be a field and A a k-algebra. Suppose A is a local ring with maximal ideal $\mathfrak{m}$ and residue field k, i.e. such that the map $k \to A/\mathfrak{m}$ is an isomorphism. Then there is an isomorphism of k-vector spaces $\Omega_{A/k} \otimes_A k \simeq \mathfrak{m}/\mathfrak{m}^2$.