PULLING BACK A GALOIS CORRESPONDENCE

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1. Definition. For G a topological group, we denote by Sub_G the set of open normal subgroups of G, which is a poset under inclusion.

2. Proposition. Let G be a profinite group. Suppose we have a morphism of topological groups $\phi\colon A \to G$ such that for every $N \in \mathrm{Sub}_G$ the induced map $\phi_N\colon A/\phi^{-1}(N) \to G/N$ is an isomorphism. Then:

(a) the preimage map $\mathrm{Sub}_G \to \mathrm{Sub}_A$, defined by $N \mapsto \phi^{-1}(N)$, is injective;

(b) $\phi^{-1}(N_1N_2) = \phi^{-1}(N_1)\phi^{-1}(N_2)$ for $N_1, N_2 \in \mathrm{Sub}_G$.

Proof. Let $N_1, N_2 \in \mathrm{Sub}_G$. To prove (a) it suffices to show that $N_1 \subseteq N_2$ if $\phi^{-1}(N_1) \subseteq \phi^{-1}(N_2)$, so assume the latter. Then

$$\phi^{-1}(N_1 \cap N_2) = \phi^{-1}(N_1) \cap \phi^{-1}(N_2) = \phi^{-1}(N_2). \tag{3}$$

Now consider the commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\phi"] & A/\phi^{-1}(N_1 \cap N_2) \arrow[r] \arrow[d, "\phi_{N_1\cap N_2}"] & A/\phi^{-1}(N_2) \arrow[d, "\phi_{N_2}"] \\
G \arrow[r] & G/(N_1 \cap N_2) \arrow[r] & G/N_2
\end{tikzcd}$$

with the horizontal maps the projections. In the right-hand square, the vertical maps are isomorphisms by hypothesis, and the top horizontal map is an isomorphism by (3); thus the bottom horizontal map is an isomorphism, which implies $N_1 \subseteq N_2$.

We now prove (b). Certainly $\phi^{-1}(N_1N_2) \supseteq \phi^{-1}(N_1)\phi^{-1}(N_2)$. And it's fairly easy to see we have the sequence of identifications

$$\begin{aligned}
\frac{\phi^{-1}(N_1)\phi^{-1}(N_2)}{\phi^{-1}(N_1)} &\simeq \frac{\phi^{-1}(N_2)}{\phi^{-1}(N_1) \cap \phi^{-1}(N_2)} = \frac{\phi^{-1}(N_2)}{\phi^{-1}(N_1 \cap N_2)} \\
&= \phi^{-1}_{N_1\cap N_2}\left(\frac{N_1}{N_1 \cap N_2}\right) \simeq \phi^{-1}_{N_1}\left(\frac{N_1N_2}{N_1}\right) = \frac{\phi^{-1}(N_1N_2)}{\phi^{-1}(N_1)}.
\end{aligned}$$

But since $\phi^{-1}(N_1)$ has finite index in A, it follows that

$$[\phi^{-1}(N_1N_2) : \phi^{-1}(N_1)\phi^{-1}(N_2)] = \left[\frac{\phi^{-1}(N_1N_2)}{\phi^{-1}(N_1)} : \frac{\phi^{-1}(N_1)\phi^{-1}(N_2)}{\phi^{-1}(N_1)}\right] = 1,$$

proving the desired claim. □

4. Example. If we take G to be a Galois group in (2), then the proposition says that when we have a suitable morphism $A \to G$, the Galois theory described by G is in fact controlled by A. This is what I meant by "pulling back a Galois correspondence" in the title. Let's state this in more detail in the motivating example.

Let K be a non-archimedean local field. Let K^{ab} be a maximal abelian extension of K. The Galois group $G\,\mathrm{Gal}(K^{\mathrm{ab}}/K)$ is a profinite group, and Galois theory tells us that the poset Sub_G is (contravariantly) equivalent to the poset Ab_K of finite abelian extensions of K, i.e. the set of finite subextensions of K^{ab} ordered by inclusion. The "reciprocity" statement in local class field theory asserts:

- existence of a morphism $\phi_K \colon K^\times \to \mathrm{Gal}(K^{\mathrm{ab}}/K)$ satisfying the hypothesis of (2);
- if $N\,\mathrm{Stab}_L \in \mathrm{Sub}_G$ is the subgroup corresponding to a finite abelian extension $K \hookrightarrow L$, then $\phi_K^{-1}(N)$ is the *norm group* $N_{L/K}(L^\times) \subseteq K^\times$.

Thus, putting (2) and Galois theory together gives us that the poset Ab_K is (contravariantly) equivalent to the poset of norm groups in $K^\times$ by the correspondence $L \mapsto N_{L/K}(L^\times)$. Then there is an "existence" theorem in local class field theory stating that the norm groups are precisely the open subgroups of finite index in $K^\times$.