MATH 216A HOMEWORK 6

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§1 Dimension pathologies

Let R be a discrete valuation ring containing its residue field k.

Let $X := \operatorname{Spec} R[t]$. Let π be a uniformizer in R and $\mathfrak{p} := (\pi t - 1) \subseteq R[t]$. Observe that $R[t]/\mathfrak{p} = R[t]/(\pi t - 1) \simeq R_\pi$ is the fraction field of R, in particular a field, so $\mathfrak{p}$ is maximal. Thus $Y := \{\mathfrak{p}\} \subseteq X$ is a closed set of dimension 0. Then observe that Y has codimension 1, i.e. $\dim R[t]_\mathfrak{p} = 1$, by Krull's hauptidealsatz. Since $\dim R = 1 \implies \dim R[t] = 2$, we have that

$$\dim R[t]_\mathfrak{p} = 1 < 2 = \dim X \quad \text{and} \quad \dim Y + \operatorname{codim}(Y, X) = 1 < 2 = \dim X$$

(note all we really needed here is that $\dim R[t] \geq 2$, which is witnessed by the chain of primes $0 \subsetneq (\pi) \subsetneq (\pi, t)$).

Lastly, consider the (nonempty) open set $U := D(\pi) \subseteq X$, which is isomorphic to $\operatorname{Spec} R[t]_\pi \simeq \operatorname{Spec} R_\pi[t]$. Since R_π is a field, this has dimension 1, unlike X which has dimension 2.