Math 216A Homework 3

"...an arithmetic theory of algebraic varieties cannot but be a theory over arbitrary ground fields and not merely the complex numbers." Zariski (1950)

Work on the following exercises, and read §3 of Chapter 2 up through Example 3.2.6.

Ch 2: 2.3, 2.4* (this explains what is really going on in 2.1), 2.5, 2.6, 2.7, 2.8*, 2.9*, 2.11, 2.13*, 2.14 (you may skip (d)), 2.16 (can you formulate a good functoriality property of the isomorphism in (d)?; note that the natural map goes from A_f to $\Gamma(X_f, \mathcal{O}_{X_f})$), 2.17*, 2.18, 2.19. We will take up a better version of 2.15 in HW4.

For 2.3, prove that the map $j : X_{\mathrm{red}} \to X$ is a closed immersion (i.e., $\mathcal{O}_X \to j_* \mathcal{O}_{X_{\mathrm{red}}}$ is surjective) and also include the following:
2.3(d) If $f, g : X \to Y$ are morphisms of schemes with X reduced, f and g coincide on topological spaces, and $f^\#$, $g^\#$ induce the same ring maps $\kappa(f(x)) = \kappa(g(x)) \to \kappa(x)$ for all $x \in X$, then $f = g$.

For 2.4, conclude in particular that for any scheme X, there is a canonical morphism of schemes $X \to \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$, suitably functorial in X, and this is an isomorphism if and only if X is an affine scheme. (See Prop. 1.8.1 in the "Errata et addenda" list for EGA I at the end of EGA II for Tate's elegant generalization with X any locally ringed space.)

For 2.7, show also the somewhat more general statement that for a scheme X and $x \in X$ and a local ring $(A, \mathfrak{m})$, there is a natural bijection between scheme morphisms $\mathrm{Spec}(A) \to X$ taking the closed point $\mathfrak{m}$ to x and local maps $\mathcal{O}_{X,x} \to A$ (hint: reduce to the case where X is affine by checking that if an open set in $\mathrm{Spec}(A)$ contains the closed point, then it must be the whole space!). Also check functoriality in the pair (X, x) and the local ring A.

Extra 1. Show that if A and B are local algebras over a field k and their residue fields coincide with k then any k-algebra map $h : A \to B$ is *automatically* local (i.e., $h^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$). Deduce that if M and N are smooth manifolds and $\mathcal{O}_M$ and $\mathcal{O}_N$ are their respective sheaves of smooth $\mathbf{R}$-valued functions then any map $(f, f^\#) : (M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ as ringed spaces of $\mathbf{R}$-algebras is *automatically* a map of locally ringed spaces, and moreover that $f^\# : \mathcal{O}_N \to f_*(\mathcal{O}_M)$ is *automatically* induced by composing smooth functions (on open subsets of N) with the map f on underlying topological spaces.

Extra 2. The following construction provides a sense of how to punch holes in a scheme and how to work with affine opens. Let A be a commutative ring, $X = \mathrm{Spec}(A)$, $U \subseteq X$ an open subset (given the induced scheme structure, as always). Define

$$S_U = \{a \in A \mid a \notin \mathfrak{p} \text{ for all } \mathfrak{p} \in U\}.$$

Show that this is a multiplicative set and that the natural map $A \simeq \Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X)$ sends S_U in $\Gamma(U, \mathcal{O}_X)^\times$, so we get a natural ring map $f_U : S_U^{-1} A \to \Gamma(U, \mathcal{O}_X)$ (an isomorphism when $U = \emptyset$ since $0 \in A = S_\emptyset$). If A is domain and $U \neq \emptyset$ prove this is injective, and if A is a principal ideal domain prove it is an isomorphism. Find a *domain* A and open U with f_U *not* an isomorphism (hint: for Dedekind A with $U = X - \{\mathfrak{p}\}$, relate this to $\mathfrak{p}$ being torsion in the class group of A; do you know a Dedekind A whose class group is not a torsion group?).