Math 216A Homework 1
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§1 Problem 1

Let R be a ring.

1.1 *Lemma* — Let A be a finitely presented R-algebra. Then A is a compact object of Alg_R, i.e. the functor $\mathrm{Map}_{\mathrm{Alg}_R}(A,-)\colon \mathrm{Alg}_R \longrightarrow \mathrm{Set}$ preserves filtered colimits.

Proof — Since A is finitely presented we may write

$$A \simeq R[x_1,\ldots,x_n]/(f_1,\ldots,f_m) \simeq \mathrm{coeq}(R[y_1,\ldots,y_m] \rightrightarrows R[x_1,\ldots,x_n]),$$

where one of the maps $R[y_1,\ldots,y_m] \longrightarrow R[x_1,\ldots,x_n]$ sends $y_i \longmapsto f_i$ and the other sends $y_i \longrightarrow 0$. It follows that we have a natural isomorphism

$$\mathrm{Map}_{\mathrm{Alg}_R}(A,B) \simeq \mathrm{eq}(B^n \rightrightarrows B^m),$$

where one of the maps $B^n \longrightarrow B^m$ sends $\underline{b} = (b_1,\ldots,b_n) \longmapsto (f_1(\underline{b}),\ldots,f_m(\underline{b}))$ and the other is the zero map. Now, filtered colimits in Alg_R are computed as filtered colimits in Set, and in Set filtered colimits commute with finite limits (equivalently finite products and equalizers), so the above description of the functor $\mathrm{Map}_{\mathrm{Alg}_R}(A,-)$ shows that it preserves filtered colimits. □

1.2 *Proposition* — Let A, B be R-algebras. Suppose A finitely presented. Let $\mathfrak{p}$ be a prime ideal in A and $\mathfrak{q}$ a prime ideal in B. Let $F\colon A_{\mathfrak{p}} \longrightarrow B_{\mathfrak{q}}$ be a local R-algebra map. Then:

(a) There exist $b \in B \setminus \mathfrak{q}$ and an R-algebra map $f\colon A \longrightarrow B_b$ such that $f^{-1}(\mathfrak{q}_b) = \mathfrak{p}$ and $f_q = F$.

(b) For any two pairs (b_1, f_1) and (b_2, f_2) satisfying the conditions of (a), there exists $b \in B$ with $b_1, b_2 \mid b$ such that f_1 and f_2 become equal after composing with the canonical maps $B_{b_1} \longrightarrow B_b$ and $B_{b_2} \longrightarrow B$.

Proof — First observe that, since F is local, a pair (b, f) satisfies the conditions of (a) if and only if the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi"'] & B_b \arrow[d, "\psi"] \\ A_{\mathfrak{p}} \arrow[r, "F"] & B_{\mathfrak{q}} \end{tikzcd}$$

commutes. Then recall that we have $B_{\mathfrak{q}} \simeq \mathrm{colim}_{b\in B\setminus\mathfrak{q}} B_b$, where the colimit is over the directed set $B \setminus \mathfrak{q}$ (ordered by divisibility). Since A is finitely presented, (1.1) implies that the canonical map

$$\operatorname*{colim}_{b\in B\setminus\mathfrak{q}} \mathrm{Map}_{\mathrm{Alg}_R}(A, B_b) \longrightarrow \mathrm{Map}_{\mathrm{Alg}_R}(A, B_{\mathfrak{q}})$$

is a bijection. Considering the element $F \circ \phi \colon A \longrightarrow B_{\mathfrak{q}}$ on the right-hand side, surjectivity gives us (a) and injectivity gives us (b). □

§2 Problem 2

Let k be an algebraically closed field.

2.1 *Lemma* — Let A, B be k-algebras. Suppose B is reduced. If A is reduced or a domain then $B \otimes_k A$ is the same.

Proof — Since subrings and filtered colimits of reduced rings or domains are the same, and tensor product commutes with colimits, writing B as the filtered colimit of its subalgebras allows us to restrict to the case that B is of finite type. We then have that $\bigcap_{\mathfrak{m} \in \mathrm{Spec_m}\, B} \mathfrak{m}$ is the nilradical of B [1], which is 0 since B is reduced. Thus the canonical map

$$B \longrightarrow \prod_{\mathfrak{m} \in \mathrm{Spec_m}\, B} B/\mathfrak{m}$$

is an injection. Since everything is flat over a field, this gives us an injection

$$B \otimes_k A \hookrightarrow \left(\prod_{\mathfrak{m} \in \mathrm{Spec_m}\, B} B/\mathfrak{m} \right) \otimes_k A.$$

By the Nullstellensatz each $B/\mathfrak{m}$ is isomorphic to k, whence the right-hand side above is isomorphic to A^I, I being the set $\mathrm{Spec_m}\, B$. If A is reduced or a domain then so is A^I, and hence so must be $B \otimes_k A$. □

2.2 *Remark* — Note that (2.1) can fail when k is not algebraically closed. E.g. if L is a nontrivial finite G-Galois extension of k then $L \otimes_k L \simeq L^G$ is not a domain; and if we take k to be non-perfect and L a nontrivial purely inseperable extension of k then $L \otimes_k L$ will not even be reduced.

2.3 *Proposition* — Let k' be an extension field of k. Let A be a k-algebra, and $A' := k' \otimes_k A$. Let P be one of the following properties of rings:

(a) is non-zero;

(b) is reduced;

(c) is a domain;

then A satisfies P if and only if A' satisfies P.

Proof — (a) If A is zero, clearly A' is zero. If A is non-zero, then since everything is flat over a field, the injection $k \hookrightarrow k'$ induces an injection $A \hookrightarrow A'$, so A' is also non-zero.

(b) If A is reduced then by (2.1) so is A'. The converse again follows from the injection $A \hookrightarrow A'$.

(c) Same as (b). □

[1] This comes down to the following fact: if $C \longrightarrow D$ is a map of k-algebras (here k can be any field) and D is of finite type over k, then the preimage of a maximal ideal is not just prime but maximal.

2.4 *Lemma* — Let k' be an extension field of k. Let A be a k-algebra, and $A' := k' \otimes_k A$. Let $\mathfrak{p}$ be a prime ideal in A, and $\mathfrak{p}' := k' \otimes_k \mathfrak{p}$. Then:

(*a*) $\mathfrak{p}'$ is a prime ideal in A';

(*b*) if we assume moreover that A is of finite type and $\mathfrak{p}$ is maximal, then $\mathfrak{p}'$ is maximal.

Proof — By flatness over a field, the exact sequence $0 \longrightarrow \mathfrak{p} \longrightarrow A \longrightarrow A/\mathfrak{p} \longrightarrow 0$ induces an exact sequence $0 \longrightarrow \mathfrak{p}' \longrightarrow A' \longrightarrow k' \otimes_k A/\mathfrak{p} \longrightarrow 0$, implying that $\mathfrak{p}'$ indeed includes into A' as an ideal, and that $A'/\mathfrak{p}' \simeq k' \otimes_k A/\mathfrak{p}$. Then:

(*a*) By (2.3)(*c*), $A/\mathfrak{p}$ being a domain implies $A'/\mathfrak{p}'$ is a domain, so $\mathfrak{p}'$ is prime.

(*b*) If A is of finite type and $\mathfrak{p}$ is maximal, then by the Nullstellensatz $A/\mathfrak{p} \simeq k$, so $A'/\mathfrak{p}' \simeq k'$ is a field, so $\mathfrak{p}'$ is also maximal. □

2.5 *Proposition* — Let k' be an extension field of k. Let A be a k-algebra, and $A' := k' \otimes_k A$. Let P be one of the following properties of rings:

(*a*) has a unique minimal prime;

(*b*) is of dimension d;

then A satisfies P if and only if A' satisfies P.

Proof — (*a*) A ring has a unique minimal prime (i.e. its spectrum is irreducible) if and only if its nilradical is a prime ideal. Consider the exact sequences

$$0 \longrightarrow A^{\mathrm{nil}} \longrightarrow A \longrightarrow A^{\mathrm{red}} \longrightarrow 0, \quad 0 \longrightarrow (A')^{\mathrm{nil}} \longrightarrow A' \longrightarrow (A')^{\mathrm{red}} \longrightarrow 0.$$

It suffices to show that A^{red} is a domain if and only if $(A')^{\mathrm{red}}$ is a domain. The "if" direction follows from the fact that the injection $A \hookrightarrow A'$ induces an injection $A^{\mathrm{red}} \hookrightarrow (A')^{\mathrm{red}}$. Let's prove "only if". By (*a*), if A^{red} is a domain, so is $k' \otimes_k A^{\mathrm{red}}$. But then in particular it is reduced, so we must have $(A')^{\mathrm{nil}} = k' \otimes_k A^{\mathrm{nil}}$, which finishes.

(*b*) We will prove that $\dim A = \dim A'$. We have

$$\dim A = \sup \dim A/\mathfrak{p}, \quad \dim A' = \sup \dim A'/\mathfrak{p}',$$

where the suprema are taken over the minimal primes in each ring. Given a prime $\mathfrak{p}'$ of A', if we let $\mathfrak{p}$ be its preimage in A, then by (2.4)(*a*) $k' \otimes_k \mathfrak{p}$ is a prime in A', evidently contained in $\mathfrak{p}'$. Thus all minimal primes of A' must be of the form $k' \otimes_k \mathfrak{p}$. It therefore suffices to show that for a (minimal) prime $\mathfrak{p}$ of A, $\dim A/\mathfrak{p} = \dim A'/\mathfrak{p}'$ for $\mathfrak{p}' := k' \otimes_k \mathfrak{p}$. Using flatness to obtain the exact sequence

$$0 \longrightarrow k' \otimes_k \mathfrak{p} \longrightarrow A' \longrightarrow k' \otimes_k (A/\mathfrak{p}) \longrightarrow 0,$$

we identify $A'/\mathfrak{p}'$ with $k' \otimes_k (A/\mathfrak{p})$ and are thus reduced to the case that A is a domain. Then by Noether normalization there exists a finite map of k-algebras $k[t_1, \ldots, t_d] \longrightarrow A$ for $d = \dim A$. Base-changing gives us a finite map of k'-algebras $k'[t_1, \ldots, t_d] \longrightarrow A'$. By (2.3)(*c*) A' is also a domain, so we must also have $d = \dim A'$. □

2.6 *Proposition* — Let k' be an algebraically closed extension field of k. Let A be a k-algebra of finite type, and $A' := k' \otimes_k A$. Then A is regular if and only if A' is regular.

Proof — We use the Jacobian criterion: by (2.5)(*b*) $\dim A = \dim A'$, and the corank of the Jacobian of a presentation $A \simeq k[t_1, \dots, t_n]/(f_1, \dots, f_m)$ is independent of whether we compute it over k or k'. □

§3 Problem 3

3.1 *Proposition* — Let R be a ring and $f \in R[t_1, \dots, t_n]$ a polynomial. There is a collection of polynomials $\chi_1(f), \dots, \chi_r(f) \in R[a_1, \dots, a_s]$ satisfying the following:

(*a*) f is irreducible over R if and only if $\chi_1(f), \dots, \chi_r(f)$ admit no common solution over R;

(*b*) the values r and s depend only on the degree of f, and $\chi_1(f), \dots, \chi_r(f)$ are functorial in the pair (R, f), in the sense that if $\phi \colon R \longrightarrow S$ is a ring map under which the polynomial f is sent to $g \in S[t_1, \dots, t_n]$, then ϕ sends $\chi_i(f)$ to $\chi_i(g)$ for $1 \le i \le r$.

(*c*) for any R-algebra K which is a field, and any algebraically closed field K' containing K, f is irreducible over K' if and only $\chi_1(f), \dots, \chi_r(f)$ generate the unit ideal in $K[a_1, \dots, a_s]$.

Proof — (*a*) Write out the condition you get on the coefficients for two polynomials of degree lower than f to multiply to f.

(*b*) Evident from the construction of (*a*).

(*c*) Apply (*a*) over K' and then the Nullstellensatz over K. □

3.2 *Remark* — Note that (3.1)(*c*) implies that if f is irreducible over an algebraically closed field containing K, it is irreducible over all algebraically closed fields containing K.

3.3 *Proposition* — Let $f \in \mathbb{Z}[t_1, \dots, t_n]$ be an integer polynomial which is irreducible over $\overline{\mathbb{Q}}$. Then for all but finitely many primes p, the reduction $\widetilde{f} \in \mathbb{F}_p[t_1, \dots, t_n]$ modulo p is irreducible over $\overline{\mathbb{F}_p}$.

Proof — Let $\chi_1(f), \dots, \chi_r(f) \in \mathbb{Z}[a_1, \dots, a_s]$ be the polynomials supplied by (3.1). Applying (3.1)(*c*), we obtain $\alpha_1, \dots, \alpha_r \in \mathbb{Q}[a_1, \dots, a_s]$ such that $\sum \alpha_i \chi_i(f) = 1$. Clearing denominators gives us $\beta_1, \dots, \beta_r \in \mathbb{Z}[a_1, \dots, a_s]$ such that $\sum \beta_i \chi_i(f) = N$ for some $N \in \mathbb{Z}$. Then for any prime p not dividing N, using the functoriality guaranteed by (3.1)(*b*), we get

$$\frac{1}{N} \sum_i \widetilde{\beta}_i \chi_i(\widetilde{f}) = 1 \in \mathbb{F}_p[t_1, \dots, t_n],$$

and we finish by once more applying (3.1)(*c*). □

§4 Problem 4

4.1 *Proposition* — Let $(A, \mathfrak{m})$ be a complete local noetherian ring. Suppose there is a subfield $k \subseteq A$ such that the map $k \longrightarrow A/\mathfrak{m}$ is an isomorphism. Then A is regular if and only if there is an isomorphism of k-algebras $A \simeq k[\![t_1, \dots, t_n]\!]$.

Proof — The "if" direction is clear, since we know $R := k[\![t_1, \dots, t_n]\!]$ is regular. So let's prove "only if". Assume A is regular, so we can pick a generating set $x_1, \dots, x_n$ of $\mathfrak{m}$ with $n := \dim A$. By the universal property of R, there's a unique k-algebra map $\phi\colon R \longrightarrow A$ sending $t_i \longmapsto x_i$ for $1 \le i \le n$.

We first show that ϕ must be surjective. Let $a \in A$. We inductively construct a sequence $f_0, f_1, \dots$ in R such that:

- f_i is a homogenous polynomial of degree i
- $\phi(\sum_{j=0}^{i} f_j) - a \in \mathfrak{m}^i$

Then $f := \sum_i f_i$ will be a well-defined element of R and by continuity we will have $\phi(f) - a \in \bigcap \mathfrak{m}^i = 0$, hence $\phi(f) = a$. For the base case, we may choose $f_0 = 0$ (we're taking $\mathfrak{m}^0 = (1) = A$ by convention). Suppose we have constructed f_i. Since $(x_1, \dots, x_n)$ generate $\mathfrak{m}$, we may write (using multi-index notation)

$$\phi(f_i) - a = \sum_{\underline{I}} a_{\underline{I}} \underline{x}^{\underline{I}},$$

where $\underline{I}$ runs over tuples in $\mathbb{Z}_{\ge 0}^n$ which sum to i. Since $k \longrightarrow A/\mathfrak{m}$ is an isomorphism, we may choose $c_{\underline{I}} \in k$ such that $\phi(c_{\underline{I}}) - a_{\underline{I}} \in \mathfrak{m}$. Then we set

$$f_{i+1} = \sum_{\underline{I}} c_{\underline{I}} \underline{t}^{\underline{I}},$$

and it's easy to see this satisfies the required properties.

So we've proven ϕ is surjective, and we now show it's in fact an isomorphism. Let $\mathfrak{p} := \ker(\phi)$, which is prime since A is a domain. Then ϕ induces an isomorphism $R/\mathfrak{p} \longrightarrow A$, so $\dim R/\mathfrak{p} = n$. But this can only happen if $\mathfrak{p} = 0$, so we're done. $\square$

4.2 *Proposition* — Let K be a field of characteristic p. Suppose $a \in K$ is not a p-th power, so K is not perfect and $t^p - a \in K[t]$ is irreducible. Let $\mathfrak{p} := (t^p - a) \in \operatorname{Spec} K[t]$. Let $A := \widehat{K[t]_{\mathfrak{p}}}$. Then:

(*a*) A is a complete DVR with residue field isomorphic to $K(a^{1/p})$ (as K-algebras).

(*b*) There is no subfield $k \subseteq A$ containing K such that the map $k \longrightarrow A/\mathfrak{m}$ is an isomorphism.

Proof — (*a*) $K[t]$ is a regular ring of dimension 1 and $\mathfrak{p}$ is maximal, so $K[t]_{\mathfrak{p}}$ is a DVR, with residue field isomorphic to $K[t]/\mathfrak{p} \simeq K(a^{1/p})$. The same is then true of its completion, A.

(*b*) Suppose there were such a subfield k. Then (by flatness over a field) we would obtain an inclusion $k \otimes_K k \longrightarrow k \otimes_K A$, and

$$k \otimes_K k \simeq K(a^{1/p})[t]/(t^p - a) \simeq K(a^{1/p})[t]/(t - a^{1/p})^p$$

is non-reduced. We claim however that $k \otimes_K A$ is reduced, giving us a contradiction.

Indeed, we have

$$\begin{aligned} k \otimes_K A &\simeq k \otimes_K K[t]_{\mathfrak{p}} \otimes_{K[t]_{\mathfrak{p}}} \widehat{K[t]_{\mathfrak{p}}} \\ &\simeq k[t]_{\mathfrak{p}'} \otimes_{K[t]_{\mathfrak{p}}} \widehat{K[t]_{\mathfrak{p}}} \\ &\simeq \widehat{k[t]_{\mathfrak{p}'}}, \end{aligned}$$

where $\mathfrak{p}' = (t - a^{1/p}) \in \operatorname{Spec} k[t]$ [2], so $k \otimes_k A$ is also a DVR, hence reduced. □

4.3 *Remark* — Let A be as in (4.2). The Cohen structure theorem implies that there is an isomorphim $A \simeq K(a^{1/p})[\![t]\!]$. This does not contradict (4.2)(*b*) because the Cohen structure theorem doesn't tell us that the isomorphism must be one of K-algebras. Hence there is a copy of $K(a^{1/p})$ inside A, but it can't contain K as a subfield!

[2] Here we've used that localizing/completing with respect to $\mathfrak{p} = (\mathfrak{p}')^p$ is the same as localizing/completing with respect to $\mathfrak{p}'$, and that completing a module or algebra over a noetherian base ring is equivalent to tensoring with the completion of the base.