MATH 216A HOMEWORK 4

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§1 Varieties and schemes

Notation — Let k be a field.

1.1 Lemma — Let A a finite k-algebra which is a domain. Then A is a field.

Proof — For any nonzero $x \in A$, multiplication by x is an injective k-linear map $m_x\colon A \longrightarrow A$, since A is a domain. Since A is finite over k, m_x is also surjective by rank-nullity. Thus x must be a unit. $\square$

1.2 Lemma — Let X be a scheme locally of finite type over k. Then $x \in X$ is a closed point if and only if the the field extension $k \longrightarrow \kappa(x)$ to its residue field is finite.

Proof — ($\Rightarrow$) Suppose given a closed point $x \in X$. Passing to an affine open, we may assume X is an affine scheme $\operatorname{Spec} A$, where A is of finite type over k. Then x is a maximal ideal $\mathfrak{m}$ in A, with $\kappa(x) \simeq A/\mathfrak{m}$; so $\kappa(x)$ is finite type over k, hence by the Nullstellensatz is finite over k.

($\Leftarrow$) Suppose given a point $x \in X$ with $\kappa(x)$ finite over k. Recall that to show a subset $Z \subseteq X$ is closed it suffices to give an open cover $\{U_i\}$ of X such that $Z \cap U_i$ is closed in U_i for each i. Thus to show x is a closed point, it suffices to show that x is closed in any affine open of X containing x. So we may again assume $X \simeq \operatorname{Spec} A$ with A of finite type over k. Then x is a prime ideal $\mathfrak{p}$ in A, with $\kappa(x) \simeq \operatorname{Frac}(A/\mathfrak{p})$ finite over k. It follows that $A/\mathfrak{p}$ is finite over k, and hence by (1.1) that $A/\mathfrak{p}$ is a field. I.e. $\mathfrak{p}$ is maximal, so x is closed. $\square$

1.3 Let X be a scheme locally of finite type over k. Let $T(X) \subseteq X$ be the subset of closed points. Put $T(X)$ in the subspace topology, let $i\colon T(X) \hookrightarrow X$ denote the inclusion, and let $\mathscr{O}_{T(X)} := i^{-1}(\mathscr{O}_X)$.

1.3.1 Lemma — Let $j\colon Y \hookrightarrow X$ be an open or closed immersion. Then $T(Y) = j^{-1}(T(X))$.

Proof — This is immediate from (1.2), since passing to an open or closed subscheme does not affect residue fields. $\square$

1.3.2 Lemma — $T(X)$ is dense in X.

Proof — We need to show any any nonempty open subset $U \subseteq X$ contains a closed point. By (1.3.1) we may pass to an nonempty open affine contained in U, hence assume $U = X$ is an affine scheme. But then we're done, since any affine scheme has a closed point. $\square$

1.3.3 PROPOSITION — The map $i^{-1}\colon \mathrm{Opens}(X) \longrightarrow \mathrm{Opens}(T(X))$ pulling back open sets from X to $T(X)$ is a bijection.

PROOF — It's equivalent to show that $i^{-1}\colon \mathrm{Closeds}(X) \longrightarrow \mathrm{Closeds}(T(X))$, pullback of closed sets, is bijective. We claim that the map $\mathrm{Closeds}(T(X)) \longrightarrow \mathrm{Closeds}(X)$ given by taking the closure inside X is a two-sided inverse.

If $Y \subseteq X$ is closed, then we may put a closed subscheme structure on Y such that it is still locally of finite type over k. By (1.3.1) we have $T(Y) = T(X) \cap Y$, and by (1.3.2) this is dense in Y. Thus $\overline{i^{-1}(Y)} = Y$.

Conversely, that $i^{-1}(\overline{Z})$ for closed subsets $Z \subseteq T(X)$ is evident, since taking closure commutes with restricting to subspaces. (Alternatively one could note that i^{-1} is obviously surjective by definition of the subspace topology.) □

1.3.4 PROPOSITION — Let A be a k-algebra of finite type and $X := \operatorname{Spec} A$. Then $(T(X), \mathscr{O}_{T(X)})$ is isomorphic to $\operatorname{MaxSpec} A$ (as a ringed space).

PROOF — By definition we see that $T(X)$ is homeomorphic to the space of maximal ideals in A in the Zariski topology. And by (1.3.3) we see that on the basic opens $D(f)$ of $T(X)$ for $f \in A$ the sheaf $\mathscr{O}_{T(X)}$ is given exactly as $\mathscr{O}_X$ is. This is also exactly how we define the structure sheaf of $\operatorname{MaxSpec} A$, so we clearly have an isomorphism. □

1.3.5 PROPOSITION — Suppose X is moreover reduced. Then $(T(X), \mathscr{O}_{T(X)})$ is an abstract algebraic set.

PROOF — If $U \subseteq X$ is an open subset then by (1.3.1) we have $T(X) \cap U = T(U)$, and since the two composite inclusions $T(U) \longrightarrow T(X) \longrightarrow X$ and $T(U) \longrightarrow U \longrightarrow X$ are equal, we must have $\mathscr{O}_{T(X)}|_{T(U)} \simeq \mathscr{O}_{T(U)}$. Since the question is local on X we are reduced to the case that X is an affine scheme $\operatorname{Spec} A$ with A reduced and finite type over k. But then we're done by (1.3.4). □

1.4 LEMMA — Let $\phi\colon A \longrightarrow B$ be a map of k-algebras, with B finite type. Let $\mathfrak{m}$ be a maximal ideal of B. Then $\phi^{-1}(\mathfrak{m})$ is a maximal ideal of A.

PROOF — Consider the induced injection $A/\phi^{-1}(\mathfrak{m}) \hookrightarrow B/\mathfrak{m}$. We know $\phi^{-1}(\mathfrak{m})$ is a prime, so the source is a domain. The target is a field, and finite type over k, so finite over k by the Nullstelensatz. Thus the source is also finite over k, so by (1.1) it is a field, i.e. $\phi^{-1}(\mathfrak{m})$ is maximal. □

1.5 LEMMA — Let A be a k-algebra of finite type. Then A is Jacobson, i.e. for any prime ideal $\mathfrak{p}$ of A, we have $\mathfrak{p} = \bigcap_{\mathfrak{m} \supseteq \mathfrak{p}} \mathfrak{m}$, where the intersection is over all maximal ideals $\mathfrak{m}$ in A containing $\mathfrak{p}$.

PROOF — Passing to $A/\mathfrak{p}$, we may reduce to the case $\mathfrak{p} = 0$. I.e. A is a domain and we need to show that given any nonzero $f \in A$, there is some maximal ideal $\mathfrak{m}$ in A such that $f \notin \mathfrak{m}$. Since A is a domain and $f \neq 0$, the localization A_f is nonzero, hence contains a maximal ideal $\mathfrak{m}'$. The localization $A_f \simeq A[t]/(1 - ft)$ is still finite type over k, so by (1.4) the preimage $\mathfrak{m}$ of $\mathfrak{m}'$ in the localization $A \longrightarrow A_f$ is maximal. And clearly $\mathfrak{m}$ does not contain f. □

1.5.1 Remark — Since $f \in A$ is nilpotent if and only if it is contained in all prime ideals $\mathfrak{p}$ in A, for a Jacobson ring we immediately see that $f \in A$ is nilpotent if and only if it is contained in all *maximal* ideals $\mathfrak{m}$ in A.