SEMISIMPLICITY OF REPRESENTATIONS

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Let k be a field. Let $\mathcal{A}$ be an abelian k-linear category. Let G be a finite group such that $\mathrm{char}(k)$ does not divide $|G|$. Denote by $\mathcal{A}^G$ the (abelian k-linear) category of G-representations in $\mathcal{A}$, i.e. the functor category $[\mathrm{B}G, \mathcal{A}]$.

1. Lemma (Averaging trick). Suppose all short exact sequences in $\mathcal{A}$ split. Then the same is true in $\mathcal{A}^G$.

Proof. Consider a short exact sequence

$$0 \to X' \to X \to X'' \to 0,$$

in $\mathcal{A}^G$. This is a short exact sequence when viewed in $\mathcal{A}$ as well, so by hypothesis there is a splitting $T\colon X'' \to X$ in $\mathcal{A}$. Then the average

$$\frac{1}{|G|} \sum_{g \in G} gTg^{-1}$$

is a morphism $X'' \to X$ in $\mathcal{A}^G$ which also splits the sequence. (Note that this average makes sense by our hypothesis that $\mathrm{char}(k)$ does not divide $|G|$.) $\square$

2. Proposition. Suppose $\mathcal{A}$ is semisimple. Then $\mathcal{A}^G$ is semisimple.

Proof. Recall [`abelian-category-linear-algebra`, 1.5] that $\mathcal{A}$ being semisimple is equivalent to all short exact sequences in $\mathcal{A}$ splitting and all objects in $\mathcal{A}$ having finite length. For an object in $\mathcal{A}^G$ we'll refer to its length in $\mathcal{A}$ as its $\mathcal{A}$-*length*.

Now, we wish to show any nonzero $X \in \mathcal{A}^G$ is semisimple. We induct on its $\mathcal{A}$-length. We may choose a nonzero subobject $\phi\colon X' \hookrightarrow X$ in $\mathcal{A}^G$ of minimal $\mathcal{A}$-length. Clearly X' must be simple in $\mathcal{A}^G$. If ϕ is an isomorphism then we're done. Otherwise, consider the short exact sequence

$$0 \to X' \to X \to X'' \to 0$$

in $\mathcal{A}^G$. By (1) it splits. Since X' is simple this reduces the semisimplicity of X to the semisimplicity of X''. But X'' has strictly smaller $\mathcal{A}$-length than X so we're done by induction. $\square$