MEASURE THEORY

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1. Measure spaces

1.1. Notation. We denote the power set of a set Ω by $\mathcal{P}(\Omega)$.

1.2. Definitions. (1.2.1) Let Ω be a set. A *σ-algebra* on Ω is a collection $\mathcal{F} \subseteq \mathcal{P}(\Omega)$ of subsets of Ω satisfying:

(1.2.1.1) $\emptyset \in \mathcal{F}$.

(1.2.1.2) If $A \in \mathcal{F}$, then $\Omega - A \in \mathcal{F}$.

(1.2.1.3) If $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{F}$.

Note that applying (1.2.1.2) to (1.2.1.1) and (1.2.1.3) implies, respectively:

(1.2.1.4) $\Omega \in \mathcal{F}$.

(1.2.1.5) If $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$, then $\bigcap_{n\in\mathbb{N}} A_n \in \mathcal{F}$.

If Ω is equipped with a σ-algebra $\mathcal{F}$, then we say a subset $A \subseteq \Omega$ is *measurable* if $A \in \mathcal{F}$.

(1.2.2) Let $\mathcal{F}$ be a σ-algebra on a set Ω. Then a *measure* on $\mathcal{F}$ (when the σ-algebra $\mathcal{F}$ is understood/implicit, we will also absusively call this a measure on Ω) is a function $\mu\colon \mathcal{F} \to [0, \infty]$ that is countably additive, i.e.:

(1.2.2.1) $\mu(\emptyset) = 0$.

(1.2.2.2) If $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}$ is a collection of (pairwise) disjoint measurable sets, then $\mu\left(\bigcup_{n\in\mathcal{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$.

Here addition with ∞ is treated as one would expect.

(1.2.3) A *measure space* is a triple $(\Omega, \mathcal{F}, \mu)$ where Ω is a set, $\mathcal{F}$ a σ-algebra on Ω, and μ a measure on $\mathcal{F}$.

1.3. Lemma. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Suppose $A, B \in \mathcal{F}$ such that $A \subseteq B$. Then $\mu(A) \leq \mu(B)$.

Proof. Note that $B - A = B \cap (\Omega - A) \in \mathcal{F}$ by (1.2.1.2, 1.2.1.5). Then $\mu(B) = \mu(A) + \mu(B - A)$ by countable additivity. This proves the claim since, by definition, $\mu(B - A) \geq 0$. □

1.4. Definition. We say a measure space $(\Omega, \mathcal{F}, \mu)$ is *finite* if $\mu(\Omega) < \infty$. By (1.3), this condition implies $\mu(A) < \infty$ for all $A \in \mathcal{F}$.

1.5. Lemma. Let Ω be a set and let $\{\mathcal{F}_i\}$ be a collection of σ-algebras on Ω. Then $\bigcap \mathcal{F}_i$ is also a σ-algebra on Ω.

Proof. Evident. □

1.6. Lemma. Let Ω be a set and let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ be any collection of subsets of Ω. Then there is a minimal (with respect to inclusion) σ-algebra $\mathcal{F}$ on Ω containing $\mathcal{A}$. We refer to $\mathcal{F}$ as the σ-algebra *generated by* $\mathcal{A}$.

Proof. Let $\{\mathcal{F}_i\}$ be the collection of σ-algebras on Ω containing $\mathcal{A}$; the collection is certainly nonempty, as it contains the σ-algebra $\mathcal{P}(\Omega)$ of *all* subsets of Ω. Then the desired minimal σ-algebra is easily seen to be the intersection $\bigcap \mathcal{F}_i$, which is a σ-algebra by (1.5). $\square$

1.7. Definition. Let X be a topological space. The *Borel σ-algebra* on X, often denoted $\mathcal{B}$, is the σ-algebra generated by the collection of open sets in X, often denoted $\mathcal{G}$.