THE DOLD-KAN CORRESPONDENCE

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1. Introduction

1.1. **Definition.** A *simplicial object* in a category $\mathcal{C}$ is a contravariant functor $\Delta \to \mathcal{C}$. We denote the category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ of simplicial objects in $\mathcal{C}$ by $\mathrm{s}\mathcal{C}$. E.g., $\mathrm{s}\mathcal{S}\mathrm{et}$ is the category of *simplicial sets* and $\mathrm{s}\mathcal{A}\mathrm{b}$ is the category of *simplicial abelian groups*.

Recall we have a functor $\mathrm{Sing} : \mathcal{T}\mathrm{op} \to \mathrm{s}\mathcal{S}\mathrm{et}$, sending $X \mapsto \mathrm{Hom}_{\mathcal{T}\mathrm{op}}(|\Delta^\bullet|, X)$. Lately we've been talking about Sing for two reasons:

(1) It's a right adjoint to geometric realisation $|-| : \mathrm{s}\mathcal{S}\mathrm{et} \to \mathcal{T}\mathrm{op}$.
(2) $\mathrm{Sing}(X)$ is a Kan complex for all $X \in \mathcal{T}\mathrm{op}$. In this sense, "Kan complexes are like spaces".

But this isn't the first place one sees Sing, probably. Indeed, the singular homology functors are essentially defined by a composition

$$\mathrm{H}_n(-;\mathbb{Z}) \coloneqq \mathcal{T}\mathrm{op} \xrightarrow{\mathrm{Sing}} \mathrm{s}\mathcal{S}\mathrm{et} \xrightarrow{\mathbb{Z}} \mathrm{s}\mathcal{A}\mathrm{b} \xrightarrow{\sum(-1)^i d_i} \mathcal{C}\mathrm{h}_{\geq 0} \xrightarrow{\mathrm{H}_n} \mathcal{A}\mathrm{b}.$$

Here $\mathbb{Z}$ denotes the functor which takes free abelian groups level-wise, from which we get the *singular chain complex* by letting the boundary map be given by $\partial \coloneqq \sum(-1)^i d_i$.

This was just to remind us that we've seen a natural functor $\mathrm{s}\mathcal{A}\mathrm{b} \to \mathcal{C}\mathrm{h}$ relating simplicial abelian groups and chain complexes. We'll look at it a bit more carefully in a second, and develop this relationship much further.

2. The correspondence

We fix $\mathcal{A}$ any abelian category—but we'll probably be imagining $\mathcal{A} = \mathcal{A}\mathrm{b}$ or $\mathcal{A} = R\text{-}\mathcal{M}\mathrm{od}$ (for some commutative ring R).

2.1. **Notation.** We denote the category of nonnegatively graded chain complexes in $\mathcal{A}$ (and chain maps) by $\mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})$.

Let's now make precise the $\partial \coloneqq \sum(-1)^i d_i$ business with which we started this discussion.

2.2. **Definition.** Let $A_\bullet \in \mathrm{s}\mathcal{A}$ a simplicial object in $\mathcal{A}$ (e.g., a simplicial abelian group). We define the *associated chain complex* $C_\bullet(A) \in \mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})$ by

$$C_n(A) \coloneqq A_n \quad \text{and} \quad \partial_n \coloneqq \textstyle\sum_{i=0}^n (-1)^i d_i : C_n(A) \to C_{n-1}(A)$$

for $n \geq 0$. Note that the simplicial identities clearly imply $\partial^2 = 0$, so $C_\bullet(A)$ is indeed a chain complex. Moreover, this evidently defines a functor $C : \mathrm{s}\mathcal{A} \to \mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})$.

This is perhaps the most natural—or familiar, at least—functor $\mathrm{s}\mathcal{A} \to \mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})$, but it turns out not to be the cleanest to use when discussing the relationship between the two categories. In fact, we will want to use the following alternative.

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2.3. **Definition.** Again let $A_\bullet \in s\mathcal{A}$ a simplicial object in $\mathcal{A}$. We define the *normalised chain complex* $N_\bullet(A) \in \mathcal{C}h_{\geq 0}(\mathcal{A})$ by

$$N_n(A) \coloneqq \bigcap_{i=0}^{n-1} \ker(d_i) \subseteq A_n \quad \text{and} \quad \partial_n \coloneqq (-1)^n d_n : N_n(A) \to N_{n-1}(A)$$

for $n \geq 0$. Again the simplicial identities imply $\partial^2 = 0$, and we have a functor $N : s\mathcal{A} \to \mathcal{C}h_{\geq 0}(\mathcal{A})$.

What is this unmotivated nonsense? Have no fear, for C and N are intimately related! For instance we can note immediately from the definitions that the natural inclusion $N_\bullet(A) \to C_\bullet(A)$ is in fact a chain map for $A \in s\mathcal{A}$. But there's more!

2.4. **Definition.** Let $A \in s\mathcal{A}$. We define the *degenerate subcomplex* $D_\bullet(A)$ of $C_\bullet(A)$ by

$$D_0(A) \coloneqq 0 \quad \text{and} \quad D_n(A) \coloneqq \sum_{i=0}^{n-1} \operatorname{im}(s_i)$$

for $n \geq 1$. That is, $D_\bullet(A)$ is generated by the images of the degeneracy maps. Note that by the simplicial identities

$$\partial s_j = \sum_{i=0}^{n} (-1)^i d_i s_j = \sum_{i=0}^{j-1} (-1)^i s_{j-1} d_i + \sum_{i=j+2}^{n} (-1)^i s_j d_i,$$

so $D_\bullet(A)$ is indeed a subcomplex.

2.5. **Proposition.** *Let $A \in s\mathcal{A}$. For $n \geq 0$ the natural map*

$$\phi : N_n(A) \oplus D_n(A) \to C_n(A)$$

induced by the inclusions is an isomorphism. Therefore we have a natural isomorphism

$$N_\bullet(A) \simeq C_\bullet(A)/D_\bullet(A).$$

Proof. Fix $n \geq 0$. For $0 \leq i \leq n-1$, the simplicial identity $d_i s_i = \mathrm{id}$ implies that we have a canonical splitting

$$A_n \simeq \ker(d_i) \oplus \operatorname{im}(s_i).$$

It follows easily that $N_n(A) \cap D_n(A) \simeq 0$, so we're just left to show that ϕ is surjective. We prove by downward induction on $0 \leq j \leq n-1$ that

$$\operatorname{im}(\phi) \supseteq N_j \coloneqq \bigcap_{i<j} \ker(d_i).$$

The base case $j = n-1$ is tautological and the final case $j = 0$ will finish the proof. Now consider the map

$$\psi \coloneqq \mathrm{id} - s_j d_j : C_n \to C_n.$$

Observe by the simplicial identities that

$$d_j \psi = d_j - d_j s_j d_j = d_j - d_j = 0 \quad \text{and} \quad d_i \psi = d_i - d_i s_j d_j = d_i - s_{j-1} d_{j-1} d_i$$

for $i < j$, implying that $\psi(N_{j+1}) \subseteq N_j$. So by induction $\operatorname{im}(\psi \circ \phi) \supseteq N_j$. But it's easy to see that $\operatorname{im}(\psi \circ \phi) \subseteq \operatorname{im}(\phi)$, since $\operatorname{im}(s_j d_j) \subseteq D_n(A)$ by definition. □

References

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