MATH 131 SECTION, IV: COMPLETION OF METRIC SPACES

ARPON RAKSIT

0.1. **Notation.** Throughout we denote by X a metric space, and by d_X the metric on X.

1. Preliminaries

Let's first remember what convergent and Cauchy sequences and completeness are in metric spaces.

1.1. **Definition.** A sequence $(x_k) \in X^{\mathbb{N}}$ is called *convergent* if there exists $x \in X$ such that for each $\epsilon > 0$ there exists some $n \in \mathbb{N}$ such that $d_X(x, x_k) < \epsilon$ for all $k \geq n$. In this case, we say (x_k) *converges to* x and write $\lim_{k\to\infty} x_k = x$, or equivalently $x_k \to x$ as $k \to \infty$.

1.2. **Definition.** A sequence $(x_k) \in X^{\mathbb{N}}$ is called *Cauchy* if for each $\epsilon > 0$ there exists some $n \in \mathbb{N}$ such that $d_X(x_j, x_k) < \epsilon$ for all $j, k \geq n$.

You might have seen Cauchy sequences defined in the past for the case $X = \mathbb{R}$. And you might have learned that a sequence being Cauchy is *equivalent* to the sequence being convergent. But this equivalence is something special to $\mathbb{R}$—it says that $\mathbb{R}$ is a complete metric space, which we will define in a second. But first check that we always have one implication.

1.3. **Exercise.** *All convergent sequences are Cauchy.*

1.4. **Definition.** We say X is *complete* if all Cauchy sequences in X are convergent in X.

1.5. **Examples.** As stated above $\mathbb{R}$ is complete. This is sort of a key fact when we're learning real analysis. So maybe it's counterintuitive that there are spaces which are *not* complete. But we don't have to stray too far for some examples!

— $(0,1)$ is not complete: it's easy to check that the sequence given by $x_k := 1/k$ for $k \in \mathbb{N}$ is Cauchy, but for any $x \in (0,1)$ there exists $n \in \mathbb{N}$ such that $1/k < x$ for all $k \geq n$, whence (x_k) can't converge to x.

— $\mathbb{Q}$ is not complete: I won't say this is as rigourously as the previous example, but rational approximations to irrational numbers (e.g., $1, 1.4, 1.41, 1.414, \ldots \to \sqrt{2}$) are Cauchy but cannot be convergent in $\mathbb{Q}$.

For another example of a complete metric space, look at the fourth problem set: we can take X the space of bounded sequences in $\mathbb{R}$, equipped with the sup metric.

1.6. **Remark.** Perhaps you've noticed something a little subtle about the difference between convergent and Cauchy sequences. Namely, the notion of convergence depends on the space X in a way that notion of Cauchyness does not. What I mean is: we said that $(1/k)_{k\in\mathbb{N}}$ doesn't converge in $(0,1)$ even though it's Cauchy—but certainly it converges in $[0,1]$ or $\mathbb{R}$! On the other hand, passing to a larger ambient space doesn't change anything about what it means to be a Cauchy sequence.

This is essentially because our definition of convergence requires us to actually produce a limit $x \in X$ (so of course it depends on X in the way described above), whereas our

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definition of Cauchy is completely intrinsic, in that it only refers to the elements of our given sequence.

Finally let's recall a couple of properties that maps between metric spaces can have.

1.7. **Definition.** Let (Y, d_Y) another metric space. A map $\phi : X \to Y$ is:

(1) *uniformly continuous* if for each $\epsilon > 0$ there exists a $\delta > 0$ such that
$$d_X(x,y) < \delta \implies d_Y(\phi(x), \phi(y)) < \epsilon$$
for all $x, y \in X$;

(2) an *isometry* (or "distance-preserving") if
$$d_X(x,y) = d_Y(\phi(x), \phi(y))$$
for all $x, y \in X$;

(3) an *isomorphism (of metric spaces)* if ϕ is a bijective isometry. (Note the inverse of an isomorphism is clearly automatically an isometry.)

1.8. **Exercise.** *Let $\phi : X \to Y$ a map of metric spaces.*

(1) *If ϕ is uniformly continuous and $(x_k) \in X^{\mathbb{N}}$ is a Cauchy sequence, then $(f(x_k)) \in Y^{\mathbb{N}}$ is also Cauchy.*

(2) *If ϕ is an isometry then ϕ is uniformly continuous and an embedding of topological spaces.*

2. Metric completion

The goal for the rest of these notes is to prove the following.

2.1. **Proposition.** *There exist a metric space X_c and an isometry $\phi : X \to X_c$ such that:*

(1) *X_c is complete;*

(2) *$\phi(X)$ is dense in X_c;*

(3) *for any complete metric space Y and uniformly continuous map $\psi : X \to Y$, there exists a unique uniformly continuous map $\psi_c : X_c \to Y$ such that $\psi_c \circ \phi = \psi$;*

(4) *if Y_c is a complete metric space and $\psi : X \to Y_c$ an isometry such that $\psi(X)$ is dense in Y_c, then there exists a unique isomorphism $\eta : X_c \to Y_c$ such that $\eta \circ \phi = \psi$.*

2.2. **Definition.** Since X_c is unique up to unique isomorphism, we are justified in calling X_c *the completion* of X.

There are a number of steps in proving the proposition, so I won't even try to put them all inside one proof environment! (Actually, as—well, if—you read along, you should try to prove the claims before reading on. This is a decent exercise in knowing how to prove fundamental things about metric spaces.) Here we go.

2.3. **Construction.** First define $C \coloneqq \{(x_k) \in X^{\mathbb{N}} \mid (x_k) \text{ is Cauchy}\}$ the set of all Cauchy sequences in X. We then define an equivalence relation on C by saying
$$(x_k) \sim (y_k) \iff \lim_{k\to\infty} d_X(x_k, y_k) = 0.$$
Reflexifity and symmetry of this relation are obvious. For transitivity we just use nonnegativity of the metric and the triangle inequality: if $(x_k) \sim (y_k)$ and $(y_k) \sim (z_k)$ then
$$\begin{aligned} 0 \le \lim_{k\to\infty} d_X(x_k, z_k) &\le \lim_{k\to\infty} d_X(x_k, y_k) + d_X(y_k, z_k) \\ &\le \lim_{k\to\infty} d_X(x_k, y_k) + \lim_{k\to\infty} d_X(y_k, z_k) = 0, \end{aligned}$$
which implies $(x_k) \sim (z_k)$.

We then define $X_c \coloneqq C/\sim$ to be the set of equivalence classes of C under this relation, and define a metric d_{X_c} on X_c as follows. For $\sigma, \tau \in X_c$, with equivalence class representatives $(x_k), (y_k) \in C$, respectively, we set

$$d_{X_c}(\sigma, \tau) \coloneqq \lim_{k\to\infty} d_X(x_k, y_k).$$

Before we define ϕ or prove properties (1) and (2) from the proposition, we should d_{X_c} is well-defined and satifies the axioms of a metric.

2.4. **The metric is well-defined.** Let $\sigma, \tau \in X_c$ with representatives $(x_n), (y_n) \in C$, repspectively. Since $(x_n), (y_n)$ are Cauchy, there exists $n \in \mathbb{N}$ such that $d_X(x_j, x_k) < \epsilon$ and $d_E(y_j, y_k) < \epsilon$ if $j, k \geq n$. By the triangle inequality we have

$$d_X(x_j, y_j) \leq d_X(x_j, x_k) + d_X(x_k, y_k) + d_X(y_j, y_k) \implies d_X(x_j, y_j) - d_X(x_k, y_k) < 2\epsilon$$

and similarly, $d_X(x_k, y_k) - d_X(x_j, y_j) < 2\epsilon$, for $j, k \geq n$. It follows that the sequence $(d_E(x_n, y_n))$ in $\mathbb{R}$ is Cauchy. Since $\mathbb{R}$ is complete, $d_{X_c}(\sigma, \tau) = \lim_{n\to\infty} d_E(x_n, y_n)$ exists.

Now suppose $(x_k) \sim (x'_k)$ and $(y_k) \sim (y'_k)$. Then by the triangle inequality we have

$$\lim_{k\to\infty} d_X(x'_k, y'_k) \leq \lim_{k\to\infty} d_X(x'_k, x_k) + \lim_{k\to\infty} d_X(x_k, y_k) + \lim_{k\to\infty} d_X(y_k, y'_k) = \lim_{k\to\infty} d_X(x_k, y_k).$$

By symmetry the reverse inequality holds as well, so d_{X_c} is well-defined.

2.5. **The metric is a metric.** Next we show d_{X_c} is in fact a metric on X_c. Let $\sigma, \tau \in X_c$ with representatives $(x_n), (y_n) \in C$, repspectively. Nonnegativity symmetry, and the triangle inequality for d_{X_c} follow immediately from these holding for d_X. And that $d_{X_c}(\sigma, \tau) = 0$ if and only if $\sigma = \tau$ follows by definition of d_{X_c} and the relation $\sim$.

2.6. **The isometric embedding.** Define $\phi : X \to X_c$ by letting $\phi(x)$ be the equivalence class of the constant sequence given by $x_k \coloneqq x$ for $k \in \mathbb{N}$. Then for $x, y \in X$ we have

$$d_X(\phi(x), \phi(y)) = \lim_{k\to\infty} d_X(x, y) = d_X(x, y),$$

so ϕ is an isometry.

Next we show that $\phi(X)$ is dense in X_c. Let $\sigma \in X_c$ with representative $(x_n) \in C$. Let $\epsilon > 0$. Since (x_n) is Cauchy there exists $n \in \mathbb{N}$ such that $d_X(x_k, x_n) < \epsilon$ for all $k \geq N$. It follows that

$$d_{X_c}(\sigma, \phi(x_n)) = \lim_{k\to\infty} d_E(x_k, x_n) < \epsilon.$$

I.e., $B_{d_{X_c}}(\sigma, \epsilon)$ intersects $\phi(X)$ for any $\sigma \in X_c$ and $\epsilon > 0$. So indeed $\phi(X)$ is dense in X_c.

2.7. **The completion is complete.** Finally we show that X_c is complete. Let $(\sigma_k) \in X_c^{\mathbb{N}}$ a Cauchy sequence in X_c. Then for each $r \in \mathbb{N}$ there exists $n_r \in \mathbb{N}$ such that $d_{X_c}(\sigma_k, \sigma_{n_r}) < 1/r$ for $k \geq n_r$. Since $\phi(X)$ is dense in X_c, there exists $x_r \in E$ such that $d_{X_c}(\sigma_{n_r}, \phi(x_r)) < 1/r$ for each $r \in \mathbb{N}$.[1] We claim $(x_r) \in X^{\mathbb{N}}$ is Cauchy. Let $\epsilon > 0$, and choose $n \in \mathbb{N}$ such that $3/n < \epsilon$. Then for $k \geq j \geq n$ we have by the triangle ineqality and the fact that ϕ is an isometry that

$$\begin{aligned} d_X(x_j, x_k) = d_{X_c}(\phi(x_j), \phi(x_k)) &\leq d_{X_c}(\phi(x_j), \sigma_{n_j}) + d_{X_c}(\sigma_{n_j}, \sigma_{n_k}) + d_{X_c}(\sigma_{n_k}, \phi(x_k)) \\ &\leq 1/j + 1/j + 1/k \leq 3/n < \epsilon, \end{aligned}$$

so indeed (x_r) is Cauchy.

Then let $\sigma \in X_c$ be the equivalence class of (x_r). We will show $\lim_{k\to\infty} \sigma_k = \sigma$, and since (σ_k) was arbitrary this will show that X_c is indeed complete. So let $\epsilon > 0$. Choose $r \in \mathbb{N}$ such that $4/r < \epsilon$. We have by the triangle inequality that

$$d_{X_c}(\sigma_{n_r}, \sigma) \leq d_{X_c}(\sigma_{n_r}, \phi(x_r)) + d_{X_c}(\phi(x_r), \sigma).$$

[1] Observe (for intuition) that we start with a sequence of sequences, which can visualise as a grid, and are producing a new sequence by taking some type of rapidly converging diagonal from this grid.

We know $d_{X_c}(\sigma_{n_r}, \phi(x_r)) < 1/r$, and we showed above that $d_X(x_k, x_r) < 3/r$ for $k \geq r$, which implies

$$d_{X_c}(\phi(x_r), \sigma) = \lim_{k\to\infty} d_X(x_r, x_k) < 3/r.$$

It follows that $d_{X_c}(\sigma_{n_r}, \sigma) < 4/r < \epsilon$. Then since (σ_{n_r}) is Cauchy, it is easy to see we must then have $\lim_{k\to\infty} \sigma_k = \sigma$. So we have proven (1) and (2) of Proposition 2.1.

2.8. **The universal property.** We now prove (3) and (4). These will follow from the following more general facts.[2]

2.9. **Lemma.** *Let A, B be topological spaces, and $A_0 \subseteq A$ a dense subspace. Let $f : A_0 \to B$ a continuous map.*

(1) *If B is Hausdorff, there exists at most one extension of f to A, that is, there exists at most one continuous map $g : A \to B$ such that $g|_{A_0} = f$.*
(2) *If A and B are metric spaces, B is complete, and f is uniformly continuous, then there exists a unique uniformly continuous extension $g : A \to B$ of f. If f is an isometry, then so is g.*

Proof. (1) Let $g_1, g_2 : A \to B$ two extensions of f. Suppose $g_1(a) \neq g_2(a)$ for some $a \in A$. Then we can choose V_1 and V_2 disjoint neighbourhoods of $g_1(a)$ and $g_2(a)$, respectively, by the Hausdorff hypothesis. Let $U_i \coloneqq g_i^{-1}(V_i)$, open by continuity of g_i, for $i \in \{1, 2\}$. Then $a \in U_1 \in U_2$, so $U_1 \cap U_2$ is a nonempty open set in X, and hence there exists $a_0 \in U_1 \cap U_2 \cap A_0$ by density of A_0. Now observe that since g_1, g_2 extend f we must have

$$V_1 \ni g_1(a_0) = f(a_0) = g_2(a_0) \in V_2,$$

contradicting the disjointness of V_1 and V_2.

(2) Uniqueness is immediate from (1), so it suffices to show existence. Define $g : A \to B$ as follows. Let $a \in A$. By density of A_0 there is a sequence $(a_k) \in A_0^{\mathbb{N}}$ such that $a_k \to a$ as $k \to \infty$. By Exercise 1.3 (a_k) is Cauchy, whence $(f(a_k)) \in B^{\mathbb{N}}$ is Cauchy by Exercise 1.8. Then $f(a_k) \to b$ as $k \to \infty$ for some $b \in B$ by completeness. We define $g(a) \coloneqq b$. We claim this is independent of the choice of sequence (a_k). Indeed let $(a'_k) \in A_0^{\mathbb{N}}$ another sequence converging to a. Then for any $\delta > 0$ there exists $n \in \mathbb{N}$ such that, by the triangle inequality,

$$d(a, a_k) < \delta/2 \text{ and } d(a, a'_k) < \delta/2 \implies d(a_k, a'_k) < \delta$$

for $k \geq n$. Therefore by uniform continuity there exists $n \in \mathbb{N}$ such that

$$d(f(a_k), f(a'_k)) < \epsilon \quad \text{for all } k \geq n.$$

It follows that $\lim_{k\to\infty} f(a_k) = \lim_{k\to\infty} f(a'_k)$, proving our claim. In particular, if $a \in A_0$ then we can choose the constant sequence $a_k \coloneqq a_0$ for $k \in \mathbb{N}$, from which we see $g(a) = \lim_{k\to\infty} f(a_0) = f(a_0)$, so g in fact extends f.

We now show g is uniformly continuous. Let $\epsilon > 0$. Since f is uniformly continuous there exists $\delta > 0$ such that $d_B(f(a), f(a')) < \epsilon/2$ whenever $a_0, a'_0 \in A_0$ are such that $d_A(a_0, a'_0) < \delta$. Suppose $a, a' \in A$ are such that $d(a, a') < \delta/3$. Choose sequences (a_k) and (a'_k) in A_0 converging to a and a', respectively. Let $n \in \mathbb{N}$ such that, by the triangle inequality,

$$\begin{aligned} d_X(a_k, a) < \delta/3 \text{ and } d_X(a'_k, a') < \delta/3 &\implies d_X(a_k, a'_k) < \delta \\ &\implies d_Y(f(a_k), f(a'_k)) < \epsilon/2 \end{aligned}$$

for all $k \geq n$. Then by continuity[3] of d_Y and definition of g we have

$$(\dagger) \qquad d_Y(g(a), g(a')) = \lim_{k\to\infty} d_Y(f(a_k), f(a'_k)) \leq \epsilon/2 < \epsilon.$$

[2]The presentation here is taken from Pete Clark's notes, `math.uga.edu/~pete/8410Chapter2v2.pdf`

[3]One could also argue more directly, but see, e.g., my solutions to problem set 4 for a proof of the continuity of the metric.

Thus g is uniformly continuous. If f is moreover an isometry, the first equality in (†) also clearly implies that g is an isometry. $\square$

We now apply the lemma to finish the proof of the proposition. Suppose Y is a complete metric space and $\psi : X \to Y$ uniformly continuous. Since $\phi : X \to X_c$ is an isometry with dense image, the lemma gives that there is a unique uniformly continuous map $\psi_c : X_c \to Y$ such that $\psi_c \circ \phi = \phi$, proving (3).

Finally, (4) is proven by the general argument giving uniqueness of objects satisfying a universal property like (3). Suppose Y_c is a complete metric space and $\psi : X \to Y_c$ an isometry with dense image. Then by (3) there is a unique uniformly continuous map $\psi_c : X_c \to Y_c$ such that $\psi_c \circ \phi = \psi$, and the lemma tells us that ψ_c is in fact an isometry. But our proof of (3) applies to Y_c equipped with ψ as well, so we symmetrically have a unique isometry $\phi_c : Y_c \to X_c$ such that $\phi_c \circ \psi = \phi$. I.e., we have the diagram

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "\psi"'] & X_c \arrow[dl, shift left, "\psi_c"] \\ Y_c. \arrow[ur, shift left, "\phi_c"] & \end{tikzcd}$$

Observe then that

$$\phi_c \circ \psi_c \circ \phi = \phi = \mathrm{id}_{X_c} \circ \phi \quad \text{and} \quad \psi_c \circ \phi_c \circ \psi = \psi = \mathrm{id}_{Y_c} \circ \psi.$$

Then the *uniqueness* statement in (3) implies that we must have $\phi_c \circ \psi_c = \mathrm{id}_{X_c}$ and $\psi_c \circ \phi_c = \mathrm{id}_{Y_c}$.[4] I.e., ϕ_c, ψ_c are inverse isomorphisms of metric spaces. And thus we are done!

[4]This argument takes a couple of reads to absorb and understand, I think (it certainly did for me!). But it's extremely general and very useful so try to do so!