MATH 131 SECTION, I: METRIC TOPOLOGY

ARPON RAKSIT

1. Some more on countability

Professor McMullen didn't get quite as far as he had planned to last Thursday, so let's talk a little more about countability. (The material for this part of section is all on pp. 8–11 of Professor McMullen's notes on the course website.)

1.1. **Exercise.** *Let S be an enemy submarine travelling indefinitely along a line $\mathbb{R}$ (underwater!). Suppose our intelligence is intelligent enough to know that, at time $t = 0$, S is at some $i \in \mathbb{Z} \subset \mathbb{R}$, and that S moves with some constant velocity $v \in \mathbb{Z}$; but not intelligent to know i or v. You need to shoot down S to win the war; luckily your set of torpedos is in bijection with $\mathbb{N}$. Give a strategy to guarantee victory in finite time.*

2. Quick review of metric spaces

I bet you'll find yourself thinking at some point that point-set topology is just absurdly, unreasonably, unmotivatedly, unnecessarily general.[1] When this happens, do two things:

(1) Stop thinking that. Bask in the generality. Like Professor McMullen said in the introductory lecture, it turns out all this generality is unbelievably useful all over mathematics. Otherwise this course probably wouldn't be taught.
(2) But seriously, if you find yourself miserable, amidst a downpour of statements about open sets, seek shelter in the land of metric spaces! Metric spaces can give you a nice picture to keep in your head when thinking about weird topological definitions. Moreover, if you're stuck in a proof, maybe try proving the statement with a metric, and see if you can't translate things back into topological terms.

In light of point (2), I thought I'd review the basic definitions of metric topology and how they relate to general topology.

2.1. **Definition.** A *metric space* is the following data:

— a set X,
— a function $d : X \times X \to \mathbb{R}$,

satisfying the following conditions:

— for $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$,
— for $x, y \in X$, $d(x, y) = d(y, x)$,
— for $x, y, z \in X$, $d(x, y) \le d(x, z) + d(z, y)$.

We refer to a metric space by the pair (X, d), or, by the usual abuse of notation when no confusion might arise, simply by X. We refer to d as the *metric* on X.

2.2. **Example.** The first example one thinks of is $X := \mathbb{R}^n$ and $d(x, y) := |x - y|$. Indeed you're probably picturing $\mathbb{R}^2$ in your head whenever you're thinking about a topological space (but be wary of this as we tread into weirder corners of topology!). Of course there are

Date: October 19, 2013.

[1] cf. `http://en.wikipedia.org/wiki/Pointless_topology`

other interesting examples, but I don't want to spend time developing the theory of metric spaces[2] right now, so I'll be a bad person and leave them out.

Of course metric spaces are usually presented as a more general place where we can talk about limits and continuous functions in terms of our trusty ϵ and δ.

2.3. **Definition.** Let X a metric space. A sequence $(x_k)_{k\in\mathbb{N}}$ of points in X is said to *converge* to $x \in X$ if for each $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $d(x_k, x) < \epsilon$ for all $k \geq n$. We write $\lim_{k\to\infty} x_k = x$, or $x_k \to x$ as $k \to \infty$.

2.4. **Exercise.** *A sequence converges to a unique point.*

2.5. **Definition.** Let $(X, d_X), (Y, d_Y)$ metric spaces. A function $\phi : X \to Y$ is

— *continuous at* $x \in X$ if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d_X(x, y) < \delta \implies d_Y(\phi(x), \phi(y)) < \epsilon$$

for $y \in X$;

— a *continuous map* (or just *map* if we're feeling lazy) if ϕ is continuous at each $x \in X$.

Ok, that's all well and good, and we can do things like differential calculus with these definitions. But in this class we are topologists, not analysts. We don't want to push these loathsome epsilons around! Even the smallest of epsilons is a burden. So let's rephrase things.

2.6. **Definition.** Let X a metric space, $x \in X$, and $r > 0$. The *open ball of radius r centred at x* is defined as $B(x, r) \coloneqq \{y \in X \mid d(x, y) < r\}$.

2.7. **Definition.** Let X a metric space. We say a subset $U \subseteq X$ is *open* if for each $x \in U$ there exists an $r > 0$ such that $B(x, r) \subseteq U$.

2.8. **Exercise.** *$B(x, r) \subseteq X$ is open for any $x \in X, r > 0$.*

2.9. **Proposition.** *Let X a metric space. Then $\mathcal{T} \coloneqq \{U \subseteq X \mid U \text{ open}\}$ is a topology on X.*

Proof. Let $\{U_\alpha\}_{\alpha\in A} \subseteq \mathcal{T}$, set $U \coloneqq \bigcup_{\alpha\in A} U_\alpha$, and let $x \in U$. Pick $\alpha \in A$ such that $x \in U_\alpha$. By definition there exists $r > 0$ such that $B(x, r) \subseteq U_\alpha \subseteq U$.

Next let $U_1, \ldots, U_n \in \mathcal{T}$, set $U \coloneqq \bigcap_{i=1}^n U_i$, and let $x \in U$. By definition there exists $r_i > 0$ such that $B(x, r_i) \subseteq U_i$ for $1 \leq i \leq n$. Then if we set $r \coloneqq \min\{r_i \mid 1 \leq i \leq n\}$ (this is where we use that the intersection is finite!) we have $B(x, r) \subseteq U_i$ for $1 \leq i \leq n$, whence $B(x, r) \subseteq U$.

Since obviously $\emptyset, X \in \mathcal{T}$, $\mathcal{T}$ is indeed a topology on X. $\square$

2.10. **Exercise.** *A subset $U \subseteq X$ is open if and only if it's a union of balls $\bigcup_{\alpha\in A} B(x_\alpha, r_\alpha)$. (Soon you'll learn that the open balls form a basis for this topology on X.)*

2.11. **Proposition.** *Let X a metric space and $A \subseteq X$. The following are equivalent:*

(1) *$X - A$ is open.*
(2) *For any sequence $(x_k)_{k\in\mathbb{N}}$ in A converging to $x \in X$, we have $x \in A$.*

Proof. Assume (1). Let $(x_k)_{k\in\mathbb{N}}$ a sequence in A converging to $x \in X$. If $x \in X - A$ then by hypothesis there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq X - A$. But one easily sees that this contradicts that $x_k \to x$ as $k \to \infty$.

Assume (2). Let $x \in X$. Assume $B(x, \epsilon) \cap A \neq \emptyset$ for each $\epsilon > 0$. In particular, for $k \in \mathbb{N}$ we have $x_k \in A$ such that $d(x_k, x) < 1/k$. Here one easily checks that $x_k \to x$ as $k \to \infty$. So $x \in A$ by hypothesis. $\square$

[2] Professor Elkies has some nice notes that do this, see `http://www.math.harvard.edu/~elkies/M55b.10/index.html`. You can find some cool problems there also.

So being closed under the operation of taking limits of sequences is the same thing as being closed in the topological sense.

2.12. **Proposition.** *Let X, Y metric spaces and $\phi : X \to Y$ a function. The following are equivalent*

(1) $\phi^{-1}(U) \subseteq X$ *is open whenever* $U \subseteq Y$ *is open.*
(2) ϕ *is a continuous map.*

Proof. Assume (1). Let $x \in X$ and $\epsilon > 0$. Since $B(\phi(x), \epsilon)$ is open in Y, by hypothesis $\phi^{-1}(B(\phi(x), \epsilon))$ is open in X. Thus there exists $\delta > 0$ such that $\phi(B(x, \delta)) \subseteq B(\phi(x), \epsilon)$, which says exactly that ϕ is continuous at x.

Assume (2). Let $U \subseteq Y$ an open set, $x \in \phi^{-1}(U)$, and $\epsilon > 0$ such that $B(\phi(x), \epsilon) \subseteq Y$. By continuity we can choose $\delta > 0$ such that

$$\phi(B(x, \delta)) \subseteq B(y, \epsilon) \implies B(x, \delta) \subseteq \phi^{-1}(U).$$

It follows that $\phi^{-1}(U)$ is open. $\square$

So a map of metric spaces is continuous in the ϵ-δ sense if and only if it's continuous in the topological sense. Ok, hopefully this provides some motivation for why things are defined the way they are for general topological spaces. (Of course, it's easy for me to say this is all motivated—the hard thing was for the actual mathematicians to come up with the correct general formulation for these notions!)