THE DOLD-KAN CORRESPONDENCE

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1. Introduction

1.1. **Definition.** A *simplicial object* in a category $\mathcal{C}$ is a contravariant functor $\Delta \to \mathcal{C}$. We denote the category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ of simplicial objects in $\mathcal{C}$ by $\mathrm{s}\mathcal{C}$. E.g., $\mathrm{s}\mathcal{S}\mathrm{et}$ is the category of *simplicial sets* and $\mathrm{s}\mathcal{A}\mathrm{b}$ is the category of *simplicial abelian groups*.

Recall we have a functor $\mathrm{Sing} : \mathcal{T}\mathrm{op} \to \mathrm{s}\mathcal{S}\mathrm{et}$, sending $X \mapsto \mathrm{Hom}_{\mathcal{T}\mathrm{op}}(|\Delta^\bullet|, X)$. Lately we've been talking about Sing for two reasons:

(1) It's a right adjoint to geometric realisation $|-| : \mathrm{s}\mathcal{S}\mathrm{et} \to \mathcal{T}\mathrm{op}$.
(2) $\mathrm{Sing}(X)$ is a Kan complex for all $X \in \mathcal{T}\mathrm{op}$. In this sense, "Kan complexes are like spaces".

But this isn't the first place one sees Sing, probably. Indeed, the singular homology functors are essentially defined by a composition

$$\mathrm{H}_n(-;\mathbb{Z}) \coloneqq \mathcal{T}\mathrm{op} \xrightarrow{\mathrm{Sing}} \mathrm{s}\mathcal{S}\mathrm{et} \xrightarrow{\mathbb{Z}} \mathrm{s}\mathcal{A}\mathrm{b} \xrightarrow{\sum(-1)^i d_i} \mathcal{C}\mathrm{h}_{\geq 0} \xrightarrow{\mathrm{H}_n} \mathcal{A}\mathrm{b}.$$

Here $\mathbb{Z}$ denotes the functor which takes free abelian groups level-wise, from which we get the *singular chain complex* by letting the boundary map be given by $\partial \coloneqq \sum(-1)^i d_i$.

This was just to remind us that we've seen a natural functor $\mathrm{s}\mathcal{A}\mathrm{b} \to \mathcal{C}\mathrm{h}$ relating simplicial abelian groups and chain complexes. We'll look at it a bit more carefully in a second, and develop this relationship much further.

2. Stating the correspondence

We fix $\mathcal{A}$ any abelian category—but we'll probably be imagining $\mathcal{A} = \mathcal{A}\mathrm{b}$ or $\mathcal{A} = R\text{-}\mathcal{M}\mathrm{od}$ (for some commutative ring R).

2.1. **Notation.** We denote the category of nonnegatively graded chain complexes in $\mathcal{A}$ (and chain maps) by $\mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})$.

Let's now make precise the $\partial \coloneqq \sum(-1)^i d_i$ business with which we started this discussion.

2.2. **Definition.** Let $A \in \mathrm{s}\mathcal{A}$ a simplicial object in $\mathcal{A}$ (e.g., a simplicial abelian group). We define the *associated chain complex* $C(A) \in \mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})$ by

$$C_n(A) \coloneqq A_n \quad \text{and} \quad \partial_n \coloneqq \textstyle\sum_{i=0}^n (-1)^i d_i : C_n(A) \to C_{n-1}(A)$$

for $n \geq 0$. Note that the simplicial identities clearly imply $\partial^2 = 0$, so $C(A)$ is indeed a chain complex. Moreover, this evidently defines a functor $C : \mathrm{s}\mathcal{A} \to \mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})$.

This is perhaps the most natural—or familiar, at least—functor $\mathrm{s}\mathcal{A} \to \mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})$, but it turns out not to be the cleanest to use when discussing the relationship between the two categories. In fact, we will want to use the following alternative.

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2.3. **Definition.** Again let $A \in \mathrm{s}\mathcal{A}$ a simplicial object in $\mathcal{A}$. We define the *normalised chain complex* $N(A) \in \mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})$ by

$$N_n(A) \coloneqq \bigcap_{i=0}^{n-1} \ker(d_i) \subseteq A_n \quad \text{and} \quad \partial_n \coloneqq (-1)^n d_n : N_n(A) \to N_{n-1}(A)$$

for $n \geq 0$. Again the simplicial identities imply $\partial^2 = 0$, and we have a functor $N : \mathrm{s}\mathcal{A} \to \mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})$.

What is this unmotivated nonsense? Have no fear, for C and N are intimately related! For instance we can note immediately from the definitions that the natural inclusion $N(A) \to C(A)$ is in fact a chain map for $A \in \mathrm{s}\mathcal{A}$. But there's more!

2.4. **Definition.** Let $A \in \mathrm{s}\mathcal{A}$. We define the *degenerate subcomplex* $D(A)$ of $C(A)$ by

$$D_0(A) \coloneqq 0 \quad \text{and} \quad D_n(A) \coloneqq \sum_{i=0}^{n-1} \mathrm{im}(s_i)$$

for $n \geq 1$. That is, $D(A)$ is generated by the images of the degeneracy maps. Note that by the simplicial identities

$$\partial s_j = \sum_{i=0}^{n} (-1)^i d_i s_j = \sum_{i=0}^{j-1} (-1)^i s_{j-1} d_i + \sum_{i=j+2}^{n} (-1)^i s_j d_i,$$

so $D(A)$ is indeed a subcomplex.

2.5. **Proposition.** *Let $A \in \mathrm{s}\mathcal{A}$. For $n \geq 0$ the natural map*

$$\phi : N_n(A) \oplus D_n(A) \to A_n = C_n(A)$$

induced by the inclusions is an isomorphism. Therefore we have a natural isomorphism

$$N(A) \simeq C(A)/D(A).$$

Proof. Fix $n \geq 0$. For $0 \leq i \leq n-1$, the simplicial identity $d_i s_i = \mathrm{id}$ implies that we have a canonical splitting $A_n \simeq \ker(d_i) \oplus \mathrm{im}(s_i)$. It follows easily that $N_n(A) \cap D_n(A) \simeq 0$, so we're just left to show that ϕ is surjective. We prove by downward induction on $0 \leq j \leq n-1$ that

$$\mathrm{im}(\phi) \supseteq N_j \coloneqq \bigcap_{i=0}^{j} \ker(d_i).$$

The base case $j = n-1$ is tautological and the final case $j = 0$ will finish the proof. Now consider the map $\psi \coloneqq \mathrm{id} - s_j d_j : C_n \to C_n$. Observe by the simplicial identities that

$$d_j \psi = d_j - d_j s_j d_j = d_j - d_j = 0 \quad \text{and} \quad d_i \psi = d_i - d_i s_j d_j = d_i - s_{j-1} d_{j-1} d_i$$

for $i < j$, implying that $\psi(N_j) \subseteq N_{j+1}$. By induction $\mathrm{im}(\phi) \supseteq N_{j+1}$, and since $\mathrm{im}(s_j d_j) \supseteq D_n(A)$ it follows that $\mathrm{im}(\phi) \supseteq N_j$. $\square$

So there's the relationship between C and N. With these definitions in hand, we can now state our main goal, the *Dold-Kan correspondence.*

2.6. **Theorem** (Dold-Kan)**.** *The functor $N : \mathrm{s}\mathcal{A} \to \mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})$ is an equivalence of categories.*

3. Proving the correspondence

3.1. **Definition.** Let $C \in \mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})$. Define

$$\Gamma_n(C) \coloneqq \bigoplus_{[n] \twoheadrightarrow [k]} C_k,$$

where the direct sum is over all surjections $\sigma : [n] \twoheadrightarrow [k]$ in the category Δ.

Let $\nu : [m] \to [n]$ a morphism in Δ. Let $\tau : [n] \twoheadrightarrow [k]$ an indexing surjection. We can factor $\tau\nu$ as a composition $[m] \twoheadrightarrow [j] \hookrightarrow [k]$ of a surjection σ and an injection ι. Then we define a map[1]

$$C_k \to C_j \quad \text{as} \quad \begin{cases} \mathrm{id}_{C_n}, & \text{if } j = k; \\ (-1)^n \partial_n, & \text{if } j = k-1 \text{ and } \iota = d^k; \\ 0, & \text{otherwise.} \end{cases}$$

Then composition with the inclusion $C_j \to \Gamma_m(C)$ of the factor with index $\sigma : [m] \twoheadrightarrow [j]$ gives a map $C_k \to \Gamma_m(C)$. Finally, the direct sum of these maps gives us an induced morphism $\nu^* : \Gamma_n(C) \to \Gamma_m(C)$.

Suppose $\mu : [l] \to [m]$ is another morphism in Δ. Factoring $\sigma\mu$ as $\rho\theta : [l] \twoheadrightarrow [i] \hookrightarrow [j]$, we have a commutative diagram

$$\begin{tikzcd} {[l]} \arrow[r, "\mu"] \arrow[d, two heads, "\rho"] & {[m]} \arrow[r, "\nu"] \arrow[d, two heads, "\sigma"] & {[n]} \arrow[d, two heads, "\tau"] \\ {[i]} \arrow[r, hook, "\theta"] & {[j]} \arrow[r, hook, "\iota"] & {[k].} \end{tikzcd}$$

It's easy to see then that $(\nu\mu)_* = \mu_*\nu_*$.

It is also evident that a chain map $C \to D$ in $\mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})$ gives rise to a simplicial map $\Gamma(C) \to \Gamma(D)$ in $\mathrm{s}\mathcal{A}$ via factor-wise application. So finally we have constructed a functor

$$\Gamma : \mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A}) \to \mathrm{s}\mathcal{A},$$

which to each chain complex in $\mathcal{A}$ gives an *associated simplicial object in* $\mathcal{A}$.

To prove Dold-Kan, we're going to show that Γ is a quasi-inverse to $N : \mathrm{s}\mathcal{A} \to \mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})$.

3.2. **Definition.** Observe that there is a natural transformation $\Phi : \Gamma \circ N \to \mathrm{id}_{\mathrm{s}\mathcal{A}}$ defined by the maps

$$\Phi_n(A) : \Gamma_n(N(A)) = \bigoplus_{[n] \twoheadrightarrow [k]} N_k(A) \to A_n$$

for $A \in \mathrm{s}\mathcal{A}$ and $n \geq 0$, which restrict to the factor indexed by $\sigma : [n] \twoheadrightarrow [k]$ as the composition

$$N_k(A) \hookrightarrow A_k \xrightarrow{\sigma^*} A_n.$$

(It is clear this defines a simplicial map $\Gamma(N(A)) \to A$ which is natural in A.)

3.3. **Lemma.** *In fact $\Phi : \Gamma \circ N \to \mathrm{id}_{\mathrm{s}\mathcal{A}}$, defined above, is a natural isomorphism.*

Proof. Fix $A \in \mathrm{s}\mathcal{A}$. We will prove by induction on $n \geq 0$ that $\Phi_n(A) : \Gamma_n(N(A)) \to A_n$ is an isomorphism, and then we will be done. Since the only surjection $[0] \twoheadrightarrow [k]$ in Δ is $\mathrm{id}_{[0]}$, and the inclusion $N_0(A) \hookrightarrow A_0$ is an isomorphism, the base case $n = 0$ is tautological.

First, surjectivity. Recall from Proposition 2.5 that we have a splitting $A_n \simeq N_n(A) \oplus D_n(A)$. From the factor $\mathrm{id}_{[n]} : [n] \twoheadrightarrow [n]$ of $\Gamma_n(N(A))$ we $\mathrm{im}(\Phi_n(A)) \supseteq N_n(A)$. By induction $\Phi_{n-1}(A)$ is surjective, so by definition we must have $\mathrm{im}(\Phi_n(A)) \supseteq D_n(A)$. Hence $\Phi_n(A)$ is surjective.

Next, injectivity. Assume we have $(x_\sigma) \in \ker(\Phi_n(A))$. Fix $0 \leq k < n$. Observe that to each surjection $\sigma : [n] \twoheadrightarrow [k]$ we can assign a section of σ,

$$\nu_\sigma : [k] \hookrightarrow [n], \quad \nu_\sigma(i) := \max\{j \in [n] \mid \sigma(j) = i\}.$$

If we have $\sigma, \sigma' : [n] \twoheadrightarrow [k]$, we say

$$\sigma \leq \sigma' \iff \nu_\sigma(i) \leq \nu_{\sigma'}(i) \text{ for all } i \in [k].$$

[1]This definition is not so random: compare it with our definition of the normalised chain complex N.

In particular, $\sigma'\nu_\sigma = \mathrm{id}_{[k]} \implies \sigma \leq \sigma'$. If there exists $\tau : [n] \twoheadrightarrow [k]$ such that $x_\tau \neq 0$, choose a *maximal* such τ (with respect to the ordering just defined). By definition of the simplicial structure on $\Gamma(N(A))$, it follows that the component of $v_\tau^*(x_\sigma)$ in the factor of $\Gamma_k(N(A))$ indexed by $\mathrm{id}_{[k]}$ is precisely x_τ. But then, by induction,

$$(x_\sigma) \in \ker(\Phi_n(A)) \implies v_\tau^*(x_\sigma) \in \ker(\Phi_k(A)) \implies x_\tau = 0,$$

contradiction.

So we must have $x_\sigma = 0$ for all $\sigma \neq \mathrm{id}_{[n]}$. But the restriction of $\Phi_n(A)$ to the factor indexed by $\mathrm{id}_{[n]}$ is just the inclusion $N_n(A) \hookrightarrow A_n$. So then $x_{\mathrm{id}_{[n]}} = 0$ too, and hence $\Phi_n(A)$ is injective. $\square$

3.4. **Lemma.** *Let $C \in \mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})$. For $n \geq 0$, the natural inclusion*

$$\Psi_n(C) : N_n(\Gamma(C)) \hookrightarrow C_n(\Gamma(C)) = \Gamma_n(C) = \bigoplus_{[n]\twoheadrightarrow[k]} C_k$$

has image the factor C_n indexed by $\mathrm{id}_{[n]}$. This of course gives a natural isomorphism

$$\Psi : N \circ \Gamma \to \mathrm{id}_{\mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})}.$$

Proof. By definition of the simplicial structure on $\Gamma(C)$ we have $C_n \subseteq \bigcap_{i=0}^{n-1} \ker(d_i) = \mathrm{im}(\Psi_n(C))$. Now note for $\sigma : [n] \twoheadrightarrow [k]$ with $k < n$, we must have a factorisation of σ as a composition

$$[n] \xrightarrow{s^i} [n-1] \longrightarrow [k],$$

so it follows that the factor of $\Gamma_n(C)$ indexed by σ lies in the image of the degeneracies $D_n(\Gamma(C))$. Then we're done, since by Proposition 2.5 we have a splitting

$$\Gamma_n(C) \simeq N_n(\Gamma(C)) \oplus D_n(\Gamma(C)).$$ $\square$

Proof of Theorem 2.6. The natural isomorphisms $\Phi : \Gamma \circ N \to \mathrm{id}_{\mathrm{s}\mathcal{A}}$ and $\Psi : N \circ \Gamma \to \mathrm{id}_{\mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})}$ of Lemmas 3.3 and 3.4 exhibit N (and Γ) as an equivalence of categories, thus proving the Dold-Kan correspondence. $\square$

4. Applying the correspondence

4.1. **Proposition.** *The natural inclusion $N(A) \to C(A)$ for $A \in \mathrm{s}\mathcal{A}$ gives a natural chain homotopy equivalence.*

References

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