THE DOLD-KAN CORRESPONDENCE

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1. Introduction

1.1. **Definition.** A *simplicial object* in a category $\mathcal{C}$ is a contravariant functor from the simplex category Δ to $\mathcal{C}$. We denote the category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ of simplicial objects in $\mathcal{C}$ by $\mathrm{s}\mathcal{C}$. E.g., $\mathrm{s}\mathcal{S}\mathrm{et}$ is the category of *simplicial sets* and $\mathrm{s}\mathcal{A}\mathrm{b}$ is the category of *simplicial abelian groups*.

Recall we have a functor $\mathrm{Sing} : \mathcal{T}\mathrm{op} \to \mathrm{s}\mathcal{S}\mathrm{et}$, sending $X \mapsto \mathrm{Hom}_{\mathcal{T}\mathrm{op}}(|\Delta^\bullet|, X)$. Lately we've been talking about Sing for two reasons:

(1) It's a right adjoint to geometric realisation $|-| : \mathrm{s}\mathcal{S}\mathrm{et} \to \mathcal{T}\mathrm{op}$.
(2) $\mathrm{Sing}(X)$ is a Kan complex for all $X \in \mathcal{T}\mathrm{op}$—this was the start of the slogan "Kan complexes are like spaces".

But this isn't the first place one sees Sing, probably. Indeed, the singular homology functors are essentially defined by a composition

$$\mathrm{H}_n(-;\mathbb{Z}) \coloneqq \mathcal{T}\mathrm{op} \xrightarrow{\mathrm{Sing}} \mathrm{s}\mathcal{S}\mathrm{et} \xrightarrow{\mathbb{Z}} \mathrm{s}\mathcal{A}\mathrm{b} \xrightarrow{\sum(-1)^i d_i} \mathcal{C}\mathrm{h}_{\geq 0} \xrightarrow{\mathrm{H}_n} \mathcal{A}\mathrm{b}.$$

Here $\mathbb{Z}$ denotes the functor which takes free abelian groups level-wise, from which we get the *singular chain complex* by letting the boundary map be given by $\partial \coloneqq \sum(-1)^i d_i$.

This was just to remind us that we've seen a natural functor $\mathrm{s}\mathcal{A}\mathrm{b} \to \mathcal{C}\mathrm{h}$ relating simplicial abelian groups and chain complexes. We'll look at it a bit more carefully in a second, and develop this relationship much further.

2. Stating the correspondence

We fix $\mathcal{A}$ any abelian category—but we'll probably be imagining $\mathcal{A} = \mathcal{A}\mathrm{b}$ or $\mathcal{A} = R\text{-}\mathcal{M}\mathrm{od}$ (for some commutative ring R).

2.1. **Notation.** We denote the category of nonnegatively graded chain complexes in $\mathcal{A}$ (and chain maps) by $\mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})$.

Let's now make precise the $\partial \coloneqq \sum(-1)^i d_i$ business with which we started this discussion.

2.2. **Definition.** Let $A \in \mathrm{s}\mathcal{A}$ a simplicial object in $\mathcal{A}$ (e.g., a simplicial abelian group). We define the *associated chain complex* $C(A) \in \mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})$ by

$$C_n(A) \coloneqq A_n \quad \text{and} \quad \partial_n \coloneqq \textstyle\sum_{i=0}^n (-1)^i d_i : C_n(A) \to C_{n-1}(A)$$

for $n \geq 0$. Note that the simplicial identities clearly imply $\partial^2 = 0$, so $C(A)$ is indeed a chain complex. Moreover, this evidently defines a functor $C : \mathrm{s}\mathcal{A} \to \mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})$.

This is perhaps the most natural—or familiar, at least—functor $\mathrm{s}\mathcal{A} \to \mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})$, but it turns out not to be the cleanest to use when discussing the relationship between the two categories. In fact, we will want to use the following alternative.

Date: October 25, 2013.

2.3. **Definition.** Again let $A \in \mathrm{s}\mathcal{A}$ a simplicial object in $\mathcal{A}$. We define the *normalised chain complex* $N(A) \in \mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})$ by

$$N_n(A) \coloneqq \bigcap_{i=0}^{n-1} \ker(d_i) \subseteq A_n \quad \text{and} \quad \partial_n \coloneqq (-1)^n d_n : N_n(A) \to N_{n-1}(A)$$

for $n \geq 0$. Again the simplicial identities imply $\partial^2 = 0$, and we have a functor $N : \mathrm{s}\mathcal{A} \to \mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})$.

What is this unmotivated nonsense? Well, let's at least see an example.

2.4. **Example.** Recall there is a functor $B : \mathcal{A}\mathrm{b} \to \mathrm{s}\mathcal{A}\mathrm{b}$ which associates to an abelian group G it's "classifying space" BG, which is constructed as the nerve of the groupoid with one object and morphisms G. In particular, we have

— $BG_n \simeq G^n$ for $n \geq 0$;
— the face map $d_i : BG_n \to BG_{n-1}$ sends

$$(g_1, \ldots, g_n) \mapsto (g_1, \ldots, g_{i-1}, g_i + g_{i+1}, g_{i+2}, \ldots, g_n),$$

where we have let $g_0 \coloneqq 0$ and $g_{n+1} \coloneqq 0$.

Let's compute the normalised chain complex $N(BG)$.

— Of course $N_0(BG) \simeq BG_0 \simeq 0$ is the trivial group.
— Thus $N_1(BG) = \ker(d_0 : BG_1 \to BG_0) = BG_1 \simeq G$, and $\partial_1 = (-1)^n d_1 : BG_1 \to BG_0$ must be the zero morphism.
— Let $n \geq 2$. Let $g \coloneqq (g_1, \ldots, g_n) \in BG_n$. Observe that by definition $g \in \ker(d_0) \implies g_2 = \cdots = g_n = 0$ and thus $g \in \ker(d_1) \implies g_1 + g_2 = 0 \implies g_1 = 0$. So then $N_n(BG) \simeq 0$.

It follows also of course that the homology of $N(BG)$ is just G concentrated in degree 1. (Perhaps this reminds you of the homotopy groups of BG! We will see why this is so in §4.)

Ok that's an example, but maybe the definition of N still seems crazy. Have no fear, for C and N are intimately related! For instance we can note immediately from the definitions that the natural inclusion $N(A) \to C(A)$ is in fact a chain map for $A \in \mathrm{s}\mathcal{A}$. But there's more!

2.5. **Definition.** Let $A \in \mathrm{s}\mathcal{A}$. We define the *degenerate subcomplex* $D(A)$ of $C(A)$ by

$$D_0(A) \coloneqq 0 \quad \text{and} \quad D_n(A) \coloneqq \textstyle\sum_{i=0}^{n-1} \mathrm{im}(s_i)$$

for $n \geq 1$. That is, $D(A)$ is generated by the images of the degeneracy maps. Note that by the simplicial identities

$$\partial s_j = \textstyle\sum_{i=0}^{n} (-1)^i d_i s_j = \sum_{i=0}^{j-1} (-1)^i s_{j-1} d_i + \sum_{i=j+2}^{n} (-1)^i s_j d_i,$$

so $D(A)$ is indeed a subcomplex.

2.6. **Proposition.** *Let $A \in \mathrm{s}\mathcal{A}$. For $n \geq 0$ the natural map*

$$\phi : N_n(A) \oplus D_n(A) \to A_n = C_n(A)$$

induced by the inclusions is an isomorphism. Therefore we have a natural isomorphism

$$N(A) \simeq C(A)/D(A).$$

Proof. When $n = 0$ we have by definition that $D_0(A) \simeq 0$ and $N_0(A) \hookrightarrow A_0$ an isomorphism, so the claim is tautological. So fix $n \geq 1$. For $0 \leq i \leq n-1$, the simplicial identity $d_i s_i = \mathrm{id}_{A_{n-1}}$ implies that we have a canonical splitting $A_n \simeq \ker(d_i) \oplus \mathrm{im}(s_i)$. It follows easily that $N_n(A) \cap D_n(A) \simeq 0$, so we're just left to show that ϕ is surjective. We prove by downward induction on $0 \leq j \leq n-1$ that

$$\mathrm{im}(\phi) \supseteq N_j \coloneqq \bigcap_{i=0}^{j} \ker(d_i).$$

The base case $j = n-1$ is tautological and the final case $j = 0$ will finish the proof. Now consider the map $\psi \coloneqq \mathrm{id} - s_j d_j : C_n \to C_n$. Observe by the simplicial identities that

$$d_j \psi = d_j - d_j s_j d_j = d_j - d_j = 0 \quad \text{and} \quad d_i \psi = d_i - d_i s_j d_j = d_i - s_{j-1} d_{j-1} d_i$$

for $i < j$, implying that $\psi(N_j) \subseteq N_{j+1}$. By induction $\mathrm{im}(\phi) \supseteq N_{j+1}$, and since $\mathrm{im}(s_j d_j) \supseteq D_n(A)$ it follows that $\mathrm{im}(\phi) \supseteq N_j$. $\square$

So there's the relationship between C and N. With these definitions in hand, we can now state our main goal, the *Dold-Kan correspondence.*

2.7. **Theorem** (Dold-Kan)**.** *The functor* $N : \mathrm{s}\mathcal{A} \to \mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})$ *is an equivalence of categories.*

3. Proving the correspondence

3.1. **Definition.** Let $C \in \mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})$. Define

$$\Gamma_n(C) \coloneqq \bigoplus_{[n] \twoheadrightarrow [k]} C_k,$$

where the direct sum is over all surjections $\sigma : [n] \twoheadrightarrow [k]$ in the category Δ.

Let $\nu : [m] \to [n]$ a morphism in Δ. Let $\tau : [n] \twoheadrightarrow [k]$ an indexing surjection. We can factor $\tau\nu$ as a composition $[m] \twoheadrightarrow [j] \hookrightarrow [k]$ of a surjection σ and an injection ι. Then we define a map[1]

$$C_k \to C_j \quad \text{as} \quad \begin{cases} \mathrm{id}_{C_n}, & \text{if } j = k; \\ (-1)^n \partial_n, & \text{if } j = k-1 \text{ and } \iota = d^k; \\ 0, & \text{otherwise.} \end{cases}$$

Then composition with the inclusion $C_j \to \Gamma_m(C)$ of the factor with index $\sigma : [m] \twoheadrightarrow [j]$ gives a map $C_k \to \Gamma_m(C)$. Finally, the direct sum of these maps gives us an induced morphism $\nu^* : \Gamma_n(C) \to \Gamma_m(C)$.

Suppose $\mu : [l] \to [m]$ is another morphism in Δ. Factoring $\sigma\mu$ as $\rho\theta : [l] \twoheadrightarrow [i] \hookrightarrow [j]$, we have a commutative diagram

$$\begin{tikzcd}
{[l]} \arrow[r, "\mu"] \arrow[d, two heads, "\rho"] & {[m]} \arrow[r, "\nu"] \arrow[d, two heads, "\sigma"] & {[n]} \arrow[d, two heads, "\tau"] \\
{[i]} \arrow[r, hook, "\theta"] & {[j]} \arrow[r, hook, "\iota"] & {[k]}.
\end{tikzcd}$$

It's easy to see then that $(\nu\mu)_* = \mu_* \nu_*$.

It is also evident that a chain map $C \to D$ in $\mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})$ gives rise to a simplicial map $\Gamma(C) \to \Gamma(D)$ in $\mathrm{s}\mathcal{A}$ via factor-wise application. So finally we have constructed a functor

$$\Gamma : \mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A}) \to \mathrm{s}\mathcal{A},$$

which to each chain complex in $\mathcal{A}$ gives an *associated simplicial object in* $\mathcal{A}$.

To prove Dold-Kan, we're going to show that Γ is a quasi-inverse to $N : \mathrm{s}\mathcal{A} \to \mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})$.

3.2. **Definition.** Observe that there is a natural transformation $\Phi : \Gamma \circ N \to \mathrm{id}_{\mathrm{s}\mathcal{A}}$ defined by the maps

$$\Phi_n(A) : \Gamma_n(N(A)) = \bigoplus_{[n] \twoheadrightarrow [k]} N_k(A) \to A_n$$

for $A \in \mathrm{s}\mathcal{A}$ and $n \geq 0$, which restrict to the factor indexed by $\sigma : [n] \twoheadrightarrow [k]$ as the composition

$$N_k(A) \hookrightarrow A_k \xrightarrow{\sigma^*} A_n.$$

[1]This definition is not so random: compare it with our definition of the normalised chain complex N.

(It is clear this defines a simplicial map $\Gamma(N(A)) \to A$ which is natural in A.)

3.3. **Lemma.** *In fact $\Phi : \Gamma \circ N \to \mathrm{id}_{s\mathcal{A}}$, defined above, is a natural isomorphism.*

Proof. Fix $A \in s\mathcal{A}$. We will prove by induction on $n \geq 0$ that $\Phi_n(A) : \Gamma_n(N(A)) \to A_n$ is an isomorphism, and then we will be done. Since the only surjection $[0] \twoheadrightarrow [k]$ in Δ is $\mathrm{id}_{[0]}$, and the inclusion $N_0(A) \hookrightarrow A_0$ is an isomorphism, the base case $n = 0$ is tautological.

First, surjectivity. Recall from Proposition 2.6 that we have a splitting $A_n \simeq N_n(A) \oplus D_n(A)$. From the factor $\mathrm{id}_{[n]} : [n] \twoheadrightarrow [n]$ of $\Gamma_n(N(A))$ we $\mathrm{im}(\Phi_n(A)) \supseteq N_n(A)$. By induction $\Phi_{n-1}(A)$ is surjective, so by definition we must have $\mathrm{im}(\Phi_n(A)) \supseteq D_n(A)$. Hence $\Phi_n(A)$ is surjective.

Next, injectivity.[2] Assume we have $(x_\sigma) \in \ker(\Phi_n(A))$. Fix $0 \leq k < n$. Observe that to each surjection $\sigma : [n] \twoheadrightarrow [k]$ we can assign a section of σ,

$$\nu_\sigma : [k] \hookrightarrow [n], \quad \nu_\sigma(i) := \max\{j \in [n] \mid \sigma(j) = i\}.$$

If we have $\sigma, \sigma' : [n] \twoheadrightarrow [k]$, we say

$$\sigma \leq \sigma' \iff \nu_\sigma(i) \leq \nu_{\sigma'}(i) \text{ for all } i \in [k].$$

In particular, $\sigma' \nu_\sigma = \mathrm{id}_{[k]} \implies \sigma \leq \sigma'$. If there exists $\tau : [n] \twoheadrightarrow [k]$ such that $x_\tau \neq 0$, choose a *maximal* such τ (with respect to the ordering just defined). By definition of the simplicial structure on $\Gamma(N(A))$, it follows that the component of $\nu_\tau^*(x_\sigma)$ in the factor of $\Gamma_k(N(A))$ indexed by $\mathrm{id}_{[k]}$ is precisely x_τ. But then, by induction,

$$(x_\sigma) \in \ker(\Phi_n(A)) \implies \nu_\tau^*(x_\sigma) \in \ker(\Phi_k(A)) \implies x_\tau = 0,$$

contradiction.

So we must have $x_\sigma = 0$ for all $\sigma \neq \mathrm{id}_{[n]}$. But the restriction of $\Phi_n(A)$ to the factor indexed by $\mathrm{id}_{[n]}$ is just the inclusion $N_n(A) \hookrightarrow A_n$. So then $x_{\mathrm{id}_{[n]}} = 0$ too, and hence $\Phi_n(A)$ is injective. □

3.4. **Lemma.** *Let $C \in \mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})$. For $n \geq 0$, the natural inclusion*

$$\Psi_n(C) : N_n(\Gamma(C)) \hookrightarrow C_n(\Gamma(C)) = \Gamma_n(C) = \bigoplus_{[n] \twoheadrightarrow [k]} C_k$$

has image the factor C_n indexed by $\mathrm{id}_{[n]}$. This of course gives a natural isomorphism

$$\Psi : N \circ \Gamma \to \mathrm{id}_{\mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})}.$$

Proof. By definition of the simplicial structure on $\Gamma(C)$ we have $C_n \subseteq \bigcap_{i=0}^{n-1} \ker(d_i) = \mathrm{im}(\Psi_n(C))$. Now note for $\sigma : [n] \twoheadrightarrow [k]$ with $k < n$, we must have a factorisation of σ as a composition

$$[n] \xrightarrow{s^i} [n-1] \longrightarrow [k],$$

so it follows that the factor of $\Gamma_n(C)$ indexed by σ lies in the image of the degeneracies $D_n(\Gamma(C))$. Then we're done, since by Proposition 2.6 we have a splitting

$$\Gamma_n(C) \simeq N_n(\Gamma(C)) \oplus D_n(\Gamma(C)). \qquad \square$$

Proof of Theorem 2.7. The natural isomorphisms $\Phi : \Gamma \circ N \to \mathrm{id}_{s\mathcal{A}}$ and $\Psi : N \circ \Gamma \to \mathrm{id}_{\mathcal{C}\mathrm{h}_{\geq 0}(\mathcal{A})}$ of Lemmas 3.3 and 3.4 exhibit N (and Γ) as an equivalence of categories, thus proving the Dold-Kan correspondence. □

[2] We won't be totally categorical and will use elements here to clarify the argument, but it's easy to see that one can get rid of them.

4. Applying the correspondence

4.1. **Proposition.** *The natural inclusion $N(A) \to C(A)$ for $A \in s\mathcal{A}$ gives a natural chain homotopy equivalence.*

4.2. **Proposition.** *If $\mathcal{A} = R\text{-}\mathcal{M}\mathrm{od}$ and $A \in s\mathcal{A}$ then for $n \geq 0$ we have natural isomorphisms*

$$\pi_n(A, 0) \simeq \mathrm{H}_n(N(A)) \simeq \mathrm{H}_n(C(A)).$$

References

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