THE DOLD-KAN CORRESPONDENCE

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1. Introduction

1.1. **Definition.** A *simplicial object* in a category $\mathcal{C}$ is a contravariant functor $\Delta \to \mathcal{C}$. We denote the category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ of simplicial objects in $\mathcal{C}$ by $\mathrm{s}\mathcal{C}$. E.g., $\mathrm{s}\mathcal{S}\mathrm{et}$ is the category of *simplicial sets* and $\mathrm{s}\mathcal{A}\mathrm{b}$ is the category of *simplicial abelian groups*.

Recall we have a functor $\mathrm{Sing} : \mathcal{T}\mathrm{op} \to \mathrm{s}\mathcal{S}\mathrm{et}$, sending $X \mapsto \mathrm{Hom}_{\mathcal{T}\mathrm{op}}(|\Delta^\bullet|, X)$. Lately we've been talking about Sing for two reasons:

(1) It's a right adjoint to geometric realisation $|-| : \mathrm{s}\mathcal{S}\mathrm{et} \to \mathcal{T}\mathrm{op}$.
(2) $\mathrm{Sing}(X)$ is a Kan complex for all $X \in \mathcal{T}\mathrm{op}$. In this sense, "Kan complexes are like spaces".

But this isn't the first place one sees Sing, probably. Indeed, the singular homology functors are essentially defined by a composition

$$\mathrm{H}_n(-;\mathbb{Z}) \coloneqq \mathcal{T}\mathrm{op} \xrightarrow{\mathrm{Sing}} \mathrm{s}\mathcal{S}\mathrm{et} \xrightarrow{\mathbb{Z}} \mathrm{s}\mathcal{A}\mathrm{b} \xrightarrow{\sum(-1)^i d_i} \mathcal{C}\mathrm{h} \xrightarrow{\mathrm{H}_n} \mathcal{A}\mathrm{b}.$$

Here $\mathbb{Z}$ denotes the functor which takes free abelian groups level-wise, from which we get the *singular chain complex* by letting the boundary map be given by $\partial \coloneqq \sum(-1)^i d_i$.

This was just to remind us that we've seen a natural functor $\mathrm{s}\mathcal{A}\mathrm{b} \to \mathcal{C}\mathrm{h}$ relating simplicial abelian groups and chain complexes. We'll look at it a bit more carefully in a second, and develop this relationship much further.

References

1. Paul G. Goerss and John F. Jardine, *Simplicial Homotopy Theory*, Birkhäuser Verlag, 1999.
2. Akhil Mathew, *The Dold-Kan correspondence*, `people.fas.harvard.edu/~amathew/doldkan.pdf`.
3. Emily Riehl, *A leisurely introduction to simplicial sets*, `www.math.harvard.edu/~eriehl/ssets.pdf`.

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