ENRICHED CATEGORIES

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1. Motivation

Recall that in a category $\mathcal{C}$ we have a *hom-set* $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ for any two objects $X, Y \in \mathrm{Ob}\,\mathcal{C}$. But sometimes we really want $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ to be more than just a set, i.e. have the structure of an object in some category. Our primary motivation is the idea that some categories should have *hom-spaces*.

1.1. **Example.** Consider $\mathcal{C} = \mathcal{T}\mathrm{op}$ the category of spaces. (Actually we'll need to restrict our class of spaces, but let's leave the details for a couple of pages.) We'll see that $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ has a natural topology, which gives us a mapping *space* $\mathrm{Map}(X, Y)$. An important property this satisfies—an incarnation of the "hom-tensor adjunction"—is that there is a natural isomorphism

$$\mathrm{Map}(X \times Y, Z) \simeq \mathrm{Map}(X, \mathrm{Map}(Y, Z)). \tag{$*$}$$

And one particularly important example of this important property is when $X = [0, 1]$. Then $(*)$ tells us that talking about *homotopies* of maps $Y \to Z$ is equivalent to talking about *paths* in $\mathrm{Map}(Y, Z)$.

2. Monoidal and enriched categories

So we want to somehow talk about categories $\mathcal{C}$ in which we have hom-objects in some other category $\mathcal{T}$. It turns out the structure we want on $\mathcal{T}$ is some sort of "tensor product". In fact it's easy to see why we need some way of taking any two objects in $\mathcal{T}$ and producing a third: we need a way to write down a composition law $\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ as a map in $\mathcal{T}$! But we'll use the notation $\otimes$ instead of $\times$.

2.1. **Definition.** A *monoidal category* $\mathcal{T}$ is a category equipped with

— a functor $- \otimes - : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$, the *tensor product*
— an object $1 \in \mathcal{T}$, the *unit*
— a natural isomorphism $\alpha : (- \otimes -) \otimes - \to - \otimes (- \otimes -)$, the *associator*
— a natural isomorphism $\lambda : 1 \otimes - \to \mathrm{id}_{\mathcal{T}}$, the *left unitor*
— a natural isomorphism $\rho : - \otimes 1 \to \mathrm{id}_{\mathcal{T}}$, the *right unitor*

satisfying coherence conditions given by two commutative diagrams: the *triangle identity*,

$$\begin{tikzcd}
(- \otimes 1) \otimes - \arrow[rr, "\alpha"] \arrow[dr, "\rho \otimes \mathrm{id}_{\mathcal{T}}"'] & & - \otimes (1 \otimes -) \arrow[dl, "\mathrm{id}_{\mathcal{T}} \otimes \lambda"] \\
& - \otimes -, &
\end{tikzcd}$$

Date: October 19, 2013.

These notes are me digesting material presented by Hiro Lee Tanaka and Sitan Chen on September 18 and 25 in a reading group on infinity categories held at Harvard in Fall 2013.

and the *pentagon identity*,

$$\begin{tikzcd}[column sep=small]
& (-\otimes -)\otimes(-\otimes -) \arrow[dr, "\alpha"] & \\
((-\otimes -)\otimes -)\otimes - \arrow[ur, "\alpha"] \arrow[d, "\alpha\otimes \mathrm{id}_{\mathcal{T}}"] & & -\otimes(-\otimes(-\otimes -)) \\
(-\otimes(-\otimes -))\otimes - \arrow[rr, "\alpha"] & & -\otimes((-\otimes -)\otimes -). \arrow[u, "\mathrm{id}_{\mathcal{T}}\otimes\alpha"']
\end{tikzcd}$$

2.2. **Remark.** The coherence conditions look sort of intimidating. Basically the point is that the tensor product should behave as one might expect. Perhaps even more concretely, as the name suggests, a symmetric monoidal category is a categorification of the notion of a commutative monoid. So just think that the tensor product gives a nice, commutative, associative multiplication on $\mathcal{T}$.

2.3. **Examples.** One thing to note immediately is that we can put different monoidal structures on the same category. We have the following familiar examples.

(1) $\mathcal{T} := \mathcal{S}\mathrm{et}$ the category of sets; $\otimes := \times$ the direct product; $1 := \{*\}$ the one-point set.
(2) $\mathcal{T} := \mathcal{S}\mathrm{et}$; $\otimes := \amalg$ the disjoint union; $1 := \emptyset$ the empty set.
(3) $\mathcal{T} := \mathcal{A}\mathrm{b}$ the category of abelian groups; $\otimes := \otimes_{\mathbb{Z}}$ the tensor product over $\mathbb{Z}$; $1 := \mathbb{Z}$.
(4) $\mathcal{T} := \mathcal{A}\mathrm{b}$; $\otimes := \oplus$ the direct sum; $1 := 0$ the trivial group.
(5) If you're feeling fancy, $\mathcal{T} := \mathcal{C}\mathrm{h}$ the category of chain complexes; $\otimes := \otimes_{\mathbb{Z}}$ the tensor product of complexes over $\mathbb{Z}$, $1 := \mathbb{Z}$ the complex with $\mathbb{Z}$ concentrated in degree 0.
(6) Most relevant to our immediate goals, $\mathcal{T} := \mathcal{T}\mathrm{op}$ the category of topological spaces; $\otimes := \times$ the direct product; $1 := \{*\}$ the one-point space.

Now we want to define precisely what it means for a category to have hom-objects in a monoidal category.

2.4. **Definition.** Let $\mathcal{T}$ be a monoidal category. Then a $\mathcal{T}$-*enriched category* (or a *category enriched over* $\mathcal{T}$) $\mathcal{C}$ is the data of

— a collection $\mathrm{Ob}\,\mathcal{C}$ of *objects*
— a *hom-object* $\mathrm{Hom}_{\mathcal{C}}(X, Y) \in \mathrm{Ob}\,\mathcal{T}$ for all pairs $X, Y \in \mathrm{Ob}\,\mathcal{C}$
— a *composition* map (in the category $\mathcal{T}$!)

$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z) \otimes \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

for all triples of objects $X, Y, Z \in \mathrm{Ob}\,\mathcal{C}$
— an *identity* $\mathrm{id}_X : 1 \to \mathrm{Hom}_{\mathcal{C}}(X, X)$ for all objects $X \in \mathcal{C}$

satisfying

— the commutative diagram for associativity for $W, X, Y, Z \in \mathrm{Ob}\,\mathcal{C}$,

$$\begin{tikzcd}
(\mathrm{Hom}_{\mathcal{C}}(Z,Y)\otimes \mathrm{Hom}_{\mathcal{C}}(X,Y))\otimes \mathrm{Hom}_{\mathcal{C}}(W,X) \arrow[r, "\circ\otimes\mathrm{id}"] \arrow[dd, "\alpha"] & \mathrm{Hom}_{\mathcal{C}}(X,Z)\otimes \mathrm{Hom}_{\mathcal{C}}(W,X) \arrow[d, "\circ"] \\
& \mathrm{Hom}_{\mathcal{C}}(W,Z) \\
\mathrm{Hom}_{\mathcal{C}}(Z,Y)\otimes(\mathrm{Hom}_{\mathcal{C}}(X,Y)\otimes \mathrm{Hom}_{\mathcal{C}}(W,X)) \arrow[r, "\mathrm{id}\otimes\circ"] & \mathrm{Hom}_{\mathcal{C}}(Z,Y)\otimes \mathrm{Hom}_{\mathcal{C}}(W,Y), \arrow[u, "\circ"']
\end{tikzcd}$$

— the commutative diagrams for the identity maps for $X, Y \in \mathrm{Ob}\,\mathcal{C}$,

$$\begin{tikzcd}
1 \otimes \mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[rr, "\mathrm{id}_Y \otimes \mathrm{id}"] \arrow[dr, "\lambda"'] & & \mathrm{Hom}_{\mathcal{C}}(Y,Y) \otimes \mathrm{Hom}_{\mathcal{C}}(X,Y) \arrow[dl, "\circ"] \\
& \mathrm{Hom}_{\mathcal{C}}(X,Y), &
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X,Y) \otimes 1 \arrow[rr, "\mathrm{id} \otimes \mathrm{id}_X"] \arrow[dr, "\rho"'] & & \mathrm{Hom}_{\mathcal{C}}(X,Y) \otimes \mathrm{Hom}_{\mathcal{C}}(X,X) \arrow[dl, "\circ"] \\
& \mathrm{Hom}_{\mathcal{C}}(X,Y). &
\end{tikzcd}$$

2.5. **Examples.**

(1) Any ordinary category is a $\mathcal{S}$et-enriched category when we put a monoidal structure on $\mathcal{S}$et as in Example 2.3.1.
(2) $\mathcal{S}$et, $\mathcal{A}$b, and $\mathcal{C}$h are all enriched over themselves.
(3) When we say a category has *hom-spaces* we mean mean the category is be enriched over $\mathcal{T}$op, given the monoidal structure in Example 2.3.6.

3. Replacing $\mathcal{T}$op with CG

Ok I just lied in that last example. As mentioned earlier, it turns out that the full category of topological spaces $\mathcal{T}$op has some pathologies that we really don't want to deal with. So we're going to make a slight modification and replace $\mathcal{T}$op with a certain subcategory CG which does not have these pathologies.[1]

3.1. **Convention.** We will follow the convention of calling a space *compact* if it is quasi-compact and Hausdorff, and the same for *locally compact*.

Before I begin defining things, a disclaimer: I honestly don't know any great motivation for these definitions besides that they turn out to give us a more convenient category to work with.

3.2. **Definitions.** Let X a space, and $A \subseteq X$ a subspace.

(1) We say A is *k-closed* if $f^{-1}(A)$ is closed in K for all compact spaces K and continuous maps $f : K \to X$.
(2) It is easy to check that the collection of k-closed subsets form the closed sets of a topology on X. We call X with this topology the *k-ification* of X, and denote it by kX.
(3) We say X is a *k-space* if $X = kX$, that is, if A is k-closed if and only if A is closed.
(4) We say X is *weakly Hausdorff* if $f(K)$ is closed in X for all compact spaces K and continuous maps $f : K \to X$.
(5) We say X is *compactly generated* if it is a weakly Hausdorff k-space.

3.3. **Remarks.** Let us note a couple of immediate facts regarding the above definitions.

(1) If $A \subseteq X$ is closed then certainly A is k-closed, i.e., $\mathrm{id} : kX \to X$ is continuous. Thus X is a k-space if and only if all k-closed sets are closed.
(2) If A is k-closed, then in particular for $K \subseteq X$ any compact subspace, with inclusion $i : K \to X$, we have $A \cap K = i^{-1}(A)$ is closed.

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[1] A much more detailed and comprehensive account of CG can be found at `neil-strickland.staff.shef.ac.uk/courses/homotopy/cgwh.pdf`, and the presentation here is mostly taken from there, in addition to May's book, *A Concise Course in Algebraic Topology*.

(3) If K is compact then a map $f : K \to X$ is continuous if and only if it is continuous as a map $K \to kX$. Hence $k(kX) = kX$, so kX is a k-space.
(4) If X is weakly Hausdorff, then so is kX, and hence kX is compactly generated by the above.
(5) Certainly if X is Hausdorff then X is weakly Hausdorff. And certainly if X is weakly Hausdorff then X is T_1 (points are closed).

3.4. **Lemma.** *Let X be a weakly Hausdorff space.*

(1) *If K is compact and $f : K \to X$ continuous then $f(K)$ is compact (in the induced topology).*
(2) *A subspace $A \subseteq X$ is compact in X if and only if it is compact in kX.*
(3) *A subspace $A \subseteq X$ is k-closed if and only if $A \cap K$ is closed in K for every compact subspace $K \subseteq X$.*

Proof. Todo. □

The following shows that many of the spaces familiar to us are compactly generated.

3.5. **Lemma.** *A locally compact space X is compactly generated and weakly Hausdorff.*

Proof. Since X is Hausdorff, X is weakly Hausdorff. So it suffices to show a k-closed set $A \subseteq X$ must be closed. Let $x \in \overline{A}$; choose a compact neighbourhood K of x. For any open neighbourhood U of x, certainly $U \cap K$ is also a neighbourhood of x, and hence $U \cap K \cap A \neq \emptyset$. But this says that $x \in \overline{K \cap A}$, and since A is k-closed we know $K \cap A$ is closed, so in fact $x \in K \cap A$. Thus $\overline{A} = A$ and we're done. □

3.6. **Lemma.** *Let $f : X \to Y$ a map, X compactly generated. The following are equivalent:*

(1) *f is continuous as a map $X \to Y$;*
(2) *the restriction $f|_K : X \to Y$ is continuous for all compact subspaces $K \subseteq X$;*
(3) *f is continuous as a map $X \to kY$.*

Proof. (1) ⇒ (2) is tautological, and (3) ⇒ (1) follows from $\mathrm{id} : kY \to Y$ being continuous (Remark 3.3.1). Assume (2). By Remark 3.3.3, this is equivalent to assuming $f|_K$ is continuous as a map $K \to kY$ for all compact $K \subseteq X$. Thus if $A \subseteq kY$ is any closed set and $K \subseteq X$ any compact set, then $f|_K^{-1}(A) = f^{-1}(A) \cap K$ is closed, whence by Lemma 3.4 this implies $f^{-1}(A)$ is closed. This gives (3). □

3.7. **Definition.** Let WH the full subcategory of $\mathcal{T}\mathrm{op}$ consisting of weakly Hausdorff spaces. Let CG the full subcategory of WH consisting of compactly generated spaces.

3.8. **Proposition.** *We have functors $j : \mathrm{CG} \to \mathrm{WH}$, the forgetful inclusion, and $k : \mathrm{WH} \to \mathrm{CG}$, given by k-ification. Moreover, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{WH}}(jX, Y) \simeq \mathrm{Hom}_{\mathrm{CG}}(X, kY),$$

i.e., k is right-adjoint to j.

Proof. Obviously j is a functor. That k is a functor, and that the adjunction given simply by the identity (on sets) is well-defined, are immediate from Lemma 3.6. □

3.9. **Definition.** Let X, Y spaces. The *compact-open topology* on $\mathrm{Hom}_{\mathcal{T}\mathrm{op}}(X, Y)$ is defined as having subbasis elements of the form

Today we're interested in $\mathcal{T} = \mathcal{T}\mathrm{op}$, $\otimes = \times$, and $1 = \{*\}$. And we're going to enrich CG compactly generatedHausdorff spaces with $\mathcal{T}$. We're interested in these spaces because we needs some conditions to get $\mathrm{Map}(X, Y) \times \mathrm{Map}(Y, Z) \to \mathrm{Map}(X, Z)$ to be continuous (i.e., a map in the category $\mathcal{T}\mathrm{op}$). There's an article on nLab about "Convenient categories of topological spaces" that deals with this.

But we jumped the gun. We need to define a topology on $\mathrm{Map}(X, Y)$. We'll define a subbasis for the compact-open topology: sets (K, U) of maps $X \to Y$ which send $K \to Y$, where $K \subseteq X$ compact and $U \subseteq Y$ open.

We claim that the following identity holds:

$$\mathrm{Map}(A \times B, C) \simeq \mathrm{Map}(A, \mathrm{Map}(B, C)).$$

Where $f \mapsto g$ where $g(a)(b) = f(a, b)$.

For continuity in forward direction, suffices to show $(K, (L, U)) \subseteq \mathrm{Map}(A, \mathrm{Map}(B, C))$ has open preimage (this is a lemma). But preimage is just $(K \times L, U)$ so clear. The other direciton is harder, and you probably need to use the compactly generated Hasudorff condition. Break K into compact boxes $K_i \times L_j$ and then do shit. "We're in the thick jungle of point-set topology."

Important example $A = [0, 1]$. This tells us homotopies are paths in the mapping space.