MATH 131 SECTION, II: INADEQUACY OF SEQUENCES

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1. Apologies

Professor McMullen has mentioned a few times that one of the reasons we talk about everything in terms of open and closed sets in general topology is that the notion of convergent sequences doesn't capture all phenomena in this generality. In this section I wanted to talk about a way we can still talk about everything in terms of convergence—but convergence of something more general than sequences, called *filters*.

Unfortunately, I don't really know too much about filters right now, and I didn't get myself to learn them well enough before section. Sorry! So I'll (optimistically) leave that for another section, and just give an example[1] to motivate them. I.e., let me prove to you that sequences aren't good enough for us.

2. The cocountable topology

2.1. **Definition.** For any set X, the *cocountable topology* on X is

$$\mathcal{T} \coloneqq \{U \subseteq X \mid X - U \text{ is countable}\} \cup \{\emptyset\}.$$

I.e., we let the closed sets of X be the countable subsets along with X (where we consider finite, and in particular empty, sets to be countable). Note that $\mathcal{T}$ is a topology because arbitrary intersections and finite unions of countable sets are countable.

Ok, now fix an uncountable set X and put it in the cocountable topology. (You can just take $X \coloneqq \mathbb{R}$ for concreteness; I'm just wrapping some notation around it since the topology we've put on X is far, far away from the intuitive topologies you know on $\mathbb{R}$.) And suppose we have $A \subsetneq X$ a proper, uncountable subset. (Again, take $A \coloneqq [0,1] \subset \mathbb{R}$ for concreteness.)

I claim that A is dense in X, but that any convergent sequence in A necessarily converges (uniquely) to a point in A. What is this saying? Recall that A being dense means that it's closure is all of X, that is $\overline{A} = X$. On the other hand, there is no sequence is A converging to any point outside of A. (Keep in mind, $A \neq X$ so that this is nontrivial!) So basically, in this topology, being closed in the topological sense is rather far from being closed under the operation of taking limits of convergent sequences. Contrast this with the familiar setting of metric spaces, where these two concepts are the same![2] Ok, let's prove it.

2.2. **Exercise.** *Show that $\overline{A} = X$ if and only if $U \cap A \neq \emptyset$ for every $\emptyset \neq U \subseteq X$ open.*

2.3. **Lemma.** $\overline{A} = X$.

Proof. We use the characterisation of denseness given in (2.2). Let $U \subseteq X$ nonempty and open. By definition of the cocountable topology $X - U$ is countable. Since A is uncountable, we cannot then have $A \subseteq X - U$, that is, $U \cap A \neq \emptyset$. $\square$

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[1]Professor McMullen also gave this example in lecture, but I did it the day before, so ha!

[2]See my notes from section I for a proof of this.

Before we examine convergent sequences in A, let's just quickly recall what it means for sequences to converge in topological spaces.

2.4. **Definition.** We say a sequence $(x_k) \in X^{\mathbb{N}}$ *converges* to $x \in X$ if for each open set $U \subseteq X$ containing x there exists $n \in \mathbb{N}$ such that $x_k \in U$ for $k \geq n$.

2.5. **Lemma.** *Let $(x_k) \in A^{\mathbb{N}}$ a convergent sequence in A. Then there exists $x \in A$ and $n \in \mathbb{N}$ such that $x_k = x$ for $k \geq n$. That is, (x_k) is* eventually constant*.*

Proof. Since (x_k) is convergent we can pick $x \in X$ such that $x_n \to x$ as $n \to \infty$. Observe that $F \coloneqq \{x_k \mid k \in \mathbb{N}, x_k \neq x\}$ is countable, and hence closed, and by definition does not contain x. I.e., $X - F$ is an open set containing x. So by definition of convergence, there exists $n \in \mathbb{N}$ such that $x_k \in X - F$ for $k \geq n$. But then by definition of F we must have $x_k = x$ for $k \geq n$. Then of course $x \in A$ since $x_k \in A$ for all $k \in \mathbb{N}$. $\square$

Note that eventually constant sequences obviously converge to a unique point.