

DEEP XVA SOLVER – A NEURAL NETWORK BASED COUNTERPARTY CREDIT RISK MANAGEMENT FRAMEWORK

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ABSTRACT. In this paper, we present a novel computational framework for portfolio-wide risk management problems, where the presence of a potentially large number of risk factors makes traditional numerical techniques ineffective. The new method utilises a coupled system of BSDEs for the valuation adjustments (xVA) and solves these by a recursive application of a neural network based BSDE solver. This not only makes the computation of xVA for high-dimensional problems feasible, but also produces hedge ratios and dynamic risk measures for xVA, and allows simulations of the collateral account.

1. INTRODUCTION

As a consequence of the 2007–2009 financial crisis, academics and practitioners have been redefining and augmenting key concepts of risk management. This made it necessary to reconsider many widely used methodologies in quantitative and computational finance.

It is now generally accepted that a reliable valuation of a financial product should account for the possibility of default of any agent involved in the transaction. Moreover, the trading activity is nowadays funded by resorting to different sources of liquidity (the interest rate multi-curve phenomenon; see, e.g., Cuchiero et al. (2019)), so that the existence of a single funding stream with a unique risk-free interest rate no longer represents a realistic assumption. Additionally, the increasingly important role of collateral agreements demands for a portfolio-wide view of valuation.

These stylized facts are incorporated into the valuation equations through value adjustments (xVA). Value adjustments are terms to be added to, or subtracted from, an idealized reference portfolio value, computed in the absence of frictions, in order to obtain the final value of the transaction.

The literature on counterparty credit risk and funding is large and we only attempt to provide insights on the main references as they relate to our work. Possibly the first contribution on the subject is a model for credit risk asymmetry in swap contracts in Duffie and Huang (1996). Before the 2007–2009 financial crisis, we have the works of Brigo and Masetti (2005) and Cherubini (2005), where the concept of credit valuation adjustment (CVA) is analyzed. The possibility of default of both counterparties involved in the transaction, represented by the introduction of the debt valuation adjustment (DVA), is investigated, among others, in Brigo et al. (2011, 2014).

Another important source of concern to practitioners apart from default risk is represented by funding costs. A parallel stream of literature emerged during and after the financial crisis to generalize valuation equations in the presence of collateralization agreements. In a Black-Scholes economy, Piterbarg (2010) gives valuation formulas both in the collateralized and uncollateralized case. Generalizations to the case of a multi-currency economy can be found in Piterbarg (2012), Fujii et al. (2010, 2011), and Gnoatto and Seiffert (2020). The funding valuation adjustment (FVA) is derived under alternative assumptions on the Credit Support Annex (CSA) in Pallavicini et al. (2011), while Brigo and Pallavicini

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(2014) also discusses the role of central counterparties for funding costs. A general approach to funding in a semimartingale setting is provided by Bielecki and Rutkowski (2015).

Funding and default risk need to be united in a single risk management framework to account for all possible frictions and their interplay. Contributions in this sense can be found in Brigo et al. (2018) by means of the so-called discounting approach. In a series of papers, see Burgard and Kjaer (2011b,a) generalize the classical Black-Scholes replication approach to include some of the aforementioned effects. A more general backward stochastic differential equation (BSDE) approach is provided by Crépey (2015a,b) and Bichuch et al. (2018a,b). The equivalence between the discounting approach and the BSDE-based replication approaches is demonstrated in Brigo et al. (2018).

A common fundamental feature of such generalized risk management frameworks is the necessity to adopt a portfolio-wide point of view in order to properly account for risk mitigation benefits arising from diversified positions. Adopting such portfolio-wide models, as is the present market practice in financial institutions, involves high-dimensional joint simulations of all positions within a portfolio.

Commonly used numerical techniques (see for instance Shöftner (2008); Karlsson et al. (2016); Broadie et al. (2015); Joshi and Kwon (2016)) make use of regression approaches, based on a modification of the Least-Squares Monte Carlo approach in Longstaff and Schwartz (2001), to alleviate the high computational cost of fully nested Monte Carlo simulations such as those initially proposed in Gordy and Juneja (2010); Broadie et al. (2011). We refer to Albanese et al. (2017) for a high-performance GPU implementation of nested Monte Carlo for bilateral xVA computations in a modern set-up including credit, margin and capital, for a large book of about 200,000 trades with 2000 counterparties. For an application of adjoint algorithmic differentiation (AAD) to xVA simulation by regression see, for instance, Capriotti et al. (2017); Fries (2019).

An alternative, hybrid, approach to counterparty risk computations is taken in de Graaf et al. (2014), where standard pricing methods are applied to the products in the portfolio and outer Monte Carlo estimators for exposures. Techniques based purely on PDEs generally suffer from the *curse of dimensionality*, a rapid increase of computational cost in presence of high dimensional problems. A PDE approach with factor-based dimension reduction has been proposed in de Graaf et al. (2018). Observe that in presence of collaterals, a PDE representation for CVA and DVA is not always available.

In the broader context of high-dimensional problems involving large amounts of data, machine learning techniques have witnessed dramatically increasing popularity. Of particular interest is the concept of an artificial neural network (ANN). From a mathematical perspective, ANNs are multiple nested compositions of relatively simple multivariate functions. The term deep neural networks refers to ANNs with several interconnected layers. One remarkable property of ANNs is given in the ‘Universal Approximation Theorem’, which essentially states that any continuous function in any dimension can be represented to arbitrary accuracy by means of an ANN, and has been proven in different versions, starting from the remarkable insight of Kolmogorov’s Representation Theorem in Kolmogorov (1956) and the seminal works of Cybenko (1989) and Hornik (1991). Recently, building heavily on earlier work of Jentzen et al. (2018), the recent results by Reisinger and Zhang (2020) have proven that deep ANNs can overcome the curse of dimensionality for approximating (nonsmooth) solutions of partial differential equations arising from (open-loop control of) SDEs. A result to the same effect has been shown for heat equations with a zero-order nonlinearity in Hutzenthaler et al. (2018). This is potentially useful in the context of risk management as simple models for CVA can be expressed in this form. For a recent literature survey of applications of neural networks to pricing, hedging and risk management problems more generally we refer the reader to Ruf and Wang (2019).

In this paper, we investigate the application of ANNs to solve high-dimensional BSDEs arising from risk management problems. Indeed, in the classical continuous-time mathematical finance literature the random behavior of the simple financial assets composing a portfolio is typically described by means multi-dimensional Brownian motions and forward stochastic differential equations (SDEs). In this setting, BSDEs naturally arise as a representation of the evolution of the hedging portfolio, where the terminal condition represents the target payoff (see, e.g., El Karoui et al. (1997)). In essence, (numerically) solving a BSDE is equivalent to identifying a risk management strategy.

Numerical BSDE methods published recently for xVA computations for single derivatives include Borovykh et al. (2018). The difficulty of extending these computational techniques to the portfolio setting is alluded to in Remark 11 of Ninomiya and Shinozaki (2019).

Here, we will consider a discretized version of the BSDE and parametrize the (high dimensional) control (i.e., hedging) process at every point in time by means of a family of ANNs. Once written in this form, BSDEs can be viewed as model-based reinforcement learning problems. The ANN parameters are then fitted so as to minimize a prescribed loss function.

The line of computational methods we follow has been initiated in the context of high-dimensional nonlinear PDEs in E et al. (2017) and further investigated in Han and Long (2020) and Fujii et al. (2019), and has led to a class of methods for the solution of BSDEs (characterised by parametrisation of the Markovian control by ANNs), which we will collectively refer to as the Deep BSDE Solver for simplicity. By way of financial applications, and xVA specifically, a primal-dual extension to the Deep BSDE Solver has been developed in Henry-Labordere (2017) and tested on stylised CVA- and IM(Initial Margin)-type PDEs; the Deep BSDE Solver has also been applied specifically to exposure computations for a Bermudan swaption and a cross-currency swap in She and Greco (2017).

Our approach goes beyond these earlier works in the following regards: we

- consider a rigorous, generic BSDE model for the dynamics of xVA, including CVA, DVA, FVA and ColVA (collateral valuation adjustment), for a derivative portfolio;
- introduce algorithms for the computation of ‘non-recursive’ xVAs – such as CVA and DVA – and ‘recursive’ xVAs – such as FVA – by (recursive) application of the Deep BSDE Solver, and deduce *a posteriori* bounds on the error of the neural network approximations;
- show how the method can be used for the simulation of xVA sensitivities and collateral, and provide careful numerical tests, showing good (i.e., basis point) accuracy for different adjustment computations, including an example with 100 underlying assets.

We will refer to our method as *Deep xVA Solver*. More recently, conditional risk measure computations (VaR and ES), based on deep learning regression, have been proposed in an xVA framework in Albanese et al. (2020), using a similar numerical approach to the one developed independently for BSDEs in Huré et al. (2020). Different from E et al. (2017), this solver approximates the value function, not the control, by means of an ANN and reconstructs it at each time step by dynamic programming techniques. A comparison of the performance and robustness of the two approaches will require comprehensive testing in industry-relevant settings. We see as a structural advantage of our algorithm that it allows to directly compute the xVA hedging strategy.

The applicability of the presented methodology is largely independent of the particular choice of the xVA framework. In particular, we do not take a position in the so-called *FVA debate* or on the question of including KVA in the pricing equation. The term *FVA debate* here refers to the possible overlap between the debt value adjustment (DVA) and the funding benefit adjustment (FBA). This overlap has been addressed in Brigo et al. (2019). The inclusion of KVA is still debated, noting, for example, the recent criticism of KVA in Andersen et al. (2019). Our approach is general enough to

accommodate different specifications of the price decomposition. In particular, our methodology can be applied immediately to the framework of Brigo et al. (2019).

We restrict the presentation of the method to a single counter-party – or ‘netting set’ – for simplicity, as is routinely done in banks. There are economic grounds for extending the computation to multiple netting sets simultaneously (see, e.g., Albanese et al. (2020)) and our method generalises accordingly. The paper is organized as follows. The financial framework is established in Section 2. In Section 3, after shortly recalling the main features of the Deep BSDE Solver presented in E et al. (2017), the algorithm for xVA computation is introduced. Numerical results for a selection of test cases are shown in Section 4, while Section 5 concludes.

2. THE FINANCIAL MARKET

For concreteness, we adopt the market setup of Biagini et al. (2019) and subsequently formulate our computational methods in the context of this model. Let us re-iterate the point elaborated in the introduction, however, that the computational framework, which is the focus of this article, is adaptable to a range of model specifications.

We fix a time horizon $T < \infty$ for the trading activity of two agents named the *bank* (B) and the *counterparty* (C). Unless otherwise stated, throughout the paper we assume the bank’s perspective and refer to the bank as the *hedger*.

All underlying processes are modeled over a probability space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{Q})$, where $\mathbb{G} = (\mathcal{G}_t)_{t \in [0, T]} \subseteq \mathcal{G}$ is a filtration satisfying the usual assumptions (\mathcal{G}_0 is assumed to be trivial). We denote by τ^B and τ^C the *time of default* of the bank and the counterparty, respectively. Specifically, we assume that $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ is a reference filtration satisfying the usual assumptions and $\mathbb{H} = \mathbb{H}^B \vee \mathbb{H}^C$, with $\mathbb{H}^j = (\mathcal{H}_t^j)_{t \in [0, T]}$ for $\mathcal{H}_t^j = \sigma(H_u | u \leq t)$, and $H_t^j := \mathbb{1}_{\{\tau^j \leq t\}}$, $j \in \{B, C\}$. We set

$$\tau = \tau^C \wedge \tau^B.$$

In the present paper we will extensively make use of the so called *Immersion Hypothesis* (see, e.g., Bielecki and Rutkowski (2004)).

Assumption 1. *Any local (\mathbb{F}, \mathbb{Q}) -martingale is a local (\mathbb{G}, \mathbb{Q}) -martingale.*

We consider the following spaces:

- $L^2(\mathbb{R}^d)$ is the space of all \mathcal{F}_T -measurable \mathbb{R}^d -valued random variables $X : \Omega \mapsto \mathbb{R}^d$ such that $\|X\|^2 = \mathbb{E} [|X|^2] < \infty$.
- $\mathbb{H}^{2, q \times d}$ is the space of all predictable $\mathbb{R}^{q \times d}$ -valued processes $\phi : \Omega \times [0, T] \mapsto \mathbb{R}^{q \times d}$ such that $\mathbb{E} \left[\int_0^T |\phi_t|^2 dt \right] < \infty$.
- \mathbb{S}^2 the space of all adapted processes $\phi : \Omega \times [0, T] \mapsto \mathbb{R}^{q \times d}$ such that $\mathbb{E} \left[\sup_{0 \leq t \leq T} |\phi_t|^2 \right] < \infty$.

2.1. Basic traded assets.

Risky assets. For $d \geq 1$, we denote by S^i , $i = 1, \dots, d$, the *ex-dividend price* (i.e. the price) of risky securities. All S^i are assumed to be càdlàg \mathbb{F} -semimartingales.

Let $W^\mathbb{Q} = (W_t^\mathbb{Q})_{t \in [0, T]}$ be a d -dimensional (\mathbb{F}, \mathbb{Q}) -Brownian motion (hence a (\mathbb{G}, \mathbb{Q}) -Brownian motion, thanks to Assumption 1). We introduce the coefficient functions $\mu : \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R}^d$, $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \mapsto \mathbb{R}^{d \times d}$, which are assumed to satisfy standard conditions ensuring existence and uniqueness of strong

solutions of SDEs driven by the Brownian motion $W^{\mathbb{Q}}$. We assume that

$$(2.1) \quad \begin{cases} dS_t = \mu(t, S_t) dt + \sigma(t, S_t) dW_t^{\mathbb{Q}}, \\ S_0 = s_0 \in \mathbb{R}^d, \end{cases}$$

on $[0, T]$. Note that we are not postulating that the processes S^i are positive.

Throughout the paper we assume that the market is complete for the sake of simplicity.

Cash accounts. Given a stochastic return process $x := (x_t)_{t \geq 0}$, which is assumed bounded, right-continuous and \mathbb{F} -adapted, we define the cash account B^x with unitary value at time 0, as the strictly positive continuous processes of finite variation

$$(2.2) \quad B_t^x := \exp \left\{ \int_0^t x_s ds \right\}, \quad t \in [0, T].$$

In particular, $B^x := (B_t^x)_{t \in [0, T]}$ is also continuous and adapted.

Defaultable bonds. Default times are assumed to be exponentially distributed random variables with time-dependent intensity

$$\Gamma_t^j = \int_0^t \lambda_s^{j, \mathbb{Q}} ds, \quad t \in [0, T], \quad j \in \{B, C\},$$

where $\lambda^{j, \mathbb{Q}}$ are non-negative bounded processes.

We introduce two risky bonds with maturity $T^* \leq T$ issued by the bank and the counterparty. We directly state their dynamics under \mathbb{Q} . We refer to Biagini et al. (2019) for more details. The risky bonds evolve according to

$$(2.3) \quad dP_t^j = r_t^j P_t^j dt - P_{t-}^j dM_t^{j, \mathbb{Q}}, \quad j \in \{B, C\},$$

where $M^{j, \mathbb{Q}}$, $j \in \{B, C\}$ are compensated Poisson random measures, see equation (3.6) in Biagini et al. (2019).

2.2. xVA framework. We consider a family of contingent claims within a portfolio with agreed dividend stream $A^m = (A_t^m)_{t \in [0, T]}$, $m = 1, \dots, M$, and set $\bar{A}_t^m := \mathbb{1}_{\{t < \tau\}} A_t^m + \mathbb{1}_{\{t \geq \tau\}} A_{\tau-}^m$. The value of the single claims within the portfolio, ignoring any counterparty risk or funding issue, that we refer to as *clean values*, are denoted by $(\hat{V}_t^m)_{m=1, \dots, M}$ and satisfy the following forward-backward stochastic differential equations (FBSDEs), for $m = 1, \dots, M$,

$$(2.4) \quad \begin{cases} -d\hat{V}_t^m = dA_t^m - r_t \hat{V}_t^m dt - \sum_{k=1}^d \hat{Z}_t^{m, k} dW_t^{k, \mathbb{Q}}, \\ \hat{V}_{T_m}^m = 0, \end{cases}$$

which reads, in integral form,

$$(2.5) \quad \hat{V}_t^m := \mathbb{E}^{\mathbb{Q}} \left[B_t^r \int_{(t, T_m]} \frac{dA_u^m}{B_u^r} \middle| \mathcal{F}_t \right], \quad t \in [0, T_m],$$

where r is a collateral rate in an idealized perfect collateral agreement.

For simplicity, we restrict ourselves to European-type contracts and write $A_t^m = \mathbb{1}_{\{t = T_m\}} g_m(S_{T_m})$, $T_m \leq T$, for a family of Lipschitz functions g_m , $m = 1, \dots, M$. Then, equation (2.4) reads

$$(2.6) \quad \begin{cases} -d\hat{V}_t^m = -r_t \hat{V}_t^m dt - \sum_{k=1}^d \hat{Z}_t^{m, k} dW_t^{k, \mathbb{Q}}, \\ \hat{V}_{T_m}^m = g_m(S_{T_m}). \end{cases}$$

Observe that the system (2.1) and (2.6) is decoupled, in the sense that the forward equation (2.1) does not exhibit a dependence on the backward component.

We continue to follow the framework of Biagini et al. (2019), where the portfolio dynamics are stated in the form of a BSDE under the enlarged filtration \mathbb{G} . We set

$$(2.7a) \quad Z_t^k := \sum_{i=1}^d \xi_t^i \sigma^{i,k}(t, S_t), \quad k = 1, \dots, d,$$

$$(2.7b) \quad U_t^j := -\xi_t^j P_{t-}^j, \quad j \in \{B, C\},$$

$$(2.7c) \quad f(t, V, C) := - \left[(r_t^{f,l} - r_t) (V_t - C_t)^+ - (r_t^{f,b} - r_t) (V_t - C_t)^- \right. \\ \left. + (r_t^{c,l} - r_t) C_t^+ - (r_t^{c,b} - r_t) C_t^- \right],$$

where

- $\xi^i, i = 1, \dots, d$, are the positions in risky assets, while ξ^B, ξ^C are the position in the bank and counterparty bond respectively;
- $r^{f,l}, r^{f,b}$ represent unsecured funding lending and borrowing rates;
- $r^{c,l}, r^{c,b}$ denote the interest on posted and received variation margin (collateral);
- C^+ and C^- represent the posted and received variation margin/collateral and $C = C^+ - C^-$.

All above processes are assumed to satisfy suitable regularity conditions ensuring existence and uniqueness for a solution to BSDE (2.8) below. Both posted and received collateral are assumed to be Lipschitz functions of the clean value of the derivative portfolio and we will write $C_t = C(V_t)$.

We denote by V the *full contract* value, i.e. the portfolio value including counterparty risk and multiple curves. The \mathbb{G} -BSDE for the portfolio's dynamics then has the form on $\{\tau > t\}$

$$(2.8) \quad \begin{cases} -dV_t = \sum_{m=1}^M d\bar{A}_t^m + (f(t, V, C) - r_t V_t) dt - \sum_{k=1}^d Z_t^k dW_t^{k,\mathbb{Q}} - \sum_{j \in \{B, C\}} U_t^j dM_t^{j,\mathbb{Q}}, \\ V_\tau = \theta_\tau(\hat{V}, C), \quad \text{with} \\ \theta_\tau(\hat{V}, C) := \hat{V}_\tau + \mathbb{1}_{\{\tau^C < \tau^B\}}(1 - R^C) (\hat{V}_\tau - C_{\tau-})^- - \mathbb{1}_{\{\tau^B < \tau^C\}}(1 - R^B) (\hat{V}_\tau - C_{\tau-})^+, \end{cases}$$

where $\hat{V}_t := \sum_{m=1}^M \hat{V}_t^m$ and R^B, R^C are two positive constants representing the recovery rate of the bank and the counterparty, respectively.

In their Theorem 3.15, Biagini et al. (2019) show that there exists a unique solution (V, Z, U) for the \mathbb{G} -BSDE (2.8), and the process V assumes the following form on $\{\tau > t\}$:

$$(2.9) \quad V_t = B_t^r \mathbb{E}^{\mathbb{Q}} \left[\sum_{m=1}^M \int_{(t, \tau \wedge T]} \frac{d\bar{A}_u^m}{B_u^r} + \int_t^{\tau \wedge T} \frac{f(u, V, C)}{B_u^r} du + \mathbb{1}_{\{\tau \leq T\}} \frac{\theta_\tau(\hat{V}, C)}{B_\tau^r} \middle| \mathcal{G}_t \right].$$

To prove existence and uniqueness for the \mathbb{G} -BSDE, Biagini et al. (2019) employ the technique introduced by Crépey (2015a) and reformulate the problem under the reduced filtration \mathbb{F} . Stated in such a form, the problem is also more amenable to numerical computations.

We consider the following \mathbb{F} -BSDE on $[0, T]$:

$$(2.10) \quad \begin{cases} -d\overline{XVA}_t = \bar{f}(t, \hat{V}_t, \overline{XVA}_t) dt - \sum_{k=1}^d \bar{Z}_t^k dW_t^{k,\mathbb{Q}}, \\ \overline{XVA}_T = 0, \end{cases}$$

where

$$\begin{aligned}
 \bar{f}(t, \hat{V}_t, \overline{\text{XVA}}_t) := & -(1 - R^C) \left(\hat{V}_t - C_t \right)^- \lambda_t^{C, \mathbb{Q}} \\
 & + (1 - R^B) \left(\hat{V}_t - C_t \right)^+ \lambda_t^{B, \mathbb{Q}} \\
 & + (r_t^{f,l} - r_t) \left(\hat{V}_t - \overline{\text{XVA}}_t - C_t \right)^+ - (r_t^{f,b} - r_t) \left(\hat{V}_t - \overline{\text{XVA}}_t - C_t \right)^- \\
 & + (r_t^{c,l} - r_t) C_t^+ - (r_t^{c,b} - r_t) C_t^- - (r_t + \lambda_t^{C, \mathbb{Q}} + \lambda_t^{B, \mathbb{Q}}) \overline{\text{XVA}}_t.
 \end{aligned}
 \tag{2.11}$$

By standard results on BSDEs, see e.g. Delong (2017, Theorem 4.1.3, Theorem 3.1.1), the existence and uniqueness of solutions $(\hat{V}^m, \hat{Z}^m) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^{2,q \times 1}$, for $m = 1, \dots, M$, and $(\overline{\text{XVA}}, \bar{Z}) \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^{2,q \times 1}$ to, respectively, (2.6) and (2.10), holds under the following conditions:

$$\begin{aligned}
 & r^{f,l}, r^{f,b}, r^{c,l}, r^{c,b}, r, \lambda^{B, \mathbb{Q}}, \lambda^{C, \mathbb{Q}} \text{ are bounded processes;} \\
 & |\mu(t, x) - \mu(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leq C|x - x'|, \\
 & |\sigma(t, x)| + |\mu(t, x)| \leq C(1 + |x|).
 \end{aligned}$$

The process $\overline{\text{XVA}}$ coincides with the *pre-default* xVA process. Indeed, given the pre-default value process \bar{V} such that $\bar{V}_t \mathbb{1}_{\{\tau > t\}} = V_t \mathbb{1}_{\{\tau > t\}}$, on $\{\tau > t\}$ the solution to (2.8) can be represented as

$$\bar{V}_t = \hat{V}_t - \overline{\text{XVA}}_t.$$

Moreover, defining the process $\tilde{r} = (\tilde{r}_t)_{t \in [0, T]}$ as $\tilde{r} := r + \lambda^{C, \mathbb{Q}} + \lambda^{B, \mathbb{Q}}$, it has been shown in Biagini et al. (2019, Corollary 3.31) that the process $\overline{\text{XVA}}$ admits the representation

$$\overline{\text{XVA}}_t = -\overline{\text{CVA}}_t + \overline{\text{DVA}}_t + \overline{\text{FVA}}_t + \overline{\text{ColVA}}_t, \tag{2.12}$$

where

$$\overline{\text{CVA}}_t := B_t^{\tilde{r}} \mathbb{E}^{\mathbb{Q}} \left[(1 - R^C) \int_t^T \frac{1}{B_u^{\tilde{r}}} \left(\hat{V}_u - C_u \right)^- \lambda_u^{C, \mathbb{Q}} du \middle| \mathcal{F}_t \right], \tag{2.13}$$

$$\overline{\text{DVA}}_t := B_t^{\tilde{r}} \mathbb{E}^{\mathbb{Q}} \left[(1 - R^B) \int_t^T \frac{1}{B_u^{\tilde{r}}} \left(\hat{V}_u - C_u \right)^+ \lambda_u^{B, \mathbb{Q}} du \middle| \mathcal{F}_t \right], \tag{2.14}$$

$$\begin{aligned}
 \overline{\text{FVA}}_t := & B_t^{\tilde{r}} \mathbb{E}^{\mathbb{Q}} \left[\int_t^T \frac{(r_u^{f,l} - r_u) \left(\hat{V}_u - \overline{\text{XVA}}_u - C_u \right)^+}{B_u^{\tilde{r}}} du \middle| \mathcal{F}_t \right] \\
 & - B_t^{\tilde{r}} \mathbb{E}^{\mathbb{Q}} \left[\int_t^T \frac{(r_u^{f,b} - r_u) \left(\hat{V}_u - \overline{\text{XVA}}_u - C_u \right)^-}{B_u^{\tilde{r}}} du \middle| \mathcal{F}_t \right],
 \end{aligned}
 \tag{2.15}$$

$$\overline{\text{ColVA}}_t := B_t^{\tilde{r}} \mathbb{E}^{\mathbb{Q}} \left[\int_t^T \frac{(r_u^{c,l} - r_u) C_u^+ - (r_u^{c,b} - r_u) C_u^-}{B_u^{\tilde{r}}} du \middle| \mathcal{F}_t \right]. \tag{2.16}$$

This representation highlights that the inclusion of different borrowing and lending rates introduces a non-zero funding adjustment which cannot be found independently of the other adjustments. An algorithm to compute all valuations adjustments systematically in the ‘non-recursive’ and ‘recursive’ setting, especially with the view of potentially large portfolios, is the focus of the next sections.

3. THE ALGORITHM

In this section, we introduce the algorithm for computing valuation adjustments by neural network approximations to the BSDE model from the previous section. We start by briefly recalling the main

features of the Deep BSDE Solver in E et al. (2017). Then, we present the application of the solver to valuation adjustments and its extensions to obtain financially important quantities. We first focus on non-recursive adjustments, namely CVA and DVA, and then extend the approach to the recursive case (see the terminology introduced at the end of the last section).

In particular, we propose to use the Deep BSDE Solver in E et al. (2017) to approximate the dynamics of \widehat{V}_u^m , $m = 1, \dots, M$, $u \in [t, T]$, which constitute the portfolio $\widehat{V}_u = \sum_{i=1}^M \widehat{V}_u^i$. Once the portfolio value has been approximated and resulting collaterals computed, the value of the adjustment can be obtained either by inserting the values in an ‘outer’ Monte Carlo computation for non-recursive adjustments, or applying a second time the Deep BSDE Solver to (2.10) in the recursive case.

3.1. The Deep BSDE Solver of E et al. (2017). For the reader’s convenience, we describe in this section the main principles of the algorithm in E et al. (2017) as they are relevant to our setting. We consider a general FBSDE framework.

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a probability space rich enough to support an \mathbb{R}^d -valued Brownian motion $W^\mathbb{Q} = (W_t^\mathbb{Q})_{t \in [0, T]}$. Let $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$ be the filtration generated by $W^\mathbb{Q}$, assumed to satisfy the standard assumptions. Let us consider an FBSDE in the following general form:

$$(3.1) \quad X_t = x + \int_0^t b(s, X_s) ds + \int_0^t a(s, X_s)^\top dW_s^\mathbb{Q}, \quad x \in \mathbb{R}^d$$

$$(3.2) \quad Y_t = \vartheta(X_T) + \int_t^T h(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s^\top dW_s^\mathbb{Q}, \quad t \in [0, T],$$

where the vector fields $b : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$, $a : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^{d \times d}$, $h : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$ and $\vartheta : \mathbb{R}^d \mapsto \mathbb{R}$ satisfy suitable assumptions ensuring existence and uniqueness results. We denote by $(X_t^x)_{t \in [0, T]} \in \mathbb{S}^2(\mathbb{R}^d)$ and $(Y_t^y, Z_t)_{t \in [0, T]} \in \mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^{2, q \times 1}$ the unique adapted solution to (3.1) and (3.2), respectively. To alleviate notations, hereafter we omit the dependency on the initial condition x of the process X^\cdot .

The above formulation of FBSDEs is intrinsically linked to the following stochastic optimal control problem:

$$(3.3) \quad \underset{y, Z=(Z_t)_{t \in [0, T]}}{\text{minimise}} \quad \mathbb{E} \left[\left| \vartheta(X_T) - Y_T^{y, Z} \right|^2 \right]$$

$$(3.4) \quad \text{subject to} \quad \begin{cases} X_t &= x + \int_0^t b(s, X_s) ds + \int_0^t a(s, X_s)^\top dW_s^\mathbb{Q}, \\ Y_t^{y, Z} &= y - \int_0^t h(s, X_s, Y_s^{y, Z}, Z_s) ds + \int_0^t Z_s^\top dW_s^\mathbb{Q}, \quad t \in [0, T]. \end{cases}$$

In particular, a solution (Y, Z) to (3.2) is a minimiser of the problem (3.3). A discretized version of the optimal control problem (3.3)–(3.4) is the basis of the Deep BSDE Solver.

Given $N \in \mathbb{N}$, consider $0 = t_0 < t_1 < \dots < t_N = T$. For simplicity, let us take a uniform mesh with step Δt such that $t_n = n\Delta t$, $n = 0, \dots, N$, and denote $\Delta W_n = W_{t_{n+1}}^\mathbb{Q} - W_{t_n}^\mathbb{Q}$. By an Euler-Maruyama approximation of (3.3)–(3.4), one has

$$(3.5) \quad \widetilde{X}_{n+1} = \widetilde{X}_n + b(t_n, \widetilde{X}_n)\Delta t + a(t_n, \widetilde{X}_n)\Delta W_n, \quad \widetilde{X}_0 = x,$$

$$(3.6) \quad \widetilde{Y}_{n+1}^{y, \widetilde{Z}} = \widetilde{Y}_n^{y, \widetilde{Z}} - h(t_n, \widetilde{X}_n, \widetilde{Y}_n^{y, \widetilde{Z}}, \widetilde{Z}_n)\Delta t + \widetilde{Z}_n^\top \Delta W_n, \quad \widetilde{Y}_0^{y, \widetilde{Z}} = y.$$

The core idea of the Deep BSDE Solver is to approximate, at each time step n , the control process \widetilde{Z}_n in (3.6) by using an artificial neural network (ANN). More specifically, in the Markovian setting, Z_t is a measurable function of X_t , which we approximate by an ANN ansatz to carry out the optimisation

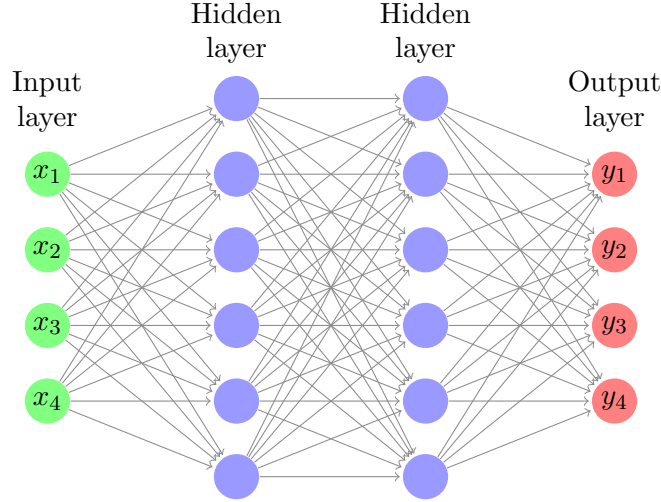


FIGURE 1. Schematic representation of a feedforward neural network with two hidden layers, i.e. $\mathcal{L} = 3$, input and output dimension $d = 4$, and $\nu = d + 2 = 6$ nodes.

above over this parametrised form. To this end, we introduce next a formalism for the description of neural networks.

ANN approximation. We consider artificial neural networks with $\mathcal{L} + 1 \in \mathbb{N} \setminus \{1, 2\}$ layers. Each layer consists of ν_ℓ nodes (also called *neurons*), for $\ell = 0, \dots, \mathcal{L}$. The 0-th layer represents the *input layer*, while the \mathcal{L} -th layer is called the *output layer*. The remaining $\mathcal{L} - 1$ layers are *hidden layers*. For simplicity, we set $\nu_\ell = \nu$, $\ell = 1, \dots, \mathcal{L} - 1$. The input and output dimensions are both d in our case. A feedforward neural network is a function $\varphi^\varrho : \mathbb{R}^d \mapsto \mathbb{R}^d$, defined via the composition

$$x \in \mathbb{R}^d \mapsto \mathcal{A}_\mathcal{L} \circ \varrho \circ \mathcal{A}_{\mathcal{L}-1} \circ \dots \circ \varrho \circ \mathcal{A}_1(x) \in \mathbb{R}^d,$$

where all \mathcal{A}_ℓ , $\ell = 1, \dots, \mathcal{L}$, are affine transformations

$$\mathcal{A}_1 : \mathbb{R}^d \mapsto \mathbb{R}^\nu, \quad \mathcal{A}_\ell : \mathbb{R}^\nu \mapsto \mathbb{R}^\nu, \quad \ell = 2, \dots, \mathcal{L} - 1, \quad \mathcal{A}_\mathcal{L} : \mathbb{R}^\nu \mapsto \mathbb{R}^d,$$

of the form $\mathcal{A}_\ell(x) := \mathcal{W}_\ell x + \beta_\ell$, $\ell = 1, \dots, \mathcal{L}$, where \mathcal{W}_ℓ and β_ℓ are matrices and vectors of suitable size called, respectively, weights and biases. The function ϱ , called *activation function* is a univariate function $\varrho : \mathbb{R} \mapsto \mathbb{R}$ that is applied component-wise to vectors. With an abuse of notation, we denote $\varrho(x_1, \dots, x_\nu) = (\varrho(x_1), \dots, \varrho(x_\nu))$. The elements of \mathcal{W}_ℓ and β_ℓ are the parameters of the neural network. We can regroup all parameters in a vector $\rho \in \mathbb{R}^R$ where $R = \sum_{\ell=0}^{\mathcal{L}} \nu_\ell(1 + \nu_\ell)$.

As indicated earlier, we use ANNs to approximate the control process Z_t . More specifically, let $R \in \mathbb{N}$ as before and let $\xi \in \mathbb{R}$, $\rho \equiv (\rho_1, \dots, \rho_R) \in \mathbb{R}^R$ be $R + 1$ parameters. We introduce a family of neural networks $\varphi_n^\rho : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $n \in \{0, \dots, N\}$ parametrized by ρ and indexed by time. We denote $\mathcal{Z}_n^\rho = \varphi_n^\rho(\tilde{X}_n)$ and consider the following parametrized version of (3.6)

$$(3.7) \quad \mathcal{Y}_{n+1}^{\xi, \rho} = \mathcal{Y}_n^{\xi, \rho} - h(t_n, X_n, \mathcal{Y}_n^{\xi, \rho}, \mathcal{Z}_n^\rho) \Delta t + (\mathcal{Z}_n^\rho)^\top \Delta W_n, \quad \mathcal{Y}_0^{\xi, \rho} = \xi,$$

meaning that, at each time step, we use a distinct neural network to approximate the control process. The Deep BSDE Solver by E et al. (2017) considers the following stochastic optimization problem

$$(3.8) \quad \underset{\xi \in \mathbb{R}, \rho \in \mathbb{R}^R}{\text{minimise}} \quad \mathbb{E} \left[\left(\vartheta(\tilde{X}_N) - \mathcal{Y}_N^{\xi, \rho} \right)^2 \right] \quad \text{subject to (3.5)–(3.7)}.$$

Observe that, in practice, one simulates $L \in \mathbb{N}$ Monte Carlo paths $(\tilde{X}_n^{(\ell)}, \mathcal{Y}_n^{\xi, \rho, (\ell)})_{n=0 \dots N}$ for $\ell = 1, \dots, L$, using (3.5)–(3.7) with N i.i.d. Gaussian random variables $(\Delta W_n)_{n=0, \dots, N-1}$ with mean 0 and variance

Δt . Replacing the expected cost functional by the empirical mean, (3.8) becomes

$$(3.9) \quad \underset{\xi \in \mathbb{R}, \rho \in \mathbb{R}^R}{\text{minimise}} \quad \frac{1}{L} \sum_{\ell=1}^L \left(\vartheta(\tilde{X}_N^{(\ell)}) - \mathcal{Y}_N^{\xi, \rho, (\ell)} \right)^2 \quad \text{subject to (3.5)–(3.7)}.$$

This minimization typically involves a huge number of parameters and it is performed by a stochastic gradient descent-type algorithm (SGD), leading to random approximations. For further details on this point we refer the reader to Section 2.6 in E et al. (2017). We will denote by \mathcal{I} the maximum number of SGD iterations. To improve the performance and stability of the ANN approximation, a *batch normalization* is also considered, see Ioffe and Szegedy (2015).

The accuracy of the solution is determined by the number of timesteps, number of samples, the chosen network architecture, and the quality of the optimiser found by the chosen optimisation routine. Our practical experience shows that quantifying and controlling the errors resulting from the latter two contributions is particularly difficult. Therefore, certain *a posteriori* error bounds as found in Bender and Steiner (2013) for decoupled FBSDEs, Han and Long (2020) for partially coupled FBSDEs, and in Reisinger et al. (2020) for fully coupled BSDEs are particularly valuable. Specifically, in Han and Long (2020, Theorem 1') the authors show that under suitable assumptions on the coefficients of the FBSDE (3.1)–(3.2), namely, in the decoupled case (see their Assumption 3, 2.), the uniform Lipschitz continuity in space, uniform 1/2-Hölder continuity in time of b, a, h and the Lipschitz continuity of ϑ one has, for Δt sufficiently small,

$$(3.10) \quad \sup_{t \in [0, T]} \mathbb{E} |Y_t - Y_t^{\xi, \rho}|^2 + \int_0^T \mathbb{E} |Z_t - Z_t^\rho|^2 dt \leq C \left(\Delta t + \mathbb{E} \left[\left(\vartheta(\tilde{X}_N) - \mathcal{Y}_N^{\xi, \rho} \right)^2 \right] \right),$$

where C is a constant independent of Δt and d possibly depending on the starting point x of the forward process and, given $(\mathcal{Y}_n^{\xi, \rho}, \mathcal{Z}_n^\rho)_{n=0, \dots, N}$ from (3.7), $Y_t^{\xi, \rho} = \mathcal{Y}_n^{\xi, \rho}$ and $Z_t^\rho = \mathcal{Z}_n^\rho$ for $t \in [t_n, t_{n+1})$. In Han and Long (2020, Theorem 2'), *a priori* estimates on the term $\mathbb{E}[(\vartheta(\tilde{X}_N) - \mathcal{Y}_N^{\xi, \rho})^2]$ appearing in the right hand side of (3.10) are also provided. However, the obtained bounds depend on the (unknown) approximation capacity of the considered ANN.

3.2. The Deep xVA Solver for non-recursive valuation adjustments. In our setting, the Deep BSDE Solver is first employed in the approximation of the clean values of the portfolio, i.e., the processes \widehat{V}_t^m for $m = 1, \dots, M$, which are the solutions of (2.6) with underlying forward dynamics given by S in (2.1). More precisely, in the notation of the previous section, we take

$$X_t = S_t \quad \text{and} \quad Y_t = \widehat{V}_t^m \quad \text{for } m = 1, \dots, M.$$

For simplicity, let us assume $T_m = T$, $\forall m = 1, \dots, M$. We now describe the algorithm for computing CVA and DVA given by formulas (2.13) and (2.14), respectively. A unifying formula for CVA and DVA can be written as

$$(3.11) \quad \mathbb{E}^{\mathbb{Q}} \left[\int_t^T \Phi_u(\widehat{V}_u) du \middle| \mathcal{F}_t \right],$$

where

- $\Phi_u(v) = (1 - R^C) \frac{B_t^{\tilde{r}}}{B_u^{\tilde{r}}} (v - C(v))^- \lambda_u^{C, \mathbb{Q}}$ for CVA;
- $\Phi_u(v) = (1 - R^B) \frac{B_t^{\tilde{r}}}{B_u^{\tilde{r}}} (v - C(v))^+ \lambda_u^{B, \mathbb{Q}}$ for DVA.

Here, $\Phi_u(v)$ indicates that Φ is a random field. One can easily observe that, thanks to the boundedness of the processes \tilde{r} and λ^j , $j \in \{B, C\}$, $\Phi_u(v)$ is uniformly Lipschitz continuous in v . We denote by L_Φ its Lipschitz constant.

Given a time discretization (uniform, for simplicity) with time step Δt , the integral in (3.11) can be approximated by a quadrature rule, i.e., taking $t = t_0 = 0$,

$$\int_0^T \Phi_u(\widehat{V}_u) du \approx \sum_{n=0}^N \eta_n \Phi_{t_n}(\widehat{V}_{t_n}).$$

For instance, one may consider the rectangle rule, i.e. $\eta_N = 0, \eta_n = \Delta t \ n = 0, \dots, N-1$,

$$(3.12) \quad \int_0^T \Phi_u(\widehat{V}_t) dt \approx \sum_{n=0}^{N-1} \Phi_{t_n}(\widehat{V}_{t_n}) \Delta t.$$

Denoting for any $m = 1, \dots, M$ by $(\widehat{\mathcal{V}}_n^{m, \xi_m^*, \rho_m^*, (p)})_{n=0, \dots, N, p=1, \dots, P}$ the approximation of the process $(\widehat{V}_{t_n}^m)_{n=0, \dots, N}$ obtained by means of the parameters (ξ_m^*, ρ_m^*) resulting from the Deep BSDE Solver optimization (3.9) and

$$\widehat{\mathcal{V}}_n^{*, (p)} := \sum_{m=1}^M \widehat{\mathcal{V}}_n^{m, \xi_m^*, \rho_m^*, (p)}, \quad n = 0, \dots, N,$$

the adjustment is then approximated by the following formula:

$$\frac{1}{P} \sum_{p=1}^P \sum_{n=0}^N \eta_n \Phi_{t_n}(\widehat{\mathcal{V}}_n^{*, (p)}).$$

Algorithms 1 and 2 summarize the main steps of the method. In what follows we will also denote by $\widehat{\mathcal{V}}^{m, \xi_m^*, \rho_m^*}$ the piecewise constant interpolation of $\widehat{\mathcal{V}}_n^{m, \xi_m^*, \rho_m^*}$.

Algorithm 1: Deep algorithm for exposure simulation

Set parameters: N, L .

▷ N time steps, L paths for inner Monte Carlo loop

Fix architecture of ANN.

▷ intrinsically defines the number of parameters R

Deep BSDE Solver (N, L) :

Simulate L paths $(\widetilde{S}_n^{(\ell)})_{n=0, \dots, N}$, $\ell = 1, \dots, L$ of the forward dynamics.

Define the neural networks $(\varphi_n^\rho)_{n=1, \dots, N}$.

for $m = 1, \dots, M$ **do**

Minimize over ξ and ρ

$$\frac{1}{L} \sum_{\ell=1}^L \left(g_m(\widetilde{S}_N^{(\ell)}) - \widehat{\mathcal{V}}_N^{m, \xi, \rho, (\ell)} \right)^2,$$

subject to

$$(3.13) \quad \begin{cases} \widehat{\mathcal{V}}_{n+1}^{m, \xi, \rho, (\ell)} = \widehat{\mathcal{V}}_n^{m, \xi, \rho, (\ell)} + r_{t_n} \widehat{\mathcal{V}}_n^{m, \xi, \rho, (\ell)} \Delta t + (\widehat{\mathcal{Z}}_n^{m, \rho, (\ell)})^\top \Delta W_n^{(\ell)}, \\ \widehat{\mathcal{V}}_0^{m, \xi, \rho, (\ell)} = \xi, \\ \widehat{\mathcal{Z}}_n^{\rho, (\ell)} = \varphi_n^\rho(\widetilde{S}_n^{(\ell)}). \end{cases}$$

Save the optimizer (ξ_m^*, ρ_m^*) .

end

end

Under reasonable assumptions, we can derive the following *a posteriori* bounds for the error associated with this approximation of the valuation adjustments in $[0, T]$, starting from (3.10). The derivation is given in Appendix A. We note that these adjustments can also be obtained from the more general framework in Section 3.3, however, we provide a simpler numerical procedure here and derive error estimates for these approximations by a more explicit computation.

Algorithm 2: Deep xVA Solver for non-recursive valuation adjustments

Apply Algorithm 1

Set parameters: P .

$\triangleright P$ paths for the outer Monte Carlo loop

Simulate, for $m = 1 \dots M$, $(\widehat{V}_n^{m, \xi_m^*, \rho_m^*, (p)})_{n=0 \dots N, p=1 \dots P}$ by means of (3.13) with $(\xi, \rho) = (\xi_m^*, \rho_m^*)$.

\triangleright approximation of the clean values

Define $\widehat{V}_n^{*, (p)} := \sum_{m=1}^M \widehat{V}_n^{m, \xi_m^*, \rho_m^*, (p)}$ for $n = 0 \dots N$, $p = 1 \dots P$.

\triangleright approximation of the clean portfolio value

Compute the adjustment as

$$\frac{1}{P} \sum_{i=1}^P \left(\sum_{n=0}^N \eta_n \Phi_{t_n}(\widehat{V}_n^{*, (p)}) \right).$$

Let $\widehat{V}_t = \sum_{m=1}^M \widehat{V}_t^m$ with \widehat{V}_t^m given by (2.6), and $\widehat{V}_n^* = \sum_{m=1}^M \widehat{V}_n^{m, \xi_m^*, \rho_m^*}$ ($n = 0, \dots, N$) its approximation from the Deep BSDE Solver. Consider the running assumptions of this paper together with uniform Hölder continuity in t of b and σ , and assume estimate (3.10) for equation (2.6).¹

Moreover, consider the specific forms of Φ above, assuming $\mathbb{E}[(\lambda_s^{j, \mathbb{Q}} - \lambda_t^{j, \mathbb{Q}})^2] \leq C(t-s)$ for $0 \leq s \leq t \leq T$ and $j \in \{B, C\}$.²

Then, for Φ as above, there exists a constant $C \geq 0$ depending only on the model inputs and the constants coming from (3.10) (in particular not on Δt and the ANN parameters), such that

$$(3.14) \quad \left| \mathbb{E} \left[\int_0^T \Phi_t(\widehat{V}_t) dt \right] - \mathbb{E} \left[\sum_{n=0}^{N-1} \Delta t \Phi_{t_n}(\widehat{V}_n^*) \right] \right| \leq C \left(\Delta t + \sum_{m=1}^M \mathbb{E} \left[|g_m(S_T) - \widehat{V}_N^{m, \xi_m^*, \rho_m^*}|^2 \right] \right)^{1/2}.$$

MC standard errors for the second expectation in (3.14) should be added to obtain a complete bound.

3.3. The Deep xVA Solver for recursive valuation adjustments. The procedure of the previous section is sufficient to perform the estimation of CVA and DVA according to (2.13) and (2.14) at time zero by means of a standard Monte Carlo estimator, given the pathwise solutions of the BSDEs for clean values. Typically, however, the bank needs to also compute risk measures on the CVA, such as Value-at-Risk. Moreover, if we consider the xVA BSDE (2.10), we observe that FVA terms introduce a recursive structure through the driver, so that a time t estimate of the process $\overline{\text{XVA}}$ requires the use of a numerical solver for a BSDE. Finally, let us observe that the bank is not only interested in computing the xVA at time t , also hedging the market risk of xVA is important, meaning that one also needs sensitivities of valuation adjustments with respect to the driving risk factors.

All above considerations motivate us to propose a two-step procedure, where we first employ the Deep BSDE Solver to estimate the clean values \widehat{V}^m , $m = 1, \dots, M$, according to Algorithm 1 and then, using the simulated paths of the M clean BSDEs obtained from the first step, we apply again the Deep BSDE Solver to numerically solve the xVA BSDE (2.10). The procedure is outlined in Algorithm 3.

Similar to Section 3.2, we can quantify the error of the Deep xVA Solver in the recursive case *a posteriori*. Let $(\overline{\text{XVA}}_t, \overline{Z}_t)$ be the solution of (2.10), $(\widetilde{\text{XVA}}_t^{\gamma, \zeta}, \widetilde{Z}_t^{\zeta})$ the corresponding approximation

¹ This is a straightforward extension of Han and Long (2020, Theorem 1') in the case of deterministic r_s in (2.6) which is Hölder-1/2 in s , by replacing their assumption on the uniform Hölder-1/2 continuity of f by $|f(t, v) - f(s, v)| \leq C|t - s|^{1/2}|v|$ for all $0 \leq s \leq t \leq T$ and all v . For stochastic rates, a more substantial extension to their analysis is needed for a direct application of Euler-Maruyama, due to the non-Lipschitz term $r_t \widehat{V}_t$ in (2.6) and accounting for the discretisation of the rates process. However, the simple transformation (A.2) from the appendix can eliminate this drift. We hence directly assume (A.1) for this analysis.

² This is immediate for deterministic Hölder-1/2 functions and a standard property of Itô diffusions with Lipschitz coefficients (see Zhang (2004, Lemma 2.4, (2.10))), but also holds, e.g., for the Cox-Ingersoll-Ross process (as follows e.g. from Hutzenthaler et al. (2014, Corollary 2.14)).

Algorithm 3: Deep xVA Solver

Apply Algorithm 1.

Set parameters: P . $\triangleright P$ paths for outer Monte Carlo loop

Fix architecture of ANN.

 \triangleright *intrinsically defines the number of parameters \bar{R} (in general $\bar{R} \neq R$)***Deep XVA-BSDE solver** (N, P):Simulate P paths $(\mathcal{V}_n^{(p)})_{n=0,\dots,N}$, $p = 1, \dots, P$, of the portfolio value.Define the neural networks $(\psi_n^\zeta)_{n=1,\dots,N}$.Minimize over γ and ζ

$$\frac{1}{P} \sum_{p=1}^P \left(\bar{\mathcal{X}}_N^{\gamma, \zeta, (p)} \right)^2,$$

subject to

$$(3.15) \quad \begin{cases} \bar{\mathcal{X}}_{n+1}^{\gamma, \zeta, (p)} = \bar{\mathcal{X}}_n^{\gamma, \zeta, (p)} - \bar{f}(t_n, \hat{\mathcal{V}}_n^{(p)}, \bar{\mathcal{X}}_n^{\gamma, \zeta, (p)}) \Delta t + (\bar{\mathcal{Z}}_n^{\zeta, (p)})^\top \Delta W_n^{(p)}, \\ \bar{\mathcal{X}}_0^{\gamma, \zeta, (p)} = \gamma, \\ \bar{\mathcal{Z}}_n^{\zeta, (p)} = \psi_n^\zeta(\hat{\mathcal{V}}_n^{(p)}). \end{cases}$$

end

from the Deep BSDE Solver with parameters γ, ζ , with \hat{V} in (2.10) replaced by $\hat{V}^{\xi, \rho}$ given by the solver with parameters ξ, ρ . We note that the result of Han and Long (2020) can be extended to multi-dimensional BSDEs (see the comment at the start of Section 2 there), or that our system is a special case of the fully-coupled McKean–Vlasov FBSDEs analysed in Reisinger et al. (2020) (where the monotonicity condition H.1.(1) imposed there is not needed here in the weakly coupled case).

Take the running assumptions of this paper. Moreover, let for simplicity all rates and intensity processes be bounded, uniformly 1/2-Hölder continuous deterministic functions of time and the functions μ, σ, \bar{f} be uniformly 1/2-Hölder continuous in time. Then there exists a constant $K \geq 0$ depending only on the model inputs (in particular not on Δt and the ANN parameters) such that

$$\begin{aligned} & \sup_{t \in [0, T]} \mathbb{E} \left[\left| \overline{\text{XVA}}_t - \widetilde{\text{XVA}}_t^{\gamma, \zeta} \right|^2 \right] + \mathbb{E} \left[\int_0^T |\bar{Z}_t - \tilde{Z}_t^\zeta|^2 dt \right] \\ & \leq K \left(\Delta t + \sum_{m=1}^M \mathbb{E} \left[|g_m(S_T) - \hat{\mathcal{V}}_N^{m, \xi_m, \rho_m}|^2 \right] + \mathbb{E} \left[|\widetilde{\text{XVA}}_T^{\gamma, \zeta}|^2 \right] \right). \end{aligned}$$

It should be possible to derive similar results for bounded or even unbounded stochastic rates, but care would have to be taken with the discretisation in the case of non-Lipschitz coefficients, such as the CIR model.

3.4. Pathwise simulation of sensitivities. One interesting feature of our approach to xVA computations is that we can easily estimate several sensitivities (i.e., partial derivatives) of pricing functions. Let us recall that, in the present Markovian setting, the control Z associated with a FBSDE of the general form (3.1)–(3.2) satisfies

$$(3.16) \quad Z_t = \frac{\partial Y}{\partial X}(t, X_t) a(t, X_t),$$

so that we can easily reconstruct the gradient of the pricing function with respect to all risk factors simply by multiplying each (vector-valued) neural network by the inverse (assuming it exists) of the matrix $a(t, X_t)$. This becomes particularly interesting in view of Algorithms 1 and 3, where we can

obtain hedge ratios both for the clean value and for the valuation adjustments without further computations.

Obtaining second order sensitivities, which may also be important for hedging purposes, is also feasible in our setting, because feedforward neural networks are compositions of simple functions and computation of gradients of neural network functions has become standard in that community. Using the notation of Section 3.1, we can write

$$(3.17) \quad \frac{\partial \mathcal{Z}_n^\rho}{\partial X_n} = \frac{\partial \varphi^\varrho(X_n)}{\partial X_n},$$

with $\varphi^\varrho(X_n) = \mathcal{A}_\mathcal{L}(\rho(\mathcal{A}_{\mathcal{L}-1} \dots \rho(\mathcal{A}_1(X_n))))$. Since $(\mathcal{A}_\ell)_{\ell=1, \dots, \mathcal{L}}$ are affine functions, their Jacobians are given by the weight matrices, i.e.

$$J_{\mathcal{A}_\ell}(\cdot) = \mathcal{W}_\ell, \quad \ell = 1, \dots, \mathcal{L}.$$

Moreover, one also has the Jacobian of ρ ,

$$J_\varrho(\cdot) = \text{diag}(\varrho'(\cdot)),$$

where, for $x \in \mathbb{R}^\nu$ we denote $\varrho'(x) = (\varrho'(x_1), \dots, \varrho'(x_\nu))$. In the present paper, we choose $\varrho(x) = \text{ReLU}(x) = \max\{x, 0\}$ so that the first derivative can be defined as

$$\varrho'(x) = \text{ReLU}'(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases} = \text{sgn}(\text{ReLU}(x)).$$

Finally, we deduce that the following explicit differentiation formula holds:

$$\frac{\partial \mathcal{Z}_n^\rho}{\partial X_n} = \mathcal{W}_\mathcal{L} \text{diag}(\varrho'(\mathcal{A}_{\mathcal{L}-1}(\dots \mathcal{A}_1(X_n)))) \dots \text{diag}(\varrho'(\mathcal{A}_1(X_n))) \mathcal{W}_1.$$

Given the availability of the derivative of \mathcal{Z}_n^ρ we can then obtain the Hessian of Y from (3.16).

4. NUMERICAL RESULTS

To test our algorithm, we start by studying two very simple examples with a similar computational structure as CVA and DVA, and for which we can easily provide reference solutions. We will then give a higher-dimensional example and illustrate further practically relevant features of the method, such as recursive xVA computations and simulation of the collateral account. The codes for the proposed experiments are available at <https://github.com/AlessandroGnoatto/Deep-xVA-Solver>.

Let S be the price of a single stock described by a Black-Scholes dynamics,

$$dS_t = rS_t dt + \sigma S_t dW_t^\mathbb{Q}, \quad S_0 = s_0,$$

and \widehat{V} a European-style contingent claim with value

$$\widehat{V}_t = \mathbb{E} \left[e^{-r(T-t)} g(S_T) | \mathcal{F}_t \right].$$

In particular, \widehat{V} solves the following BSDE:

$$(4.1) \quad \begin{cases} -d\widehat{V}_t = -r\widehat{V}_t dt - \widehat{Z}_t dW_t^\mathbb{Q}, \\ \widehat{V}_T = g(S_T). \end{cases}$$

The discounted expected positive and negative exposure of \widehat{V} are defined, respectively, by

$$(4.2) \quad \text{DEPE}(s) = \mathbb{E}^\mathbb{Q} \left[e^{-r(s-t)} \left(\widehat{V}_s \right)^+ \middle| \mathcal{F}_t \right],$$

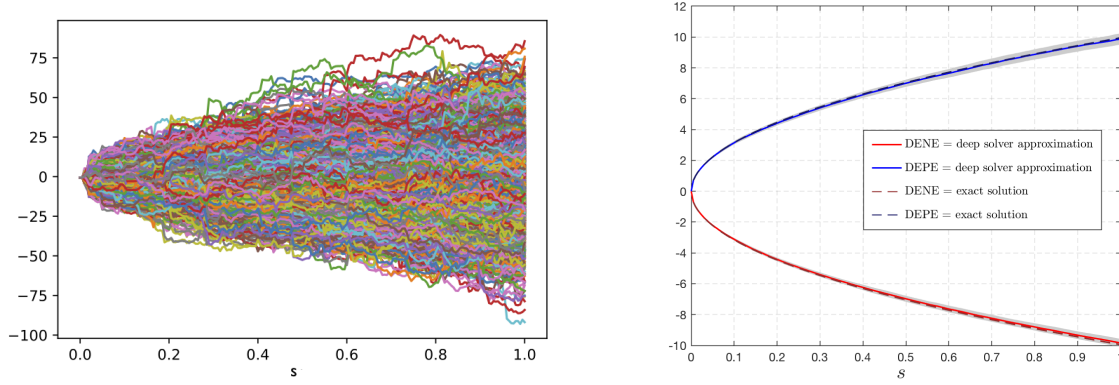


FIGURE 2. Forward contract: approximated exposure (left) and EPE, ENE (right). Parameters used: outer MC paths $P = 2048$, inner MC paths $L = 64$, internal layers $\mathcal{L} - 1 = 2$, $\nu = d + 20 = 21$, $\mathcal{I} = 4000$, time steps $N = 200$.

$$(4.3) \quad \text{DENE}(s) = -\mathbb{E}^{\mathbb{Q}} \left[e^{-r(s-t)} \left(\widehat{V}_s \right)^- \middle| \mathcal{F}_t \right].$$

In order to take into account the randomness of the algorithm (through the inner and outer Monte Carlo estimation and stochastic gradient descent), in the plots below we report with solid lines the average DEPE (in blue) and DENE (in red) obtained after 100 runs of the algorithm and the gray region represent the obtained standard deviation from the average value.

4.1. **A forward on S .** In this case, we consider

$$g(S_T) = S_T - K$$

with $K = s_0$. The pathwise exposure \widehat{V} at time $s \in [t, T]$ is given by

$$\widehat{V}_s = \mathbb{E}^{\mathbb{Q}} \left[e^{-r(T-s)} (S_T - K) \middle| \mathcal{F}_s \right] = S_s - K e^{-r(T-s)}.$$

Substituting in (4.2), one has

$$(4.4) \quad \text{DEPE}(s) = S_t \Phi(d_1) - K e^{-r(T-t)} \Phi(d_2),$$

$$(4.5) \quad \text{DENE}(s) = S_t \Phi(-d_1) - K e^{-r(T-t)} \Phi(-d_2),$$

where $\Phi(\cdot)$ denotes the standard normal cumulative distribution function and, as usual,

$$d_1 = \frac{\ln(e^{r(t-s)} S_t / K) + (r + \sigma^2/2)(s - t)}{\sigma \sqrt{s - t}} \quad \text{and} \quad d_2 = d_1 - \sigma \sqrt{s - t}.$$

σ	K	T
0.25	100	1

TABLE 1. Parameters used in numerical experiments.

We report in Figure 2 the plot of the numerical results obtained by Algorithm 2 using the parameters in Table 1 and $r = 0$. In particular, on the left we plot the simulated pathwise exposure, i.e. the paths $t_n \rightarrow \mathcal{V}_n^{(p)}$ for $p = 1, \dots, P$, while on the right we compare the approximated EPE and ENE (solid lines) with the exact expected exposures given by (4.4)–(4.5) (dashed lines).

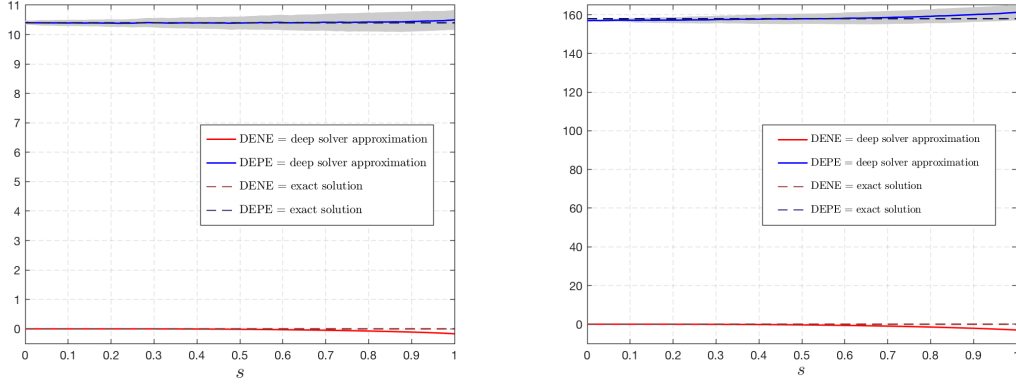


FIGURE 3. DEPE and DENE for a European call option (left) and a European basket option with 100 underlyings (right). Parameters used: outer MC paths $P = 2048$, inner MC paths $L = 64$, internal layers $\mathcal{L} - 1 = 2$, $\nu = d + 20 = 21$ (left) and $\nu = d + 10 = 110$ (right), $\mathcal{I} = 4000$ (left) and $\mathcal{I} = 10000$ (right), time steps $N = 200$ (left) and $N = 100$ (right).

4.2. A European call option. In this case we consider

$$g(S_T) = (S_T - K)^+,$$

where we set $K = s_0$. The pathwise exposure \widehat{V} at time $s \in [t, T]$ is given by the Black-Scholes formula

$$\widehat{V}_s = \mathbb{E}^{\mathbb{Q}} \left[e^{-r(T-s)} (S_T - K)^+ \middle| \mathcal{F}_s \right] = S_s \Phi(d_1) - K e^{-r(T-s)} \Phi(d_2) > 0.$$

It follows immediately that

$$\text{DEPE}(s) = \mathbb{E}^{\mathbb{Q}} \left[e^{-r(s-t)} \widehat{V}_s \middle| \mathcal{F}_t \right] = \widehat{V}_t,$$

and

$$\text{DENE}(s) = 0.$$

The results obtained using Algorithm 2 with the parameters in Table 1 and $r = 0.01$ are reported in Figure 3 (left). The exact European call price is 10.4036, while the approximation of the positive and negative exposure obtained by the solver and reported in Figure 3 (left) take values, for $t \in [0, T]$, within the interval $[10.4072, 10.4963]$ and $[-0.1692, 0]$, respectively. The accuracy of the time zero option value for this architecture and simulation parameters is hence 0.36 bps, and that of DEPE and DENE in the worst case (over s) is 9.3 bps and 17 bps, respectively.

4.3. A basket call option. Let us now consider the case of several underlying assets (S^1, \dots, S^d) :

$$dS_t^i = r^i S_t^i dt + \sigma^i S_t^i dW_t^{\mathbb{Q},i}, \quad S_0^i = s_0^i > 0, \quad i = 1, \dots, d,$$

where $W^{\mathbb{Q}} = (W^{\mathbb{Q},1}, \dots, W^{\mathbb{Q},d})$ is a standard Brownian motion in \mathbb{R}^d with correlation matrix $(\rho_{i,j})_{1 \leq i,j \leq d}$. We set $d = 100$. A European basket call option is associated with the payoff

$$g(S_T^1, \dots, S_T^d) = \left(\sum_{i=1}^d S_T^i - d \cdot K \right)^+.$$

The results obtained by Algorithm 2 using the parameters in Table 1 with $\sigma^i = \sigma$ for all $i = 1, \dots, d$, zero correlation, $s_0^i = 100$ for all $i = 1, \dots, d$ and $r^i = r = 0.01$ are reported in Figure 3 (right).

The distinctive feature of the present example is the high dimension of the vector of risk factors. While the two previous one-dimensional examples mainly served as a validation for the methodology, the present example highlights the ability of the proposed methodology to provide an accurate numerical

approximation in a high-dimensional context. For this example, we used the feedforward neural network with two layers and $d + 10$ nodes, with a ReLU activation function. The approximation parameters used are reported in the caption of Figure 3 (right). We increase the number of nodes ν roughly linearly with the dimension d , which turned out to be a useful rule-of-thumb for consistent accuracy across dimensions in this case.

A detailed study of deep learning values of basket derivative (on six underlying assets) from simulated values, not based on BSDEs, see Ferguson and Green (2018).

For the case of the basket call option, we observe that the exposure profile corresponds to the present value of the contract. As a consequence, we obtain a simple method to validate the exposure profile by computing an estimate of the basket call option price by means of a standard Monte Carlo simulation with 10^5 paths. We regard this as the ‘exact’ price. The Monte Carlo price we obtained is 157.99 with confidence interval $[157.63, 158.34]$. The average values of the expected exposures produced by the deep solver reported in Figure 3 (right) vary with time between the values 156.98 and 161.24 in the positive case, and 0 and -2.9824 in the negative one, achieving at the terminal time $t = T$ the maximum distance 3.25 to the Monte Carlo price in the first case and 2.85 to the exact zero solution in the second case. The accuracy of the time 0 option price is therefore 1bp, and hence of the same order of magnitude as for the single underlying.

Remark 4.1. It is noticeable that the error of DEPE and DENE is relatively low at time zero and eventually increases with time. This is because the time zero value is determined solely by the obtained optimiser for ξ , which is decoupled from the harder optimisation problem for ρ . The optimal ξ which minimises the idealised objective function without time stepping and sampling error is the expected payoff, while ρ determines the ANN hedge which minimises the variance. A suboptimal ANN leads to larger hedging errors, and hence increasing DEPE and DENE, as time increases. One could use this observation to set ξ to be an accurate MC estimator for the option price, and then minimize over ρ only. This by construction gives accurate time zero values for DEPE and DENE, but from our tests (not reported here) lead to similar results to above for larger t .

For this product, next, we also perform an xVA calculation with the objective to validate Algorithm 2 and Algorithm 3 in a case where both are applicable. To perform this comparison, we need the xVA BSDE to be non-recursive: this can be achieved by assuming that there is a unique risk-free interest rate, so that FVA and ColVA are identically zero, i.e., xVA consists only of the CVA and DVA term. The idea is then to compare a Monte Carlo estimate of xVA according to Algorithm 2 with the initial value of the BSDE as produced by a full application of Algorithm 3.

We assume that the default intensities of the bank and the counterparty are $\lambda^{C,\mathbb{Q}} = 0.10$ and $\lambda^{B,\mathbb{Q}} = 0.01$, respectively. For the recovery rates we set $R^C = 0.3$ and $R^B = 0.4$, while the unique risk-free interest rate is $r = 0.01$. Using the same network setting (see again the caption of Figure 3, right), the Deep xVA Solver produced an xVA estimate of 0.6966 by means of Algorithm 3, whereas the estimate produced by Algorithm 2 is 0.6987 with an associated confidence interval $[0.6959, 0.7016]$.

4.4. Recursive FVA computation. In this section, we provide an FVA calculation that serves as a further validation of Algorithm 3 for recursive valuation adjustments. For the sake of illustration, we simplify the framework of Biagini et al. (2019) so that we recover the funding equations of Piterbarg (2010). More specifically, we assume that there is no default risk, i.e. $\tau^C = \tau^B = +\infty$. We consider the case of a bank trading a forward on a single underlying stock, in line with Example 4.1. We set $r^{c,b} = r^{c,l} = r = 0.02$, and $r^{f,b} = r^{f,l} = 0.04$. Due to the different interest rates for funding and collateral, the clean value of the contract is not at par. We assume that the claim is perfectly

uncollateralized, i.e. $C_t \equiv 0$ $d\mathbb{Q} \otimes dt$ -a.s. In this case, as first shown in Piterbarg (2010) and then Biagini et al. (2019) among others, one can employ a risk neutral valuation formula where the discount rate is given by the unsecured funding rate $r^f = r^{f,b} = r^{f,l}$. Precisely, we can write the solution of the pricing problem as

$$V_t = \widehat{V}_t - \overline{\text{FVA}}_t, \quad \text{where}$$

$$\overline{\text{FVA}}_t = B_t^r \mathbb{E}^{\mathbb{Q}} \left[\int_t^T \frac{(r_u^f - r_u) (\widehat{V}_u - \overline{\text{FVA}}_u)}{B_u^r} du \middle| \mathcal{F}_t \right].$$

The analytic computation of the clean value of the forward contract at time t yields $\widehat{V}_0^{\text{exact}} = 1.9801$. The claim is however uncollateralized, hence, by applying a risk neutral valuation formula where the discounting rate is now r^f , we obtain $V_0^{\text{exact}} = 1.9409$. The difference between the two analytic computations provides us with the exact value of the FVA, i.e. $\overline{\text{FVA}}_0^{\text{exact}} = 0.0392$. For this experiment we apply Algorithm 3 with the following parameters that are the same both for the estimation of the clean value and the FVA: we use $N = 100$ and $L = 64$ and $P = 2048$. We use two neural networks for the clean value and the FVA both having 2 hidden layers with $d + 20$ nodes.

We then applied Algorithm 3 to the xVA BSDE associated with FVA and obtained an initial value of $\overline{\text{FVA}}_0 = 0.0395$, thus a validation of our proposed numerical procedure.

4.5. Realistic simulation of the collateral account. A useful feature of our proposed approach consists in the possibility of performing realistic simulations of the collateral account without resorting to simplifying assumptions. We can in fact compute the overall outstanding exposure between the bank and the counterparty by the following steps. Algorithm 1 allows us to simulate paths for all processes \widehat{V}^m , $m = 1, \dots, M$. Such paths can then be aggregated so as to produce a simulation of the portfolio process $\widehat{V} = \sum_{m=1}^M \widehat{V}^m$, that corresponds to the *pre-collateral exposure*. After this, we compute the value of the collateral balance C corresponding to the simulated paths of \widehat{V} , which in turn allows us to compute the *post-collateral exposure* process $\widehat{V} - C$ that enters the xVA formulas. For illustration, we consider $M = 1$ and the equity forward from the first example. We introduce a simple example of a collateral agreement where collateral is exchanged between the counterparties at every point in time (a margin call frequency that does not coincide with the simulation time discretization can of course be treated as well). Collateral is exchanged only in case the pre-collateral exposure is above (below) a receiving (posting) threshold which are both set equal to 5, i.e.

$$C_t := C(\widehat{V}_t) = (\widehat{V}_t - 5)^+ - (\widehat{V}_t + 5)^-.$$

An illustration for a single path is provided in Figure 4.

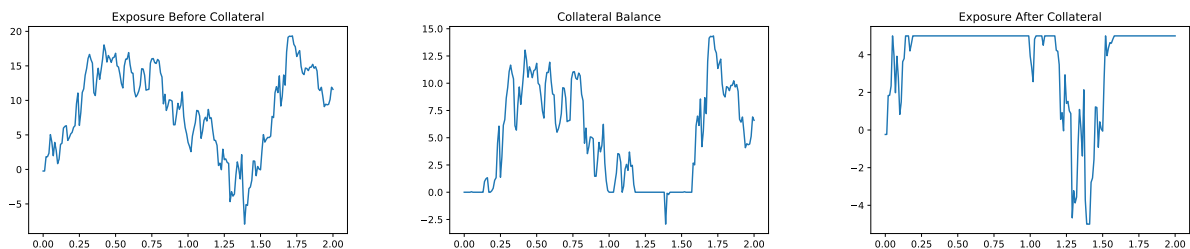


FIGURE 4. Pathwise simulation of a collateralized exposure. Left: \widehat{V} . Middle: C . Right: $\widehat{V} - C$. Posting and receiving threshold are 5 EUR.

5. CONCLUSIONS AND EXTENSIONS

The proposed xVA algorithm exploits two useful complementary aspects of the Deep BSDE Solver of E et al. (2017). First, the formulation as an optimisation problem over a parametrisation of the (Markovian) control of the xVA BSDE, which is carried out by SDE discretisation and path sampling, directly gives both the hedge ratios in approximate functional form and model-based derivative prices along the sample paths. This is amenable to the simulation of exposure profiles, the computation of higher-order Greeks by pathwise differentiation, and allows for the computation of funding and margin variation adjustments as well as xVA hedging. A second aspect of the Deep BSDE Solver is the use of neural networks specifically as parametrisation for the Markovian control. A key advantage results from the approximation power of neural networks in high dimensions, which has the potential to make risk management computations on portfolio level feasible. Moreover, the simple functional form allows standard pathwise sensitivity computations.

Our numerical examples provide a proof of concept, but further systematic testing in realistic application settings is needed. An additional difficulty arises from the non-linear, non-convex parametric form, which, combined with the large number of parameters, leads to challenging optimisation problems. The expression power of the ANN and the practicalities of the learning process, are extremely active research areas and further developments of the proposed Deep xVA Solver will be informed by the rapidly developing understanding of neural networks in a broader sense.

The application of our proposed scheme is not restricted to the chosen xVA framework. For example, one could in principle apply our methodology to the balance-sheet based model computed in Albanese et al. (2020). In this case, the xVA computation involves multiple recursive valuations (illustrated succinctly in Abbas-Turki et al. (2018, Figure 1)), which can be approached by means of multiple applications of the Deep xVA Solver.

We also emphasise that the Deep xVA Solver can be combined with an existing analytics library: the computation of the mark-to-market cube (i.e., the simulation of all possible scenarios for the clean values over different points in time) represents a classical numerical problem to be solved in order to compute traditional risk figures such as Value-at-Risk or Expected Shortfall (this is often referred to as “Monte Carlo full revaluation approach”). Since most products individually depend on a limited number of risk factors, it may be best to use a traditional numerical scheme, such as a finite difference solver, for at least some of the more vanilla products, and then revalue the products over different Monte Carlo paths by means of a look-up table over the pre-computed numerical solution. This provides an alternative route with respect to our Algorithm 1 for the simulation of the clean values. However, once we aggregate all mark-to-markets, we end up with an object that depends on a high number of risk factors, so for the computation of xVA our proposed methodology provides a useful tool which allows the recursive computation of valuation adjustments, their hedging strategy, and simulation of collateral.

APPENDIX A. A POSTERIORI ERROR ESTIMATES FOR NON-RECURSIVE ADJUSTMENTS

The estimates provided in Han and Long (2020) can be applied as follows to the adjustment computation in Subsection 3.2, Algorithm 2. We assume the existence of some constant C such that

$$(A.1) \quad \sup_{t \in [0, T]} \mathbb{E} \left[|\hat{V}_t - \hat{V}_t^{\xi, \rho}|^2 \right] \leq C \left(\Delta t + \sum_{m=1}^M \mathbb{E} \left[|g_m(S_T) - \hat{V}_T^{m, \xi_m, \rho_m}|^2 \right] \right),$$

where $\widehat{V}_t^{\xi, \rho}$ is the ANN approximation associated with parameters $\xi = (\xi_1, \dots, \xi_M)$ and $\rho = (\rho_1, \dots, \rho_M)$ (and extended to $[0, T]$ by piecewise constant interpolation) of the clean portfolio value \widehat{V}_t .

Under the assumed conditions on Φ (uniformly Lipschitz with constant L_Φ), one directly obtains the following estimates

$$\begin{aligned}
& \left| \mathbb{E} \left[\int_0^T \Phi_t(\widehat{V}_t) dt \right] - \mathbb{E} \left[\sum_{n=0}^N \eta_n \Phi_{t_n}(\widehat{V}_{t_n}^{\xi, \rho}) \right] \right| \\
& \leq \left| \mathbb{E} \left[\int_0^T \Phi_t(\widehat{V}_t) dt - \sum_{n=0}^N \eta_n \Phi_{t_n}(\widehat{V}_{t_n}) \right] \right| + \left| \mathbb{E} \left[\sum_{n=0}^N \eta_n (\Phi_{t_n}(\widehat{V}_{t_n}) - \Phi_{t_n}(\widehat{V}_{t_n}^{\xi, \rho})) \right] \right| \\
& \leq \left| \int_0^T \mathbb{E}[\Phi_t(\widehat{V}_t)] dt - \sum_{n=0}^N \eta_n \mathbb{E}[\Phi_{t_n}(\widehat{V}_{t_n})] \right| + \left| \sum_{n=0}^N \eta_n \mathbb{E}[(\Phi_{t_n}(\widehat{V}_{t_n}) - \Phi_{t_n}(\widehat{V}_{t_n}^{\xi, \rho}))] \right| \\
& \leq Q(\Delta t) + \left(\sum_{n=0}^N |\eta_n|^2 \right)^{1/2} \left(\sum_{n=0}^N \left| \mathbb{E} [\Phi_{t_n}(\widehat{V}_{t_n}) - \Phi_{t_n}(\widehat{V}_{t_n}^{\xi, \rho})] \right|^2 \right)^{1/2} \\
& \leq Q(\Delta t) + L_\Phi \left(\sum_{n=0}^N |\eta_n|^2 \right)^{1/2} \left(\sum_{n=0}^N \mathbb{E} [|\widehat{V}_{t_n} - \widehat{V}_{t_n}^{\xi, \rho}|^2] \right)^{1/2},
\end{aligned}$$

where $Q(\Delta t)$ is the error associated with the quadrature rule for the function $\varphi(t) := \mathbb{E}[\Phi_t(\widehat{V}_t)]$. The function φ can be proven to be $1/2$ -Hölder continuous. Indeed, for $\Phi_t(\widehat{V}_t) = (B_t^r)^{-1} \Psi(\widehat{V}_t) \lambda_t^{C, \mathbb{Q}}$ (the CVA case, and similar for DVA), denoting $\Psi(\widehat{V}_t) = (1 - R^C)(\widehat{V}_t - C(\widehat{V}_t))^-$ Lipschitz in \widehat{V}_t ,

$$\begin{aligned}
|\phi(t) - \phi(s)| & \leq \\
& \mathbb{E} \left[\left| (B_t^r)^{-1} \Psi(\widehat{V}_t) (\lambda_t^{C, \mathbb{Q}} - \lambda_s^{C, \mathbb{Q}}) \right| + \left| (B_t^r)^{-1} (\Psi(\widehat{V}_t) - \Psi(\widehat{V}_s)) \lambda_s^{C, \mathbb{Q}} \right| + \left| ((B_t^r)^{-1} - (B_s^r)^{-1}) \Psi(\widehat{V}_s) \lambda_s^{C, \mathbb{Q}} \right| \right] \\
& \leq C \left\{ \mathbb{E}[(\lambda_t^{C, \mathbb{Q}} - \lambda_s^{C, \mathbb{Q}})^2]^{1/2} + \mathbb{E}[(\widehat{V}_t - \widehat{V}_s)^2]^{1/2} + \mathbb{E} \left[\left(1 - \exp \left(- \int_s^t \tilde{r}_u du \right) \right)^2 \right]^{1/2} \right\},
\end{aligned}$$

for some constant C , using the boundedness of r , $\lambda^{C, \mathbb{Q}}$, $\lambda^{B, \mathbb{Q}}$, and of $\mathbb{E}[(\Psi(\widehat{V}_t))^2]$. The first and last term are of order $|t - s|^{1/2}$ by the assumptions made, and it remains to estimate the middle term. Recalling that $\widehat{V}_t = \sum_{m=1}^M \widehat{V}_t^m$ with \widehat{V}_t^m the solution of the FBSDE (2.1), (2.6), under the regularity assumptions on the coefficients μ and σ of the forward SDE one gets

$$\begin{aligned}
|\varphi(t) - \varphi(s)| & \leq C \left(|t - s|^{1/2} + \sum_{m=1}^M \mathbb{E} [|\widehat{V}_t^m - \widehat{V}_s^m|^2]^{1/2} \right) \\
& \leq C \left(|t - s|^{1/2} + \sum_{m=1}^M \mathbb{E} [g_m(S_T)^2 |t - s| + \int_s^t |\widehat{Z}_u^m|^2 du]^{1/2} \right) \\
& \leq C \left(|t - s|^{1/2} + |t - s|^{1/2} \sum_{m=1}^M \mathbb{E} [g_m(S_T)^2 + 1 + \sup_{u \in [s, t]} |S_u|^2]^{1/2} \right) \\
& \leq C |t - s|^{1/2}.
\end{aligned}$$

To obtain the estimates for the increment of the BSDE solution and of the control in terms of the forward process, in the second and third line, respectively, we can apply Zhang (2004, Lemma 2.4, (2.11)) and Zhang (2017, Theorem 5.2.2(i)) to the equivalent BSDE

$$(A.2) \quad d\widetilde{V}_t^m = \widetilde{Z}_t^m dW_t^\mathbb{Q}, \quad \widetilde{V}_T^m = g_m(S_T), \quad \text{where} \quad \widetilde{V}_t^m = \widehat{V}_t^m B_T^r / B_t^r, \quad \widetilde{Z}_t^m = \widehat{Z}_t^m B_T^r / B_t^r.$$

Above and in the following, we do not keep track of constants and C denotes any non-negative constant depending only on T, M, s_0 and the regularity constants of the coefficients.

Then, if we consider the rectangle quadrature rule we get

$$(A.3) \quad Q(\Delta t) \leq C\Delta t^{1/2}.$$

Therefore, observing that

$$\left(\sum_{n=0}^N |\eta_n|^2 \right)^{1/2} \leq \Delta t N^{1/2}$$

and using (A.1) and (A.3), one has

$$\begin{aligned} & \left| \mathbb{E} \left[\int_0^T \Phi(t, \widehat{V}_t) dt \right] - \mathbb{E} \left[\sum_{n=0}^{N-1} \Delta t \Phi(t_n, \widehat{V}_{t_n}^{\xi, \rho}) \right] \right| \\ & \leq C\Delta t^{1/2} + C\Delta t N^{1/2} N^{1/2} \left(\sup_{n=0, \dots, N-1} \mathbb{E} \left[|\widehat{V}_{t_n} - \widehat{V}_{t_n}^{\xi, \rho}|^2 \right] \right)^{1/2} \\ & \leq C\Delta t^{1/2} + TC \left(\Delta t + \sum_{m=1}^M \mathbb{E} \left[|g_m(S_T) - \widehat{V}_T^{m, \xi_m, \rho_m}|^2 \right] \right)^{1/2}, \end{aligned}$$

from which the claim (3.14) follows just taking $\xi = (\xi_1^*, \dots, \xi_M^*)$ and $\rho = (\rho_1^*, \dots, \rho_M^*)$.

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