Rough volatility project

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1 Estimation of the parameter H

1.1 Question 1 : Simulation of a path of he variance process on the interval [0,1]

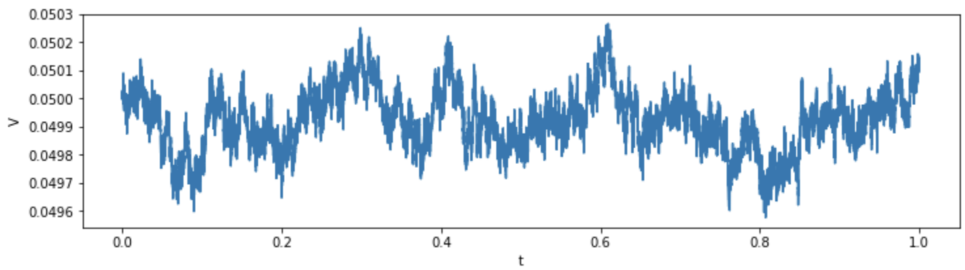

FIGURE 1 – Variance process path

1.2 Question 2 : Estimation de H

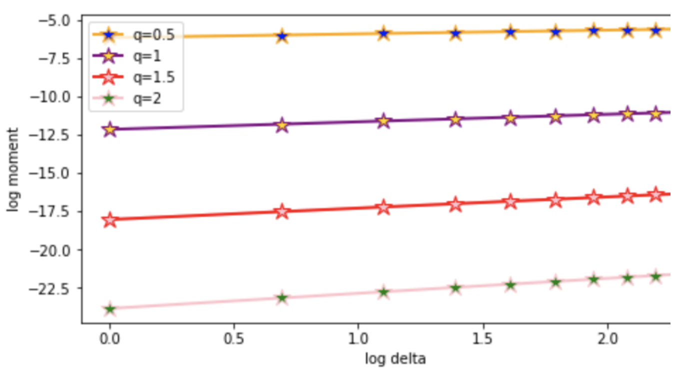

FIGURE 2 – First Regression

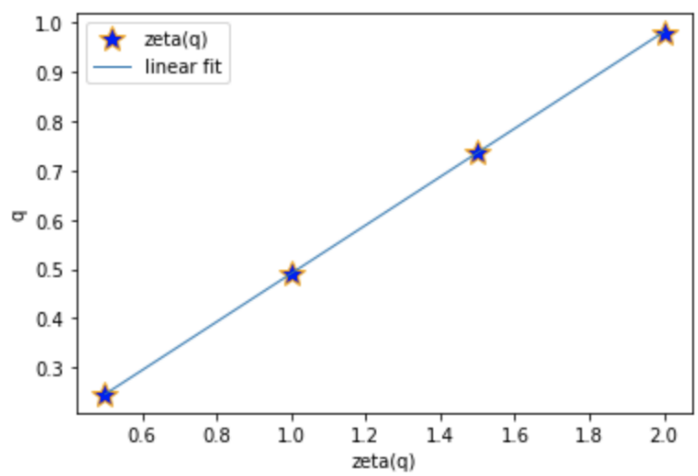

FIGURE 3 – Second regression

```
]: from scipy.stats import linregress
s05, intercept, r_value, p_value, std_err=linregress(logdelta,m05)
s1, intercept, r_value, p_value, std_err=linregress(logdelta,m1)
s15, intercept, r_value, p_value, std_err=linregress(logdelta,m15)
s2, intercept, r_value, p_value, std_err=linregress(logdelta,m2)
q=[0.5,1,1.5,2]
zetaq=[s05,s1,s15,s2]
print("the parameter H is estimated at",np.polyfit(q, zetaq, 1)[0])

the parameter H is estimated at 0.48968813602881206
```

FIGURE 4 – Estimation of H

We observe that the estimation of H is far from the real value. The real value is 0.1 but the method used above give an estimation of 0.48

1.3 Question 3 estimation of H with sampled path

	l	H(l)
0	1	0.487636
1	2	0.478013
2	3	0.464368
3	4	0.457148
4	5	0.450655
5	6	0.445328
6	7	0.441173
7	8	0.436698
8	9	0.431287
9	10	0.429385

FIGURE 5 – Values of H w.r.t l

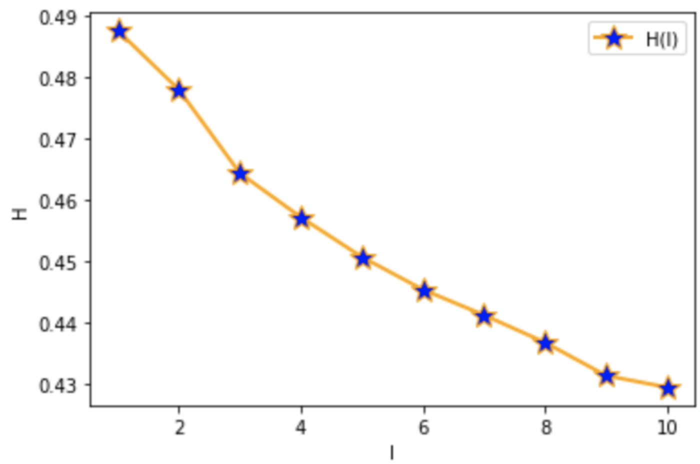

FIGURE 6 – plot of the estimation of w.r.t l

We observe that as l gets bigger the estimation of H decreases but it still is far from the real value

1.4 Question 4 : Estimation of H for a classical brownian motion

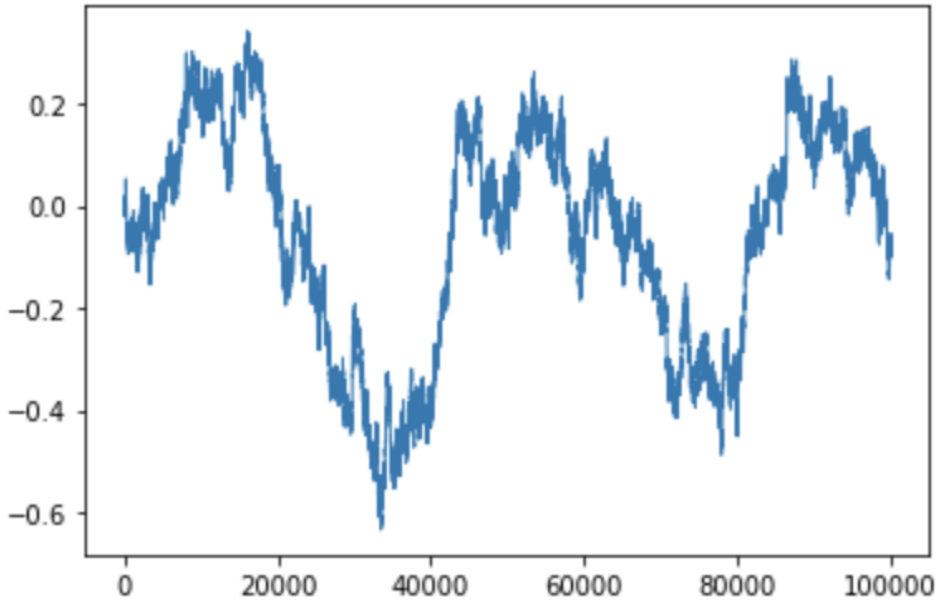

FIGURE 7 – Classical brownian motion path

1.4.1 Estimation de H

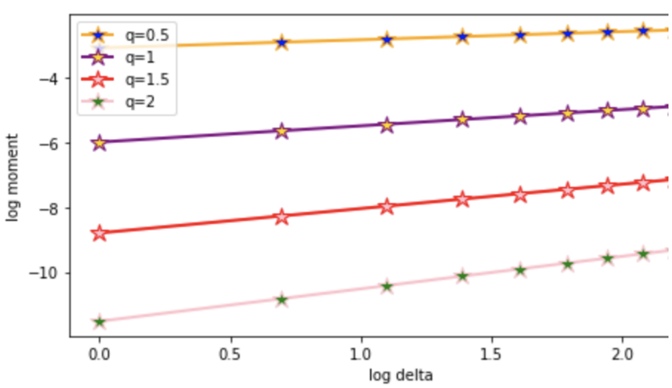

FIGURE 8 – First Regression

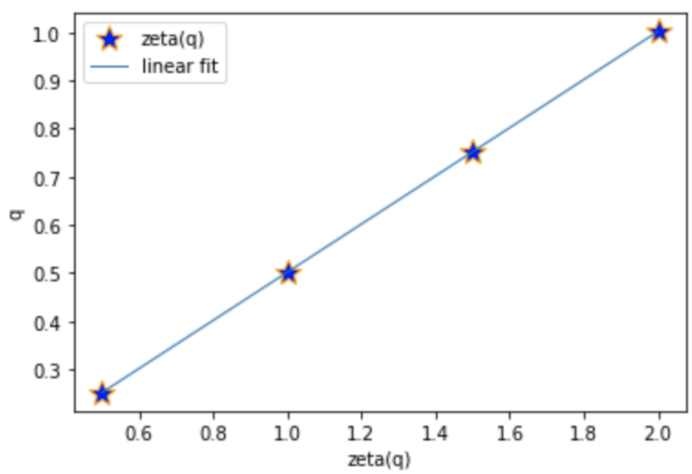

FIGURE 9 – Second regression

```
q=[0.5,1,1.5,2]
zetaq=[s05,s1,s15,s2]
np.polyfit(q, zetaq, 1)

print("the parameter H is estimated at",np.polyfit(q, zetaq, 1)[0])
```

```
the parameter H is estimated at 0.5020294525224289
```

FIGURE 10 – Estimation of H

We observe that for the classical brownian motion the method gives quite a good result. Indeed the estimation is equal of 0.502 which is really close to the hurst parameter of a classical brownian motion which is equal to 0.5.

1.4.2 estimation of H with sampled path

	l	H(l)
0	1	0.499974
1	2	0.500718
2	3	0.503077
3	4	0.502867
4	5	0.501174
5	6	0.499397
6	7	0.499690
7	8	0.498757
8	9	0.493057
9	10	0.491918

FIGURE 11 – Values of H w.r.t l

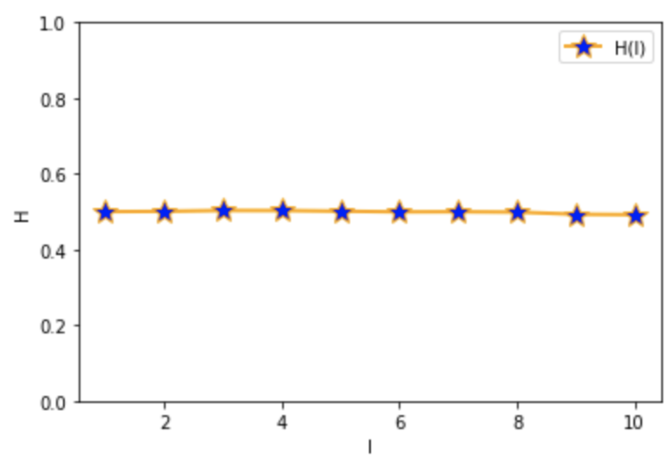

FIGURE 12 – plot of the estimation of w.r.t l

For the classical BM, the H is estimated at 0.502. The estimation is pretty stable, meaning it does not vary much even when we estimate using sampled paths.

1.5 Estimation of H for a fractional brownian motion

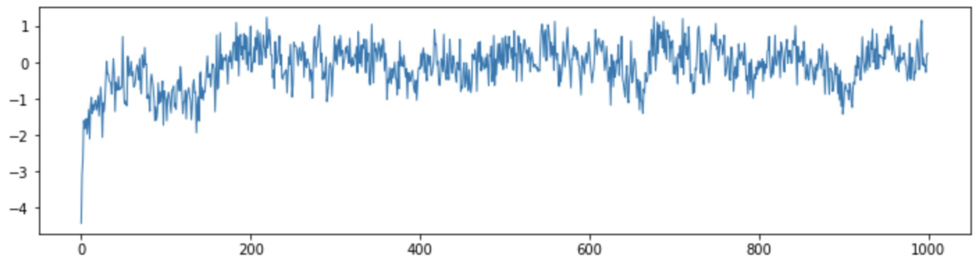

FIGURE 13 – fractionale brownian motion path

1.5.1 Estimation de H

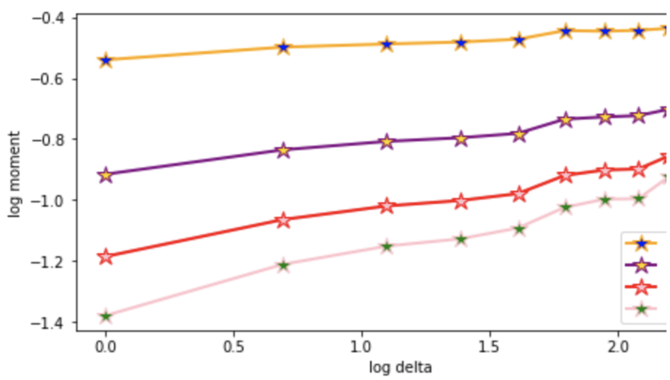

FIGURE 14 – First Regression

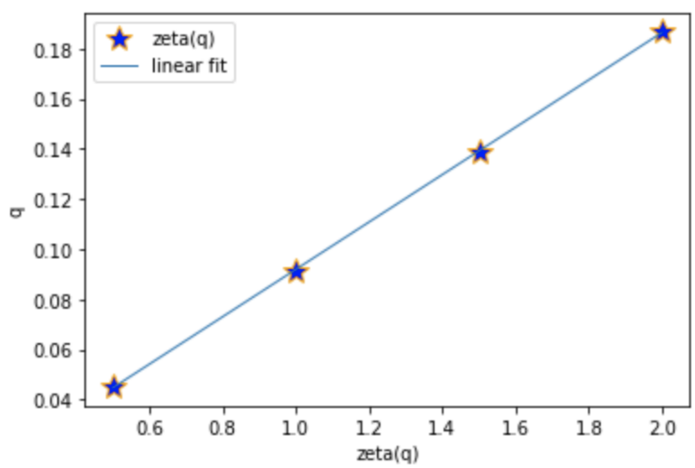

FIGURE 15 – Second regression

```
q=[0.5,1,1.5,2]
zetaq=[s05,s1,s15,s2]
np.polyfit(q, zetaq, 1)
print("the parameter H is estimated at",np.polyfit(q, zetaq, 1)[0])
```

```
the parameter H is estimated at 0.09458551694286746
```

FIGURE 16 – Estimation of H

We observe that for the fractionanal brownian motion the method gives also quite a good result. Indeed the estimation is equal of 0.09 which is really close to the hurst parameter we chose for our fBM whixh is equal to 0.1

1.5.2 estimation of H with sampled path

	l	H(l)
0	1	0.094586
1	2	0.101281
2	3	0.090847
3	4	0.131480
4	5	0.139460
5	6	0.116395
6	7	0.133463
7	8	0.158025
8	9	0.100582
9	10	0.104910

FIGURE 17 – Values of H w.r.t l

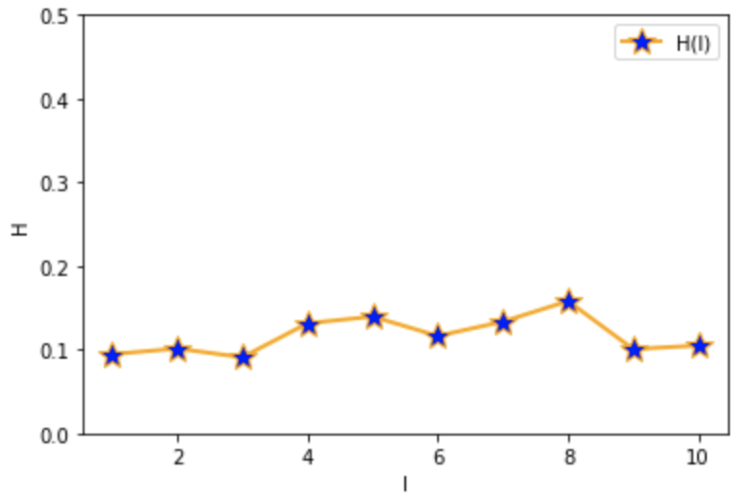

FIGURE 18 – plot of the estimation of w.r.t l

For the fractionnal BM, the H is estimated at 0.094. The estimation is also pretty stable, meaning it does much vary much even when we estimate using sampled paths.

2 Implied volatility in the lifted heston model

2.1 Question 1 : Carr Madan formula of a european call option

Let C_0 be the price of a European call option with strike K, maturity T and interest rate r_{int} let $\phi_T(u)$ be the characteristic function of the log price of the underlying asset $X_t = log(S_t)$

$$\phi_T(u) = \mathbb{E}[exp(iuX_T)]$$

$$C_0 = \mathrm{e}^{-r_{int}T}\mathbb{E}\left[(S_T - K)_+\right]$$

$$= \mathrm{e}^{-r_{int}T}\mathbb{E}\left[\left(\mathrm{e}^{X_T} - \mathrm{e}^{k}\right)_+\right]$$

where $k = \log(K)$

$$C_0 = \mathrm{e}^{-r_{int}T}\int_{-\infty}^{+\infty}(\mathrm{e}^{x} - \mathrm{e}^{k})_+\phi_T(x)dx$$

Now Let introduce C_{α_2}

$$\begin{aligned}
C_{\alpha_2}(k) &= \mathrm{e}^{\alpha_2 k} C_0 \\
&= \mathrm{e}^{-r_{int}T+\alpha_2 k}\mathbb{E}\left[(S_T - K)_+\right] \\
&= \mathrm{e}^{-r_{int}T+\alpha_2 k}\mathbb{E}\left[\left(\mathrm{e}^{X_T} - \mathrm{e}^{k}\right)_+\right] \\
&= \mathrm{e}^{-r_{int}T}\int_k^{\infty}\left(\mathrm{e}^{x+\alpha_2 k} - \mathrm{e}^{(\alpha_2+1)k}\right)\phi_T(x)dx
\end{aligned}$$

Now let's take the fourrier transform of $C_{\alpha_2}(k)$, which we note $\widehat{C_\alpha}(\nu)$

$$\begin{aligned}
\widehat{C_{\alpha_2}}(\nu) &= \int_{\mathbb{R}} C_{\alpha_2}(k)\mathrm{e}^{ik\nu}dk \\
&= \mathrm{e}^{-r_{int}T}\int_{\mathbb{R}}\int_k^{\infty}\left(\mathrm{e}^{x+\alpha_2 k} - \mathrm{e}^{(\alpha_2+1)k}\right)\mathrm{e}^{ik\nu}\phi_T(x)dxdk \\
&= \mathrm{e}^{-r_{int}T}\int_{\mathbb{R}}\phi_T(x)dx\int_{-\infty}^{x}\left(\mathrm{e}^{x+\alpha_2 k} - \mathrm{e}^{(\alpha_2+1)k}\right)\mathrm{e}^{ik\nu}dk \\
&= \mathrm{e}^{-r_{int}T}\int_{\mathbb{R}}\left(\frac{\mathrm{e}^{ix(\nu-i(\alpha+1))}}{\alpha_2+i\nu} - \frac{\mathrm{e}^{ix(\nu-i(\alpha+1))}}{(\alpha_2+1)+i\nu}\right)\phi_T(x)dx \\
&= \mathrm{e}^{-r_{int}T}\phi(\nu - i(\alpha_2+1))\left(\frac{1}{\alpha_2+i\nu} - \frac{1}{(\alpha_3+1)+i\nu}\right) \\
&\boxed{= \mathrm{e}^{-r_{int}T}\frac{\phi(\nu - i(\alpha_2+1))}{(\alpha_2+i\nu)(\alpha_2+1+i\nu)}}
\end{aligned}$$

Thus

$$\begin{aligned}
C_0 &= e^{-\alpha_2 k}C_{\alpha_2}(k) \\
&= \frac{e^{-\alpha_2 k}}{2\pi}\int_{\mathbb{R}} e^{-ik\nu}\widehat{C}_{\alpha_2}(\nu)d\nu \\
&= \frac{e^{-\alpha_2 k}}{2\pi}\int_{\mathbb{R}} \mathrm{e}^{-r_{int}T}\frac{\phi(\nu - i(\alpha_2+1))}{(\alpha_2+i\nu)(\alpha_2+1+i\nu)}d\nu \\
&= \frac{e^{-\alpha_2 k - r_{int}T}}{2\pi}\int_{\mathbb{R}}\frac{\phi(\nu - i(\alpha_2+1))}{(\alpha_2+i\nu)(\alpha_2+1+i\nu)}d\nu
\end{aligned}$$

Knowing that the price is real i.e $C_0 \in \mathbb{R}$ it implies that the function inside the integral is odd in its imaginary part and even in its real part which enables us to write

$$\boxed{C_0 = \frac{\mathrm{e}^{-r_{int}T-\alpha_2 k}}{\pi}\int_0^{\infty}\operatorname{Re}\left(\frac{\Phi_T\left(\nu-(\alpha_2+1)\,i\right)}{(\alpha_2+i\nu)\,(\alpha_2+1+i\nu)}\mathrm{e}^{-ik\nu}\right)\mathrm{d}\nu} \quad CQFD$$

2.2 Question 2 : Implementation of Characteristic Function of the Lifted Heston model

See attached notebook

2.3 Question 3 : Implementation of the pricing function of the lifted heston model

See attached notebook

2.4 Question 4 : Volatility smile

Implied volatility is considered an important quantity in finance. Given an observed market option price V^{mkt}, the Black-Scholes implied volatility σ^* can be determined by solving $BS\left(\sigma^*; S, K, \tau, r\right) = V^{mkt}$. The monotonicity of the Black-Scholes equation with respect to the volatility guarantees the existence of $\sigma^* \in [0, +\infty]$. We can write the implied volatility as an implicit formula,

$$\sigma^*(K, T) = BS^{-1}\left(V^{mkt}; S, K, \tau, r\right)$$

where BS^{-1} denotes the inverse Black-Scholes function. Moreover, by adopting moneyness, $m = \frac{S_t}{K}$, and time to maturity, $\tau = T - t$, one can express the implied volatility as $\sigma^*(m, \tau)$, .

For simplicity, we denote here $\sigma^*(m, \tau)$ by σ^*. An analytic solution for the above Equation does not exist. The value of σ^* is determined by means of a numerical iterative technique, since the equation can be converted into a root-finding problem,

$$g\left(\sigma^*\right) = BS\left(S, \tau, K, r, \sigma^*\right) - V^{mkt}(S, \tau; K) = 0 \quad [1]$$

To compute the implied volatility we are going to match the price obtain with the heston pricer with the one we obtain with the classic black scholes pricer. The implied volatility is the value of sigma to insert in the black scholes formula so that its price is equal to the market price which is, in this case the price, given by the Heston model.

2.4.1 a) for T=1 and log strike $\in [-1.2, 0.2]$

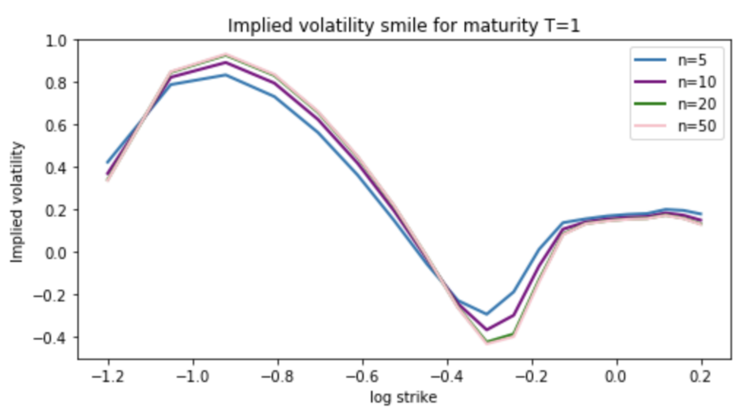

Figure 19 – Volatility smile for n (number of factors : 5,10,20,50

We observe that the implied volatility smiles for each value of n superimpose almost perfectfly. The implied volatility converges towars the same value for every n.

2.4.2 a) for T=1/26 and log strike $\in [-0.15, 0.05]$

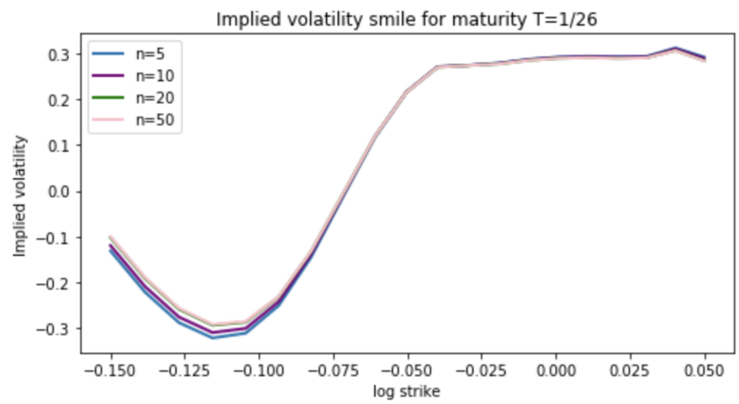

FIGURE 20 – Volatility smile for n (number of factors : 5,10,20,50

We observe that the implied volatility smiles for each value of n superimpose almost perfectfly. The implied volatility converges towars the same value for every n.

3 Bibliography

Références

[1] S. Liu, C. W. Oosterlee, and S. M. Bohte, “Pricing options and computing implied volatilities using neural networks,” *Risks*, vol. 7, no. 1, p. 16, 2019.