MSE Unbiased estimator

Remind b_1 is linear combination of y_i, it is $b_1 = \sum_{i=1}^n c_i y_i$, where $c_i = (x_i - \bar{x})/(\sum_{i=1}^n (x_i - \bar{x})^2)$, then

$$\begin{aligned}
b_1 &= \sum_{i=1}^n c_i y_i = \sum_{i=1}^n c_i(\beta_0 + \beta_1 x_i + \varepsilon_i) \\
&= \sum_{i=1}^n c_i \beta_0 + \sum_{i=1}^n c_i \beta_1 x_i + \sum_{i=1}^n c_i \varepsilon_i \\
&= \beta_0 \sum_{i=1}^n c_i + \beta_1 \sum_{i=1}^n c_i x_i + \sum_{i=1}^n c_i \epsilon_i \\
&= \beta_1 + \sum_{i=1}^n c_i \varepsilon_i
\end{aligned}$$

above is because $\sum_{i=1}^n c_i = 0$

$$\sum_{i=1}^n \frac{(x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2} = \frac{n\bar{x} - n\bar{x}}{\sum_{i=1}^n (x_i - \bar{x})^2} = 0$$

and $\sum_{i=1}^n c_i x_i = 1$.

$$\sum_{i=1}^n \frac{(x_i - \bar{x})x_i}{\sum_{i=1}^n (x_i - \bar{x})^2} = \frac{\sum_{i=1}^n (x_i - \bar{x})x_i}{\sum_{i=1}^n (x_i - \bar{x})x_i - \sum_{i=1}^n (x_i - \bar{x})\bar{x}} = \frac{\sum_{i=1}^n (x_i - \bar{x})x_i}{\sum_{i=1}^n (x_i - \bar{x})x_i - (n\bar{x} - n\bar{x})\bar{x}} = 1$$

Then expect value,

$$
\begin{aligned}
&\mathbb{E}\left[-2(b_1-\beta_1)\sum_{i=1}^{n}(x_i-\bar{x})(\varepsilon_i-\bar{\varepsilon})\right]\\
&=\mathbb{E}\left[-2(\beta_1+\sum_{i=1}^{n}c_i\varepsilon_i-\beta_1)\sum_{i=1}^{n}(x_i-\bar{x})(\varepsilon_i-\bar{\varepsilon})\right]\\
&=-2\mathbb{E}\left[\sum_{i=1}^{n}c_i\varepsilon_i\sum_{i=1}^{n}(x_i-\bar{x})\varepsilon_i\right]\\
&=-2\mathbb{E}\left[\frac{(\sum_{i=1}^{n}(x_i-\bar{x})\varepsilon_i)^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\right]\\
&=\frac{-2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\mathbb{E}\left[\left((x_1-\bar{x})\varepsilon_1+\sum_{i=2}^{n}(x_i-\bar{x})\varepsilon_i\right)^2\right]\\
&=\frac{-2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\mathbb{E}\left[(x_1-\bar{x})^2\varepsilon_1^2+\left(\sum_{i=2}^{n}(x_i-\bar{x})\varepsilon_i\right)^2+2\left(x_i-\bar{x}\right)\left(\sum_{i=2}^{n}\varepsilon_i(x_i-\bar{x})\right)\right]\\
&=\frac{-2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\mathbb{E}\left[(x_1-\bar{x})^2\varepsilon_1^2\right]+\mathbb{E}\left[\left(\sum_{i=2}^{n}(x_i-\bar{x})\varepsilon_i\right)^2\right]+\mathbb{E}\left[2\left(x_i-\bar{x}\right)\left(\sum_{i=2}^{n}\varepsilon_i(x_i-\bar{x})\right)\right]\\
&=\frac{-2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\left((x_1-\bar{x})^2\mathbb{E}\left[\varepsilon_1^2\right]+\mathbb{E}\left[\left(\sum_{i=2}^{n}(x_i-\bar{x})\varepsilon_i\right)^2\right]+2\left(x_i-\bar{x}\right)\left(\sum_{i=2}^{n}(x_i-\bar{x})\mathbb{E}\left[\varepsilon_i\right]\right)\right)\\
&=\frac{-2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\left((x_1-\bar{x})^2\mathbb{E}\left[\varepsilon_1^2\right]+\mathbb{E}\left[\left(\sum_{i=2}^{n}(x_i-\bar{x})\varepsilon_i\right)^2\right]\right)\\
&...\\
&=\frac{-2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\left((x_1-\bar{x})^2\mathbb{E}\left[\varepsilon_1^2\right]+(x_2-\bar{x})^2\mathbb{E}\left[\varepsilon_2^2\right]+...+(x_n-\bar{x})^2\mathbb{E}\left[\varepsilon_n^2\right]\right)\\
&=\frac{-2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\left((x_1-\bar{x})^2\sigma^2+(x_2-\bar{x})^2\sigma^2+...+(x_n-\bar{x})^2\sigma^2\right)\\
&=\frac{-2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\left((x_1-\bar{x})^2+(x_2-\bar{x})^2+...+(x_n-\bar{x})^2\right)\sigma^2\\
&=\frac{-2\sum_{i=1}^{n}(x_i-\bar{x})^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}\sigma^2=-2\sigma^2
\end{aligned}
$$