Quantum algorithms for multivariate Monte Carlo estimation

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UNIVERSITY OF AMSTERDAM

QuSoft
Research Center for Quantum Software

CWI

Overview

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1. Prior work on Monte Carlo estimation:

	Classically	Quantumly
Univariate	Textbook	Textbook
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4. Improvements & further directions

Univariate Monte Carlo estimation

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To obtain precision ε, we need $N = \mathcal{O}(1/\varepsilon^2)$.

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Quadratic speed-up!

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Can we get a similar quadratic quantum speed-up here?

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Difficulty is that

$$U : |0\rangle |0\rangle \mapsto \sqrt{\mathbb{E}[X]} |\psi_1\rangle |1\rangle + \sqrt{1 - \mathbb{E}[X]} |\psi_0\rangle |0\rangle$$

has no clear multidimensional generalization.

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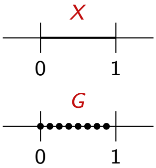
Oracle conversion graph:

$$\begin{tikzcd}
 & B_{f,\varepsilon} & \\
O_f \arrow[ur, "\text{Phase estimation} : \mathcal{O}(\frac{1}{\varepsilon})"] & & U_f \arrow[ul, "\mathcal{O}(\frac{1}{\varepsilon}) : \text{Amplitude estimation}"'] \arrow[ll, "\widetilde{\mathcal{O}}(1)"', "\text{[GAW18]}"]
\end{tikzcd}$$

Modification of the quantum univariate Monte Carlo estimation algorithm

Let $X : \Omega \to [0, 1]$.
Let $G = \{\frac{j}{2^n} : j \in \{0, \dots, 2^n - 1\}\}$, and $f : G \to [0, 1]$, defined by $f(a) = a\mathbb{E}[X]$.



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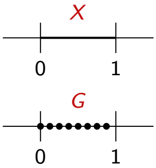

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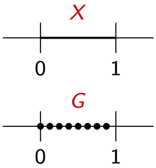

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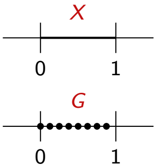

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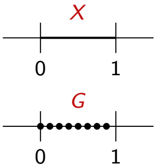

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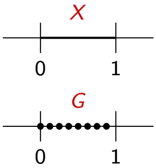

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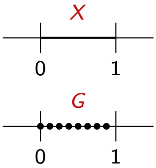

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Modification of the quantum univariate Monte Carlo estimation algorithm

Let $X : \Omega \to [0,1]$.
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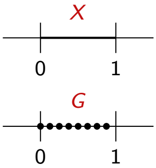

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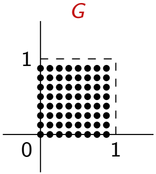
Tiny tweak of amplitude estimation.

Quantum multivariate Monte Carlo estimation

Let $G = \{\frac{j}{2^n} : j \in \{0, \ldots, 2^n - 1\}\}^d$, and $f : G \to [0, 1]$, defined by $f(\vec{a}) = \vec{a}^T \mathbb{E}[X]$.



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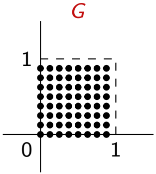

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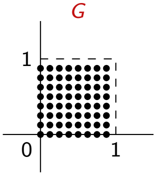

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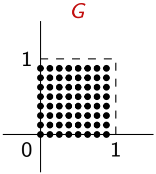

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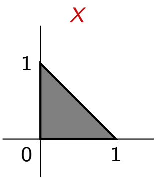

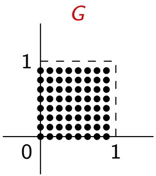

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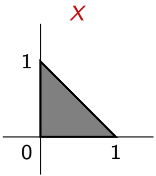

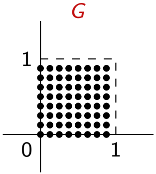

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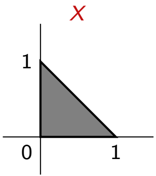

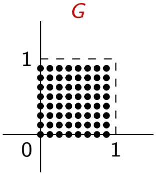

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Thanks for your attention!

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