Quantum gradient estimation

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April 24th, 2019

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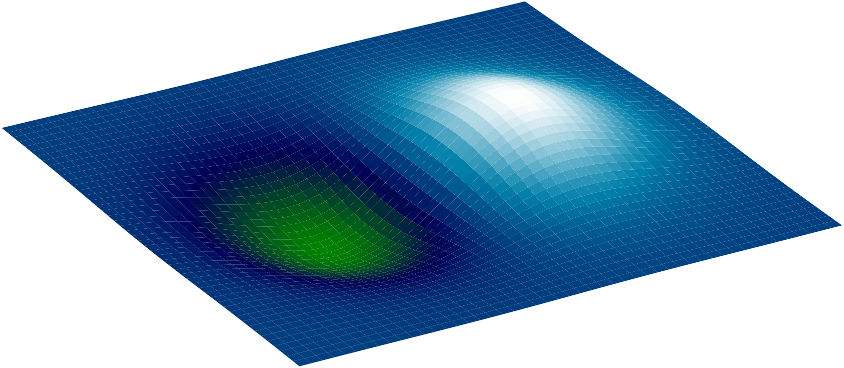

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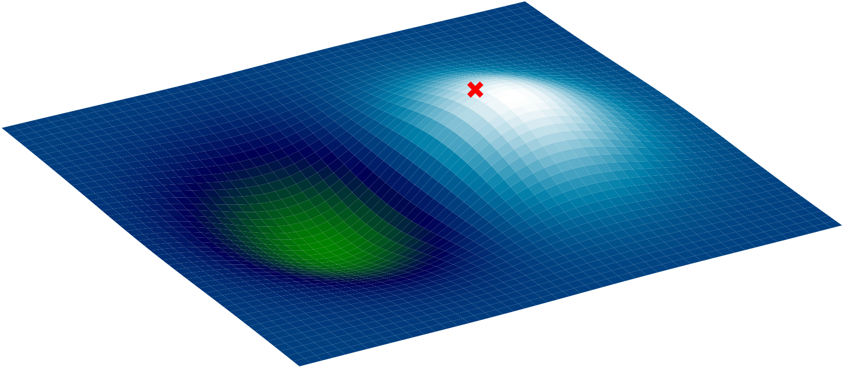

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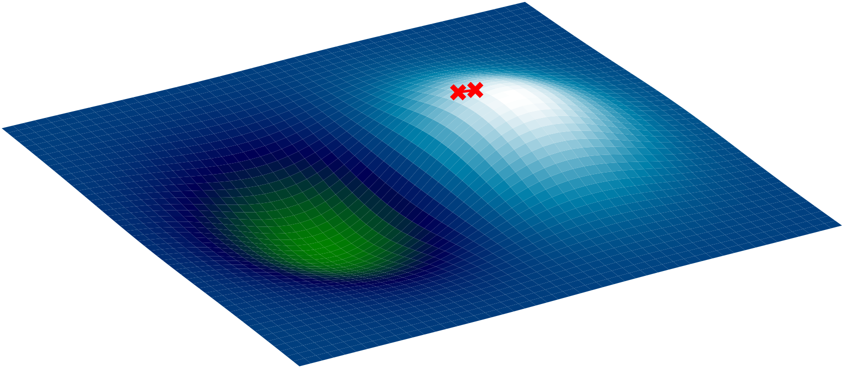

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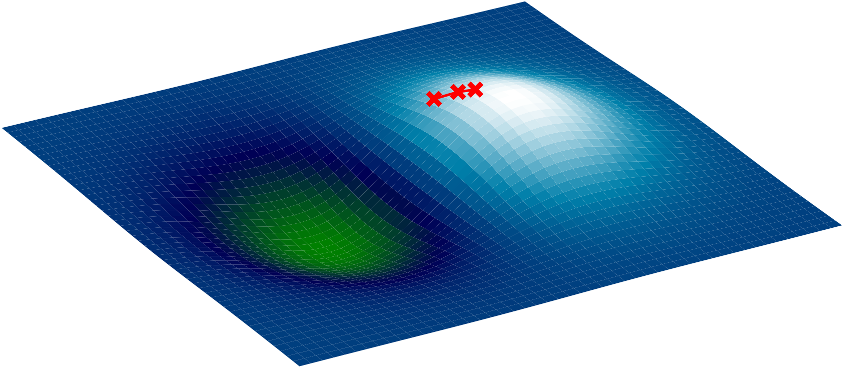

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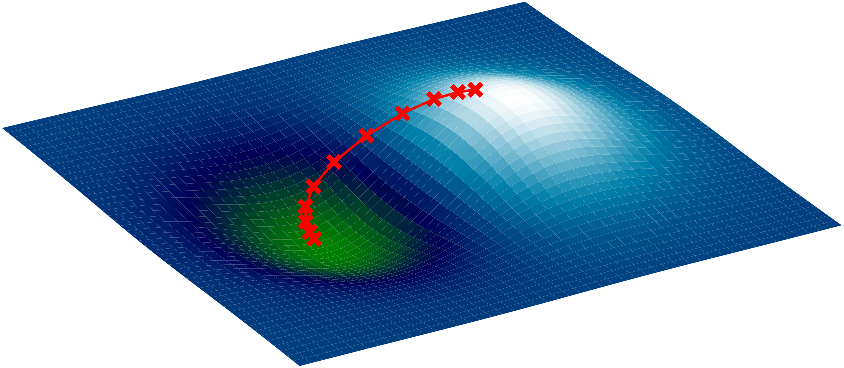

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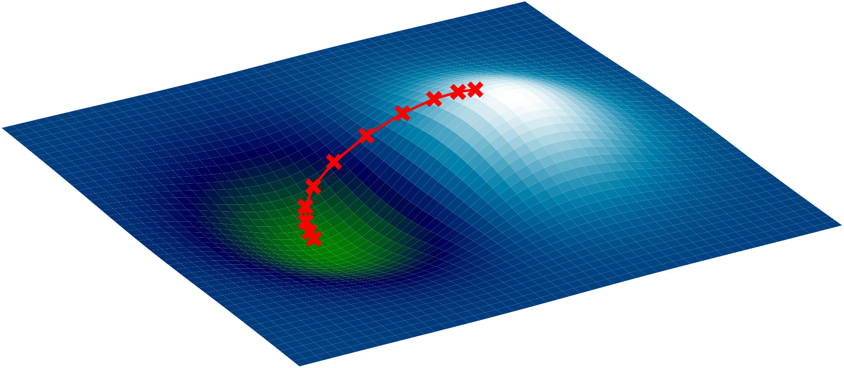

Can we speed up the gradient calculation step when d is large?

Classical gradient estimation

Classical gradient estimation

- Easiest case: let $f : \mathbb{R}^d \to \mathbb{R}$ be linear.

$$f(\mathbf{x}) = a + g_1 x_1 + \cdots + g_d x_d, \qquad \nabla f = \begin{bmatrix} g_1 \\ \vdots \\ g_d \end{bmatrix}$$

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- **Can we do better with a quantum computer?**

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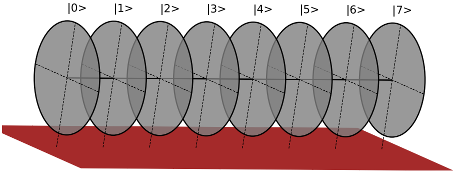

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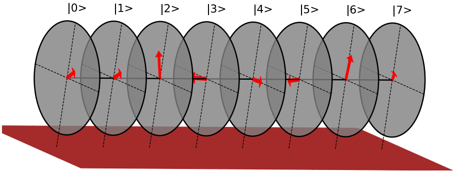

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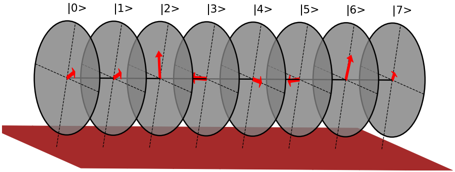

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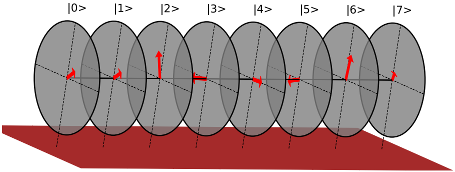

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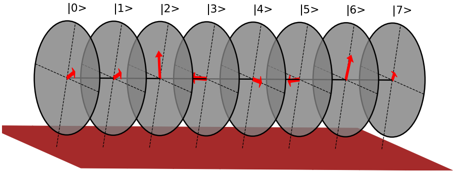

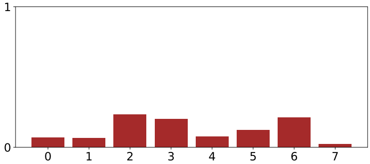

Quantum Fourier transform

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- The n-qubit quantum Fourier transform is defined as:

$$\mathrm{QFT}_{2^n} : |j\rangle \mapsto \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} e^{\frac{2\pi ijk}{2^n}} |k\rangle$$

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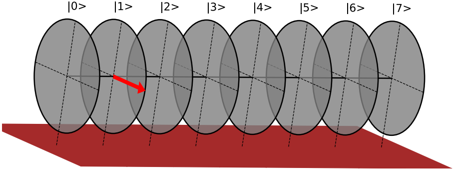

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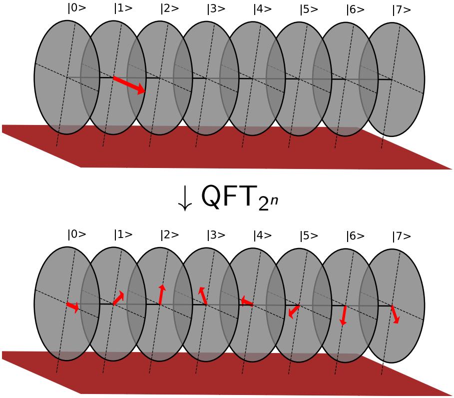

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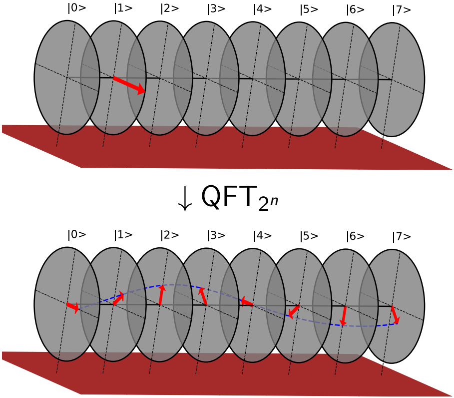

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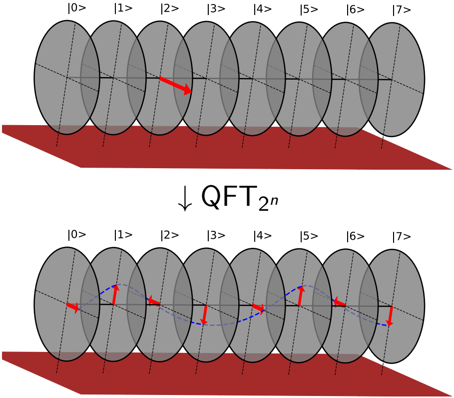

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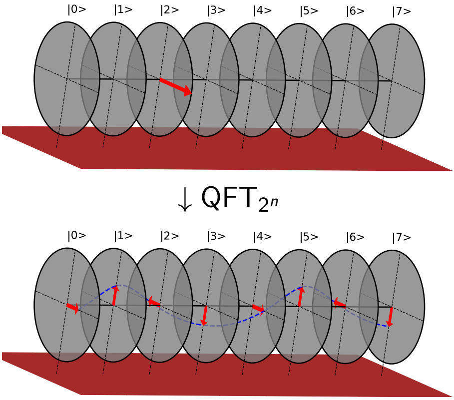

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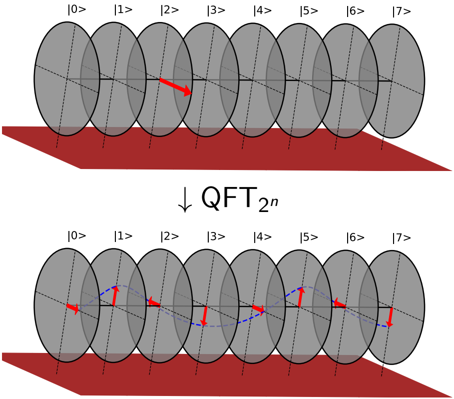

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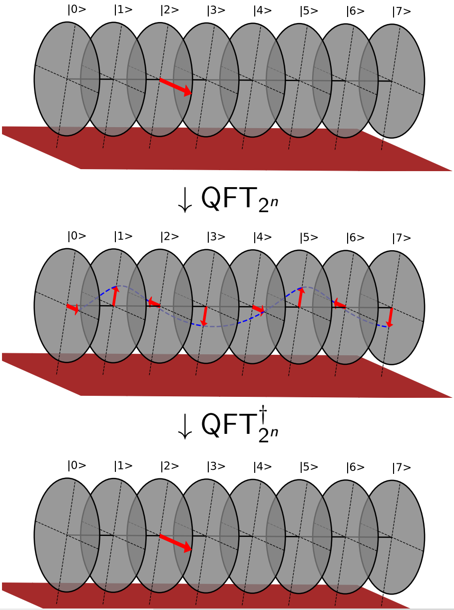

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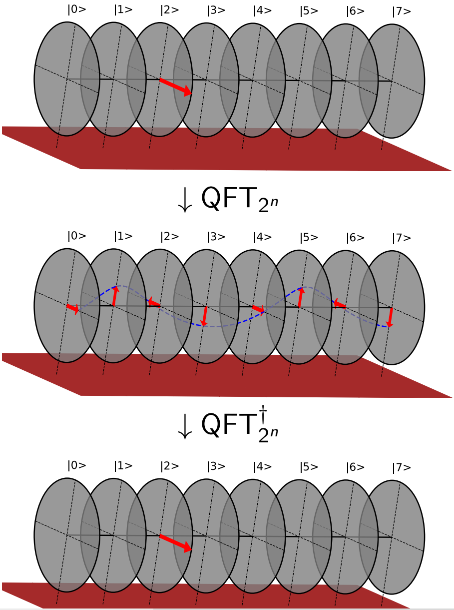

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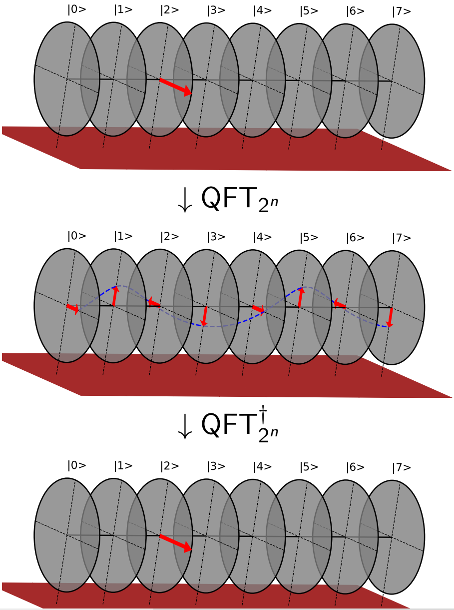
- Also works for non-integer revolutions.



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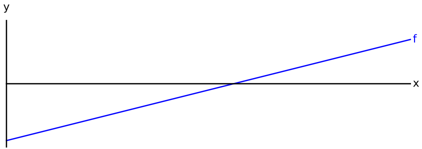

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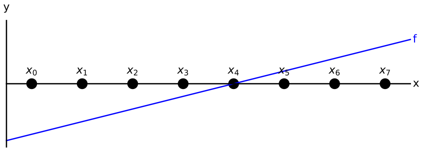

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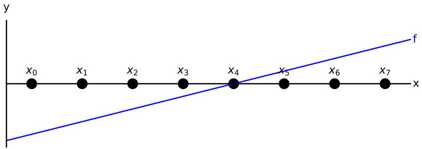

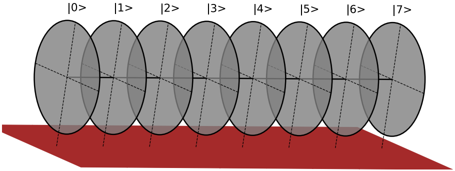

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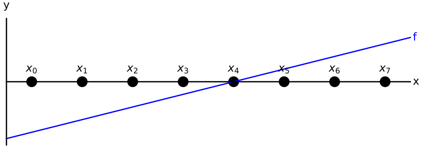

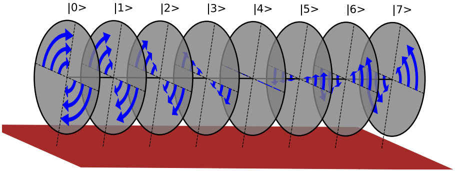

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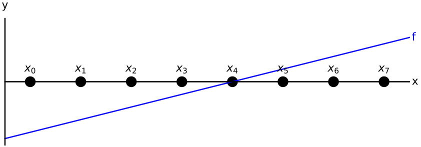

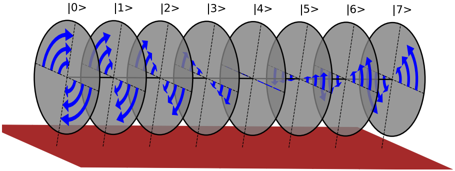

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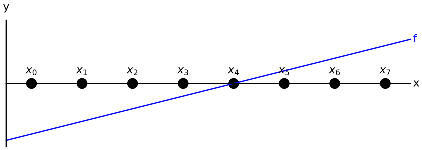
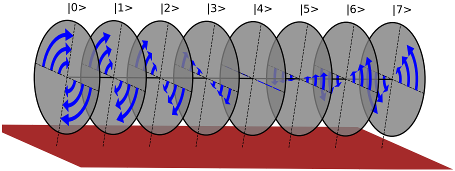
- One application of this phase oracle is one *quantum function evaluation*.





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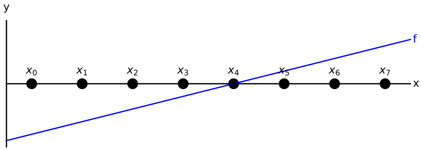

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Let $f : \mathbb{R} \to \mathbb{R}$ linear with $|f'| \leq C$.

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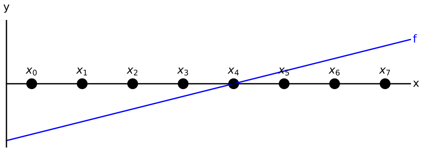

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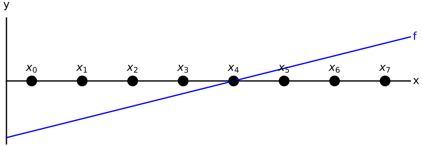

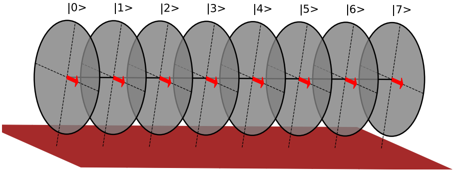

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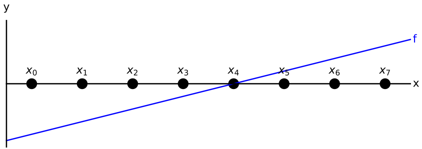

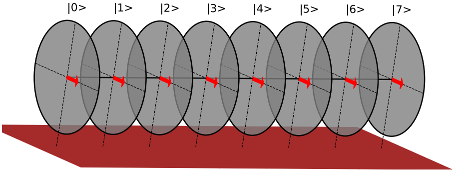

Quantum derivative estimation algorithm for linear functions

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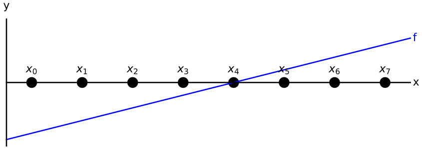

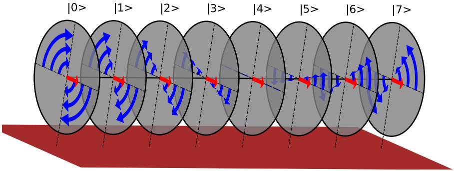

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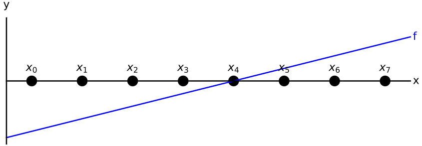

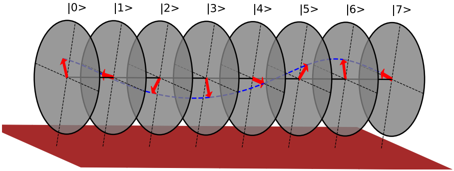

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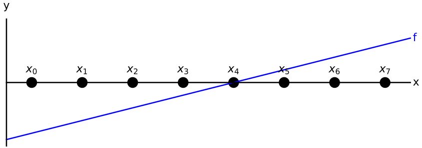
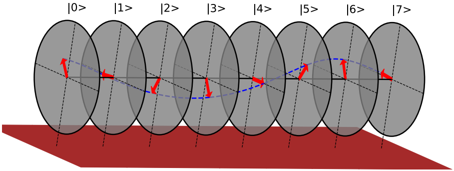
4. Measure.





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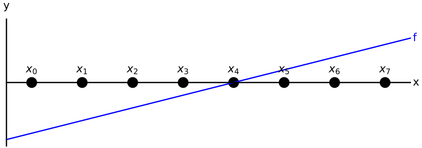

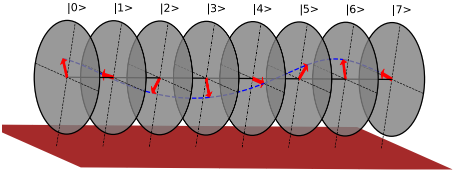

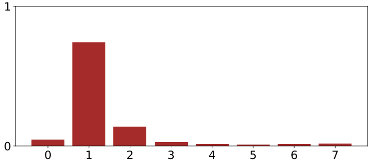

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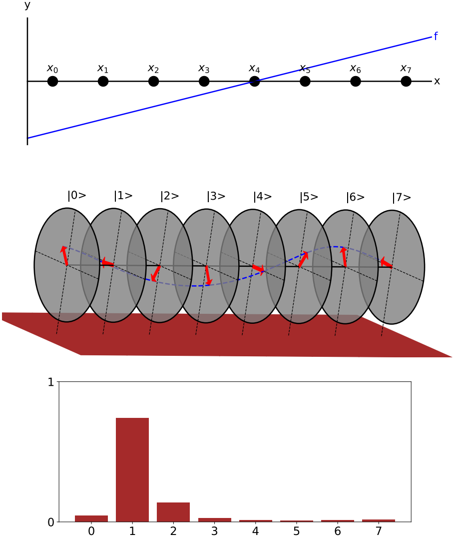
Generalizes to $f : \mathbb{R}^d \to \mathbb{R}$.
(Jordan, 2004)



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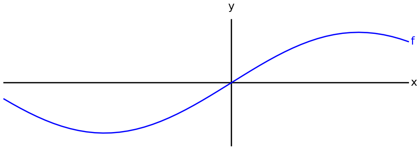

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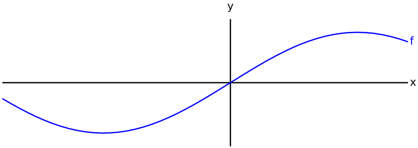

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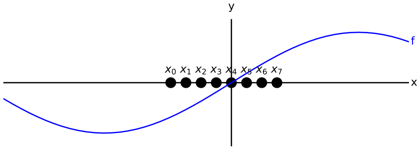

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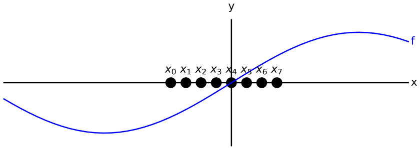

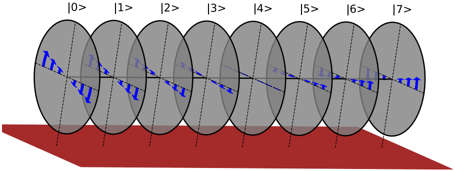

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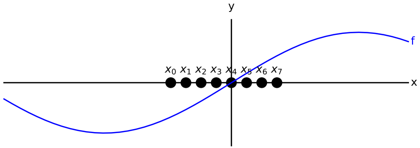

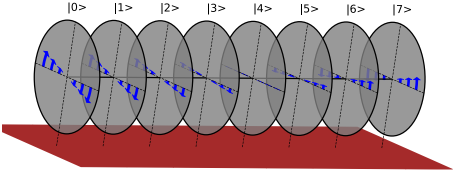

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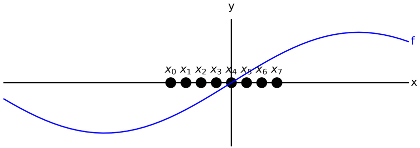

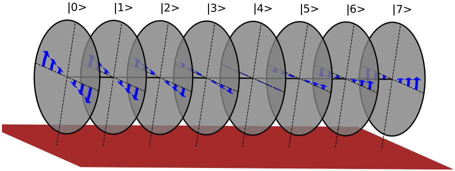

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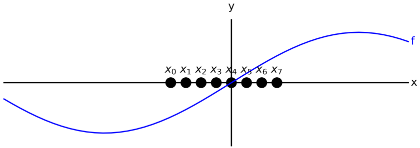

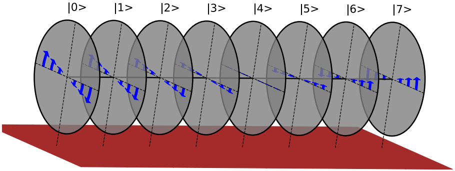

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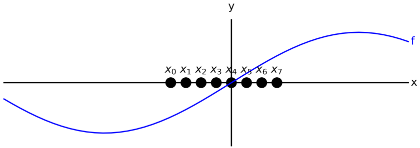
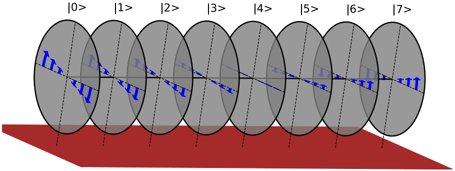
- Key idea: central difference method to extend region of approximate linearity.





Central difference method

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- Let $f : \mathbb{R}^d \to \mathbb{R}$ and $m > 0$. We define:

$$\widetilde{f}_{(2m)}(\mathbf{x}) = \sum_{\ell=-m}^{m} a_\ell^{(2m)} f(\ell \mathbf{x})$$

- such that:

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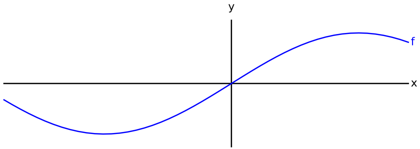

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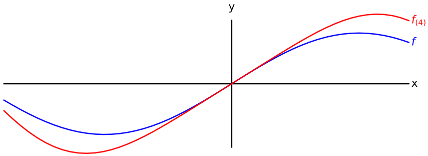

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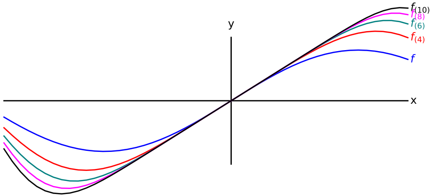

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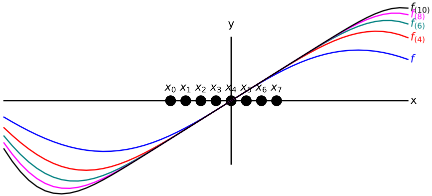

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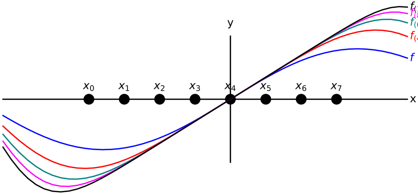

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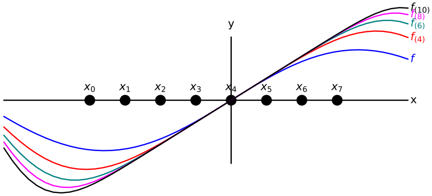

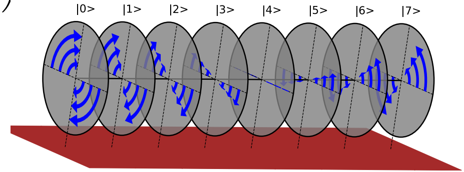

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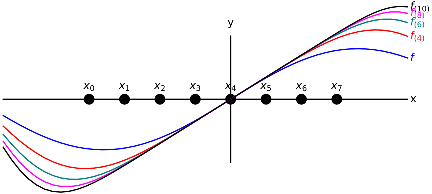
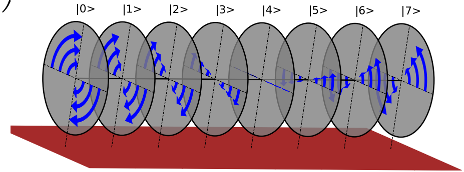
- One can implement $O_{\widetilde{f}_{(2m)}}$ using $\widetilde{\mathcal{O}}(m)$ queries to O_f. (Gilyén, Arunachalam, Wiebe, 2018)





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Thanks for your attention!

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