A self-contained, simplified analysis of span program algorithms

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[2]QuSoft – CWI

September 18th, 2020

Span programs – overview

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Boolean function
$f : \{0,1\}^n \to \{0,1\}$

Quantum algorithm
$\mathcal{A}$

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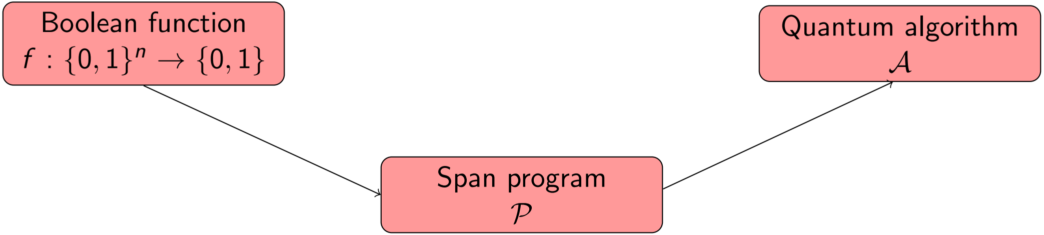

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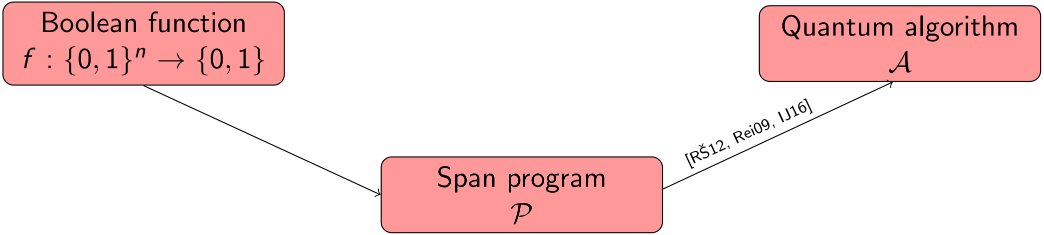

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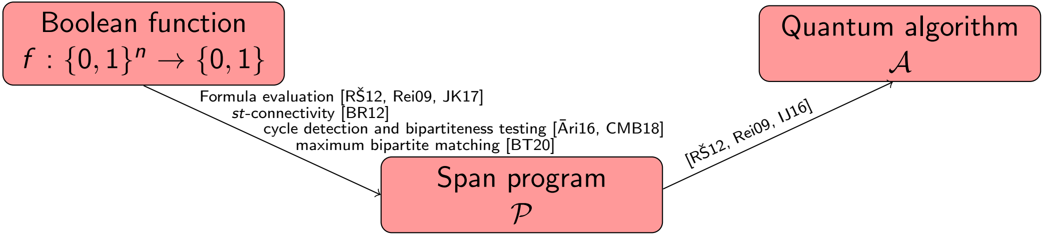

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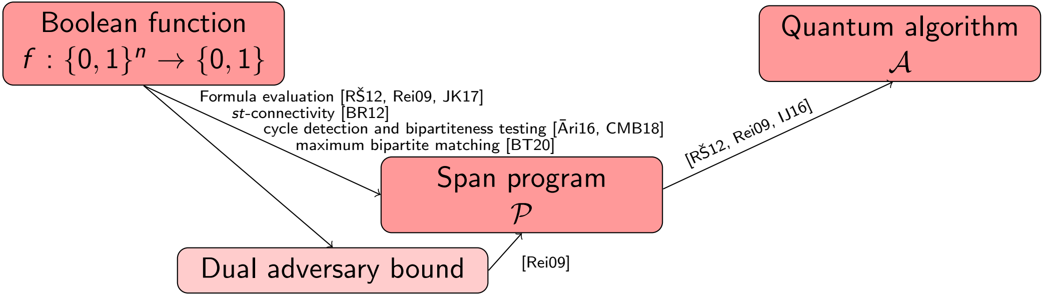

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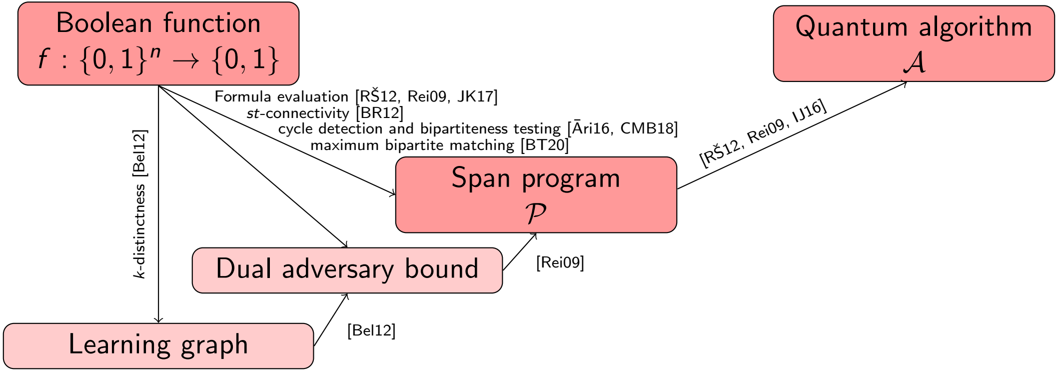

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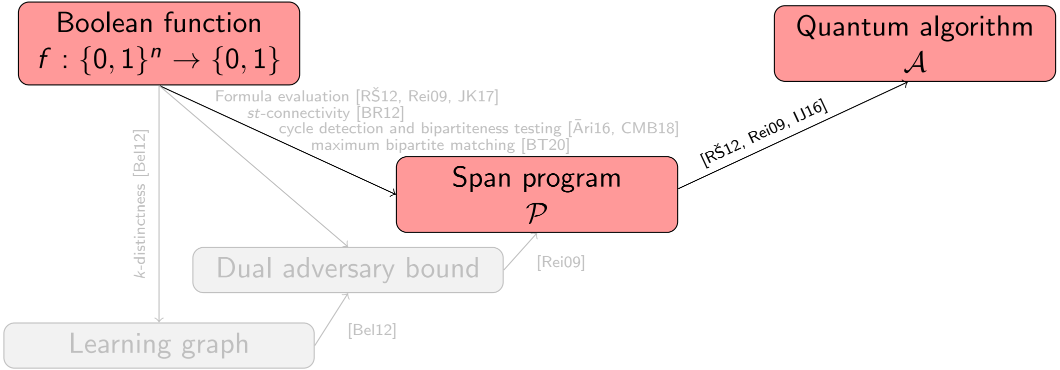

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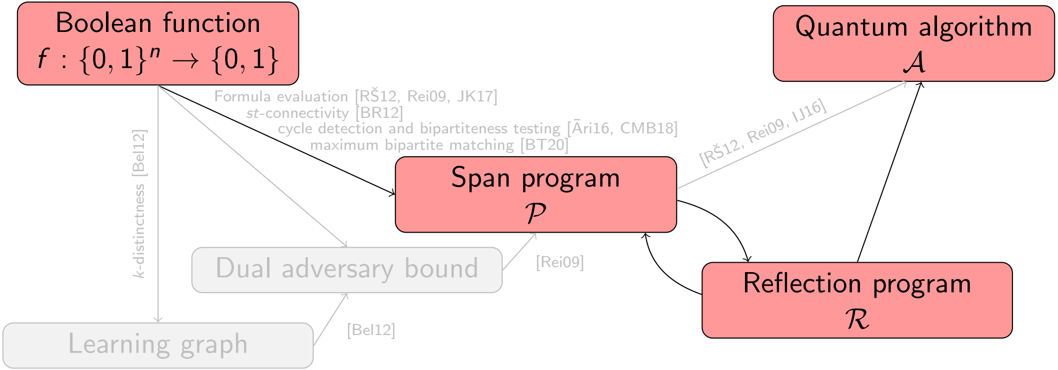

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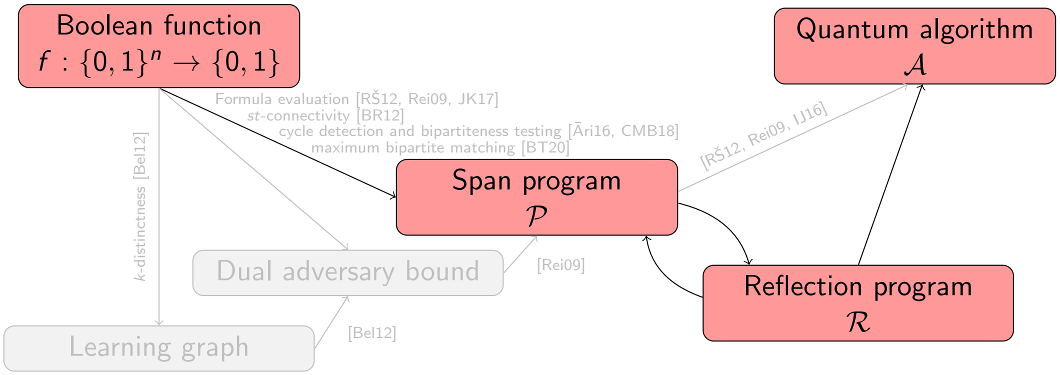

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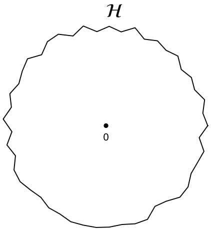

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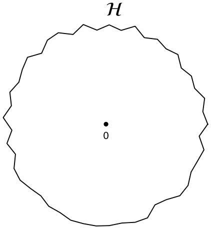

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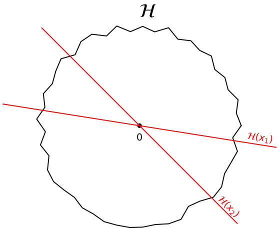

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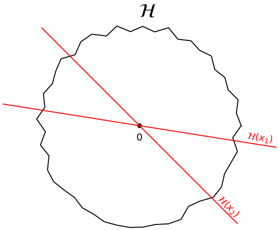

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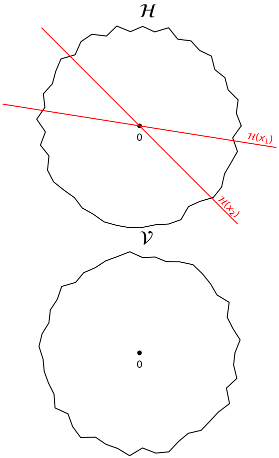

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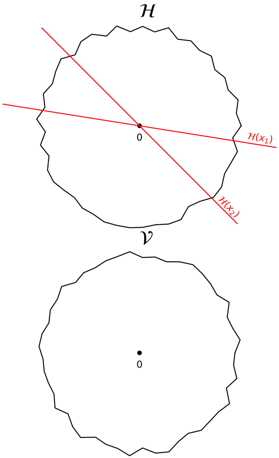

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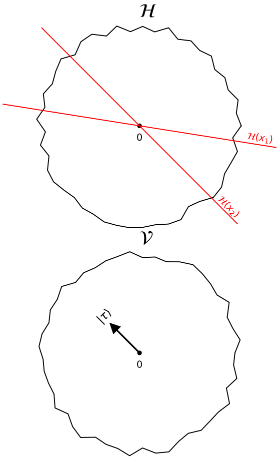

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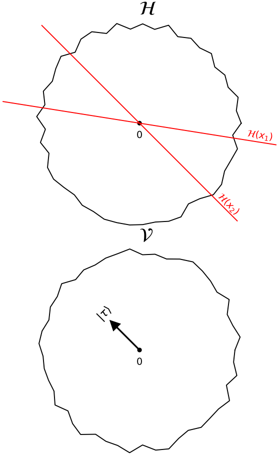

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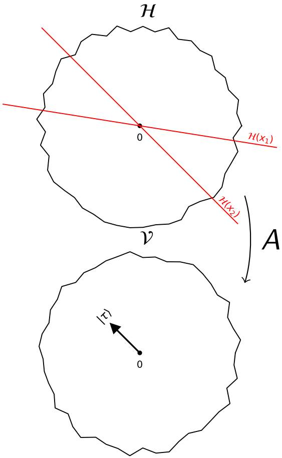

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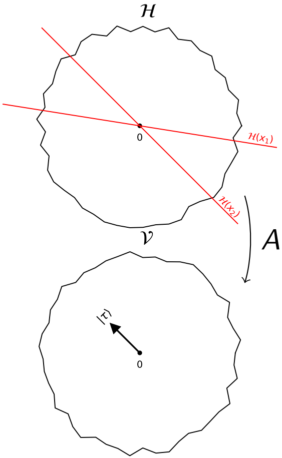

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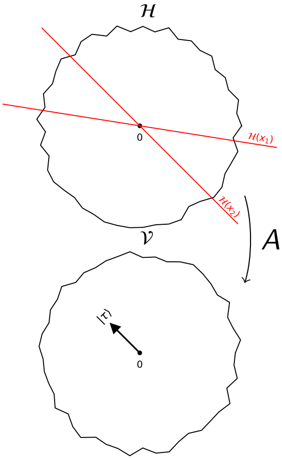

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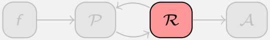

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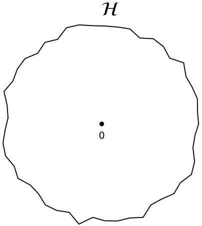

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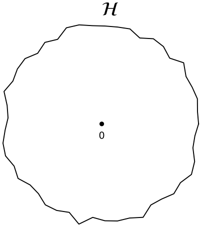

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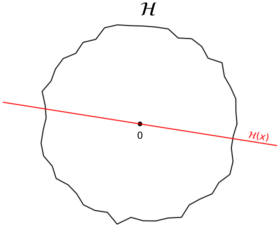

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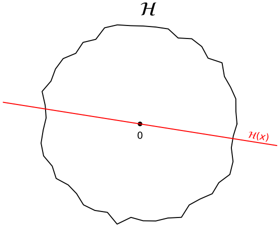

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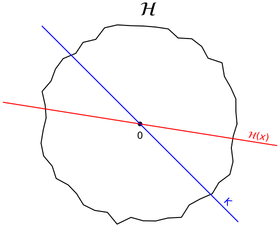

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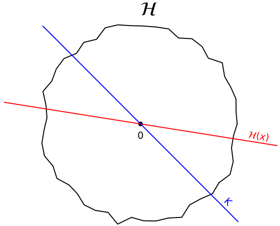

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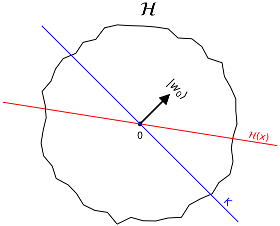

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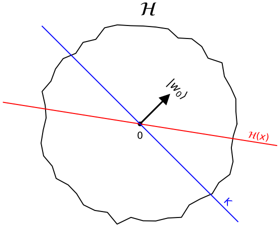

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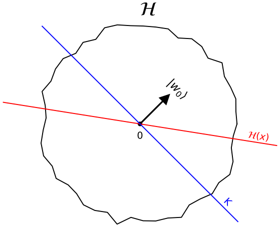

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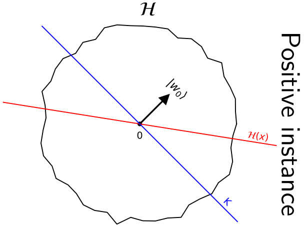

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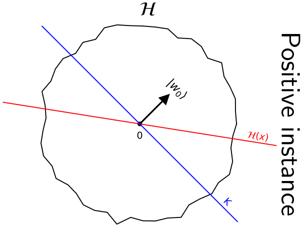

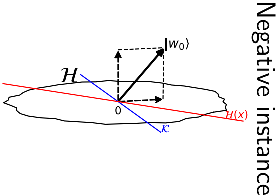

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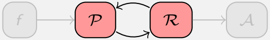

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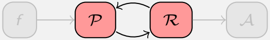

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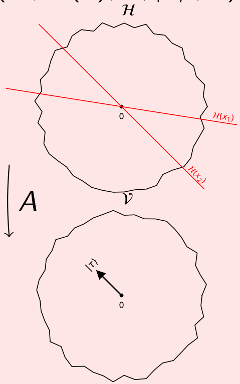

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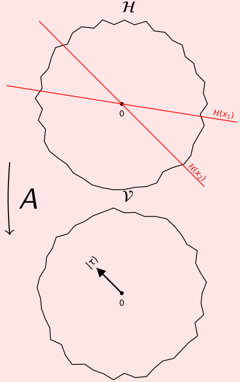

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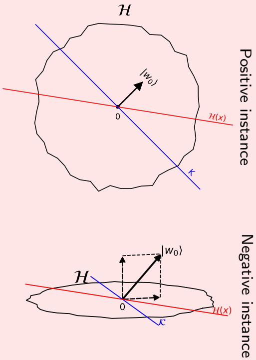

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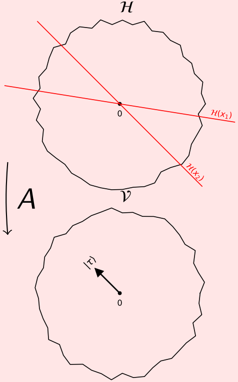
Neg	$|w_0\rangle \notin \mathcal{K} + \mathcal{H}(x)$

Conversions between span and reflection programs

Span program $\mathcal{P}$

$(\mathcal{H}, \mathcal{H}(x), \mathcal{V}, |\tau\rangle, A)$

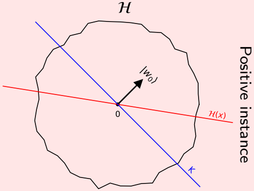


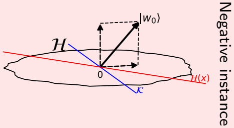
Pos	$\|\tau\rangle \in A(\mathcal{H}(x))$
Neg	$\|\tau\rangle \notin A(\mathcal{H}(x))$

$$\mathcal{K} = \mathrm{Ker}(A)$$
$$|w_0\rangle = \frac{A^+|\tau\rangle}{\|A^+|\tau\rangle\|}$$

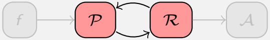
Reflection program $\mathcal{R}$

$(\mathcal{H}, \mathcal{H}(x), \mathcal{K}, |w_0\rangle)$



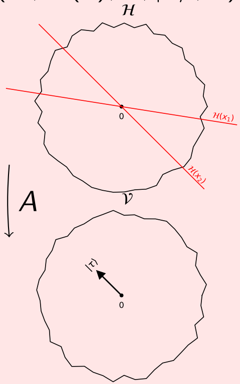


Pos	$\|w_0\rangle \in \mathcal{K} + \mathcal{H}(x)$
Neg	$\|w_0\rangle \notin \mathcal{K} + \mathcal{H}(x)$



Span program $\mathcal{P}$

$(\mathcal{H}, \mathcal{H}(x), \mathcal{V}, |\tau\rangle, A)$



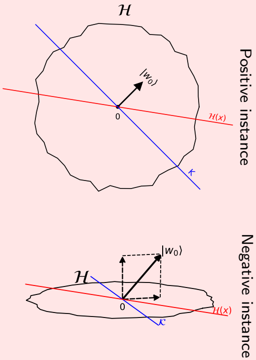
Pos	$\|\tau\rangle \in A(\mathcal{H}(x))$
Neg	$\|\tau\rangle \notin A(\mathcal{H}(x))$

$$\mathcal{K} = \mathrm{Ker}(A)$$
$$|w_0\rangle = \frac{A^+|\tau\rangle}{\|A^+|\tau\rangle\|}$$

$$\mathcal{V} = \mathcal{H},$$
$$|\tau\rangle = |w_0\rangle$$
$$A = I - \Pi_{\mathcal{K}}.$$

Reflection program $\mathcal{R}$

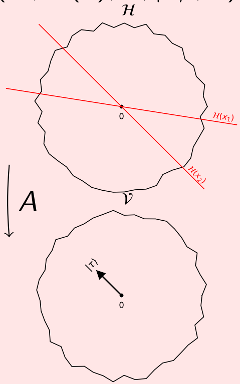
$(\mathcal{H}, \mathcal{H}(x), \mathcal{K}, |w_0\rangle)$



Pos	$\|w_0\rangle \in \mathcal{K} + \mathcal{H}(x)$
Neg	$\|w_0\rangle \notin \mathcal{K} + \mathcal{H}(x)$

Span program $\mathcal{P}$

$$(\mathcal{H}, \mathcal{H}(x), \mathcal{V}, |\tau\rangle, A)$$



Pos	$\|\tau\rangle \in A(\mathcal{H}(x))$
Neg	$\|\tau\rangle \notin A(\mathcal{H}(x))$

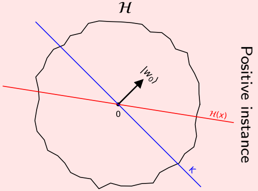
$$\mathcal{K} = \mathrm{Ker}(A)$$
$$|w_0\rangle = \frac{A^+|\tau\rangle}{\|A^+|\tau\rangle\|}$$

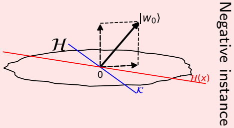
These conversions don't change positive and negative instances.

$$\mathcal{V} = \mathcal{H},$$
$$|\tau\rangle = |w_0\rangle$$
$$A = I - \Pi_{\mathcal{K}}.$$

Reflection program $\mathcal{R}$

$$(\mathcal{H}, \mathcal{H}(x), \mathcal{K}, |w_0\rangle)$$

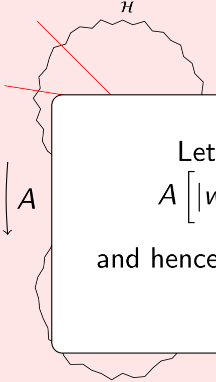




Pos	$\|w_0\rangle \in \mathcal{K} + \mathcal{H}(x)$
Neg	$\|w_0\rangle \notin \mathcal{K} + \mathcal{H}(x)$

Span program $\mathcal{P}$

$(\mathcal{H}, \mathcal{H}(x), \mathcal{V}, |\tau\rangle, A)$

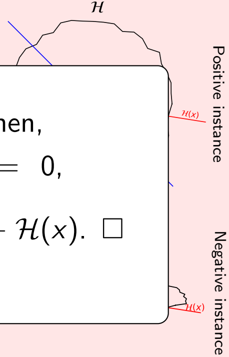


Pos	$\|\tau\rangle \in A(\mathcal{H}(x))$
Neg	$\|\tau\rangle \notin A(\mathcal{H}(x))$

$$\mathcal{K} = \mathrm{Ker}(A)$$
$$|w_0\rangle = \frac{A^+|\tau\rangle}{\|A^+|\tau\rangle\|}$$

Reflection program $\mathcal{R}$

$(\mathcal{H}, \mathcal{H}(x), \mathcal{K}, |w_0\rangle)$



Pos	$\|w_0\rangle \in \mathcal{K} + \mathcal{H}(x)$
Neg	$\|w_0\rangle \notin \mathcal{K} + \mathcal{H}(x)$

Positive instance $\to$ positive instance.

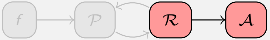
Let $|w_x\rangle \in \mathcal{H}(x)$ be such that $A|w_x\rangle = |\tau\rangle$. Then,

$$A\left[|w_0\rangle - \frac{|w_x\rangle}{\|A^+|\tau\rangle\|}\right] = \frac{AA^+|\tau\rangle - A|w_x\rangle}{\|A^+|\tau\rangle\|} = \frac{|\tau\rangle - |\tau\rangle}{\|A^+|\tau\rangle\|} = 0,$$

and hence $|w_0\rangle = \underbrace{|w_0\rangle - \frac{|w_x\rangle}{\|A^+|\tau\rangle\|}}_{\in \mathrm{Ker}(A) = \mathcal{K}} + \underbrace{\frac{|w_x\rangle}{\|A^+|\tau\rangle\|}}_{\in \mathcal{H}(x)} \in \mathcal{K} + \mathcal{H}(x)$. $\square$

$$|\tau\rangle = |w_0\rangle$$
$$A = I - \Pi_{\mathcal{K}}.$$

Szegedy's spectral lemma [Sze04]



Szegedy's spectral lemma [Sze04]

Reflection program: $\mathcal{R} = (\mathcal{H}, x \mapsto \mathcal{H}(x), \mathcal{K}, |w_0\rangle)$.

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Reflection program: $\mathcal{R} = (\mathcal{H}, x \mapsto \mathcal{H}(x), \mathcal{K}, |w_0\rangle)$.

Reflection program operator: $U(x) = (2\Pi_{\mathcal{H}(x)} - I)(2\Pi_{\mathcal{K}} - I)$.

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Reflection program operator: $U(x) = (2\Pi_{\mathcal{H}(x)} - I)(2\Pi_{\mathcal{K}} - I)$.

1. Acts as I on $(\mathcal{K} \cap \mathcal{H}(x)) \oplus (\mathcal{K}^\perp \cap \mathcal{H}(x)^\perp)$,
2. Acts as $-I$ on $(\mathcal{K} \cap \mathcal{H}(x)^\perp) \oplus (\mathcal{K}^\perp \cap \mathcal{H}(x))$.

Szegedy's spectral lemma [Sze04]

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3. The remainder decomposes into 2-dimensional rotation spaces with angles $\varphi_1, \dots, \varphi_n$.

Szegedy's spectral lemma [Sze04]

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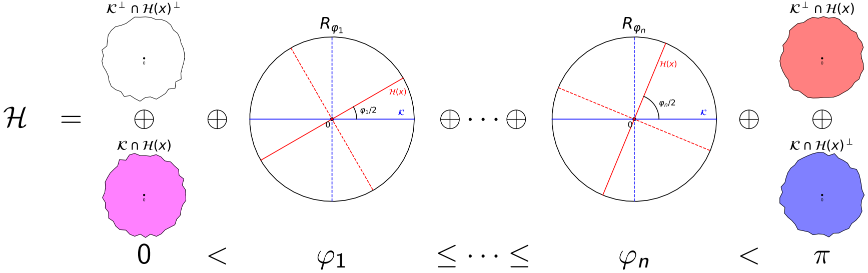
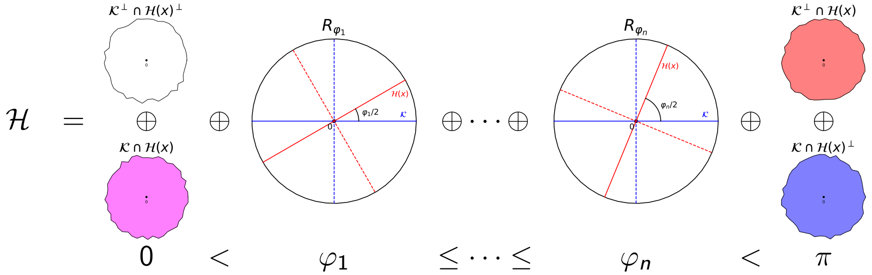


Szegedy's spectral lemma [Sze04]

Reflection program: $\mathcal{R} = (\mathcal{H}, x \mapsto \mathcal{H}(x), \mathcal{K}, |w_0\rangle)$.

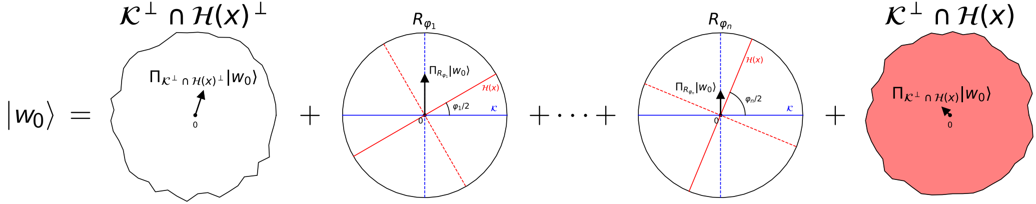
Reflection program operator: $U(x) = (2\Pi_{\mathcal{H}(x)} - I)(2\Pi_{\mathcal{K}} - I)$.

1. Acts as I on $(\mathcal{K} \cap \mathcal{H}(x)) \oplus (\mathcal{K}^\perp \cap \mathcal{H}(x)^\perp)$,
2. Acts as $-I$ on $(\mathcal{K} \cap \mathcal{H}(x)^\perp) \oplus (\mathcal{K}^\perp \cap \mathcal{H}(x))$.
3. The remainder decomposes into 2-dimensional rotation spaces with angles $\varphi_1, \dots, \varphi_n$.
4. $\Pi_{\mathcal{K}}$ and $\Pi_{\mathcal{H}(x)}$ commute with the projectors on all these spaces.

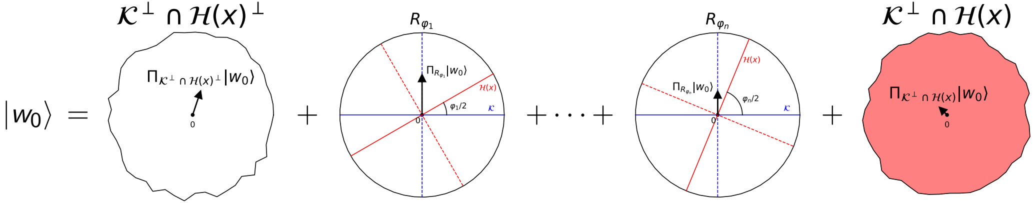


Decomposition of $|w_0\rangle$

Decomposition of $|w_0\rangle \in \mathcal{K}^\perp$:



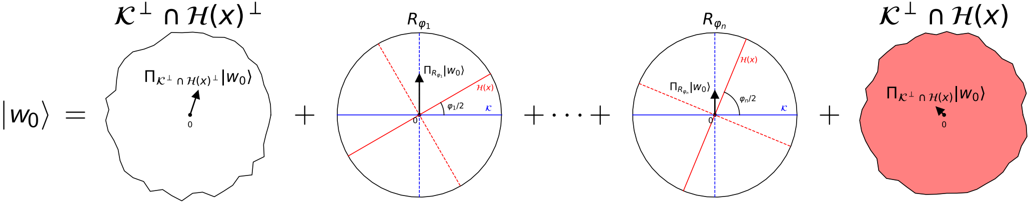
Decomposition of $|w_0\rangle \in \mathcal{K}^\perp$:



Positive instance:	$\|w_0\rangle \in \mathcal{K} + \mathcal{H}(x)$
Negative instance:	$\|w_0\rangle \notin \mathcal{K} + \mathcal{H}(x)$

Decomposition of $|w_0\rangle$

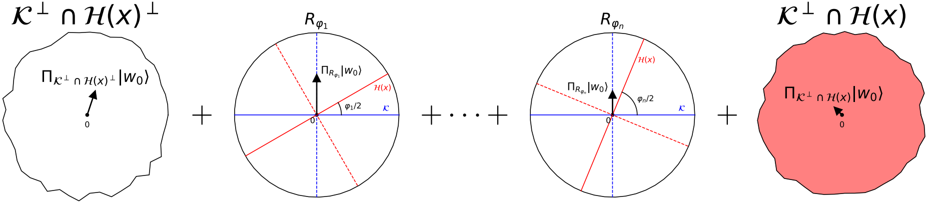
Decomposition of $|w_0\rangle \in \mathcal{K}^\perp$:



Positive instance:	$\|w_0\rangle \in \mathcal{K} + \mathcal{H}(x)$	$\Leftrightarrow$	$\Pi_{\mathcal{K}^\perp \cap \mathcal{H}(x)^\perp} \|w_0\rangle = 0$
Negative instance:	$\|w_0\rangle \notin \mathcal{K} + \mathcal{H}(x)$	$\Leftrightarrow$	$\Pi_{\mathcal{K}^\perp \cap \mathcal{H}(x)^\perp} \|w_0\rangle \neq 0$

Decomposition of $|w_0\rangle$

Decomposition of $|w_0\rangle \in \mathcal{K}^\perp$:

$|w_0\rangle =$ 

Positive instance:	$\|w_0\rangle \in \mathcal{K} + \mathcal{H}(x)$	$\Leftrightarrow$	$\Pi_{\mathcal{K}^\perp \cap \mathcal{H}(x)^\perp} \|w_0\rangle = 0$
Negative instance:	$\|w_0\rangle \notin \mathcal{K} + \mathcal{H}(x)$	$\Leftrightarrow$	$\Pi_{\mathcal{K}^\perp \cap \mathcal{H}(x)^\perp} \|w_0\rangle \neq 0$

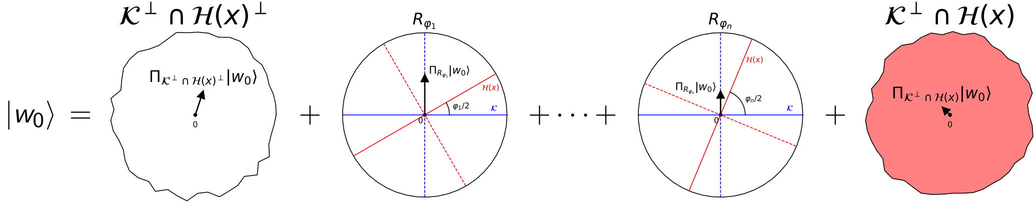
Thought experiment: run phase estimation

1. With operator $U(x)$,
2. With initial state $|w_0\rangle$,
3. With *infinite precision*,

call the outcome Φ.

Decomposition of $|w_0\rangle$

Decomposition of $|w_0\rangle \in \mathcal{K}^\perp$:



Positive instance:	$\|w_0\rangle \in \mathcal{K} + \mathcal{H}(x)$	$\Leftrightarrow$	$\Pi_{\mathcal{K}^\perp \cap \mathcal{H}(x)^\perp}\|w_0\rangle = 0$
Negative instance:	$\|w_0\rangle \notin \mathcal{K} + \mathcal{H}(x)$	$\Leftrightarrow$	$\Pi_{\mathcal{K}^\perp \cap \mathcal{H}(x)^\perp}\|w_0\rangle \neq 0$

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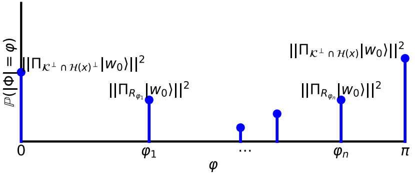
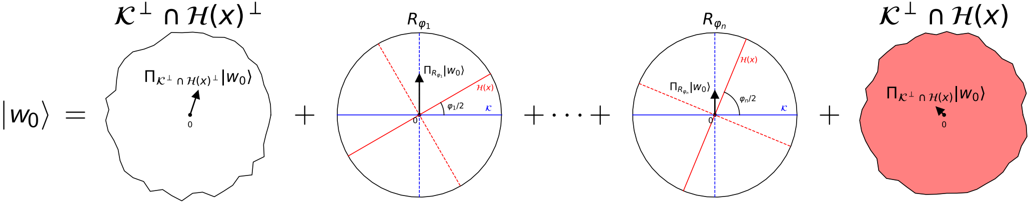


Decomposition of $|w_0\rangle$

Decomposition of $|w_0\rangle \in \mathcal{K}^\perp$:



Positive instance:	$\|w_0\rangle \in \mathcal{K} + \mathcal{H}(x)$	$\Leftrightarrow$	$\Pi_{\mathcal{K}^\perp \cap \mathcal{H}(x)^\perp}\|w_0\rangle = 0$	$\Leftrightarrow$	$\mathbb{P}(\Phi = 0) = 0$
Negative instance:	$\|w_0\rangle \notin \mathcal{K} + \mathcal{H}(x)$	$\Leftrightarrow$	$\Pi_{\mathcal{K}^\perp \cap \mathcal{H}(x)^\perp}\|w_0\rangle \neq 0$	$\Leftrightarrow$	$\mathbb{P}(\Phi = 0) > 0$

Thought experiment: run phase estimation

1. With operator $U(x)$,
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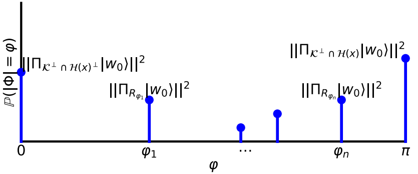
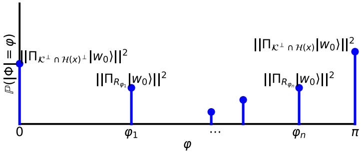


Finite precision

Positive instance:	$\mathbb{P}(\Phi = 0) = 0,$
Negative instance:	$\mathbb{P}(\Phi = 0) > 0.$



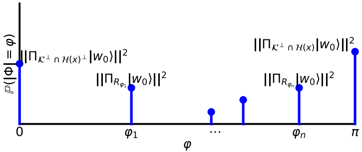
Finite precision

Positive instance:	$\mathbb{P}(\Phi = 0) = 0,$
Negative instance:	$\mathbb{P}(\Phi = 0) > 0.$

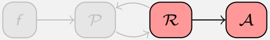
Finite precision algorithm:

1. Run phase estimation
    1. With operator $U(x)$,
    2. With initial state $|w_0\rangle$,
    3. With precision $\delta > 0$,

    call the outcome Φ_δ.



Finite precision

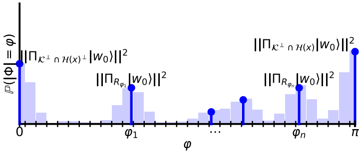


Positive instance:	$\mathbb{P}(\Phi = 0) = 0,$
Negative instance:	$\mathbb{P}(\Phi = 0) > 0.$

Finite precision algorithm:

1. Run phase estimation
   1. With operator $U(x)$,
   2. With initial state $|w_0\rangle$,
   3. With precision $\delta > 0$,

   call the outcome Φ_δ.



Finite precision

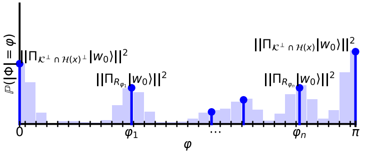
Positive instance:	$\mathbb{P}(\Phi = 0) = 0$,
Negative instance:	$\mathbb{P}(\Phi = 0) > 0$.

Finite precision algorithm:

1. Run phase estimation
   1. With operator $U(x)$,
   2. With initial state $|w_0\rangle$,
   3. With precision $\delta > 0$,

   call the outcome Φ_δ.
2. Distinguish between
   1. $\mathbb{P}(\Phi_\delta = 0) \leq \varepsilon/2$ (output $f(x) = 1$),
   2. $\mathbb{P}(\Phi_\delta = 0) \geq \varepsilon$ (output $f(x) = 0$),

   by running amplitude estimation with precision $\Theta(\sqrt{\varepsilon})$.



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1. Run phase estimation
   1. With operator $U(x)$,
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   3. With precision $\delta > 0$,

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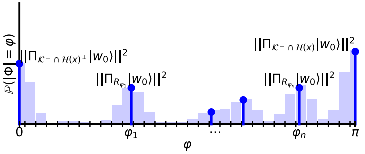
2. Distinguish between
   1. $\mathbb{P}(\Phi_\delta = 0) \leq \varepsilon/2$ (output $f(x) = 1$),
   2. $\mathbb{P}(\Phi_\delta = 0) \geq \varepsilon$ (output $f(x) = 0$),

   by running amplitude estimation with precision $\Theta(\sqrt{\varepsilon})$.

Total cost: $\Theta\left(\frac{1}{\delta\sqrt{\varepsilon}}\right)$ calls to $U(x)$.



Finite precision

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1. Run phase estimation
   1. With operator $U(x)$,
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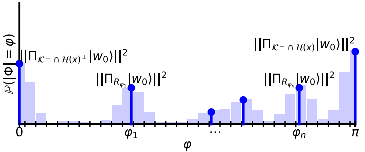
Total cost: $\Theta\left(\frac{1}{\delta\sqrt{\varepsilon}}\right)$ calls to $U(x)$.



Analysis of phase estimation:

$$\mathbb{P}(\Phi = 0) \leq \mathbb{P}(\Phi_\delta = 0) \leq \delta^2 \mathbb{E}\left[\frac{1}{\sin^2\left(\frac{\Phi}{2}\right)}\right],$$

Finite precision

Positive instance:	$\mathbb{P}(\Phi = 0) = 0$,
Negative instance:	$\mathbb{P}(\Phi = 0) > 0$.

Finite precision algorithm:

1. Run phase estimation
   1. With operator $U(x)$,
   2. With initial state $|w_0\rangle$,
   3. With precision $\delta > 0$,

   call the outcome Φ_δ.
2. Distinguish between
   1. $\mathbb{P}(\Phi_\delta = 0) \leq \varepsilon/2$ (output $f(x) = 1$),
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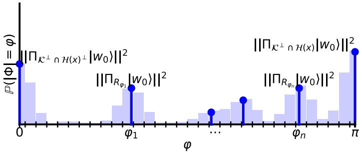
Total cost: $\Theta\left(\frac{1}{\delta\sqrt{\varepsilon}}\right)$ calls to $U(x)$.



Analysis of phase estimation:

$$\mathbb{P}(\Phi = 0) \leq \mathbb{P}(\Phi_\delta = 0) \leq \delta^2 \mathbb{E}\left[\frac{1}{\sin^2\left(\frac{\Phi}{2}\right)}\right],$$

We need to ensure that:

1. For positive instances:

$$\mathbb{E}\left[\frac{1}{\sin^2\left(\frac{\Phi}{2}\right)}\right] \leq \frac{\varepsilon}{2\delta^2}.$$

Finite precision

Positive instance:	$\mathbb{P}(\Phi = 0) = 0$,
Negative instance:	$\mathbb{P}(\Phi = 0) > 0$.

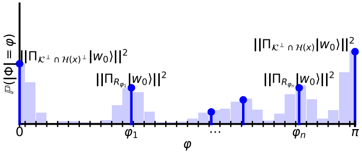
Finite precision algorithm:

1. Run phase estimation
    1. With operator $U(x)$,
    2. With initial state $|w_0\rangle$,
    3. With precision $\delta > 0$,

    call the outcome Φ_δ.
2. Distinguish between
    1. $\mathbb{P}(\Phi_\delta = 0) \leq \varepsilon/2$ (output $f(x) = 1$),
    2. $\mathbb{P}(\Phi_\delta = 0) \geq \varepsilon$ (output $f(x) = 0$),

    by running amplitude estimation with precision $\Theta(\sqrt{\varepsilon})$.

Total cost: $\Theta\left(\frac{1}{\delta\sqrt{\varepsilon}}\right)$ calls to $U(x)$.



Analysis of phase estimation:

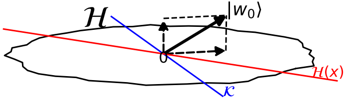
$$\mathbb{P}(\Phi = 0) \leq \mathbb{P}(\Phi_\delta = 0) \leq \delta^2 \mathbb{E}\left[\frac{1}{\sin^2\left(\frac{\Phi}{2}\right)}\right],$$

We need to ensure that:

1. For positive instances:
$$\mathbb{E}\left[\frac{1}{\sin^2\left(\frac{\Phi}{2}\right)}\right] \leq \frac{\varepsilon}{2\delta^2}.$$
2. For negative instances:
$$\mathbb{P}(\Phi = 0) \geq \varepsilon.$$

Negative witnesses

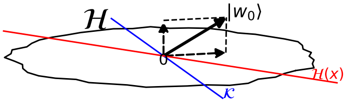
For all negative instances, we need to ensure that $\mathbb{P}(\Phi = 0) \geq \varepsilon$.



Negative witnesses

For all negative instances, we need to ensure that $\mathbb{P}(\Phi = 0) \geq \varepsilon$.

$$\mathbb{P}(\Phi = 0) = \left\| \Pi_{\mathcal{K}^\perp \cap \mathcal{H}(x)^\perp} \left| w_0 \right\rangle \right\|^2$$



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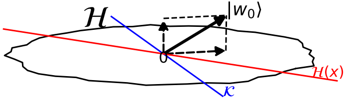

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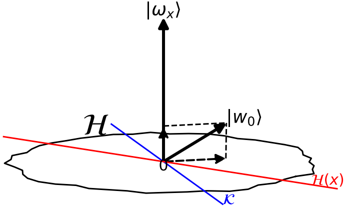

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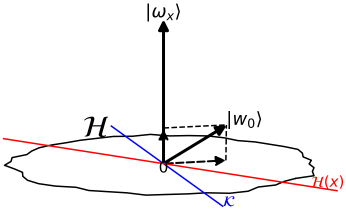

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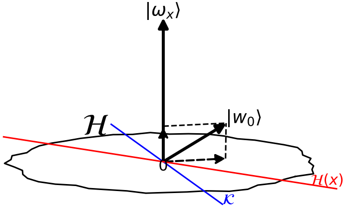

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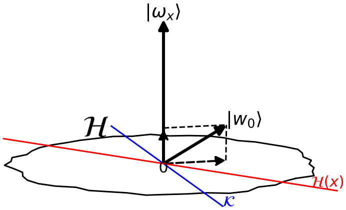

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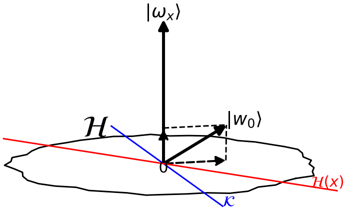

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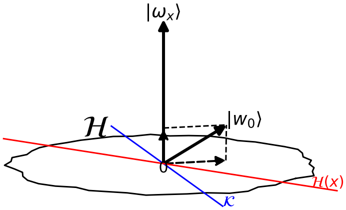

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Shorter negative witnesses give better bounds.

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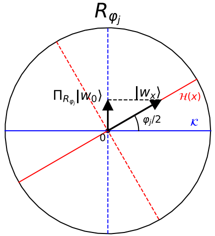

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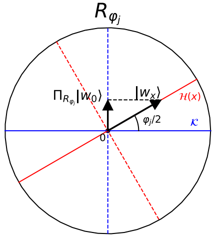

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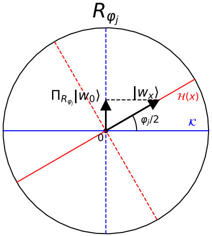

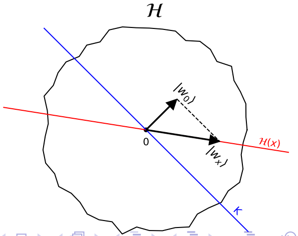

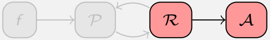

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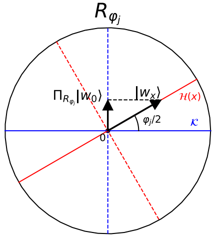

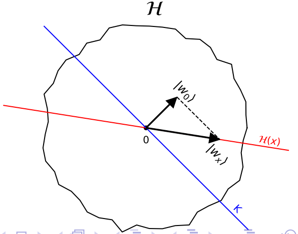

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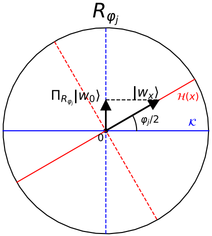

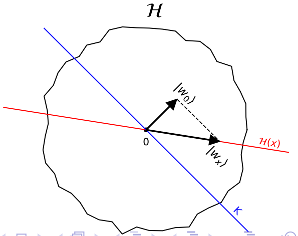

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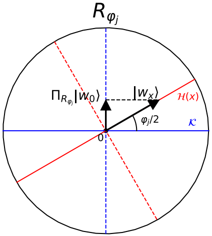

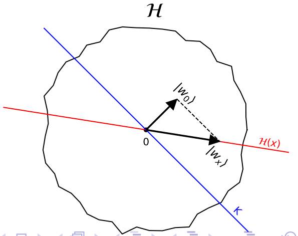

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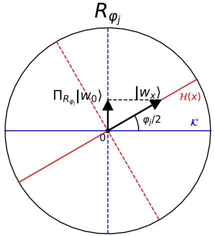

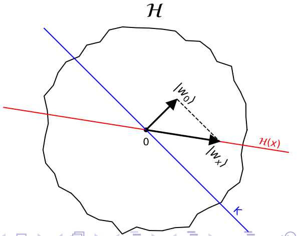

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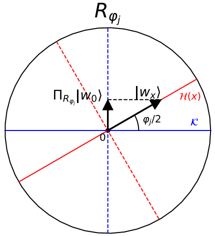

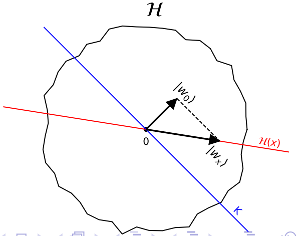

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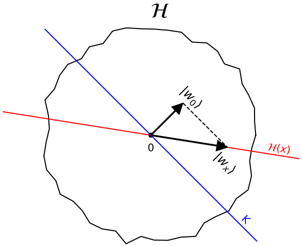

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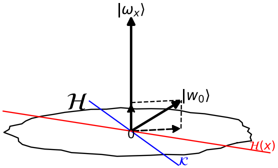

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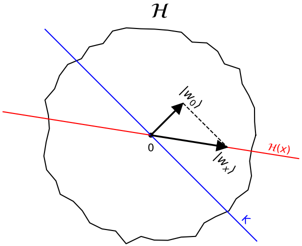

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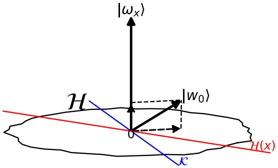

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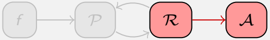

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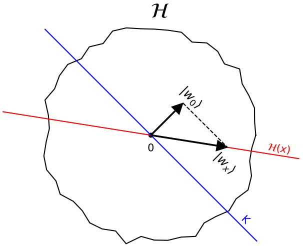
   2. with initial state $|w_0\rangle$,
   3. with precision δ,

   call the outcome Φ_δ.



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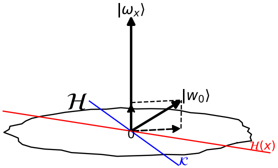

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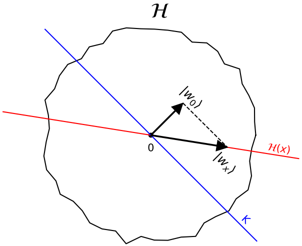
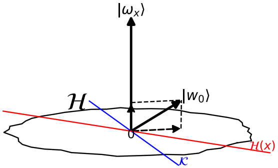
   1. with operator $U(x) = (2\Pi_{\mathcal{H}(x)} - I)(2\Pi_{\mathcal{K}} - I)$,
   2. with initial state $|w_0\rangle$,
   3. with precision δ,

   call the outcome Φ_δ.
2. Distinguish between
   1. $\mathbb{P}(\Phi_\delta = 0) \leq \varepsilon/2$ (output $f(x) = 1$),
   2. $\mathbb{P}(\Phi_\delta = 0) \geq \varepsilon$ (output $f(x) = 0$),

   with amplitude estimation with precision $\Theta(\sqrt{\varepsilon})$.



$$|w_0\rangle \in \mathcal{H}(x),$$
$$\Pi_{\mathcal{K}^\perp} |w_x\rangle = |w_0\rangle.$$



$$|\omega_x\rangle \in \mathcal{K}^\perp \cap \mathcal{H}(x)^\perp,$$
$$\langle \omega_x | w_0 \rangle = 1.$$

Summary of the algorithm

Reflection program: $\mathcal{R} = (\mathcal{H}, x \mapsto \mathcal{H}(x), \mathcal{K}, |w_0\rangle)$.

1. For all $x \in f^{(-1)}(1)$, find a *positive witness* $|w_x\rangle$,
2. For all $x \in f^{(-1)}(0)$, find a *negative witness* $|\omega_x\rangle$,

$$W_+ = \max_{x \in f^{(-1)}(1)} \| |w_x\rangle \|^2, \quad W_- = \max_{x \in f^{(-1)}(0)} \| |\omega_x\rangle \|^2.$$

Choose the finite precision parameters:

1. $\varepsilon := 1/W_-$,
2. $W_+ = \frac{\varepsilon}{2\delta^2} \quad \Rightarrow \quad \delta := 1/\sqrt{2W_- W_+}$,

Run the algorithm:

1. Run phase estimation
   1. with operator $U(x) = (2\Pi_{\mathcal{H}(x)} - I)(2\Pi_{\mathcal{K}} - I)$,
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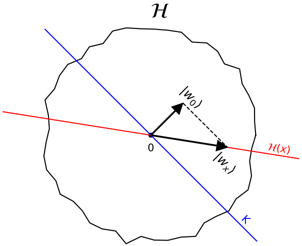
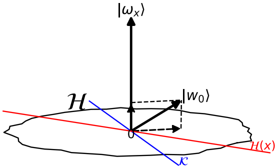
   with amplitude estimation with precision $\Theta(\sqrt{\varepsilon})$.

Total calls to $U(x)$: $\mathcal{O}\left(\frac{1}{\delta\sqrt{\varepsilon}}\right) = \mathcal{O}(W_-\sqrt{W_+})$,



$$|w_0\rangle \in \mathcal{H}(x), \quad \Pi_{\mathcal{K}^\perp} |w_x\rangle = |w_0\rangle.$$



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      Span program $\mathcal{P}$: W_+, W_-. $\xrightarrow{\text{Span program renormalization}}$ Span program $\mathcal{P}'$: $W_+' = \Theta(W_+W_-)$, $W_-' = \Theta(1)$.

   2. How does this technique look in the reflection program case?

      Reflection program $\mathcal{R}_1$ $\xrightarrow{\text{Negation}}$ Reflection program $\mathcal{R}_2$ $\xrightarrow{\text{Composition}}$ Reflection program $\mathcal{R}_3$ $\xrightarrow{\text{Negation}}$ Reflection program $\mathcal{R}_4$

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2. Other relations seem to play a non-trivial role:

$$\chi(y) := \mathbb{E}\left[\frac{1}{y - \sin^2\left(\frac{\Phi}{2}\right)}\right] = \langle w_0| \left(\Pi_{\mathcal{K}^\perp}\Pi_{\mathcal{H}(x)^\perp}\Pi_{\mathcal{K}^\perp} - (1-y)I\right)^{-1} |w_0\rangle .$$

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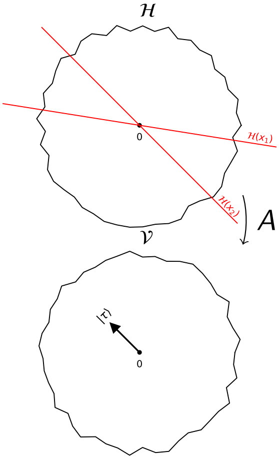
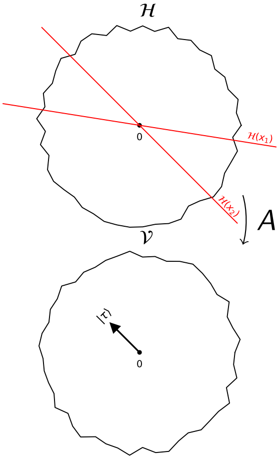
To be continued...

Thanks for your attention!

arjan@cwi.nl


$\mathcal{H}$
0
$\mathcal{H}(x_1)$
$\mathcal{H}(x_2)$
A
$\mathcal{V}$
$|\tau\rangle$
0

Search function: $f(x_1, \dots, x_n) = x_1 \vee \cdots \vee x_n$.



Span programs – example

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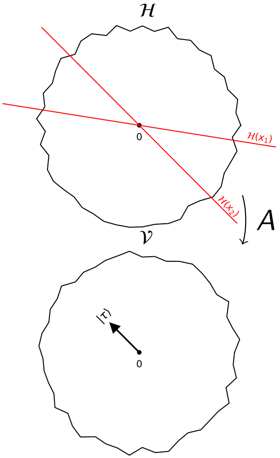

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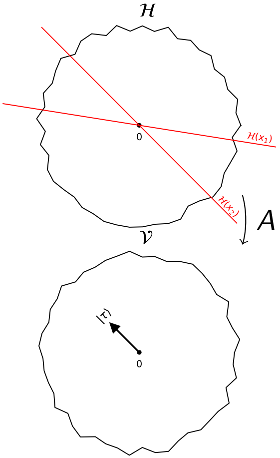

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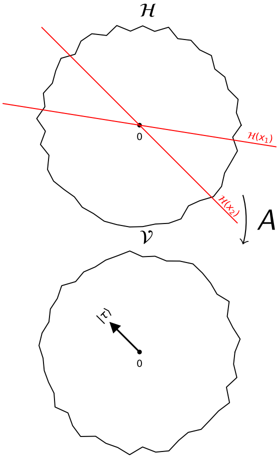

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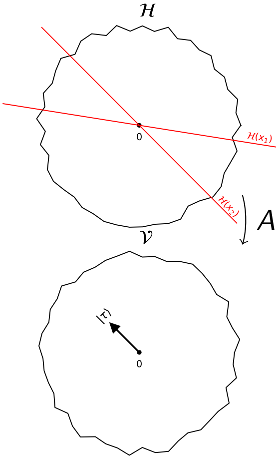

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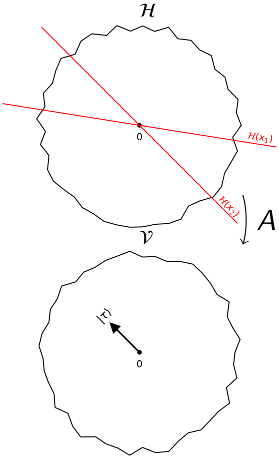

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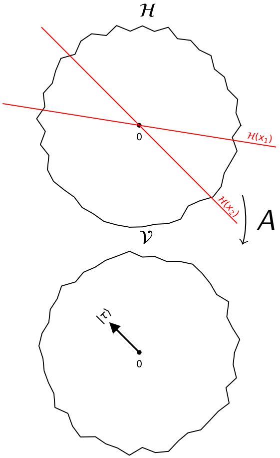
$\mathcal{P} = (\mathcal{H}, x \mapsto \mathcal{H}(x), \mathcal{V}, |\tau\rangle, A)$ evaluates f, as:



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Positive instance:	$x \neq 0^n \quad \Rightarrow$
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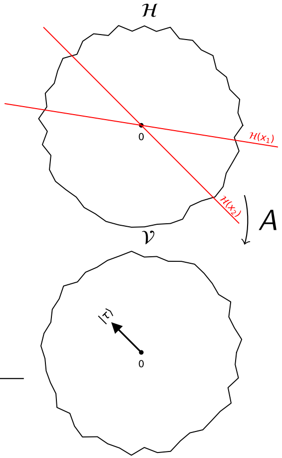

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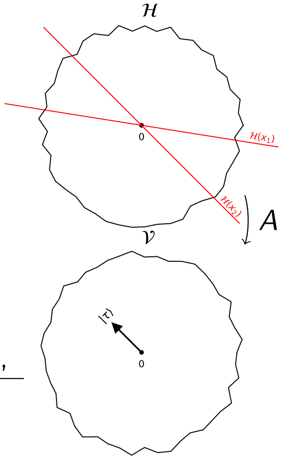

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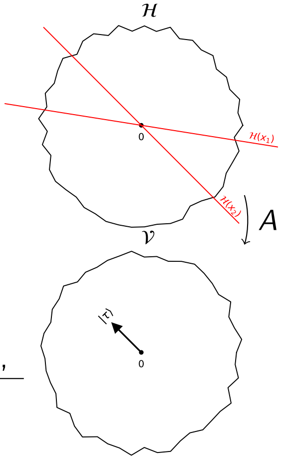
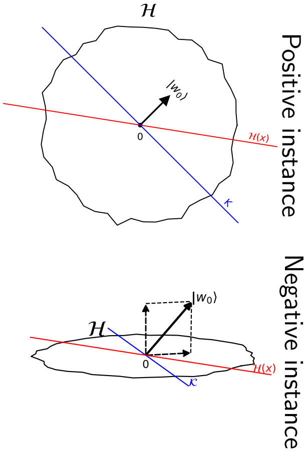


Reflection programs – example



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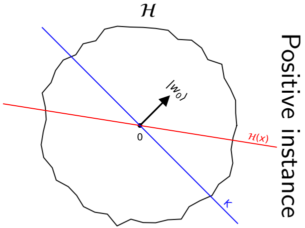

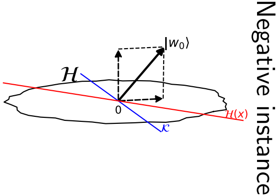

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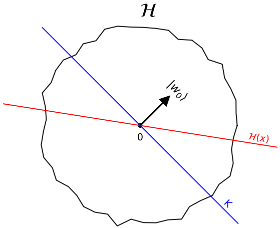
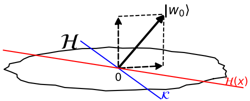


Positive instance



Negative instance

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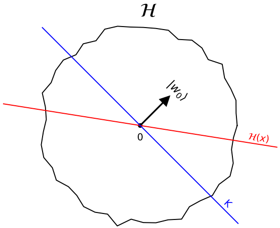
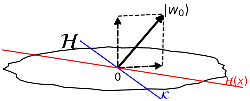


Positive instance



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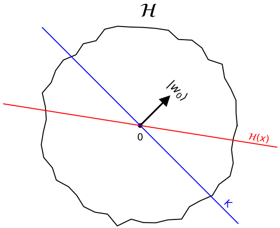

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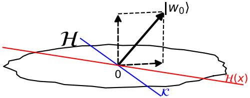

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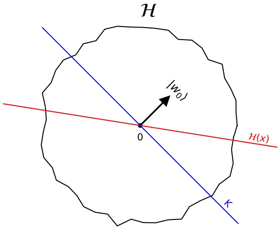

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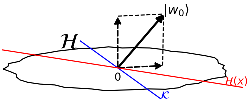

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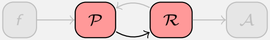

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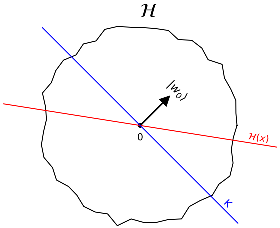

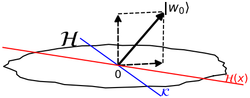

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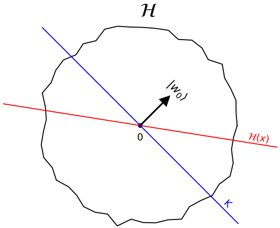

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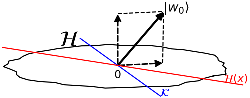

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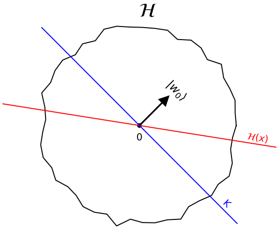
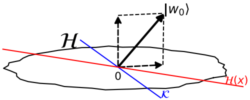
3. $|w_0\rangle = \frac{A^+|\tau\rangle}{\|A^+|\tau\rangle\|} = \frac{1}{\sqrt{n}} \sum_{j=1}^n |j\rangle$.
4. $\mathcal{K} = \mathrm{Ker}(A) = \mathrm{Ker}(\langle w_0|) = \mathrm{Span}\{|w_0\rangle\}^\perp$.

$\mathcal{R} = (\mathcal{H}, x \mapsto \mathcal{H}(x), \mathcal{K}, |w_0\rangle)$ evaluates f, as:

Positive instance:	$x \neq 0^n \quad \Rightarrow$
Negative instance:	$x = 0^n \quad \Rightarrow$

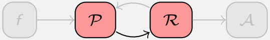


Positive instance



Negative instance

Reflection programs – example



Search function: $f(x_1, \ldots, x_n) = x_1 \vee \cdots \vee x_n$.

1. $\mathcal{H} = \mathbb{C}^n$.
2. $\mathcal{H}(x) = \text{Span}\{|j\rangle : x_j = 1\}$.
3. $\mathcal{V} = \mathbb{C}$.
4. $|\tau\rangle = 1$.
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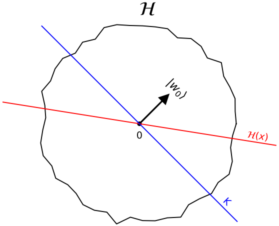

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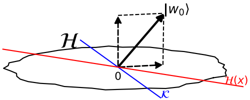
Positive instance: $x \neq 0^n \quad \Rightarrow \quad$ Let $x_j = 1$,

$$|w_0\rangle = \underbrace{\sqrt{n}\,|j\rangle}_{\in\mathcal{H}(x)} + \underbrace{\frac{1}{\sqrt{n}} \sum_{k=1}^n |k\rangle - \sqrt{n}\,|j\rangle}_{\in\mathcal{K}} \in \mathcal{K} + \mathcal{H}(x)$$

Negative instance: $x = 0^n \quad \Rightarrow$



Negative instance

Reflection programs – example

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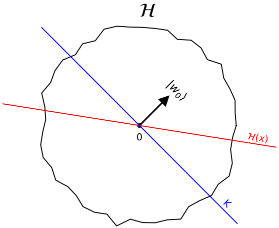

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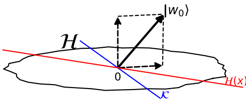
Negative instance: $x = 0^n \quad \Rightarrow \quad \mathcal{K} + \mathcal{H}(x) = \mathcal{K} = \mathrm{Span}\{|w_0\rangle\}^\perp \not\ni |w_0\rangle$.



Negative instance