Lower bound on quantum full mixed-state tomography

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   2. $\Omega(\frac{dr}{\varepsilon})$. [This talk – yet to be included]

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Specific observables and other norms:

1. $O_j = |j\rangle\langle j|$, for $j = 1, \dots, d$.
2. $p_j = |\langle j|\psi\rangle|^2$.

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Given unitary access to $|\psi\rangle$:

$$\widetilde{\Theta}\left(\min\left\{\frac{d^{\frac{1}{q}}}{\varepsilon}, \frac{1}{\varepsilon^{\frac{1}{1-\frac{1}{q}}}}\right\}\right).$$ [vA'21; This work]

Probably many more...

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1. *Parameters:* $1 \leq r \leq d$, and $\varepsilon \in [0, 1/256]$.
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Remainder of this talk: $\Omega(\frac{dr}{\varepsilon})$ queries are required.

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1. Absolute constants $c, C \in (0, 1)$.
2. $D \subseteq \{0, 1\}^{dr}$ with $|D| \geq C \cdot 2^{dr}$.
3. For all $b \in D$, let $S_b \subseteq |D|$, with $b \in S_b$.
   (horizontal edges)
4. For all $\widetilde{b} \in D$, $|\{b \in D : \widetilde{b} \in S_b\}| \leq 2^{cdr}$.
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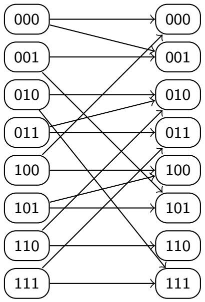

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Approximate string recovery (ε, D, $b \mapsto S_b$):

1. *Input:* $O_b^{(\varepsilon)} : |j\rangle \mapsto e^{2\pi i \varepsilon b_j} |j\rangle$ with $b \in D$.
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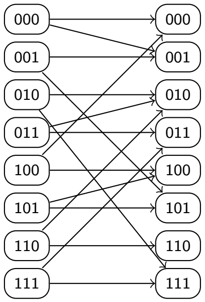

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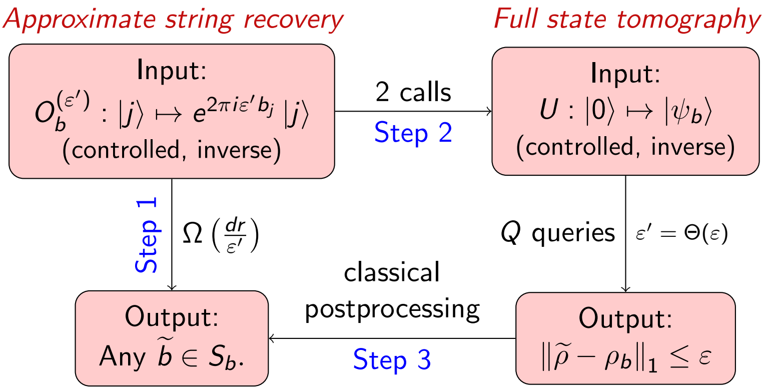

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2. Adversary method for relations: [Bel'15]
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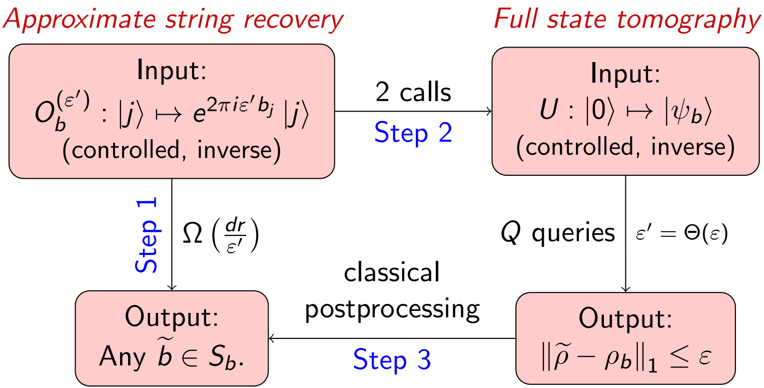
2. Adversary method for relations: [Bel'15]
    1. Approximate string recovery is a relation.
3. Combine both: [CJ'21].

Thus, $Q = \Omega(\frac{dr}{\varepsilon'})$.

Lower bound – proof overview

Idea: "Embed the approximate string recovery problem into the full state-tomography problem."

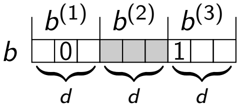


Lower bound – embedding

Embedding: ($\varepsilon < 1/256$)

1. Let $U^{(1)}, \ldots, U^{(r)} \in \mathbb{C}^{d \times d}$ unitaries.
2. Let $b \in \{0,1\}^{dr}$.
3. Define
$$\left|\psi_b^{(j)}\right\rangle = \frac{1}{\sqrt{d}} \sum_{k=1}^{d} \sum_{c \in \{0,1\}} \sqrt{\frac{1}{2} + 128\varepsilon(-1)^{c+b_k^{(j)}}} \left|c\right\rangle \left|k\right\rangle.$$
4. Let $\left|\psi_b\right\rangle = \frac{1}{\sqrt{r}} \sum_{j=1}^{r} (I \otimes U^{(j)}) \left|\psi_b^{(j)}\right\rangle_A \left|j\right\rangle_B$
5. Let $\rho_b = \mathrm{Tr}_B[\left|\psi_b\right\rangle\!\left\langle\psi_b\right|]$.

Lower bound – embedding



Embedding: $(\varepsilon < 1/256)$

1. Let $U^{(1)}, \ldots, U^{(r)} \in \mathbb{C}^{d \times d}$ unitaries.
2. Let $b \in \{0,1\}^{dr}$.
3. Define
$$\left|\psi_b^{(j)}\right\rangle = \frac{1}{\sqrt{d}} \sum_{k=1}^{d} \sum_{c \in \{0,1\}} \sqrt{\frac{1}{2} + 128\varepsilon(-1)^{c+b_k^{(j)}}} \, |c\rangle \, |k\rangle .$$
4. Let $|\psi_b\rangle = \frac{1}{\sqrt{r}} \sum_{j=1}^{r} (I \otimes U^{(j)}) \left|\psi_b^{(j)}\right\rangle_A |j\rangle_B$
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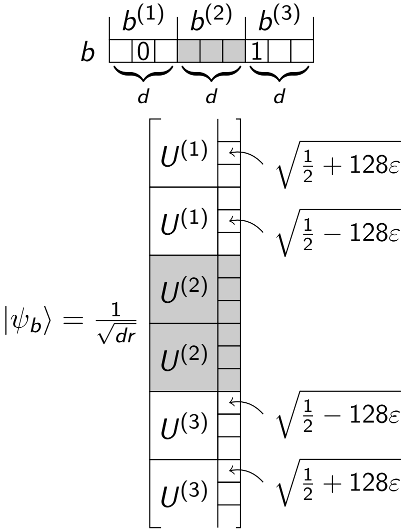
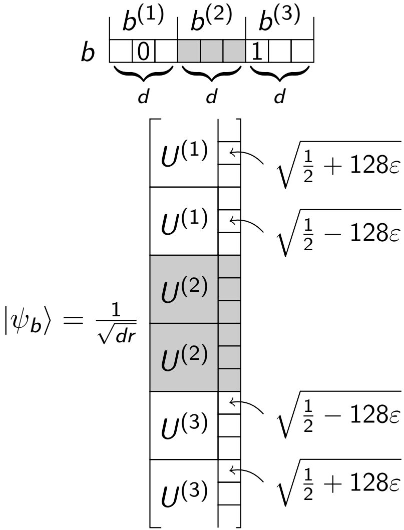


Lower bound – embedding

For a single bit $b \in \{0,1\}$, we can perform:

1. Let ξ, ε' to be fixed later.
2. Start with state
$$\frac{1}{\sqrt{2}}\begin{bmatrix} e^{\pi i(\xi+\varepsilon')} \\ e^{-\pi i(\xi+\varepsilon')} \end{bmatrix}$$



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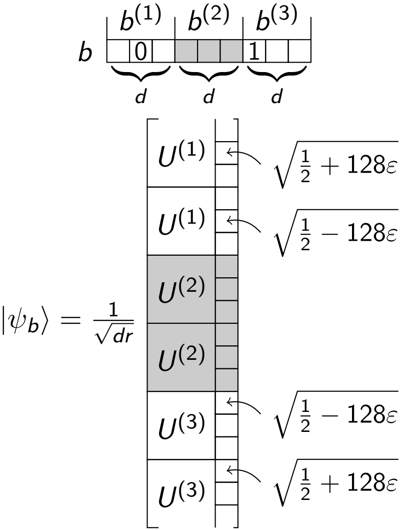

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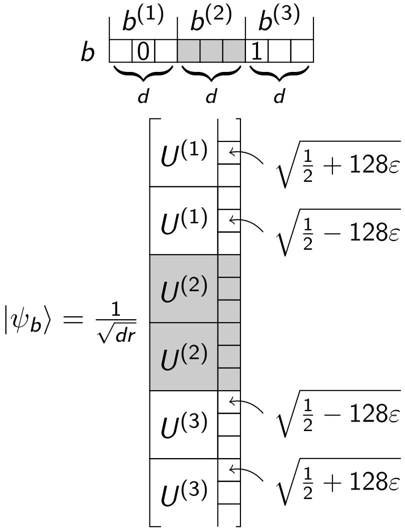
4. Apply H and then S
$$\begin{bmatrix} \cos(\pi(\xi+(-1)^b\varepsilon')) \\ \sin(\pi(\xi+(-1)^b\varepsilon')) \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{1}{2}+(-1)^b 128\varepsilon} \\ \sqrt{\frac{1}{2}-(-1)^b 128\varepsilon} \end{bmatrix}$$



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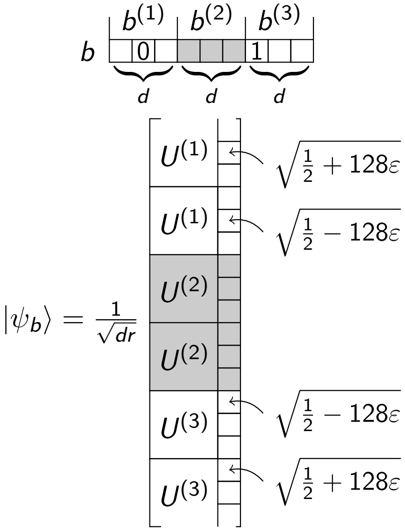
5. Solve for ξ, ε', then $\varepsilon' = \Theta(\varepsilon)$.



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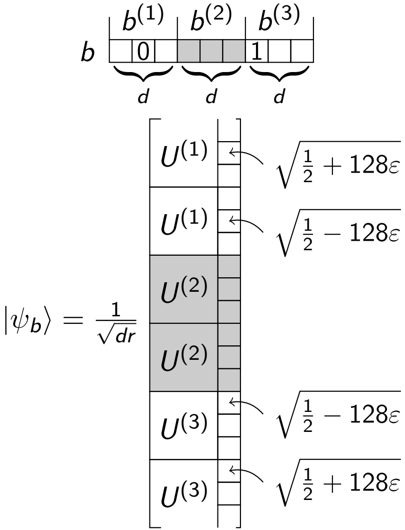
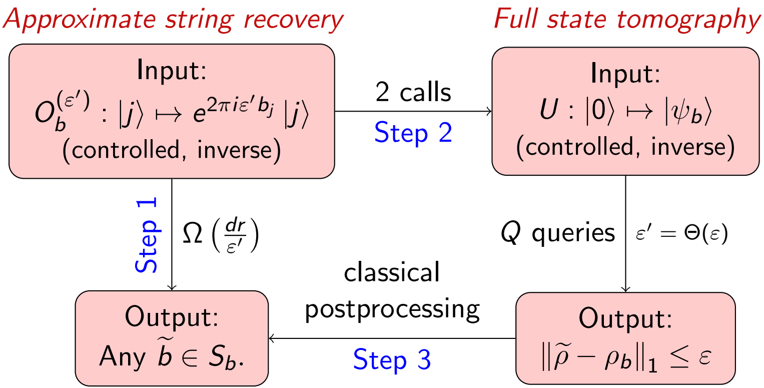
Now perform in parallel to obtain $|\psi_b\rangle$.



Idea: "Embed the approximate string recovery problem into the full state-tomography problem."



Lower bound – postprocessing

Lower bound – postprocessing

1. Postprocessing algorithm:
   1. Run algorithm to obtain $\widetilde{\rho}$ s.t. $\|\widetilde{\rho} - \rho_b\|_1 \leq \varepsilon$.
   2. Output any $\widetilde{b} \in \{0,1\}^{dr}$ such that $\left\|\widetilde{\rho} - \rho_{\widetilde{b}}\right\|_1 \leq \varepsilon$.

   It follows that $\left\|\rho_b - \rho_{\widetilde{b}}\right\|_1 \leq 2\varepsilon$.

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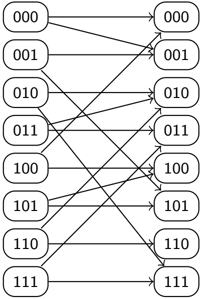
2. Thus, let $S_b = \{\widetilde{b} \in \{0,1\}^{dr} : \left\|\rho_b - \rho_{\widetilde{b}}\right\|_1 \leq 2\varepsilon\}$.

Input: $b \in \{0,1\}^{dr}$ Output: $\widetilde{b} \in \{0,1\}^{dr}$



Edge when $\left\|\rho_b - \rho_{\widetilde{b}}\right\|_1 \leq 2\varepsilon$.

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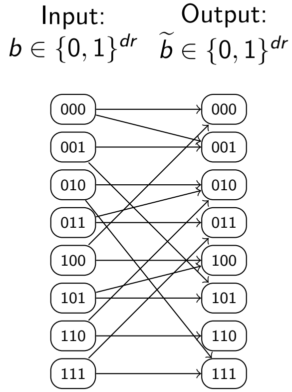
3. Then $b \in S_b$.
   (horizontal edges)



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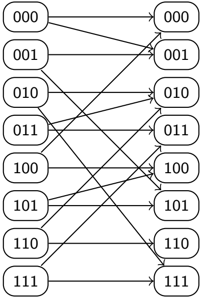
3. Then $b \in S_b$.
   (horizontal edges)
4. Remains to show that $\exists c \in (0,1)$ s.t.
   $|\{b \in \{0,1\}^{dr} : \widetilde{b} \in S_b\}| \leq 2^{cdr}$.
   (bounded right degree)

Input: $b \in \{0,1\}^{dr}$ Output: $\widetilde{b} \in \{0,1\}^{dr}$



Edge when $\left\|\rho_b - \rho_{\widetilde{b}}\right\|_1 \leq 2\varepsilon$.

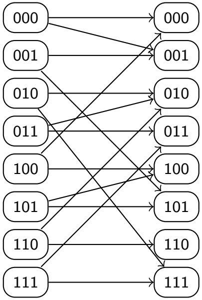
Lower bound – postprocessing

Let $b, \widetilde{b} \in \{0,1\}^{dr}$ uniformly at random.

Suppose that $\exists c \in (0,1)$ s.t.

$$\mathbb{P}\left[\left\|\rho_b - \rho_{\widetilde{b}}\right\|_1 \leq 2\varepsilon\right] \leq 2^{-cdr}.$$

Input: $b \in \{0,1\}^{dr}$ Output: $\widetilde{b} \in \{0,1\}^{dr}$



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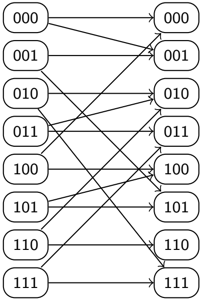

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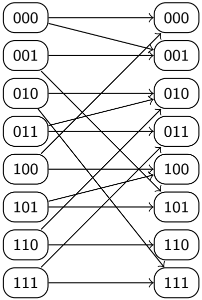
2. $\sum \deg(\widetilde{b}) \leq 2^{-cdr} \cdot 2^{2dr}$.

Input: $b \in \{0,1\}^{dr}$ Output: $\widetilde{b} \in \{0,1\}^{dr}$



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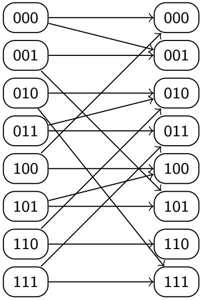

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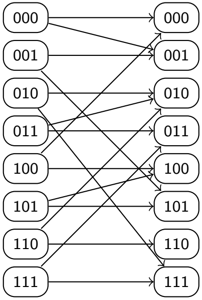

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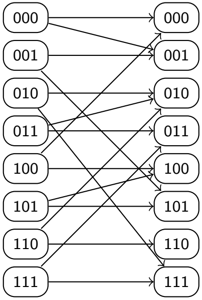
5. Let $D \subseteq \{0,1\}^{dr}$ be the set for all these $\widetilde{b}$'s: $|D| \geq C \cdot 2^{dr}$, with $C \in (0,1)$.

Input: $b \in \{0,1\}^{dr}$ Output: $\widetilde{b} \in \{0,1\}^{dr}$



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Thus it remains to show $\exists c \in (0,1)$ s.t.

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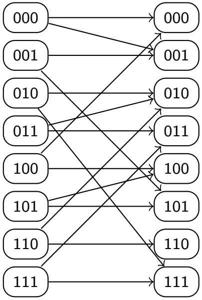
Input: $b \in \{0,1\}^{dr}$ Output: $\widetilde{b} \in \{0,1\}^{dr}$



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Lower bound – proximity probability analysis

Remains to prove: $\exists c \in (0,1)$ s.t.
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Lower bound – proximity probability analysis

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Approximation: $\sqrt{\frac{1}{2} \pm 128\varepsilon} \approx \frac{1}{\sqrt{2}}(1 \pm 128\varepsilon)$.

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1. *Total error:* $2(256\varepsilon)^2$.
2. *Simplification:*
   $\left\|\rho_b - \rho_{\widetilde{b}}\right\|_1 = \frac{32\varepsilon}{rd}\left\|XY^\dagger\right\|_1$, where
   1. $X = [U^{(1)}\delta^{(1)} \ \cdots \ U^{(r)}\delta^{(r)}]$,
   2. $Y = [U^{(1)}\mathbb{1} \ \cdots \ U^{(r)}\mathbb{1}]$,
   3. $\delta^{(j)} = (-1)^{b^{(j)}} - (-1)^{\widetilde{b}^{(j)}}$.

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3. *Interpretation:* overlaying point clouds.

$$\left\|XY^\dagger\right\|_1 = \max_{U \text{ unitary}} \left|\mathrm{Tr}[Y^\dagger UX]\right| = \max_{U \text{ unitary}} \sum_{j=1}^r |y_j^\dagger U x_j|.$$

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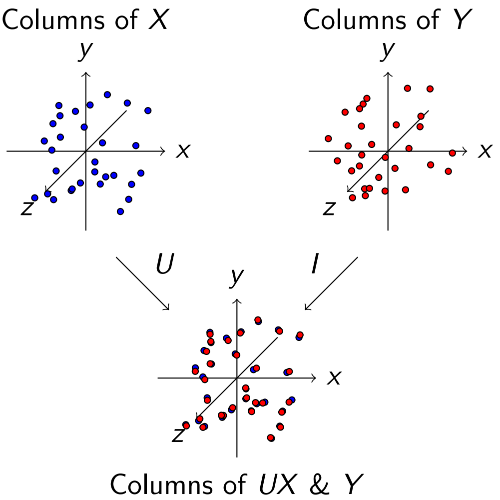

Lower bound – proximity probability analysis II

Recall: $Y = \begin{bmatrix} U^{(1)}\mathbb{1} & \cdots U^{(r)}\mathbb{1} \end{bmatrix}$.

Lower bound – proximity probability analysis II

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Idea: Let

$$U^{(1)} = \frac{1}{\sqrt{d}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_d & \omega_d^2 & \cdots & \omega_d^{d-1} \\ 1 & \omega_d^2 & \omega_d^4 & \cdots & \omega_d^{2(d-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_d^{d-1} & \omega_d^{2(d-1)} & \cdots & \omega_d^{(d-1)^2} \end{bmatrix}$$

$$U^{(2)} = \frac{1}{\sqrt{d}} \begin{bmatrix} 1 & \omega_d^{d-1} & \omega_d^{2(d-1)} & \cdots & \omega_d^{(d-1)^2} \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_d & \omega_d^2 & \cdots & \omega_d^{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_d^{d-2} & \omega_d^{2(d-2)} & \cdots & \omega_d^{(d-2)(d-1)} \end{bmatrix}$$

Then $Y = \sqrt{d}I$.

Lower bound – proximity probability analysis II

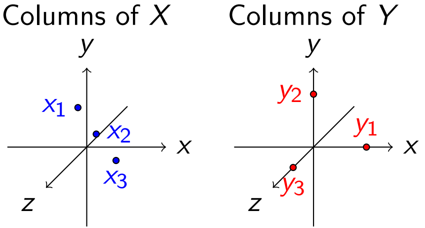
Recall: $Y = \begin{bmatrix} U^{(1)}\mathbb{1} & \cdots U^{(r)}\mathbb{1} \end{bmatrix}$.

Idea: Let

$$U^{(1)} = \frac{1}{\sqrt{d}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_d & \omega_d^2 & \cdots & \omega_d^{d-1} \\ 1 & \omega_d^2 & \omega_d^4 & \cdots & \omega_d^{2(d-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_d^{d-1} & \omega_d^{2(d-1)} & \cdots & \omega_d^{(d-1)^2} \end{bmatrix}$$

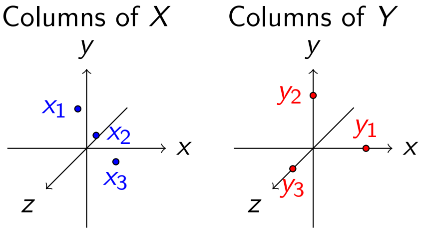
$$U^{(2)} = \frac{1}{\sqrt{d}} \begin{bmatrix} 1 & \omega_d^{d-1} & \omega_d^{2(d-1)} & \cdots & \omega_d^{(d-1)^2} \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_d & \omega_d^2 & \cdots & \omega_d^{d-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_d^{d-2} & \omega_d^{2(d-2)} & \cdots & \omega_d^{(d-2)(d-1)} \end{bmatrix}$$

Then $Y = \sqrt{d} I$.



Lower bound – proximity probability analysis II

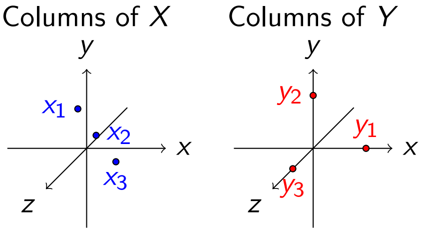
Recall $X = \begin{bmatrix} U^{(1)}\delta^{(1)} & \cdots & U^{(r)}\delta^{(r)} \end{bmatrix}$.



Lower bound – proximity probability analysis II

Recall $X = \begin{bmatrix} U^{(1)}\delta^{(1)} & \cdots & U^{(r)}\delta^{(r)} \end{bmatrix}$.

1. For any unitary U:
   $\left\|\rho_b - \rho_{\tilde{b}}\right\|_1 \geq \frac{32\varepsilon}{r\sqrt{d}} \sum_{j=1}^r |e_j^\dagger U x_j|$.



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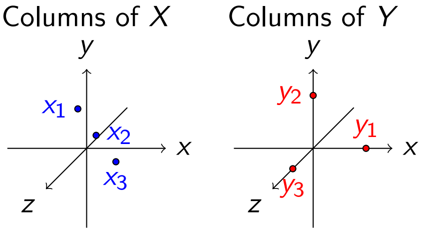

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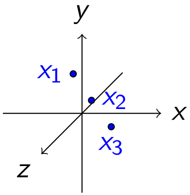
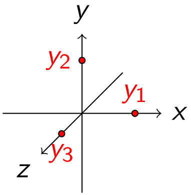
3. Let $S_j = \mathrm{Span}\{x_1, \dots, x_{j-1}\}$. Then:
   1. $\dim(S_j) = j - 1$.
   2. $|e_j^\dagger U x_j| = \left\|\Pi_{S_j^\perp} x_j\right\|$.
   3. S_j is independent from x_j.

Columns of X



Columns of Y



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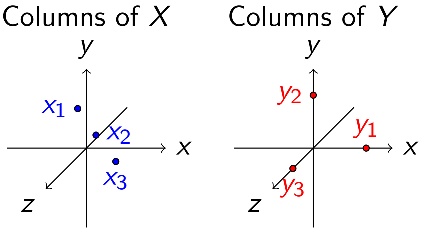
   3. S_j is independent from x_j.
4. Since $\delta^{(j)}$ has independent entries, $\exists c \in (0,1)$ s.t.
   $\mathbb{P}\left[\left|\left\|\Pi_{S_j^\perp} x_j\right\| - \sqrt{d-j+1}\right| \geq \frac{1}{4}\sqrt{d}\right] \leq 2^{-cd}$.[RV'13]



Lower bound – proximity probability analysis II

Recall $X = \begin{bmatrix} U^{(1)}\delta^{(1)} & \cdots & U^{(r)}\delta^{(r)} \end{bmatrix}$.

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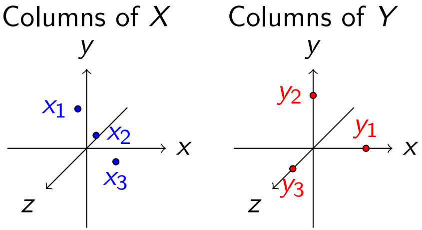
5. If $\left\|\rho_b - \rho_{\tilde{b}}\right\|_1 \leq 2\varepsilon$, then the above must hold for at least $r/4$ terms.



Lower bound – proximity probability analysis II

Recall $X = \begin{bmatrix} U^{(1)}\delta^{(1)} & \cdots & U^{(r)}\delta^{(r)} \end{bmatrix}$.

1. For any unitary U:
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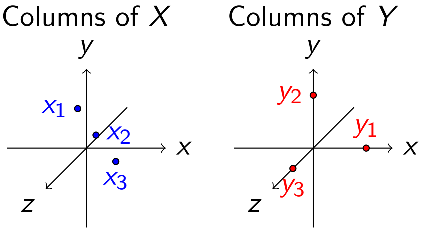
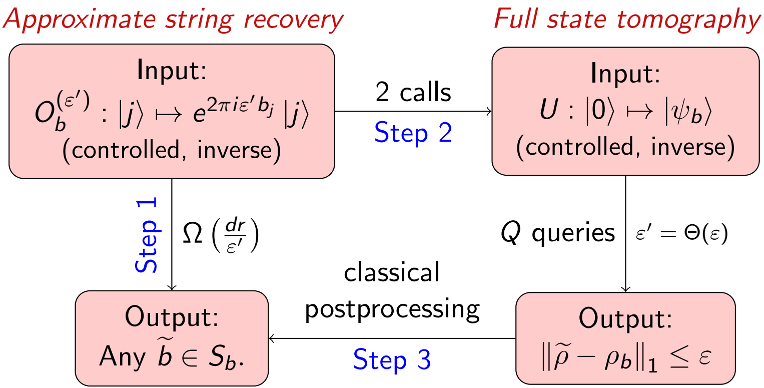
5. If $\left\|\rho_b - \rho_{\widetilde{b}}\right\|_1 \leq 2\varepsilon$, then the above must hold for at least $r/4$ terms.
6. Thus $\mathbb{P}\left[\left\|\rho_b - \rho_{\widetilde{b}}\right\|_1 \leq 2\varepsilon\right] \leq 2^{-cdr/4}$.



Lower bound – proof overview

Idea: "Embed the approximate string recovery problem into the full state-tomography problem."



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