# Quantum tomography using state-preparation unitaries

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"Quantum state tomography is learning a classical description of a quantum state"

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Model	Output	
$-\ket{\psi}$ $ j$	$[ \alpha_j ]_{j=1}^d$	
$ \ket{\psi}$	$e^{i\chi}\ket{\psi}$	
$rac{ 0 angle 0 angle+ 1 angle \psi angle}{\sqrt{2}}$	$ \psi angle$	

#### "Quantum state tomography is learning a classical description of a quantum state"

$$|\psi\rangle = \sum_{j=1}^d \alpha_j |j\rangle \in \mathbb{C}^d$$

\*Tildes hide polylogarithmic factors in d,  $1/\varepsilon$ .

		Approximation ( $\ell_q$ -norms)			
Model	Output	$\ \cdot\ _{\infty} \leq \varepsilon$	$\left\  \cdot \right\ _2 \leq \varepsilon$	$\left\ \cdot\right\ _q \leq \varepsilon, \ q \in [2, \infty]$	
$-\ket{\psi}$ $ j$	$[ \alpha_j ]_{j=1}^d$				
$ \ket{\psi}$	$e^{i\chi}\ket{\psi}$	$\widetilde{\mathcal{O}}\left(rac{1}{arepsilon^2} ight) [ ext{KP20}] \ \Omega\left(rac{1}{arepsilon^2} ight)$	$\widetilde{\mathcal{O}}\left(rac{d}{arepsilon^2} ight) [ ext{KP20}] \ \Omega\left(rac{d}{arepsilon^2} ight)$	$\widetilde{\Theta}\left(\min\left\{rac{1}{rac{1}{arepsilon^{rac{1}{2}-rac{1}{q}}},rac{d^{rac{2}{q}}}{arepsilon^{2}} ight\} ight)$	
$\frac{\ket{0}\ket{0}+\ket{1}\ket{\psi}}{\sqrt{2}}$	$ \psi angle$	$(\varepsilon^2)$	$\frac{12}{\varepsilon^2}$	$\left(\begin{array}{cc} \left(\varepsilon^{\frac{7}{2}-q}\right)\right)$	

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$-\ket{\psi}$ $ j$	$[ \alpha_j ]_{j=1}^d$	~	~	/ / /	
$ \ket{\psi}$	$e^{i\chi}\ket{\psi}$	$\begin{array}{c} \widetilde{\mathcal{O}}\left(\frac{1}{\varepsilon^2}\right) [\text{KP20}] \\ \Omega\left(\frac{1}{\varepsilon^2}\right) \end{array}$	$\widetilde{\mathcal{O}}\left(rac{d}{arepsilon^2} ight) [ ext{KP20}] \ \Omega\left(rac{d}{arepsilon^2} ight)$	$\widetilde{\Theta}\left(\min\left\{rac{1}{arepsilon^{rac{1}{2}-rac{1}{q}}},rac{d^{rac{2}{q}}}{arepsilon^{2}} ight\} ight)$	
$\frac{\ket{0}\ket{0}+\ket{1}\ket{\psi}}{\sqrt{2}}$	$ \psi angle$	(ε²)	(ε²)	\ \(\(\epsilon^2 q \)\	
$ 0\rangle$ $\overline{-U}$ $ \psi\rangle$ , $ \psi\rangle$ $\overline{-U^{\dagger}}$ $ 0\rangle$	$ \psi angle$	$\widetilde{\Theta}\left(\min\left\{\frac{1}{\varepsilon^2}, \frac{\sqrt{d}}{\varepsilon}\right\}\right)$	$\widetilde{\Theta}\left(\frac{d}{arepsilon} ight)$	$\widetilde{\Theta}\left(\min\left\{\frac{1}{\frac{1}{\varepsilon^{\frac{1}{2}-\frac{1}{q}}}},\frac{d^{\frac{1}{2}+\frac{1}{q}}}{\varepsilon}\right\}\right)$	

# Quantum state tomography (2/2) – mixed states

$$ho = \sum_{j=1}^{r} p_j |\psi_j\rangle\langle\psi_j| \in \mathbb{C}^{d \times d}$$

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$$\rho = \sum_{j=1}^{r} p_j |\psi_j\rangle\langle\psi_j| \in \mathbb{C}^{d\times d}$$

\*Tildes hide polylogarithmic factors in d, r,  $1/\varepsilon$ .

Model	Output				ns) $\left\  \cdot  ight\ _{a} \leq arepsilon$
- ρ	ρ	-	_	$O\left(rac{dr^2}{arepsilon^2} ight)  ext{[GLF+10]}  onumber  Onumber \Omega\left(rac{dr^2}{arepsilon^2} ight)  ext{[HHJ+17; CHL+22]}$	<u> </u>
_ ρ _ ρ	ho	_	-	$\widetilde{\mathcal{O}}\left(\frac{dr}{arepsilon^2}\right)$ [OW16; HHJ+17] $\Omega\left(\frac{dr}{arepsilon^2}\right)$ [HHJ+17; Yue22]	-

# Quantum state tomography (2/2) – mixed states

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,		Approximation (Schatten norms)			
Model	Output	$\ \cdot\ _{\infty} \leq \varepsilon$	$\left\  \cdot \right\ _2 \leq \varepsilon$	$\left\  \cdot \right\ _1 \leq \varepsilon$	$\left\ \cdot\right\ _{q}\leq arepsilon$
- ρ	ρ	_	- Ω (	$\mathcal{O}\left(rac{dr^2}{arepsilon^2} ight)$ [GLF+ $\left(rac{dr^2}{arepsilon^2} ight)$ [HHJ+17; $0$	+10]
ρ _ ρ	ρ	_	- Ĉ	$\widetilde{O}\left(\frac{dr}{arepsilon^2}\right)$ [OW16; HE $O\left(\frac{dr}{arepsilon^2}\right)$ [HHJ+17;	HJ+17] Yue22]
$ \begin{vmatrix} 0 \rangle_{A} \\ 0 \rangle_{B} \end{vmatrix} = \begin{vmatrix} \psi \rangle_{AB} $ $ \begin{vmatrix} 0 \rangle_{A} \\ 0 \rangle_{B} \end{vmatrix} = \begin{vmatrix} \psi \rangle_{AB} $	$Tr_{\mathcal{B}}[ \psi\rangle\!\langle \psi ]$	$\widetilde{\Theta}(\frac{d}{\varepsilon})$ $\widetilde{\Theta}$	$\left(\min\left\{rac{d\sqrt{r}}{arepsilon},rac{d}{arepsilon^2} ight\}$	$\Theta\left(\frac{dr}{\varepsilon}\right)$	$\widetilde{\Theta}\left(\min\left\{\frac{dr^{\frac{1}{q}}}{\varepsilon}, \frac{d}{\frac{1}{\varepsilon^{1-\frac{1}{q}}}}\right\}\right)$

 $O_1, \ldots, O_M$  observables, with  $||O_i|| \leq 1$ 

 $O_1, \ldots, O_M$  observables, with  $||O_j|| \leq 1$ 

Model	Output	Complexity	
1.7	54.1.5.1.3344	$\mathcal{O}\left(\frac{\log(M)}{arepsilon^2} ight)$	If the observables commute
$-\ket{\psi}$	$[\langle \psi   O_j   \psi \rangle]_{j=1}^M$	$\mathcal{O}\left(\frac{\log(M)}{\varepsilon^2}\right)$	Shadow tomography [HKP20]

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$-\ket{\psi}$	$[\langle \psi   O_j   \psi \rangle]_{j=1}^M$	$\mathcal{O}\left(\frac{\log(M)}{\varepsilon^2}\right)$	Shadow tomography [HKP20]
$ 0 angle \stackrel{\longleftarrow}{-U}  \psi angle$	$[\langle \psi   O_j   \psi \rangle]_{j=1}^M$	$\widetilde{\mathcal{O}}\left(\frac{\sqrt{\sum_{j=1}^{M}\ O_j\ ^2}}{\varepsilon}\right)$	[HWC+22]
$ \psi angle \stackrel{-}{-U^{\dagger}}  0 angle$	$[\langle \psi   \mathcal{O}_j   \psi \rangle]_{j=1}$	$\widetilde{\mathcal{O}}\left(\frac{\sqrt{\left\ \sum_{j=1}^{M}O_{j}^{2}\right\ }}{\varepsilon}\right)$	$f(\mathbf{x}) = \mathbf{x}^T [\langle \psi   \ O_j \   \psi  angle]_{j=1}^M$ Compute $ abla_{\mathbf{x}} f$

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$ \psi angle \stackrel{\longleftarrow}{-U^{\dagger}} -  0 angle$	$[\langle \psi   O_j   \psi \rangle]_{j=1}^{\infty}$	$\left  \widetilde{\mathcal{O}} \left( \frac{\sqrt{\left\  \sum_{j=1}^{M} O_{j}^{2} \right\ }}{\varepsilon} \right) \right $	$f(\mathbf{x}) = \mathbf{x}^T [\langle \psi   \ O_j \   \psi  angle]_{j=1}^M$ Compute $ abla_{\mathbf{x}} f$

#### Density matrix:

$$\rho = i \begin{bmatrix} | & | \\ -\rho_{ij} & - | \\ | & | \end{bmatrix}$$



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Model	Output	Complexity	
1.7	54.11.5.11.714	$\mathcal{O}\left(\frac{\log(M)}{\varepsilon^2}\right)$	If the observables commute
$-\ket{\psi}$	$[\langle \psi   O_j   \psi \rangle]_{j=1}^M$	$\mathcal{O}\left(\frac{\log(M)}{\varepsilon^2}\right)$	Shadow tomography [HKP20]
$ 0 angle \stackrel{\longleftarrow}{-U} -  \psi angle$	$[/2/1] O \cdot  2/1 M$	$\widetilde{\mathcal{O}}\left(rac{\sqrt{\sum_{j=1}^{M} \lVert O_{j}  Vert^{2}}}{arepsilon} ight)$	[HWC+22]
$ \psi angle \stackrel{\longleftarrow}{-U^{\dagger}-} 0 angle$	$\left[ \langle \psi   O_j   \psi \rangle \right]_{j=1}^M$	$\left  \widetilde{\mathcal{O}} \left( \frac{\sqrt{\left\  \sum_{j=1}^{M} O_{j}^{2} \right\ }}{\varepsilon} \right) \right $	$f(\mathbf{x}) = \mathbf{x}^T [\langle \psi   \ O_j \   \psi  angle]_{j=1}^M$ Compute $ abla_{\mathbf{x}} f$

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#### Observables:

$$\rho = i \begin{bmatrix} | & Observables. \\ | & O_{ij}^{+} = \frac{|i\rangle\langle j| + |j\rangle\langle i|}{2} \\ | & O_{ij}^{-} = \frac{|i\rangle\langle j| - |j\rangle\langle i|}{2i} \\ | & \langle \psi | O_{ij}^{+} | \psi \rangle = \operatorname{Re}[\rho_{ij}] \\ | & \langle \psi | O_{ij}^{-} | \psi \rangle = \operatorname{Im}[\rho_{ij}] \end{bmatrix}$$

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$ 0 angle - U -  \psi angle$	$[\langle \psi   O_j   \psi \rangle]_{j=1}^M$	$\widetilde{\mathcal{O}}\left(rac{\sqrt{\sum_{j=1}^{M}\ O_j\ ^2}}{arepsilon} ight)$	[HWC+22]
$ \psi angle \ \overline{-U^\dagger-} \  0 angle$	$[ [ [ \psi ] \ \mathcal{O}_{J} \ ] \psi / ]_{j=1}$	$\left  \widetilde{\mathcal{O}} \left\langle \frac{\sqrt{\left\  \sum_{j=1}^{M} O_{j}^{2} \right\ }}{\varepsilon} \right\rangle \right $	$f(\mathbf{x}) = \mathbf{x}^T [\langle \psi   \ O_j \   \psi  angle]_{j=1}^M$ Compute $ abla_{\mathbf{x}} f$

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#### Norm bound:

$$\rho = i \begin{bmatrix} | & O_{ij}^{+} = \frac{|i\rangle\langle j| + |j\rangle\langle i|}{2} & \sum_{\substack{i \leq j \\ i \neq j}}^{d} (O_{ij}^{+})^{2} \\ \langle \psi | O_{ij}^{-} | \psi \rangle = \operatorname{Re}[\rho_{ij}] & + \sum_{\substack{i \leq j \\ i \neq j}}^{d} (O_{ij}^{+})^{2} = dI_{d} \end{bmatrix}$$



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$$\rho = \int_{i} \begin{bmatrix} 1 \\ -\rho_{ij} \end{bmatrix} \qquad \begin{array}{c} O_{ij}^{+} = \frac{|i\rangle\langle j| + |j\rangle\langle i|}{2} \\ O_{ij}^{-} = \frac{|i\rangle\langle j| - |j\rangle\langle i|}{2i} \\ \langle \psi | O_{ij}^{+} | \psi \rangle = \operatorname{Re}[\rho_{ij}] \\ \langle \psi | O_{ij}^{-} | \psi \rangle = \operatorname{Im}[\rho_{ij}] \end{array} \qquad \begin{array}{c} \sum_{\substack{i \leq j \\ i,j=1}}^{d} (O_{ij}^{+})^{2} \\ + \sum_{\substack{i \leq j \\ i,j=1}}^{d} (O_{ij}^{-})^{2} = dI_{d} \end{array} \qquad \begin{array}{c} \|\widetilde{\rho} - \rho\|_{\max} \leq \varepsilon \text{ costs} \\ \widetilde{\mathcal{O}}(\frac{\sqrt{d}}{\varepsilon}) \text{ queries.} \end{array}$$

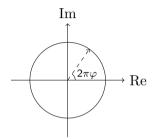
#### Result:

$$\|\widetilde{
ho} - 
ho\|_{\max} \leq arepsilon \ \operatorname{costs} \ \widetilde{\mathcal{O}}(rac{\sqrt{d}}{arepsilon}) \ \operatorname{queries}.$$

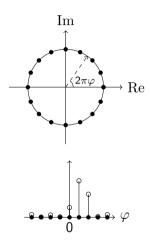


#### Phase estimation:

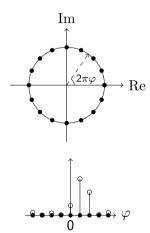
• Given a copy of  $|\psi\rangle$ , and U s.t.  $U|\psi\rangle = e^{2\pi i \varphi} |\psi\rangle$ , determine  $\varphi$ .



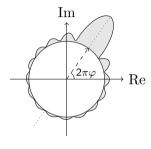
- Given a copy of  $|\psi\rangle$ , and U s.t.  $U|\psi\rangle = e^{2\pi i \varphi} |\psi\rangle$ , determine  $\varphi$ .
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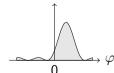


- Given a copy of  $|\psi\rangle$ , and U s.t.  $U|\psi\rangle = e^{2\pi i \varphi} |\psi\rangle$ , determine  $\varphi$ .
- Standard approach: finite outcome set.
- Symmetrization [LdW21]:
  - Let  $\theta \in [0,1)$  unif. at random.
  - Run PE with  $e^{2\pi i\theta}U$ .
  - Correct for choice of  $\theta$ .

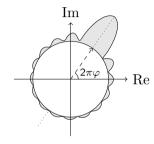


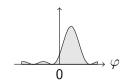
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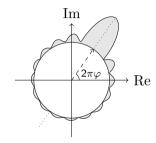


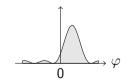
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- Exponentially small bias.



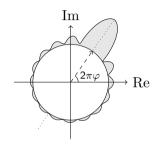


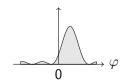
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#### Result:

$$\|\widetilde{\rho} - \rho\|_{\max} \leq \varepsilon \quad \stackrel{[\mathsf{RV}10]}{\Rightarrow} \quad \|\widetilde{\rho} - \rho\|_{\infty} \leq \sqrt{d}\varepsilon$$





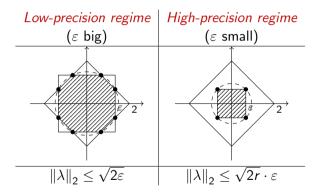
- Two bounds:
  - $\|\widetilde{\rho} \rho\|_{\infty} \le \varepsilon$ .
  - $\|\widetilde{\rho} \rho\|_1^{\infty} \le \|\widetilde{\rho}\|_1 + \|\rho\|_1 = 2.$

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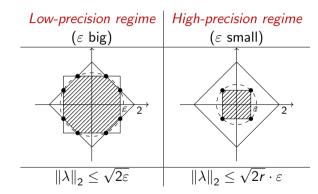
Low-precision regime	High-precision regime
( $arepsilon$ big)	$(arepsilon \ small)$
	₹ 2 ·
	$\ \lambda\ _2 \le \sqrt{2r} \cdot \varepsilon$

- Two bounds:
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  - $\|\widetilde{\rho} \rho\|_1 \le \|\widetilde{\rho}\|_1 + \|\rho\|_1 = 2$ .
- Let  $\sigma(\widetilde{\rho} \rho) = \{\lambda_j\}_{j=1}^{2r}$ , then  $\|\widetilde{\rho} - \rho\|_{a} = \|\lambda\|_{a}$ .

$$\textit{Result:} \quad \|\widetilde{\rho} - \rho\|_q \lesssim \min \Big\{ \varepsilon^{1 - \frac{1}{q}}, r^{\frac{1}{q}} \varepsilon \Big\}.$$

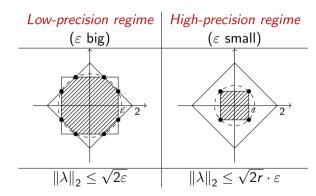


### Setting:

- Two bounds:
  - $\|\widetilde{\rho} \rho\|_{\infty} \le \varepsilon$ .
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$$\textit{Result:} \quad \|\widetilde{\rho} - \rho\|_{q} \lesssim \min \Big\{ \varepsilon^{1 - \frac{1}{q}}, r^{\frac{1}{q}} \varepsilon \Big\}.$$

• Source of all regimes.



### Setting:

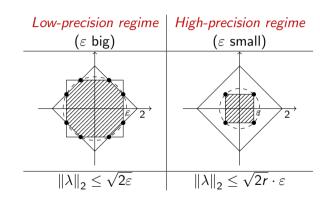
- Two bounds:
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• 
$$\|\widetilde{\rho} - \rho\|_1 \le \|\widetilde{\rho}\|_1 + \|\rho\|_1 = 2.$$

• Let  $\sigma(\widetilde{\rho} - \rho) = \{\lambda_j\}_{j=1}^{2r}$ , then  $\|\widetilde{\rho} - \rho\|_{a} = \|\lambda\|_{a}$ .

$$\textit{Result:} \quad \|\widetilde{\rho} - \rho\|_q \lesssim \min \Big\{ \varepsilon^{1 - \frac{1}{q}}, r^{\frac{1}{q}} \varepsilon \Big\}.$$

- Source of all regimes.
- Also relevant for lower bound proofs.



$$\begin{array}{c|c} |0\rangle_A & \hline \\ |0\rangle_B & \hline \end{array} |\psi\rangle_{AB} \quad \begin{array}{c|c} |0\rangle_A & \hline \\ |0\rangle_B & \hline \end{array} |\psi\rangle_{AB}$$

Want to learn:  $\rho = \operatorname{Tr}_B[|\psi\rangle\langle\psi|]$ 

$$\begin{array}{c|c} |0\rangle_{A} & \hline U & |\psi\rangle_{AB} & |0\rangle_{A} & \hline U^{\dagger} & |\psi\rangle_{AB} \\ \downarrow & & Learning \ observables \\ \|\widetilde{\rho} - \rho\|_{\max} \leq \varepsilon' \end{array}$$

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$$ho={\rm Tr}_B[|\psi\rangle\!\langle\psi|]$$
  $\widetilde{\mathcal{O}}(rac{\sqrt{d}}{\varepsilon'})$  queries

$$\begin{array}{c|c} |0\rangle_{A} & & & |0\rangle_{A} & |0\rangle_{A} \\ |0\rangle_{B} & & |0\rangle_{B} & |0\rangle_{B} \\ & \downarrow & & \text{Learning observables} \\ \|\widetilde{\rho} - \rho\|_{\text{max}} \leq \varepsilon' & & \downarrow & \text{Unbiased phase estimation} \\ \|\widetilde{\rho} - \rho\|_{\infty} \lesssim \sqrt{d}\varepsilon' & & \end{array}$$

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Want to learn: 
$$\rho = \mathrm{Tr}_B[|\psi\rangle\!\langle\psi|]$$
  $\widetilde{\mathcal{O}}(\frac{\sqrt{d}}{\varepsilon^L})$  queries

$$\begin{aligned} &|0\rangle_{A} \\ &|0\rangle_{B} \end{aligned} \qquad |\psi\rangle_{AB} \quad |0\rangle_{A} \\ &|0\rangle_{B} \end{aligned} \qquad |\psi\rangle_{AB} \\ &|\psi\rangle_{AB} \quad |\psi\rangle_{AB} \end{aligned}$$
 
$$|\widetilde{\rho} - \rho|_{\max} \leq \varepsilon'$$
 
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Want to learn: 
$$\rho = \text{Tr}_B[|\psi\rangle\langle\psi|]$$

$$\widetilde{\mathcal{O}}(rac{\sqrt{d}}{arepsilon'})$$
 queries

$$\Rightarrow \widetilde{\mathcal{O}}\left(\min\left\{\frac{\frac{d}{1-\frac{1}{q}},\frac{dr^{\frac{1}{q}}}{\varepsilon}\right\}\right)$$
 queries.

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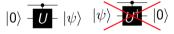
#### Open problems:

Sampling complexities in other Schatten norms.

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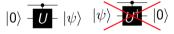
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Thanks for your attention! arjan@cwi.nl

