EE 25088 Convex Optimization II

Problem Set III

Due Date: Khordad 10, 1401

1) Derive the Lagrange dual of the optimization problem

$$\begin{aligned} \underset{\boldsymbol{x}\in\mathbf{R}^n}{\text{minimize}} \quad & \sum_{i=1}^{n} \phi(x_i) \\ \text{subject to} \quad & A\boldsymbol{x} = \boldsymbol{b}, \end{aligned}$$

where $\phi(x) = \frac{|x|}{c-|x|}$ and $\mathbf{dom}\phi = (-c, c)$. while c is a positive parameter.

2) *Step sizes that guarantee moving closer to the optimal set.* Consider the subgradient method iteration $x_{k+1} = x_k - \alpha g$, where $g \in \partial f(x)$. Show that if $\alpha < 2(f(x) - f^*)/||g||_2^2$, we have

$$||x_{k+1} - x^*||_2 < ||x_k - x^*||_2, \tag{2}$$

for any optimal point x^*.

3) *Numerical perturbation analysis.* Consider the following problem

$$\begin{aligned} \text{minimize} \quad & x_1^2 + 2x_2^2 - x_1x_2 - x_1 \\ \text{subject to} \quad & x_1 + 2x_2 \le u_1 \\ & x_1 - 4x_2 \le u_2 \\ & 5x_1 + 76x_2 \le 1 \end{aligned}$$

with variables x_1, x_2, and parameters u_1, u_2.

a) Solve this problem, for parameter values $u_1 = -2$, $u_2 = -3$, to find optimal primal variable values x_1^* and x_2^*, and optimal dual variable values λ_1^*, λ_2^*, and λ_3^*. Let p^* denote the optimal objective value. Verify that the KKT conditions hold for the optimal primal and dual variables you found. (You need to find the answers via `CVX` and also verify your results using KKT conditions).

Hint: To specify the quadratic-from functions, you can use `quad_form()`

δ_1	δ_2	$p^\star_{\text{pred}}$	$p^\star_{\text{exact}}$
0	0		
0	-0.1		
0	0.1		
-0.1	0		
-0.1	-0.1		
-0.1	0.1		
0.1	0		
0.1	-0.1		
0.1	0.1		

Table I: Sample Table for problem 3

b) We will now solve some perturbed versions of this problem, with

$$u_1 = -2 + \delta_1, u_2 = -3 + \delta_2$$

where δ_1 and δ_2 each take values from $\{-0.1, 0, 0.1\}$. (There are a total of nine such combinations, including the original problem with $\delta_1 = \delta_2 = 0$.) For each combination of δ_1 and δ_2, make a prediction p^*_{pred} of the optimal value of the perturbed QP, and compare it to p^* exact, the exact optimal value of the perturbed problem. Put your results in the two righthand columns in a table with the form shown below. Check that the inequality $p^*_{\text{pred}} \leq p^*_{\text{exact}}$ holds. Express your insights about this problem.

4) *When strong duality fails!* Consider the optimization problem

$$\begin{aligned} \text{minimize} \quad & e^{-x} \\ \text{subject to} \quad & x^2/y \leq 0 \end{aligned}$$

with variables x and y, and domain $\mathcal{D} = \{(x, y) | y > 0\}$.

a) Verify that this is a convex optimization problem. Then find the optimal value and verify your answers using `CVX`.
b) Derive the Lagrange dual problem, and find the optimal solution λ^*. Also find the optimal value d^* of the dual problem. What is the optimal duality gap? Verify your answers using `CVX`.
c) Does Slater's condition hold for this problem?

d) *(Bonus) What is the optimal value $p^*(u)$ of the perturbed problem

$$\begin{array}{ll} \text{minimize} & e^{-x} \\ \text{subject to} & x^2/y \leq u \end{array}$$

as a function of u? Verify that the following inequality does not hold.

$$p^*(u) \geq p^*(0) - \lambda^* u.$$

5) *A distributed approach to the bi-commodity network flow problem!* Consider a network, i.e., a directed graph, with n arcs and p nodes, described by the incidence matrix $A \in \mathbf{R}^{p \times n}$, where

$$A_{ij} = \begin{cases} 1, & \text{if arc } j \text{ enters node } i \\ -1 & \text{if arc } j \text{ leaves node } i \\ 0 & \text{otherwise} \end{cases}$$

Two commodities flow in the network. Commodity 1 has source vector $s \in \mathbf{R}^p$, and commodity 2 has source vector $t \in \mathbf{R}^p$, which satisfy $\mathbf{1}^T s = \mathbf{1}^T t = 0$. The flow of commodity 1 on arc i is denoted by x_i, and the flow of commodity 2 on arc i is denoted by y_i. Each of the flows must satisfy flow conservation, which can be expressed as $Ax + s = 0$ (for commodity 1), and $Ay + t = 0$ (for commodity 2).

Arc i has associated flow cost $\phi_i(x_i, y_i)$, where $\phi_i : \mathbf{R}^2 \to \mathbf{R}$ is convex. We can impose constraints such as nonnegativity of the flows by restricting the domain of ϕ_i to $\mathbf{R}^2_+$. One natural form for ϕ_i is a function of only the total traffic on the arc, i.e., $\phi_i(x_i, y_i) = f_i(x_i + y_i)$, where $f_i : \mathbf{R} \to \mathbf{R}$ is convex. In this form, however, ϕ is not strictly convex, which will complicate things. Therefore, to avoid this complications, we assume that ϕ_i is strictly convex.

The problem of choosing the minimum cost flows that satisfy the flow conservation can be expressed as

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^n \phi_i(x_i, y_i) \\ \text{subject to} & Ax + s = 0, \quad Ay + t = 0 \end{array}$$

with variables $x,\ y \in \mathbf{R}^n$. This is the *bi-commodity network flow problem.*

a) Propose a distributed solution to the bi-commodity flow problem using dual decomposition.

b) Use your algorithm to solve the particular problem instance with

$$\phi_i(x_i, y_i) = (x_i + y_i)^2 + \epsilon(x_i^2 + y_i^2), \quad \textbf{dom}\ \phi_i = \mathbf{R}_+^2,$$

with $\epsilon = 0.1$. The other data for this problem can be found in `bicommodity_data.m`. To check that your method works, compute the optimal value p^*, using `CVX`. For the subgradient updates use a constant step-size of 0.1. Run the algorithm for 200 iterations and plot the dual lower bound versus iteration. With a logarithmic vertical axis, plot the norms of the residuals for each of the two flow conservation equations, i.e., $||Ax + s||_2$ and $||Ay + t||_2$, respectively, versus iteration number, on the same plot.

Hint: A function `[x,y] = quad2_min(eps,alpha,beta)` is posted, which analytically computes

$$(x^*, y^*) = \underset{x \geq 0, y \geq 0}{\operatorname{argmin}} \big((x + y)^2 + \epsilon(x^2 + y^2) + \alpha x + \beta y\big).$$

You might find this function useful.

6) Problem 9.10 of the text book. [1]

7) (Bonus points*) Problem 9.30 (a) of the text book.

[1]Boyd, Stephen, Stephen P. Boyd, and Lieven Vandenberghe. Convex optimization. Cambridge university press, 2004.