Convex Optimization II

Lecture 11: Generalized Network Utility Maximization

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References

[1] M. Chiang, S. H. Low, A. R. Calderbank, and J. C. Doyle, “Layering as optimization decomposition: A mathematical theory of network architectures,” *Proceedings of the IEEE*, vol. 95, no. 1, pp. 255-312, January 2007.

[2] M. Chiang, “Balancing transport and physical layers in wireless multihop networks: Jointly optimal congestion control and power control,” *IEEE Journal of Selected Areas in Communications*, vol. 23, no. 1, pp. 104-116, January 2005.

Acknowledgment

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Motivations

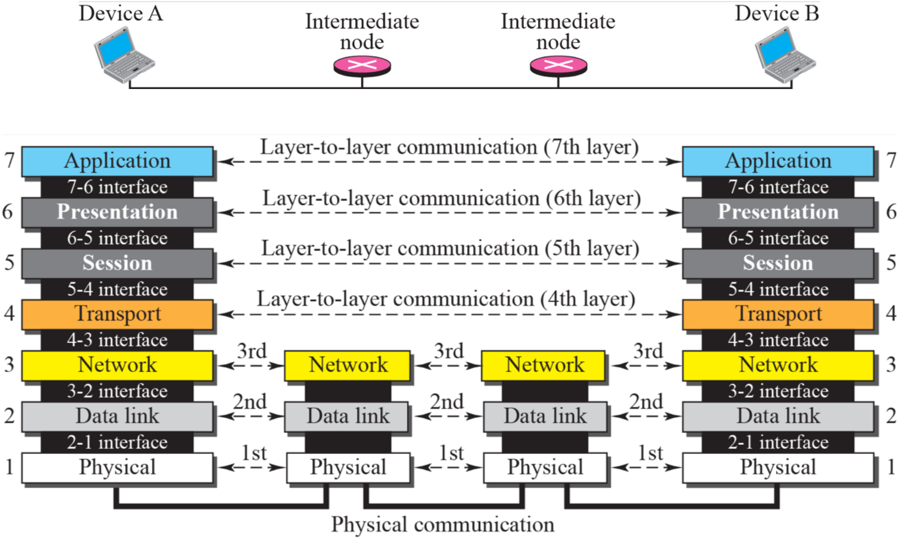
- Recall the network utility maximization problem for congestion control

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{maximize}} & \sum_{s \in \mathcal{S}} U_s(x_s) \\ \text{subject to} & R\boldsymbol{x} \preceq \boldsymbol{c} \end{array} \tag{1}$$

  where R is the fixed routing matrix, $\boldsymbol{c}$ is the fixed link capacity vector, and $\boldsymbol{x}$ is the variable vector of data rates.

- Problem (1) formulates the protocol stack only at the transport layer.
- Question: Is it possible to extend the basic network utility maximization problem to design a layered architecture?

Network OSI Model



[1] BA. Forouzan, *TCP/IP Protocol Suite*. McGraw-Hill Inc., 4th edition, Jun, 2002.

LAYERING AS OPTIMIZATION DECOMPOSITION [1]

- Network architecture determines functionality allocation: *Who does what*, and *how to connect them*

  Systematic Approach

- Network: Generalized network utility maximization
- Layering architecture: Vertical decomposition
- Layer: Decomposed subproblems
- Interfaces: Functions of primal or dual variables
- Distributed computation and control of a functionality module over geographically disparate network elements: Horizontal decomposition

Generalized Network Utility Maximization

- Generalized NUM (GNUM) can be formulated as [1]:

$$\begin{array}{ll} \underset{\boldsymbol{x},\boldsymbol{w},R}{\text{maximize}} & \sum_{s\in\mathcal{S}} U_s(x_s) \\ \text{subject to} & R\boldsymbol{x} \leq \boldsymbol{c}(\boldsymbol{w}), \\ & R \in \mathcal{R}, \quad \boldsymbol{w} \in \mathcal{W} \end{array}$$

  where

- $\boldsymbol{x}$ is vector of data rates (variable)
- $\boldsymbol{w}$ is vector of network (lower layer) resources (variable)
- R is routing matrix (variable)
- $\mathcal{W}$ is feasible set for network resources
- $\mathcal{R}$ is feasible set for routing matrices

Generalized Network Utility Maximization

- For the choices of routing matrix, we distinguish two different routing models:
  - Single-path routing
  - Multipath routing
- Question: What are the differences between the two cases above in terms of the corresponding feasible sets?
- We will see a lot about the routing problem in the next lecture.

Cross-Layer Design

From cross-layer design point of view, $\boldsymbol{x}$, $\boldsymbol{w}$, and R are usually associated to different network layers.

- Data rate $\boldsymbol{x}$:
  - Transport Layer
- Routing matrix R:
  - Network Layer
- Network resources $\boldsymbol{w}$:
  - Data Link or MAC Layer
  - Physical Layer
- Question: Does GNUM provide a framework for cross-layer design?

Case Study: Joint Congestion Control & Power Control [2]

- Assume that $\boldsymbol{w}$ denotes the vector of transmission power in a wireless network.

- In particular, we replace $\boldsymbol{w}$ by vector $\boldsymbol{p} = (p_l, \forall\, l \in \mathcal{L})$, where p_l denotes the transmission power of link l.

- Assume that routing paths are fixed. That is, the matrix R is given.

- We would like to solve the following GNUM [2]:

$$\begin{aligned} \underset{\boldsymbol{x}\succeq 0,\ \boldsymbol{p}\succeq 0}{\text{maximize}} \quad & \sum_{s\in\mathcal{S}} U_s(x_s) \\ \text{subject to} \quad & \sum_{s\,:\,l\in\mathcal{L}_s} x_s \le c_l(\boldsymbol{p}), \quad \forall\, l \in \mathcal{L}. \end{aligned} \tag{2}$$

- Question: What is set $\mathcal{W}$ (or $\mathcal{P}$) in problem (2)?

Capacity Model

- For each link $l \in \mathcal{L}$, the link capacity can be obtained as

$$\begin{aligned} c_l(\boldsymbol{p}) &= B \log(1 + K \ \mathrm{SINR}_l(\boldsymbol{p})) \\ &\approx B \log(K \ \mathrm{SINR}_l(\boldsymbol{p})) \end{aligned}$$

  where

SINR_l	:	signal to interference plus noise ratio for link l
K	:	a constant, depending on the modulation and required bit-error rate
B	:	channel bandwidth, assumed to be one unit (i.e., $B = 1$)

- Here, we assume that $K \ \mathrm{SINR}_l$ is much larger than one for all links.

SINR Model

- For each link $l \in \mathcal{L}$, we have

$$\mathrm{SINR}_l(\boldsymbol{p}) = \frac{p_l G_{ll}}{\sum\limits_{k \in \mathcal{L},\, k \neq l} p_k G_{lk} + n_l}$$

  where

p_l	:	transmission power of link l
G_{lk}	:	path loss from the transmitter of link k to the receiver of link l
n_l	:	noise power at the receiver of link l

Problem Formulation

- Joint congestion control and power control problem can be formulated as

$$\begin{aligned} \underset{\boldsymbol{x}\succeq 0,\ \boldsymbol{p}\succeq 0}{\text{maximize}} \quad & \sum_{s\in\mathcal{S}} U_s(x_s) \\ \text{subject to} \quad & \sum_{s\,:\,l\in\mathcal{L}_s} x_s \le \log\left(\frac{p_l K G_{ll}}{\sum_{k\in\mathcal{L},\ k\neq l} p_k G_{lk} + n_l}\right), \quad \forall\, l\in\mathcal{L}. \end{aligned}$$

- Question: Is this problem convex?

- Question: Is this problem separable?

Is the Problem Convex?

- Let us rewrite the constraint as

$$\sum_{s\,:\,l\in\mathcal{L}_s} x_s \leq \log\left(p_l K G_{ll}\right) - \log\left(\sum_{k\in\mathcal{L},\ k\neq l} p_k G_{lk} + n_l\right), \quad \forall\, l \in \mathcal{L}.$$

- After reordering the terms, we have

$$\sum_{s\,:\,l\in\mathcal{L}_s} x_s - \log\left(p_l K G_{ll}\right) + \log\left(\sum_{k\in\mathcal{L},\ k\neq l} p_k G_{lk} + n_l\right) \leq 0, \quad \forall\, l \in \mathcal{L}.$$

- Question: Is this set of constraints convex over $\boldsymbol{x}$ and $\boldsymbol{p}$?

Change of Variables

- Question: How to solve the joint congestion control and power control problem? In particular, how to solve it in a distributed manner?

- Let us define some new variables

$$\tilde{p}_l = \log p_l, \quad \forall\, l \in \mathcal{L}.$$

- In that case, we have

$$p_l = \exp \tilde{p}_l = e^{\tilde{p}_l}, \quad \forall\, l \in \mathcal{L}.$$

- Question: In case we can solve the problem with respect to $\tilde{\boldsymbol{p}}$, then would it be possible to recover optimal $\boldsymbol{p}$ from optimal $\tilde{\boldsymbol{p}}$? Why?

Change of Variables

- Let us rewrite the constraint for the new choice of variables

$$\sum_{s\,:\,l\in\mathcal{L}_s} x_s - \log\left(e^{\tilde{p}_l} K G_{ll}\right) + \log\left(\sum_{k\in\mathcal{L},\ k\neq l} e^{\tilde{p}_k} G_{lk} + n_l\right) \leq 0, \quad \forall\, l \in \mathcal{L}.$$

- After reordering the terms, the constraints become

$$\sum_{s\,:\,l\in\mathcal{L}_s} x_s - \tilde{p}_l - \log\left(K G_{ll}\right) + \log\left(\sum_{k\in\mathcal{L},\ k\neq l} e^{\tilde{p}_k} G_{lk} + n_l\right) \leq 0, \quad \forall\, l \in \mathcal{L}.$$

- Question: Are these new constraints convex in $\boldsymbol{x}$ and $\tilde{\boldsymbol{p}}$?

Modified Problem Formulation

- Joint congestion control and power control problem can be reformulated as

$$
\begin{aligned}
\underset{\boldsymbol{x}\succeq 0,\ \tilde{\boldsymbol{p}}}{\text{maximize}} \quad & \sum_{s\in\mathcal{S}} U_s(x_s) \\
\text{subject to} \quad & \sum_{s\,:\,l\in\mathcal{L}_s} x_s - \tilde{p}_l - \log\left(KG_{ll}\right) + \log\left(\sum_{k\in\mathcal{L},\ k\neq l} e^{\tilde{p}_k} G_{lk} + n_l\right) \leq 0, \ \forall\ l \in \mathcal{L}.
\end{aligned}
$$

- Question: Is this problem convex?
- Question: Is this problem separable?

Dual Function

- We can write the Lagrangian as

$$L(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{\lambda}) = \sum_{s \in \mathcal{S}} U_s(x_s) - \sum_{l \in \mathcal{L}} \lambda_l \sum_{s\,:\, l \in \mathcal{L}_s} x_s + \sum_{l \in \mathcal{L}} \lambda_l \left(\tilde{p}_l + \log\left(K G_{ll}\right) - \log\left(\sum_{k \in \mathcal{L},\, k \neq l} e^{\tilde{p}_k} G_{lk} + n_l \right) \right)$$

- The dual function is obtained as

$$D(\boldsymbol{\lambda}) = \underset{\boldsymbol{x} \succeq 0,\, \tilde{\boldsymbol{p}}}{\text{maximize}} \;\; L(\boldsymbol{x}, \tilde{\boldsymbol{p}}, \boldsymbol{\lambda})$$

- Question: Is the above maximization separable?

Dual Decomposition

- We can write

$$
\begin{aligned}
D(\boldsymbol{\lambda}) = & \underset{\boldsymbol{x}\succeq 0}{\text{maximize}} \sum_{s\in\mathcal{S}} \left(U_s(x_s) - \sum_{l\in\mathcal{L}_s} \lambda_l x_s \right) \\
& + \underset{\tilde{\boldsymbol{p}}}{\text{maximize}} \sum_{l\in\mathcal{L}} \lambda_l \left(\tilde{p}_l + \log\left(K G_{ll}\right) - \log\left(\sum_{k\in\mathcal{L},\, k\neq l} e^{\tilde{p}_k} G_{lk} + n_l \right) \right)
\end{aligned}
$$

- In fact, the maximization problem is decomposed into two subproblems.

Congestion Control Subproblem

- The congestion control subproblem becomes

$$\underset{\boldsymbol{x}\succeq 0}{\text{maximize}} \sum_{s\in\mathcal{S}} \left(U_s(x_s) - \sum_{l\in\mathcal{L}_s} \lambda_l x_s \right)$$

- Question: Is this problem different from the problem we saw in TCP forward engineering? How can we solve this problem?

- Also, notice that

$$\underset{\boldsymbol{x}\succeq 0}{\text{maximize}} \sum_{s\in\mathcal{S}} \left(U_s(x_s) - \sum_{l\in\mathcal{L}_s} \lambda_l x_s \right) = \sum_{s\in\mathcal{S}} \underset{x_s\geq 0}{\text{maximize}} \left(U_s(x_s) - \sum_{l\in\mathcal{L}_s} \lambda_l x_s \right)$$

Power Control Subproblem

- The power control subproblem becomes

$$\underset{\tilde{\boldsymbol{p}}}{\text{maximize}} \sum_{l \in \mathcal{L}} \lambda_l \left(\tilde{p}_l + \log\left(KG_{ll}\right) - \log\left(\sum_{k \in \mathcal{L},\ k \neq l} e^{\tilde{p}_k} G_{lk} + n_l \right) \right)$$

- Question: How do we solve the above problem?
- To answer this question, let us replace $\tilde{p}_l = \log p_l$. The power control subproblem becomes

$$\underset{\boldsymbol{p}}{\text{maximize}} \sum_{l \in \mathcal{L}} \lambda_l c_l(\boldsymbol{p}).$$

- Question: Which links do you think should assign a higher capacity?

Power Control Subproblem

- Question: How can we solve the power control subproblem?
- We can use the gradient projection method.
- In particular, we can update the transmission power of link $l \in \mathcal{L}$ as

$$p_l(t+1) = \left[p_l(t) + \kappa \left(\frac{\lambda_l}{p_l(t)} - \sum_{j \in \mathcal{L},\, j \neq l} \frac{\lambda_j G_{jl}}{\sum_{k \in \mathcal{L},\, k \neq j} G_{jk} p_k + n_j} \right) \right]^+$$

where $\kappa > 0$ is a fixed stepsize. The above is equivalent to

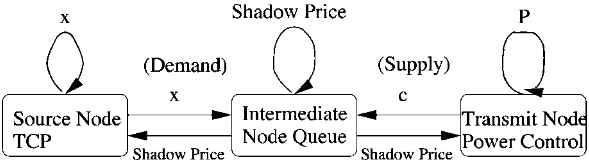
$$p_l(t+1) = \left[p_l(t) + \frac{\kappa \lambda_l}{p_l(t)} - \kappa \sum_{j \in \mathcal{L} \setminus \{l\}} G_{jl} m_j(t) \right]^+$$

where $m_j(t)$ are messages passed from link j:

$$m_j(t) = \frac{\lambda_j \mathrm{SINR}_j(t)}{p_j(t) G_{jj}}.$$

Coupling Between Congestion and Power Control

- Congestion control and power control algorithms are coupled together as [2]:



Summary

- Network utility maximization problem can be extended to include various other network design problems in various other layers of the network protocol stack.
- Joint congestion control and power control problem
- Congestion control subproblem
- Power control subproblem
- Layering as optimization decomposition