EE 25088 Convex Optimization II

Problem Set II

Due Date: Ordibehesht 18, 1400

1) Let C be a nonempty convex subset of $\mathbb{R}^n$. Let also $f = (f_1, \cdots, f_m)$, where $f_i : C \to \mathbb{R}$, $i = 1, \cdots, m$, are convex functions, and let $g : \mathbb{R}^m \to \mathbb{R}$ be a function that is convex and monotonically nondecreasing over a convex set that contains the set $\{f(x) | x \in C\}$, in the sense that for all u_1, u_2 in this set such that $u_1 \leq u_2$, we have $g(u_1) \leq g(u_2)$. Show that the function h defined by $h(x) = g(f(x))$ is convex over C. If in addition, $m = 1$, g is monotonically increasing and f is strictly convex, then h is strictly convex.

2) Consider a convex function $f(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}$ and an affine function $h(\mathbf{x}) = \boldsymbol{x}^T \boldsymbol{w} - h_0$ with $h_0 = \sup_{\boldsymbol{x}} \boldsymbol{x}^T \boldsymbol{w} - f(\boldsymbol{x})$. Show that the function $h(\boldsymbol{x})$ minorizes the convex function $f(\boldsymbol{x})$, i.e., for any $\boldsymbol{x} \in \mathbb{R}^n$ we have $f(\boldsymbol{x}) \geq h(\boldsymbol{x})$.

3) Show that the function
$$f(x) = \frac{||Ax - b||_2^2}{1 - x^T x}$$
is convex on $\{x | \ ||x||_2 < 1\}$.

4) Suppose $\lambda_1, \cdots, \lambda_n$ are positive. Show that the function $f : \mathbf{R}^n \to \mathbf{R}$, given by
$$f(x) = \prod_{i=1}^{n} (1 - e^{-x_i})^{\lambda_i},$$
is concave on **dom** $f = \{x \in \mathbf{R}^n_{++} | \sum_{i=1}^n \lambda_i e^{-x_i} \leq 1\}$.

5) Problem 3.15 [1]

6) Problem 3.22 (a)–(c) [1].

[1] S. Boyd and L. Vandenberghe, Convex Optimization, Cambridge University Press, 2004.

7) Use KKT conditions to solve
$$\underset{\boldsymbol{x} \in \mathbf{R}^n}{\text{minimize}} f(\boldsymbol{x}) \quad \text{subject to} \quad h(\boldsymbol{x}) = 0$$

for the following cases

a) $f(\boldsymbol{x}) = ||\boldsymbol{x}||^2$ and $h(\boldsymbol{x}) = \sum_{i=1}^n x_i - 1$.

b) $f(\boldsymbol{x}) = \sum_{i=1}^n x_i - 1$ and $h(\boldsymbol{x}) = ||\boldsymbol{x}||^2 - 1$.

8) Derive the dual of the projection problem defined as

$$\begin{aligned} \underset{\boldsymbol{x}\in\mathbf{R}^n}{\text{minimize}} \quad & ||\boldsymbol{z}-\boldsymbol{x}||^2 \\ \text{subject to} \quad & A\boldsymbol{x} = \mathbf{0}, \end{aligned}$$

where the $m \times n$ matrix A and the vector $\boldsymbol{z} \in \mathbf{R}^n$ are given. Show that the dual problem is also a problem of projection onto a subspace.

9) *Relative entropy*. The relative entropy between two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_{++}$ is defined as $\sum_{k=1}^n x_k \log(x_k/y_k)$, which is a convex function. Through the following problem, we calculate the vector $\boldsymbol{x}$ that minimizes the relative entropy with a given vector $\boldsymbol{y}$, subject to equality constraints on $\boldsymbol{x}$:

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \; & \sum_{k=1}^n x_k \log(x_k/y_k) \\ \text{subject to} \; & A\boldsymbol{x} = \boldsymbol{b}, \\ & \mathbf{1}^T\boldsymbol{x} = 1. \end{aligned}$$

The optimization variable is $\boldsymbol{x} \in \mathbb{R}_{++}$. The parameters $\boldsymbol{y} \in \mathbb{R}_{++}$, $A \in \mathbb{R}_{++}^{m\times n}$, and $\boldsymbol{b} \in \mathbb{R}_{++}^m$ are given.

a) Derive the Lagrangian and the dual of this problem.

b) Simplify your answer to obtain the following optimization problem

$$\text{maximize } \boldsymbol{b}^T\boldsymbol{z} - \log\sum_{k=1}^n y_k \exp(\boldsymbol{a}_k^T\boldsymbol{z}),$$

with $\boldsymbol{a}_k$ be the k-th column of A.