((به نام خدا))

ارسلان فیروزی - ۹۷۱۰۲۲۲۵

a-1)

f Convex $\longrightarrow f_i(\theta U_1+(1-\theta)U_2) \leqslant \theta f_i(U_1)+(1-\theta)f_i(U_2) \;,\; i=1,\dots,m$

$\xrightarrow{U_1<U_2 \to g(U_1)\leqslant g(U_2)} g\circ f(\theta U_1+(1-\theta)U_2) \leqslant g(\theta f(U_1)+(1-\theta)f(U_2))$

$\xrightarrow{g \text{ is Convex}} g\circ f(\theta U_1+(1-\theta)U_2) \leqslant \theta\, g\circ f(U_1)+(1-\theta)\, g\circ f(U_2)$

$\Longrightarrow g\circ f$ is Convex

b)

f Convex sorictly $\longrightarrow f(\theta U_1+(1-\theta)U_2) < \theta f(U_1)+(1-\theta)f(U_2)$

$\xrightarrow{U_1<U_2 \to g(U_1)<g(U_2)} g\circ f(\theta_1 U_1+(1-\theta)U_2) < g(\theta f(U_1)+(1-\theta)f(U_2))$

$\xrightarrow{g \text{ is Convex}} g\circ f(\theta_1 U_1+(1-\theta)U_2) < \theta\, g\circ f(U_1)+(1-\theta)\, g\circ f(U_2)$

$\Longrightarrow g\circ f$ is sorictly Convex

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$f(x)-h(x) = f(x)-x^T w+\sup(\underbrace{x^T w-f(x)}_{g(x)}) = -g(x)+\sup(g(x))$

بنابر تعریف $g(x) \underset{\text{همواره}}{\leqslant} \sup g(x) \Longrightarrow f(x)-h(x) = \sup(g(x))-g(x) \geqslant 0$

$\Longrightarrow f(x) \geqslant h(x)$

-۳

$\underset{\text{Convex}}{x^T x} \longrightarrow \underset{\text{Concave}}{1-x^T x}$

از طرفی میدانیم $\|x\|_2^2$ Convex است و اگر با تابع affine $Ax-b$ ترکیب کنیم، $\|Ax-b\|_2^2$ Convex است

ادامه صفحه بعد

میدانیم که تابع h برای $y>0$ ، Convex است:

$h = \frac{x^2}{y}$ ، $y>0$ Convex $\quad \frac{\partial h}{\partial x} \geq 0$ ، $y>0$ ، $\|x\|_2<1$

$\frac{\partial h}{\partial y} \leq 0 \qquad f = \log(x)$

$g(x) = (\underbrace{\|Ax-b\|^2}_{\text{Convex}} , \underbrace{1-x^T x}_{\text{Concave}})$

$$f'' = g'(x)^T \nabla^2 \log(x)\, g'(x) + \nabla \log(x)^T g''(x)$$

g_1 تابع : $\quad \geq 0 \qquad \geq 0 \qquad \geq 0 \implies f'' \geq 0$

h Convex

$\implies$ Convex f است

g_2 تابع : $\quad \geq 0 \qquad \leq 0 \qquad \leq 0 \implies f'' \geq 0$

$\Leftarrow \frac{\|Ax-b\|^2}{1-x^T x}$ Convex

۳-

$$f(x) = \prod_{i=1}^{n} (1-e^{-x_i})^{\lambda_i}$$

$$\frac{\partial f}{\partial x_i} = \frac{\lambda_i \cancel{(1-e^{-x_i})^{\lambda_i-1}} e^{-x_i}}{1-e^{-x_i}} \prod_{i=1}^{n} (1-e^{-x_i})^{\lambda_i} = \frac{\lambda_i}{-1+e^{x_i}} \prod_{i=1}^{n} (1-e^{-x_i})^{\lambda_i}$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\lambda_i \lambda_j}{(e^{x_i}-1)(e^{x_j}-1)} \prod_{i=1}^{n} (1-e^{-x_i})^{\lambda_i}$$

$$\frac{\partial^2 f}{\partial x_i^2} = \frac{\lambda_i^2}{(e^{x_i}-1)^2} \prod_i (1-e^{-x_i})^{\lambda_i} - \frac{\lambda_i}{(e^{x_i}-1)^2} \prod_i (1-e^{-x_i})^{\lambda_i} = \frac{\lambda_i(\lambda_i-1)}{(e^{x_i}-1)^2} \prod_i (1-e^{-x_i})^{\lambda_i}$$

(a-5

$$\lim_{\alpha\to 0} U_\alpha(x) = \lim_{\alpha\to 0} \frac{x^\alpha - 1}{\alpha} \underset{\text{هوپیتال}}{=} \lim_{\alpha\to 0} \frac{\frac{\partial}{\partial\alpha}(x^\alpha - 1)}{\frac{\partial}{\partial\alpha}(\alpha)} = \lim_{\alpha\to 0} x^\alpha \ln x = \underline{\underline{\ln x}} = U_0(x)$$

(b

$$\frac{d}{dx}(U_\alpha(x)) = \frac{d}{dx}\left(\frac{x^\alpha - 1}{\alpha}\right) = \frac{1}{\alpha}\frac{\alpha}{x}x^\alpha = x^{\alpha-1} \underset{\substack{x>0 \\ 0<\alpha\leq 1}}{>} 0 \Longrightarrow \text{Monotone Increasing}$$

$$\frac{d^2}{dx^2}(U_\alpha(x)) = \frac{d}{dx}\left(x^{\alpha-1}\right) \overset{\text{اگر } \alpha\neq 1}{=} \frac{x^{\alpha-2}}{\alpha-1} \leq 0 \Longrightarrow \text{Concave}$$

$\alpha \leq 1 \to \alpha - 1 \leq 0$

$\overset{\text{اگر } \alpha=1}{=} 0 \leq 0$

(a-6 میدانیم $\log \sum_i \exp(g_i(x))$ درصورت محدب بودن $g_i(x)$، محدب است. از طرفی $a_i^T x + b_i$ محدب است پس $\log \sum_i \exp(a_i^T x + b_i)$ محدب است $-\log \sum \exp(a_i^T x + b_i)$ مقعراست.

$h(x) = -\log(x)$ ، Concave ، nonincreasing

$g(x) = -\log \sum \exp(a_i^T x + b_i)$ ، Concave

$\Longrightarrow$ Convex

(b

$h(x) = -\sqrt{yx}$ ، Convex

$\frac{\partial h}{\partial x} = -\frac{\sqrt{y}}{2\sqrt{yx}} \leq 0 \to$ decreasing

$\frac{\partial h}{\partial y} = -\frac{\sqrt{x}}{2\sqrt{xy}}$

$g(x) = \left(u , v - \frac{x^T x}{u}\right)$

Concave

مقعراست $v - \frac{x^T x}{u}$

$\Rightarrow h \circ g = -\sqrt{uv - x^T x}$ Convex

میدانیم $\underset{u>0}{\frac{x^T x}{u}}$ محدب است پس $-\frac{x^T x}{u}$ مقعراست وچون v مقعراست،

$h(x) = -\log(xy)$ ، Concave ، nonincreasing

$g(x) = \left(u , v - \frac{x^T x}{u}\right)$

Concave Concave

$\Rightarrow h \circ g = -\log(uv - x^T x)$ (c

Convex

$\frac{\partial h}{\partial x} = -\frac{1}{x}$ ، $\frac{\partial^2 h}{\partial x^2} = \frac{1}{x^2}$

$\frac{\partial^2 h}{\partial y^2} = \frac{1}{y^2}$ ، $\frac{\partial h}{\partial x y} = 0$

$\Rightarrow \nabla^2 h > 0 \Rightarrow$ Convex

$\nabla h < 0 \longrightarrow$ nonincreasing (a - ۷

$f(x) = \|x\|^2$ ، $h(x) = \sum_i x_i - 1$

$L(x) = \|x\|^2 + v(1^T x - 1)$

$$\begin{cases} \dfrac{\partial L}{\partial x_i} = 2x_i^* + v = 0 \ ، \ i = 1,\dots,n \longrightarrow x_i^* = -\dfrac{v}{2} \ ، \ i = 1 \dots n \\ 1^T x^* - 1 = 0 \longrightarrow \sum_i x_i^* = 1 \longrightarrow -\dfrac{nv}{2} = 1 \longrightarrow v = -\dfrac{2}{n} \Longrightarrow \boxed{x_i^* = \dfrac{1}{n} \ ، \ i = 1 \dots n} \end{cases}$$

(b

$f(x) = \sum_i x_i - 1$ ، $h(x) = \|x\|^2 - 1$

$L(x) = 1^T x - 1 + v(x^T x - 1)$

$$\begin{cases} \dfrac{\partial L}{\partial x_i} = 1 + 2v x_i^* = 0 \ ، \ i = 1 \dots n \longrightarrow x_i^* = -\dfrac{1}{2v} \\ x^T x - 1 = 0 \longrightarrow \sum_i x_i^{*2} = 1 \longrightarrow \dfrac{n}{4v^2} = 1 \longrightarrow v = \dfrac{\sqrt{n}}{2} \Longrightarrow \boxed{x_i^* = -\dfrac{1}{\sqrt{n}} \ ، \ i = 1 \dots n} \end{cases}$$

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$\underset{x \in \mathbb{R}^n}{\text{minimize}} \ \|z - x\|^2$ ($z \in \mathbb{R}^n$)

s.t. $Ax = 0$ (A: $m \times n$)

$L(x, v) = (z - x)^T (z - x) + \underbrace{v}_{1 \times m} Ax \longrightarrow$ Convex و بدون قید

$\dfrac{\partial L}{\partial x} = 2(x - z) + A^T v^T = 0 \longrightarrow x = z - \dfrac{A^T v^T}{2}$

$g(v) = \inf_x L(x, v) = -\dfrac{1}{4} v A A^T v^T + v A z$

dual Problem $\longrightarrow$ maximize $-\dfrac{1}{4} v A A^T v^T + v A z$

$= -\dfrac{1}{4} \|vA\|_2^2 + vAz$

$-\frac{1}{2}(A^T v^T + 2z)^T (A^T v^T + 2$

$-z^T z$

$\|z\|^2$

(a-9

$$\underset{x}{\text{minimize}} \sum_k x_k \log\left(\frac{x_k}{y_k}\right)$$

s.t. $\quad Ax = b \longrightarrow \sum_k a_k x_k = b$

ستون k ام A

$\quad 1^T x = 1 \longrightarrow \sum_k x_k = 1$

$$L(x,\nu) = \sum_k x_k \log\left(\frac{x_k}{y_k}\right) + \underset{1\times m}{\nu_1}\left(\sum_k a_k x_k - b\right) + \nu_2\left(\sum_k x_k - 1\right)$$

$$\frac{\partial L}{\partial x_i} = \log\left(\frac{x_i}{y_i}\right) + y_i + \nu_1 a_k + \nu_2 = 0 \quad i = 1,\dots,n \longrightarrow x_i^* = y_i e^{-y_i - \nu_1 a_k - \nu_2}$$

$$g(\nu) = \sum_k y_k(-y_k - \nu_1 a_k - \nu_2) e^{-y_k - \nu_1 a_k - \nu_2} + \nu_1\left(\sum_k a_k y_k e^{-y_k - \nu_1 a_k - \nu_2} - b\right)$$
$$+ \nu_2\left(\sum_k y_k e^{-y_k - \nu_1 a_k - \nu_2} - 1\right)$$

$$= -\sum_k y_k^2 e^{-y_k - \nu_1 a_k - \nu_2} - \nu_1 b - \nu_2$$

$\sum_k y_k^2 \qquad - \log - \sum y_k^2 e^{-y_k - \nu_1 a_k}$

Dual Problem $\longrightarrow$ $\quad \text{maximize} \quad -\sum_k y_k^2 e^{-y_k - \nu_1 a_k - \nu_2} - \nu_1 b - \nu_2$

(b

$$\frac{\partial}{\partial \nu_2} g(\nu_1, \nu_2) = +e^{-\nu_2} \sum_k y_k^2 e^{-y_k - \nu_1 a_k} - 1 = 0 \longrightarrow \nu_2^* = +\log \sum_k y_k^2 e^{-y_k - \nu_1 a_k}$$

با جایگذاری $\longrightarrow$

$$-\sum_k y_k^2 e^{-y_k - \nu_1 a_k - \log \sum_k y_k^2 e^{-y_k - \nu_1 a_k}} - \nu_1 b - \log \sum_k y_k^2 e^{-y_k - \nu_1 a_k}$$

$$= -\frac{\sum_k y_k^2 e^{-y_k - \nu_1 a_k}}{\sum_k y_k^2 e^{-y_k - \nu_1 a_k}} - \nu_1 b - \log \sum_k y_k^2 e^{-y_k - \nu_1 a_k}$$

$$= -1 \underset{}{\overset{z^T}{(-\nu_1)}} b - \log \sum_k \underset{Y_k}{(y_k^2 e^{-y_k})} e^{\overset{z^T}{(-\nu_1)} a_k}$$

$$\Longrightarrow \quad \text{maximiz} \quad b^T z - \log \sum_k Y_k e^{a_k^T z}$$