GET1028 Logic

Arguments, Validity, and Contradiction

- **Declarative sentence**: sentences that can be true or false.
- **Argument**: consists of a set of declarative sentences (the **premises**) and a declarative sentence (the **conclusion**) marked as the concluded sentence.
  - There can be any number of premises in an argument.
  - The conclusion may be found before (some of) the premises.
- **Logical/Deductive validity**: an argument is logically/formally/deductively valid iff there is no interpretation (of the *subject-specific vocabulary*) under which the premises are all true and the conclusion is false.
  - In a logically valid argument:
    - **Topic neutrality**: the conclusion follows from the premises independently of the meaning of subject-specific expressions.
    - **Formality**: the conclusion follows from the premises purely in virtue of the form of the argument.
    - **Necessity**: it is not possible that the premises are T and the conclusion is F.
  - In a logically invalid argument, (there exists some assignment of sentence letters in which) the premises are true but the conclusion is false.
  - It is possible that the premises of a logically valid argument are false, and also that the conclusion is false.
- **Soundness**: an argument is sound iff (1) it is valid, and (2) it has all true premises.
- **Logical truth**: a sentence is a logical truth iff it is true under every interpretation (of the subject-specific vocabulary).
- **Logical falsity/contradiction**: a sentence is a logical falsity iff it is false under every interpretation (of the subject-specific vocabulary).
- **Contingent sentence**: a sentence is contingent iff it is true under *some* interpretation and false under *some* interpretation (of the subject-specific vocab).
- A **logical system/logic** typically consists of:
  1. A formal language + indications of how to translate English expressions into it.
  2. Definitions of logical validity and other logical notions for arguments and sentences in the formal language of the system.

Propositional Logic

- Basic aspects to (formal) languages:
  - **Syntax**: what counts as an expression of the language and what doesn't.
  - **Semantics**: studies what the meanings of the expressions of the language are.
  - *Use* vs *mention* in languages.
- Types of definitions:
  1. **Explicit**: e.g., "something is an X iff it is a Φ", where Φ is an expression we already know the meaning of.
  2. **Recursive/Inductive**: consists of 3 kinds of clauses:
     (a) **Base clause**: if something is a Φ, then it is an X.
     (b) **Recursive clause(s)**: applying some operation(s) to Xs gives us new Xs.
     (c) **Closure clause**: only things declared to be X by the 2 clauses above are X.

$\mathcal{L}_1$-sentences

- Definition of a $\mathcal{L}_1$**-sentence**:
  1. All sentence letters are (atomic) sentences of $\mathcal{L}_1$.
  2. If ϕ, ψ are sentences of $\mathcal{L}_1$, then so are $\neg\phi$, $(\phi \vee \psi)$, $(\phi \wedge \psi)$, $(\phi \to \psi)$, and $(\phi \leftrightarrow \psi)$.
  3. Nothing else is a sentence of $\mathcal{L}_1$.
- $\mathcal{L}_1$ has 3 sorts of basic expressions:
  1. **Sentence letters**: 'P', 'Q', 'R', etc., plays the role of sentences of English.
     - They stand for natural language declarative sentences not containing logical connectives.
     - We **interpret** sentence letters by assigning them **truth-values** (i.e., T/F).
  2. **Connectives**: '$\neg$', '$\wedge$', '$\vee$', '$\to$', and '$\leftrightarrow$'.
     - They are **truth-functional**: the truth/falsity of the atomic components (i.e., sentence letters) *completely determines* whether the sentence is true or false.
  3. **Brackets**: '(' and ')'.
- Conventions in $\mathcal{L}_1$:
  - $\wedge, \vee$ bind more strongly than $\to, \leftrightarrow$ (so brackets around $\wedge, \vee$ can be dropped).
  - One may drop brackets on strings of $\wedge$s or strings of $\vee$s (but not when mixed).
  - One may drop outer brackets.
- **Main connective** of an $\mathcal{L}_1$-sentence: the last connective that has been added in the 'construction' of the sentence.
- **Metavariables**: ϕ, ψ are not sentences of $\mathcal{L}_1$; they stand for arbitrary sentences, and are part of the metalanguage.
  - **Object-language**: the language we are theorising *about*.
  - **Metalanguage**: the language we are theorising *in*.

$\mathcal{L}_1$-structures

An $\mathcal{L}_1$-structure is an *assignment* of exactly one truth-value (i.e., T/F) to *every* sentence letter of $\mathcal{L}_1$.

- In **classical logic** (e.g., **propositional logic**), every sentence must be either true or false, but not both.
  - In **paracomplete logic**, sentences can be neither true nor false (indeterminate).
  - In **paraconsistent logic**, sentences can be both true and false.
- **Truth-value**, $|\phi|_{\mathcal{A}}$ of ϕ under $\mathcal{A}$, an $\mathcal{L}_1$-structure:
  Let ϕ, ψ be $\mathcal{L}_1$-sentences and $\mathcal{A}$ an $\mathcal{L}_1$-structure:
  1. If ϕ is a sentence letter, $|\phi|_{\mathcal{A}} = T$ iff $\mathcal{A}$ assigns T to ϕ.
  2. **Negation**: $|\neg\phi|_{\mathcal{A}} = T$ iff $|\phi|_{\mathcal{A}} = F$.
  3. **Conjunction**: $|\phi \wedge \psi|_{\mathcal{A}} = T$ iff $|\phi|_{\mathcal{A}} = T$ and $|\psi|_{\mathcal{A}} = T$.
  4. **(Inclusive) disjunction**: $|\phi \vee \psi|_{\mathcal{A}} = T$ iff $|\phi|_{\mathcal{A}} = T$ or $|\psi|_{\mathcal{A}} = T$.
  5. **(Material) conditional**: $|\phi \to \psi|_{\mathcal{A}} = T$ iff $|\phi|_{\mathcal{A}} = F$ or $|\psi|_{\mathcal{A}} = T$.
  6. **Biconditional**: $|\phi \leftrightarrow \psi|_{\mathcal{A}} = T$ iff $|\phi|_{\mathcal{A}} = |\psi|_{\mathcal{A}}$.
- **Indicative conditional**: a conditional sentence such as "if snow is white, then grass is green", whose grammatical form restricts it to discussing what could be true; it is *not truth-functional*.
- $\mathcal{L}_1$ is **truth-functionally complete**: all 2^4 different binary truth functions are *definable/expressible* in L_1 just in terms of $\neg$ and $\wedge$, or $\neg$ and $\vee$, or $\neg$ and $\to$.
  - **(Exclusive) disjunction**: $(\phi \veebar \psi) := \neg(\phi \leftrightarrow \psi)$.
  - **Nor**: $(\phi \downarrow \psi) := \neg(\phi \vee \psi)$.
  - **Nand**: $(\phi \uparrow \psi) := \neg(\phi \wedge \psi)$.
  - **Only if**: $(\phi \leftarrow \psi) := (\psi \to \phi)$.

Logical notions

- Methods of proof:
  1. Let $\mathcal{A}$ be an arbitrary $\mathcal{L}_1$-structure, so either $|P|_{\mathcal{A}} = T$ or $|P|_{\mathcal{A}} = F$.
  2. **Truth table method**.
  3. **Backwards truth table method**: assume the relevant sentence is *not* a tautology/contradiction, and try to show that this assumption entails an inconsistency.
     (a) Put a T/F in the main column.
     (b) Work backwards to show that either this leads to an inconsistency, or there is a counterexample.
- **Logical truth**, $\models \phi$: a sentence ϕ of $\mathcal{L}_1$ is logically true iff it is true under every $\mathcal{L}_1$-structure.
  - Every instance of a tautology schema is a tautology (same for contradiction and valid argument schemata).
  - Examples of tautology schemata:
    - **Law of excluded middle**: $\phi \vee \neg\phi$.
    - **Law of non-contradiction**: $\neg(\phi \wedge \neg\phi)$.
    - **Reflexivity**: $\phi \to \phi$.
- **Logical falsity**: a sentence ϕ of $\mathcal{L}_1$ is a contradiction iff it is false under every $\mathcal{L}_1$-structure.
- **Contingent sentence**: a sentence ϕ of $\mathcal{L}_1$ is contingent iff it is true under ≥ 1 $\mathcal{L}_1$-structure and false under ≥ 1 $\mathcal{L}_1$-structure.
  - There are no contingent schemata, as some instances are bound to be tautologies or contradictions.
- **Validity**, $\Gamma \models \phi$: the argument with all sentences in Γ as premises and ϕ as conclusion is valid iff there is no $\mathcal{L}_1$-structure in which all sentences in Γ are true and ϕ is false.
  - We write an argument as Γ/ϕ, or $\psi_1, \ldots, \psi_n/\phi$.
  - If Γ consists of multiple premises (e.g., $p_1, \ldots, p_n$), we write $p_1, \ldots, p_n \models \phi$ (or equivalently, $\models p_1 \wedge \ldots \wedge p_n \to \phi$).
- **Logical equivalence**: sentences ϕ, ψ of $\mathcal{L}_1$ are logically equivalent iff they are true under exactly the same $\mathcal{L}_1$ structures.
- **Satisfiability**: Γ is satisfiable/semantically consistent iff there is an $\mathcal{L}_1$-structure in which all sentences in Γ are true.

Theorem 4.1 *ϕ is a tautology iff $\neg\phi$ is a contradiction.*
Theorem 4.2 *ϕ is contingent iff $\neg\phi$ is also contingent.*
Theorem 4.3 *$\phi \models \psi$ iff $\models \phi \to \psi$.*
Theorem 4.4 *ϕ, ψ are logically equivalent iff $\models \phi \leftrightarrow \psi$.*
Theorem 4.5 *$\phi_1, \ldots, \phi_n \models \psi$ iff $\{\phi_1, \ldots, \phi_n, \neg\psi\}$ is unsatisfiable.*

Formalization

- **Translating** English sentences into $\mathcal{L}_1$-expressions:
  1. Find the logical structure/form of the sentence.
     (a) Identify the **main connective**.
     (b) Replace it with the corresponding **standard connective(s)**. If not negation, enclose the resulting expression within brackets.
        - **Negation**: "it is not the case that".
        - **Conjunction**: "and".
        - **Disjunction**: "or".
        - **Conditional**: "if ... then ...".
        - **Biconditional**: "if and only if".
     (c) Identify the **direct subsentences**, and restart the process for each of them.
        - **Direct subsentence** of a conjunction, disjunction, conditional, or biconditional are the conjuncts, disjuncts, antecedent and consequent, and the left- and right-hand side, respectively.
  2. Replace English connectives with appropriate connectives of $\mathcal{L}_1$, and the remaining English sentences with sentence letters (according to a dictionary).
- **Formalizing** the logical form of an English sentence to its formalization in $\mathcal{L}_1$:
  1. Replace standard connectives by their respective symbols.
     - P is **sufficient for/only if** $Q \Leftrightarrow$ **if** P **then** $Q \Leftrightarrow P \to Q$.
     - P is **necessary for** $Q \Leftrightarrow Q \to P$.
  2. Uniformly replace every English sentence by a sentence letter.
  3. Provide a dictionary.
- Issues with formalization:
  1. Logical form cannot be uniquely determined because sentence is **ambiguous**.
  2. Some English connectives are not **truth-functional**.
     - **Truth-functional** connective: the truth-value of the *compound sentence* cannot be changed by replacing a *direct subsentence* with another having the same truth-value.
     - Only truth-functional connectives can be expressed in $\mathcal{L}_1$.
     - Most English conditionals are not truth functional, e.g.:
       - "it is possible/could have been the case that..."
       - "because..."
       - "believes/knows/thinks/said that..."
       - **Indicative conditionals**: conditional sentences such as "If Leona is at home, she isn't in Paris", whose grammatical form restricts it to discussing what could be true.
       - **Counterfactual/Subjunctive conditionals**: conditional sentences which discuss what would have been true under different circumstances.
  3. Not clear how much force to apply when reformulating an English sentence.
     - If the truth-value depends on the **presence of a causal or explanatory link** between the antecedent and consequent, it cannot be formalized with "$\to$".
     - Some uses of "and" should not be formalized in terms of "$\wedge$".
       - Try substituting "and" with "but": if it works, this "and" can be formalized with "$\wedge$".

Set Theory

- **Set**: collection of objects.
  - **Elements/Members** of a set: objects in the set.
  - **Extensionality**: sets are **identical** iff they have the same elements.
    - **Intensionality**: objects are identical iff the internal definitions of objects are the same.
  - Edge cases:
    - $\varnothing = \{\}$.
    - $\{a, a\} = \{a\}$.
- Set operations:
  - **Subset**: $S \subseteq T$ iff every member of S is also a member of T.
  - **Union**, $S \cup T$: the set that contains exactly all members of S and all members of T.
  - **Intersection**, $S \cap T$: is the set whose elements are exactly the members that S shares with T.
  - **Power set**, $\mathcal{P}(S)$: the set that contains exactly all subsets of S.
- **Ordered tuples** are *identical* iff they agree in each of their components (i.e., $\langle a, b\rangle = \langle c, d\rangle$ iff $a = c$ and $b = d$).
  - **Cartesian product**, $S \times M$: the set of all ordered pairs with their first component in S and second component in M.
    - Cartesian products are not associative.
  - **Binary relation**: a set is a binary relation from S (i.e., *departure set*) to M (i.e., *destination set*) iff it is a subset of $S \times M$.
    - The **empty set** is always a relation between any two sets.
    - The **Cartesian product** of two sets is always a relation between those sets.
    - Any set of ordered pairs is a binary relation.
  - n**-ary relation**: a set is an n-ary relation between sets $S_1, \ldots, S_n$ iff its only members are ordered n-tuples with their i^{th} component in S_i for each $i \in [1, n]$.

- **Function**: a binary relation R is a function iff $\forall x, y, z$, if $\langle x, y\rangle \in R$ and $\langle x, z\rangle \in R$, then $y = z$.

First-Order Predicate Logic

- Preserves a more fine-grained structure of atomic sentences, distinguishing between **designators** and **predicates**.

$\mathcal{L}_2$-sentences

- $\mathcal{L}_2$ has 6 sorts of basic expressions:
  1. **Sentence letters**: $P, Q, R, P_1, \ldots$.
     - They play the role of simple sentences *without reference to objects*.
  2. **Predicate/Relation letters**: $P_n^m, Q_n^m, R_n^m, \ldots$, where $m, n \in \mathbb{Z}^+$.
     - They play the role of predicates in English, expressing a **property/concept**.
     - **Arity**: value of the upper index, indicates that m objects are linked.
     - e.g., P^2: $\ldots_1$ hates $\ldots_2$ (subindexes are used to fix the order of arguments).
  3. **Constants**: $a, b, c, a_1, \ldots$.
     - Function as **proper names/designators**, denoting single **particular** objects.
  4. **Variables**: $x, y, z, x_1, \ldots$.
     - They stand for **arbitrary** objects, allowing us to generalize about objects.
  5. **Logical operators**: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \forall$ (universal quantifier), $\exists$ (existential q.).
  6. **Brackets**: '(' and ')'.

  Sentence letters, *predicate letters*, and *constants* are **subject-specific expressions**.
- **Term** of $\mathcal{L}_2$: a constant or a variable.
- In the **metalanguage**, we use:
  - $k, k_1, \ldots$ as schematic letters for constants,
  - $u, v, v_1, \ldots$ for individual variables,
  - $t, t_1, \ldots$ for terms, and
  - $P^n, P_1^n, \ldots$ for n-ary predicate letters.
- Definition of a **formula** of $\mathcal{L}_2$:
  1. Sentence letters are (atomic) formulae of $\mathcal{L}_2$.
     - Atomic sentences stand for natural language sentences without logical expressions but *with objects*.
  2. If P^n is an n-ary predicate letter and $t_1, \ldots, t_n$ are terms, then $P^n(t_1, \ldots, t_n)$ is an (atomic) formula of $\mathcal{L}_2$.
  3. If ϕ, ψ are formulae of $\mathcal{L}_2$, then so are $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$, and $(\phi \leftrightarrow \psi)$.
  4. If ϕ is a formula of $\mathcal{L}_2$, then so are $\forall v\phi$ and $\exists v\phi$.
  5. Nothing else is a formula of $\mathcal{L}_2$.
  - **Main operator** of a formula: the last operator that has been added in the construction of the formula.
  - Conventions:
    * We may drop arity indexes when there is no room for confusion.
    * We may also drop brackets according to the bracketing conventions.
- **Quantifier scope**: the scope of an occurrence of a quantifier in a formula ϕ, is the formula we attach it to in the construction process of ϕ.
  - **Free occurrence**: an *occurrence* of v is free in a formula iff it does not occur in the scope of any quantifier with v.
  - **Free variable**: a variable is free in a formula iff there is ≥ 1 free occurrence of the variable in the formula.
  - **$\mathcal{L}_2$-sentence**: ϕ is a sentence of $\mathcal{L}_2$ iff ϕ is a formula of $\mathcal{L}_2$ with no free variables.
    * Otherwise, ϕ is an **open formula**.
    * **Substitution**, $\phi[k/v]$: the result of substituting all *free occurrences* of v with k in ϕ.

$\mathcal{L}_2$-structures

An $\mathcal{L}_2$**-structure** $\mathcal{A}$ is an ordered pair $\langle D_\mathcal{A}, I_\mathcal{A}\rangle$ where $D_\mathcal{A}$ is a set (i.e., the **domain**), $I_\mathcal{A}$ is the **interpretation function** of $\mathcal{A}$, and:

1. $D_\mathcal{A} \neq \varnothing$ (i.e., the domain contains at least one thing).
2. For each constant k, $I_\mathcal{A}(k) \in D_\mathcal{A}$ (i.e., $\mathcal{A}$ interprets k with an object in the domain).
3. For each sentence letter P, $I_\mathcal{A}(P) = T$ or $I_\mathcal{A}(P) = F$ (i.e., $\mathcal{A}$ interprets P with a truth-value T or F).
4. For each n-place predicate letter P^n, $I_\mathcal{A}(P^n) \subseteq D_\mathcal{A}^n$ (i.e., $\mathcal{A}$ interprets P^n with a set of n-tuples of objects in the domain).

- It is not always possible to interpret sentences by either T/F, since quantifiers are *not always truth-functional*.
  - *Assigning a name* to every object in a given context is not always possible.
  - Specifically, names cannot be assigned to each object in y infinite sets.
- To interpret all sentences as T/F, we turn to **expansions**.
  - An **expansion** of $\mathcal{L}_2$ is a language identical to $\mathcal{L}_2$, except it may contain additional *constants*.
    * If $\mathcal{L}$ and $\mathcal{L}'$ are expansions of $\mathcal{L}_2$, we say that $\mathcal{L}'$ expands $\mathcal{L}$ if all constants of $\mathcal{L}$ are constants of $\mathcal{L}'$.
  - Let $\mathcal{L}_2^+$ be an arbitrary expansion of $\mathcal{L}_2$. An $\mathcal{L}_2^+$**-structure** is identical to an $\mathcal{L}_2$-structure, except it also assigns suitable interpretations to the new constants.
    * An $\mathcal{L}_2^+$-structure $\mathcal{A}^+$ is an **expansion** of an $\mathcal{L}_2^+$-structure $\mathcal{A}$ iff $D_{\mathcal{A}^+} = D_\mathcal{A}$ and the interpretation of every subject-specific term of $\mathcal{L}_2$ in $\mathcal{A}^+$ is identical to the interpretation assigned by $\mathcal{A}$.
- **Truth** in an $\mathcal{L}_2^+$-structure:

  Let $\mathcal{A}$ be an $\mathcal{L}_2^+$-structure:
  1. $|P^n k_1 \ldots k_n|_\mathcal{A} = T$ iff $\langle I_\mathcal{A}(k_1), \ldots, I_\mathcal{A}(k_n)\rangle \in I_\mathcal{A}(P^n)$.
  2. $|\neg\phi|_\mathcal{A} = T$ iff $|\phi|_\mathcal{A} = F$.
  3. $|\phi \wedge \psi|_\mathcal{A} = T$ iff $|\phi|_\mathcal{A} = T$ and $|\psi|_\mathcal{A} = T$.
  4. $|\phi \vee \psi|_\mathcal{A} = T$ iff $|\phi|_\mathcal{A} = T$ or $|\psi|_\mathcal{A} = T$.
  5. $|\phi \rightarrow \psi|_\mathcal{A} = T$ iff $|\phi|_\mathcal{A} = F$ or $|\psi|_\mathcal{A} = T$.
  6. $|\phi \leftrightarrow \psi|_\mathcal{A} = T$ iff $|\phi|_\mathcal{A} = |\psi|_\mathcal{A}$.
  7. $|\forall v\phi|_\mathcal{A} = T$ iff $\forall d \in D_\mathcal{A}$, there is a k and an expansion $\mathcal{A}^+$ of $\mathcal{A}$ to $\mathcal{L}_2^+ + k$ s.t. $I_{\mathcal{A}^+}(k) = d$ and $|\phi[k/v]|_{\mathcal{A}^+} = T$.
  8. $|\exists v\phi|_\mathcal{A} = T$ iff $\exists d \in D_\mathcal{A}$, there is a k and an expansion $\mathcal{A}^+$ of $\mathcal{A}$ to $\mathcal{L}_2^+ + k$ s.t. $I_{\mathcal{A}^+}(k) = d$ and $|\phi[k/v]|_{\mathcal{A}^+} = T$.

Logical notions

- **Logical truth**, $\models \phi$: ϕ is logically true iff it is true in every $\mathcal{L}_2$-structure (i.e., $|\phi|_\mathcal{A} = T$ in every $\mathcal{L}_2$-structure $\mathcal{A}$).
- **Logical falsity**: ϕ is logically false iff it is false in every $\mathcal{L}_2$-structure.
- **Validity**, $\Gamma \models \phi$: the argument with the members of Γ as premises and ϕ as conclusion is valid iff there is no $\mathcal{L}_2$-structure in which all sentences in Γ are true and ϕ is false.
- **Logical equivalence**: ϕ, ψ are logically equivalent iff they have the same truth value under every $\mathcal{L}_2$-structure.
- **Satisfiability**: Γ is satisfiable/semantically consistent iff there is an $\mathcal{L}_2$-structure in which all members of Γ are true.

Theorem 9.1 *ϕ is a logical truth iff $\neg\phi$ is a logical falsity.*

Theorem 9.2 *$\phi \models \psi$ iff $\models \phi \rightarrow \psi$.*

Theorem 9.3 *ϕ, ψ are logically equivalent iff $\models \phi \leftrightarrow \psi$.*

Theorem 9.4 *$\phi_1, \ldots, \phi_n \models \psi$ iff $\{\phi_1, \ldots, \phi_n, \neg\psi\}$ is unsatisfiable.*

Natural Deduction

- **Fitch-style natural deduction calculus**:
  - **Proofs** are finite lists of sentences of $\mathcal{L}_2$.
  - Every proof begins with **premises** or ***assumptions*** *that must be discharged* later.
  - Rules allow us to append only a single sentence in each step.
  - The last sentence ϕ that has been appended is the sentence that the proof proves from the premises $\phi_1, \ldots, \phi_n$.
    * We write $\{\phi_1, \ldots, \phi_n\} \vdash \phi$ or $\phi_1, \ldots, \phi_n \vdash \phi$ to indicate that there is such a proof.
- Admissible inference steps are:
  - **Valid** arguments.
  - **Sufficient** so that it is possible to offer a proof for every valid argument.
- For each connective, there are:
  - **Introduction rules** which allow us to infer sentences with it as their *main connective*.
  - **Elimination rules** which allow us to extract information from sentences with it as the *main connective*.
- Rules:
  - **$\wedge$Intro**: $\phi, \psi \Rightarrow \phi \wedge \psi$.
  - **$\wedge$Elim1**: $\phi \wedge \psi \Rightarrow \phi$.
    $\wedge$Elim2: $\phi \wedge \psi \Rightarrow \psi$.
  - **$\rightarrow$Intro**: ψ, assumption that $\phi \Rightarrow \phi \rightarrow \psi$ and discharge this assumption.
    * **Closed subproof**: list of sentences wrapped around the square bracket.
    * A sentence ϕ is **available** in a step of a proof if it occurs before this step outside all closed subproofs.
    * Once we established our conditional statement, the assumption is **no longer in place**.
  - **$\rightarrow$Elim**: $\phi, \phi \rightarrow \psi \Rightarrow \psi$.
  - **$\vee$Intro1**: $\phi \Rightarrow \phi \vee \psi$.
    $\vee$Intro2: $\psi \Rightarrow \phi \vee \psi$.
  - **$\vee$Elim**: $\phi \vee \psi, \phi \rightarrow \chi, \psi \rightarrow \chi \Rightarrow \chi$.
  - **$\neg$Intro**: $\psi, \neg\psi$, assumption that $\phi \Rightarrow \neg\phi$ and discharge this assumption.
  - **$\neg$Elim**: $\neg\neg\phi \Rightarrow \phi$.
  - **$\leftrightarrow$Intro**: $\phi \rightarrow \psi, \psi \rightarrow \phi \Rightarrow \phi \leftrightarrow \psi$.
  - **$\leftrightarrow$Elim1**: $\phi \leftrightarrow \psi, \phi \Rightarrow \psi$.
    $\leftrightarrow$Elim2: $\phi \leftrightarrow \psi, \psi \Rightarrow \phi$.
  - **$\forall$Elim**: $\forall v\phi \Rightarrow \phi[k/v]$.
  - **$\forall$Intro**: $\phi[k/v] \Rightarrow \forall v\phi$, if k does not occur in any premise, undischarged assumption, or ϕ.
    * If ϕ holds of an **arbitrary** object k, ϕ holds of every object.
  - **$\exists$Intro**: $\phi[k/v] \Rightarrow \exists v\phi$.
  - **$\exists$Elim**: $\exists v\phi, \phi[k/v] \rightarrow \psi \Rightarrow \psi$, if k does not occur in any premise, undischarged assumption, ϕ, or ψ.
    * If ϕ and the assumption that an **arbitrary** object is such that ϕ implies ψ, then ψ is true.

Proof-Theoretic Validity

- **Proof** of ϕ from Γ: a finite list of $\mathcal{L}_2$ ending with ϕ, each of which:
  1. Is a member of Γ (i.e., a premise), or
  2. Is an *assumption* that has been *discharged* by the application of one of the discharging inference rules, or
  3. Follows by one of the inference rules from sentences that occur previously in the list, and are available at the time in which the rule is applied (i.e., they do not occur in a closed subproof).
- **Proof-theoretic validity**: the argument with premises Γ and ϕ as conclusion is *proof-theoretically valid* iff there is a proof of ϕ from Γ (i.e., if $\Gamma \vdash \phi$).
- **Theoremhood**: ϕ is a *theorem* iff there is a proof of ϕ without premises (i.e., if $\varnothing \vdash \phi$, or $\vdash \phi$).
- Consistency:
  - **Negation consistency**: Γ is *negation consistent* iff there is no ϕ s.t. $\Gamma \vdash \phi$ and $\Gamma \vdash \neg\phi$.
  - **Absolute consistency**: Γ is *absolutely consistent* iff there is a ϕ s.t. $\Gamma \nvdash \phi$.

Theorem 12.1 *Γ is negation consistent iff it is absolutely consistent.*

Theorem 12.2 **(Soundness and completeness)** *$\Gamma \models \phi$ iff $\Gamma \vdash \phi$.*

Theorem 12.3 *$\models \phi$ iff $\vdash \phi$.*

Theorem 12.4 *ϕ is a logical falsity iff $\vdash \neg\phi$.*

Theorem 12.5 *ϕ and ψ are logically equivalent iff $\vdash \phi \leftrightarrow \psi$.*

Theorem 12.6 *Γ is satisfiable iff Γ is consistent.*