№1

$\xi \sim R(0, \theta)$ $\quad M\xi = \frac{\theta}{2} \quad M[\xi^2] = \frac{\theta^2}{3} \quad D\xi = \frac{\theta^2}{12}$

$\vec{X}_n = (x_1, \dots, x_n)$

1) Оценка $\tilde{\theta}_1 = 2\bar{x} = 2 \cdot \frac{1}{n} \cdot \sum_{i=1}^{n} x_i$

• несмещенность: ✓

$$M[\tilde{\theta}_1] = M\left[\frac{2}{n}\sum_{i=1}^{n} x_i\right] = \frac{2}{n} \cdot \sum_{i=1}^{n} M[x_i] = \frac{2}{n} \cdot \sum_{i=1}^{n} M\xi = \frac{2}{n} \cdot n \cdot M\xi = \frac{2}{n} \cdot n \cdot \frac{\theta}{2} = \theta$$

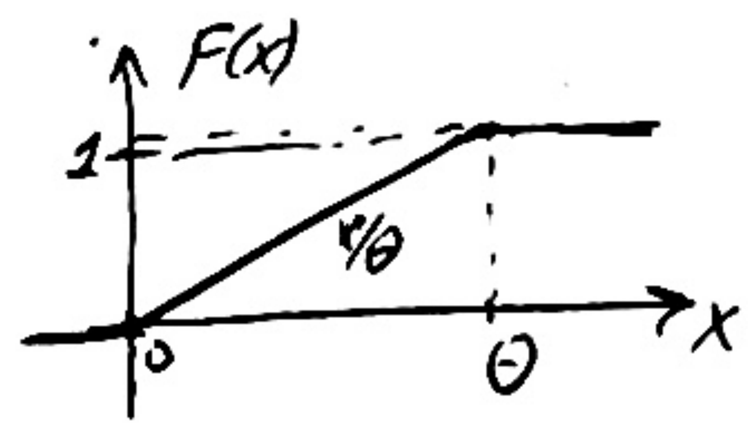
• состоятельность: ✓

$$D[\tilde{\theta}_1] = D\left[\frac{2}{n}\sum_{i=1}^{n} x_i\right] = \frac{4}{n^2}\sum_{i=1}^{n} Dx_i = \frac{4}{n^2}\sum_{i=1}^{n} D\xi = \frac{4}{n^2} \cdot n \cdot \frac{\theta^2}{12} = \frac{\theta^2}{3n} \xrightarrow[n \to \infty]{} 0$$

2) Оценка $\tilde{\theta}_2 = x_{min}$

$\xi \sim F(x)$

$\min(\xi_1, \dots, \xi_n) \sim 1 - (1 - F(y))^n = \Phi(y)$



$\varphi(y) = \Phi'(y) = n(1 - F(y))^{n-1} p(y)$, где

$p(y)$ – ф-ия распределения

$F(x)$ – величины ξ

$p(y) = F'(y) = \frac{1}{\theta}\ (0, \theta)$

• несмещенность ✓ (fixed)

$$M[\tilde{\theta}_2] = \int_0^\theta y\,\varphi(y)dy = \int_0^\theta y \cdot n(1 - F(y))^{n-1} p(y)\, dy = \int_0^\theta yn \cdot \frac{1}{\theta}\left(1 - \frac{y}{\theta}\right)^{n-1} dy = \left\{ t = \frac{y}{\theta} \right\} =$$

$$= \int_0^1 t\, n(1-t)^{n-1}\theta\, dt = n \cdot \theta \cdot B(2, n) = n\theta \cdot \frac{\Gamma(2)\,\Gamma(n)}{\Gamma(n+2)} = \frac{\theta}{n+1} \text{ – смещ.}$$

Исправим $\tilde{\theta}_2' = \tilde{\theta}_2(n+1) = (n+1) \cdot x_{min}$

~~Далее $\tilde{\theta}_2 = \tilde{\theta}_2'$~~

• состоятельность ✗

по опр. $\tilde{\theta}_2' \xrightarrow{P} \theta$

$\forall \varepsilon > 0 \quad P(|\tilde{\theta}_2' - \theta| \geq \varepsilon) \xrightarrow[n \to +\infty]{} 0$

$$P(|\tilde{\theta}_2' - \theta| \geq \varepsilon) \geq P(\tilde{\theta}_2' \geq \theta + \varepsilon) = P(\cancel{\min}(n+1)\tilde{\theta}_2 \geq \theta + \varepsilon) = P\left(\tilde{\theta}_2 \geq \frac{\theta + \varepsilon}{n+1}\right) =$$

$$= P\left(x_1 \geq \frac{\theta + \varepsilon}{n+1}, \dots, x_n \geq \frac{\theta + \varepsilon}{n+1}\right) = \prod_{i=1}^{n} P\left(x_i \geq \frac{\theta + \varepsilon}{n+1}\right) = \left(P\left(\xi \geq \frac{\theta + \varepsilon}{n+1}\right)\right)^n =$$

$$= \left(1 - P\left(\xi < \frac{\theta + \varepsilon}{n+1}\right)\right)^n = \left(1 - F\left(\frac{\theta + \varepsilon}{n+1}\right)\right)^n = \left(1 - \frac{\theta + \varepsilon}{\theta(n+1)}\right)^n \longrightarrow e^{-\frac{\theta + \varepsilon}{\theta}} \neq 0$$

не сост.

3) $\tilde{\theta}_3 = X_{max}$

$\xi \sim F(x)$

$\max(\xi_1, \dots, \xi_n) \sim (F(y))^n = \Phi(y)$

$\varphi(y) = \Phi'(y) = n\,(F(y))^{n-1} \cdot p(y) = n \cdot \left(\frac{y}{\theta}\right)^{n-1} \cdot \frac{1}{\theta}(0,\theta)$

• несмещенность $\vee$ (fixed)

$M[\tilde{\theta}_3] = \int_0^{\theta} n \cdot \left(\frac{y}{\theta}\right)^{n-1} \cdot \frac{1}{\theta}\, y \cdot dy = n \cdot \int_0^{\theta} \left(\frac{y}{\theta}\right)^{n} dy = \theta n \cdot \int_0^1 t^n dt = \frac{n\theta}{n+1}$ - смещ

исправим: $\tilde{\theta}_3' = \frac{n+1}{n}\tilde{\theta}_3 = \frac{n+1}{n} X_{max}$

• состоятельность $\vee$

$M[\tilde{\theta}_3^2] = \int_0^{\theta} y^2 \left(\frac{y}{\theta}\right)^{n-1} \frac{n}{\theta}\, dy = \frac{n}{n+2}\theta^2$

$D[\tilde{\theta}_3] = \frac{n}{n+2}\theta^2 - \frac{n^2}{(n+1)^2}\theta^2 = \theta^2 \cdot \left\{ \frac{n^3 + 2n^2 + n - n^3 - 2n^2}{(n+2)(n+1)^2} \right\} = \frac{n\theta^2}{(n+2)(n+1)^2}$

$D[\tilde{\theta}_3'] = D\left[\frac{n+1}{n}\tilde{\theta}_3\right] = \frac{\cancel{(n+1)^2}}{n^2} \cdot \frac{n}{(n+2)\cancel{(n+1)^2}}\theta^2 = \frac{\theta^2}{n(n+2)} \xrightarrow[n \to +\infty]{} 0$ - сост.

4) $\tilde{\theta}_4 = X_{min} + X_{max}$

• несмещенность $\vee$

$M[\tilde{\theta}_4] = M[\tilde{\theta}_2 + \tilde{\theta}_3] = M[\tilde{\theta}_2] + M[\tilde{\theta}_3] = \frac{\theta}{n+1} + \frac{\theta n}{n+1} = \theta$ - несм

• состоятельность $\vee$

$D[\tilde{\theta}_4] = D[\tilde{\theta}_2] + D[\tilde{\theta}_3] + 2\operatorname{cov}(\tilde{\theta}_2, \tilde{\theta}_3)$

$\operatorname{cov}(\tilde{\theta}_2, \tilde{\theta}_3) = M[\tilde{\theta}_2 \cdot \tilde{\theta}_3] - M[\tilde{\theta}_2] \cdot M[\tilde{\theta}_3]$

$K(y,z) = \begin{cases} F^n(z) - (F(z) - F(y))^n, & z \geq y \\ F^n(z), & z < y \end{cases}$

$\varkappa(y,z) = \frac{\partial^2 K}{\partial y\, \partial z} = \frac{\partial}{\partial z}\left(\frac{\partial K}{\partial y}\right) = \frac{\partial}{\partial z}\left(n(F(z) - F(y))^{n-1} F'(y)\right) = n(n-1)\big(\underbrace{F(z)}_{z/\theta} - \underbrace{F(y)}_{y/\theta}\big)^{n-2} \underbrace{F'(z)}_{\frac{1}{\theta}(0,\theta)} \underbrace{F'(y)}_{\frac{1}{\theta}(0,\theta)}$

$z \geq y$

$M[\tilde{\theta}_2 \cdot \tilde{\theta}_3] = \iint_{-\infty}^{+\infty} yz \cdot \varkappa(y,z)\, dy dz = \dots = \frac{1}{n+2}\theta^2$

$\operatorname{cov}(\tilde{\theta}_2, \tilde{\theta}_3) = \frac{\theta^2}{n+2} - \theta^2 \frac{n}{(n+1)^2} = \theta^2 \frac{1}{(n+2)(n+1)^2}$

$D[\tilde{\theta}_4] = \theta^2\left(\frac{n}{(n+1)^2(n+2)} + \frac{n}{(n+1)^2(n+2)} + 2 \cdot \frac{1}{(n+2)(n-1)^2}\right) = \frac{2\theta^2}{(n+1)(n+2)} \xrightarrow[n \to \infty]{} 0$ - сост

5) $\tilde{\theta}_5 = X_1 + \frac{1}{n+1}\sum_{i=2}^{n} X_i$

• несмещенность ✓

$$M[\tilde{\theta}_5] = M[X_1] + \frac{1}{n+1}\cdot(n-1)\cdot M[X_2] = \frac{\theta}{2} + \frac{n-1}{n-1}\frac{\theta}{2} = \theta$$ — смещ

• состоятельность ✗

$$D[\tilde{\theta}_5] = D[X_1] + \frac{1}{(n-1)^2}(n-1)\cdot D[X_2] = \frac{\theta^2}{12}\left(1+\frac{1}{n+1}\right) \underset{n\to\infty}{\not\to} 0$$

по опр: $\underset{\downarrow\, X_1}{\tilde{\theta}_5 = X_2} + \underbrace{\frac{1}{n-1}\sum_{i=2}^{n} X_i}_{M\xi=\frac{\theta}{2}} \xrightarrow[\times]{P} X_1 + \frac{\theta}{2} \neq 0$ — не сост

$\xi_n + \eta_n \xrightarrow{P} \xi + \eta$

Эффективности оценок:

① $D[\tilde{\theta}_1] = \frac{\theta^2}{3n} \sim \frac{1}{n}$ ✗

② $D[\tilde{\theta}_2'] = \frac{n\theta^2}{n+2}$ точно нет

$\frac{1}{n(n+2)} < \frac{2}{(n+1)(n+2)}$

$\frac{1}{n} < \frac{2}{n+1}$

⑤ $D[\hat{\theta}_5] = \frac{\theta^2}{12}\left(1+\frac{1}{n+1}\right)$ точно нет

③ $D[\tilde{\theta}_3'] = \frac{\theta^2}{n(n+2)} \sim \frac{1}{n^2}$

④ $D[\tilde{\theta}_4] = \frac{2\theta^2}{(n+1)(n+2)} \sim \frac{1}{n^2}$

$\Rightarrow$ $\tilde{\theta}_3'$ — самая эффективная.