№5

$$\xi \sim p(x) = \begin{cases} \dfrac{\theta-1}{x^{\theta}}, & x \geq 1 \\ 0, & x < 1 \end{cases}, \quad \theta > 1$$

$\vec{X}_n$ — выборка

а) По $\vec{X}_n$ найти оценку θ методом МП.

$$L(\theta) = \prod_{i=1}^{n} p(x_i, \theta)$$

$$L(\theta) = \frac{(\theta-1)^n}{(x_1 \dots x_n)^{\theta}} \quad (x_{min} \geq 1)$$

$$\ln L(\theta) = n \ln(\theta-1) - \theta(\ln x_1 + \dots + \ln x_n)$$

$$\frac{\partial \ln L(\theta)}{\partial \theta} = \frac{n}{\theta-1} - (\ln x_1 + \dots + \ln x_n) = 0 \Rightarrow \tilde{\theta} = 1 + \frac{n}{\ln x_1 + \dots + \ln x_n} \geq 1$$

в) $\cancel{\int_1^m \frac{\theta-1}{x^{\theta}} dx = 1/2}$, ~~$m$ — медиана~~

$$\cancel{(\theta-1) \cdot \left. \frac{x^{-\theta+1}}{-\theta+1} \right|_1^m = 1/2}$$

$$\cancel{1 - \frac{1}{m^{\theta-1}} = 1/2}$$

$$\cancel{\frac{1}{m^{\theta-1}} = 1/2 \Rightarrow m = 2^{\frac{1}{\theta-1}} \Rightarrow \tilde{m} = 2^{\frac{1}{\tilde{\theta}-1}}}$$

с) ~~Байесовская оценка θ, байесовский дов. инт.~~

в) Медиана: $\int_1^m \frac{\theta-1}{x^{\theta}} dx = 1/2 \Rightarrow m = 2^{\frac{1}{\theta-1}}$ и оценка $\tilde{m} = 2^{\frac{1}{\tilde{\theta}-1}}$

Ас. дов. инт по ОМП:

$$\frac{m(\tilde{\theta}) - m(\theta)}{\sigma} \sqrt{n} \sim N(0,1)$$

$$\cancel{\sigma = \sqrt{(\nabla m(\tilde{\theta}))^T \cdot I^{-1}(\theta) \, (\nabla m(\tilde{\theta}))}}$$

$$\sigma = \sqrt{(\nabla m(\theta))^T \, I^{-1}(\theta) \, (\nabla m(\theta))}$$

$$\nabla m(\theta) = \left(2^{\frac{1}{\theta-1}} \ln 2 \cdot -\frac{1}{(\theta-1)^2} \right) = (\nabla m(\theta))^T$$

проверка на сильную регул. далее

$$I(\theta) = -M\left[\frac{\partial^2 \ln p}{\partial \theta^2}\right] = -M\left[\frac{\partial^2}{\partial \theta^2}\left(\ln(\theta-1) - \theta \ln x\right)\right] = M\left[\frac{1}{(\theta-1)^2}\right] = \frac{1}{(\theta-1)^2}$$

$$\frac{m(\theta) - 2^{\frac{1}{\tilde{\theta}-1}}}{2^{\frac{1}{\tilde{\theta}-1}} \cdot \ln 2} \cdot +(\tilde{\theta}-1)^2 \cdot \sqrt{n} \sqrt{\frac{1}{(\tilde{\theta}-1)^2}} \sim N(0,1)$$

$$(\tilde{\theta}-1)\sqrt{n} \cdot \frac{m(\theta) - 2^{\frac{1}{\tilde{\theta}-1}}}{\ln 2 \cdot 2^{\frac{1}{\tilde{\theta}-1}}} \sim N(0,1)$$

Доверительный интервал найдем из

$$P\left(u_{\frac{1-\beta}{2}} < (\tilde\theta-1)\sqrt{n}\,\frac{m(\theta)-2^{\frac{1}{\tilde\theta-1}}}{2^{\frac{1}{\tilde\theta-1}}\cdot\ln 2} < u_{\frac{1+\beta}{2}}\right)=\beta$$

$$u_{\frac{1-\beta}{2}}\cdot\frac{\ln 2\cdot 2^{\frac{1}{\tilde\theta-1}}}{\sqrt{n}(\tilde\theta-1)} < \underbrace{m(\theta)}_{m}-2^{\frac{1}{\tilde\theta-1}} < u_{\frac{1+\beta}{2}}\cdot\frac{\ln 2\cdot 2^{\frac{1}{\tilde\theta-1}}}{\sqrt{n}(\tilde\theta-1)}$$

$$u_{\frac{1-\beta}{2}}\cdot\frac{\ln 2\cdot 2^{\frac{1}{\tilde\theta-1}}}{\sqrt{n}(\tilde\theta-1)}+2^{\frac{1}{\tilde\theta-1}} < m < u_{\frac{1+\beta}{2}}\cdot\frac{\ln 2\cdot 2^{\frac{1}{\tilde\theta-1}}}{\sqrt{n}(\tilde\theta-1)}+2^{\frac{1}{\tilde\theta-1}}$$

$$I=\left(u_{\frac{1-\beta}{2}}\cdot\frac{\ln 2\cdot 2^{\frac{1}{\tilde\theta-1}}}{\sqrt{n}\cdot(\tilde\theta-1)}+2^{\frac{1}{\tilde\theta-1}};\ u_{\frac{1+\beta}{2}}\cdot\frac{\ln 2\cdot 2^{\frac{1}{\tilde\theta-1}}}{\sqrt{n}(\tilde\theta-1)}+2^{\frac{1}{\tilde\theta-1}}\right)$$

Проверка на регул. / сильную рег-ть:

① $\frac{\partial}{\partial\theta}\int_{-\infty}^{+\infty}p(x)\,dx=\int_{-\infty}^{+\infty}\frac{\partial p(x)}{\partial\theta}dx \Rightarrow \int_{-\infty}^{+\infty}\frac{\partial p(x)}{\partial\theta}dx=0$

$$\frac{\partial p(x)}{\partial\theta}=\frac{1}{x^\theta}-\frac{(\theta-1)\ln x}{x^\theta}$$

$$\underbrace{\int_1^{+\infty}\frac{dx}{x^\theta}}_{ⓐ}-(\theta-1)\underbrace{\int_1^{+\infty}\frac{\ln x\,dx}{x^\theta}}_{ⓑ}=\frac{1}{\theta-1}-\frac{1}{\theta-1}=0$$

ⓐ $\int_1^{+\infty}\frac{dx}{x^\theta}=\frac{1}{-\theta+1}\cdot(0-1)=\frac{-1}{1-\theta}$

ⓑ $\int_1^{+\infty}\frac{\ln x\,dx}{x^\theta}=\int_0^{+\infty}\frac{t\,e^t dt}{e^{\theta t}}=\int_0^{+\infty}t\,e^{-(\theta-1)t}dt=\frac{1}{\theta-1}\int_0^{+\infty}\tau e^{-\tau}d\tau=\frac{1}{\theta-1}$

② $\frac{\partial^2}{\partial\theta^2}\int_{-\infty}^{+\infty}p(x)\,dx=\int_{-\infty}^{+\infty}\frac{\partial^2 p(x)}{\partial\theta^2}dx \Rightarrow \int_{-\infty}^{+\infty}\frac{\partial^2 p(x)}{\partial\theta^2}dx=0$

$$\int_1^{+\infty}\Big(\underbrace{-\frac{2\ln x}{x^\theta}}_{ⓐ}+\underbrace{\frac{(\theta-1)\ln^2 x}{x^\theta}}_{ⓑ}\Big)dx=-\frac{2}{(\theta-1)^2}+\frac{2}{(\theta-1)^2}=0$$

ⓐ $-2\int_1^{+\infty}\frac{\ln x}{x^\theta}dx=-2\int_0^{+\infty}\frac{t\,e^t dt}{e^{\theta t}}=-2\cdot\int_0^{+\infty}t\,e^{-(\theta-1)t}dt=-\frac{2}{(\theta-1)^2}\cdot\int_0^{+\infty}\tau e^{-\tau}d\tau=-\frac{2}{(\theta-1)^2}$

ⓑ $(\theta-1)\int_1^{+\infty}\frac{\ln^2 x}{x^\theta}dx=(\theta-1)\int_0^{+\infty}t^2e^{-(\theta-1)t}dt=\frac{\theta-1}{(\theta-1)^3}\cdot\int_0^{+\infty}\tau^2e^{-\tau}d\tau=\frac{1}{(\theta-1)^2}\cdot 2=\frac{2}{(\theta-1)^2}$

$\Rightarrow$ модель сильно регулярна.

d) $\xi \sim p(x) = \begin{cases} \frac{\ldots}{x^{\theta}}, & x \geq 1 \\ 0, & x < 1 \end{cases} \quad \theta > 1$

$F(x) = \int_{1}^{x} \frac{\theta-1}{t^{\theta}} dt = (\theta-1) \cdot \frac{1}{-\theta+1} \left(t^{-\theta+1}\right)\Big|_{1}^{x} = -\left(\frac{1}{x^{\theta-1}} - \frac{1}{(1)^{\theta-1}}\right) = 1 - \frac{1}{x^{\theta-1}}$

$\ln \xi \sim \Phi(y)$

$\Phi(y) = P(\ln \xi < y) = P(\xi < e^{y}) = F(e^{y}) = 1 - \frac{1}{e^{y(\theta-1)}}$

$\varphi(y) = \left(-e^{-y(\theta-1)}\right)'_{y} = -e^{-y(\theta-1)} \cdot (-(\theta-1)) = (\theta-1) e^{-y(\theta-1)}$

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d) $\frac{\tilde{\theta} - \theta}{\sigma} \sqrt{n} \sim N(0,1)$ – асимпт. по ОМП

$f(\theta) = \theta \qquad f'(\theta) = 1 \qquad J(\theta) = \frac{1}{(\theta-1)^2}$

$(\tilde{\theta} - \theta) \cdot \frac{\sqrt{n}}{(\theta-1)} \sim N(0,1)$

$P\left(u_{\frac{1-\beta}{2}} < (\tilde{\theta}-\theta) \cdot \frac{\sqrt{n}}{\tilde{\theta}-1} < u_{\frac{1+\beta}{2}}\right) = \beta$

$\frac{\tilde{\theta}-1}{\sqrt{n}} u_{\frac{1-\beta}{2}} < \tilde{\theta} - \theta < \frac{\tilde{\theta}-1}{\sqrt{n}} u_{\frac{1+\beta}{2}}$

$\tilde{\theta} + \frac{\tilde{\theta}-1}{\sqrt{n}} u_{\frac{1+\beta}{2}} < \theta < \tilde{\theta} - \frac{\tilde{\theta}-1}{\sqrt{n}} u_{\frac{1-\beta}{2}}$

$I = \left(\tilde{\theta} - \frac{\tilde{\theta}-1}{\sqrt{n}} u_{\frac{1+\beta}{2}} ; \tilde{\theta} - \frac{\tilde{\theta}-1}{\sqrt{n}} u_{\frac{1-\beta}{2}}\right)$, где $\tilde{\theta} = 1 + \frac{n}{\sum_{i=1}^{n} \ln x_i}$

c) Априорная пл-ть $p(y) = \begin{cases} e^{1-y}, & y \geq 1 \\ 0, & y < 1 \end{cases}$