№4

$\xi \sim R(\theta, 2\theta)$, $\theta > 0$

$\vec{X_n}$ - выборка

а) • ОММ (по 1 моменту)

$\tilde{\alpha}_1 = \frac{1}{n}\sum_{i=1}^{n} x_i = \overline{X}$ $\quad \alpha_1 = M[\xi] = \int_{\theta}^{2\theta} \frac{x}{\theta} dx = \frac{3}{2}\theta \to \alpha_1(\tilde{\theta})$

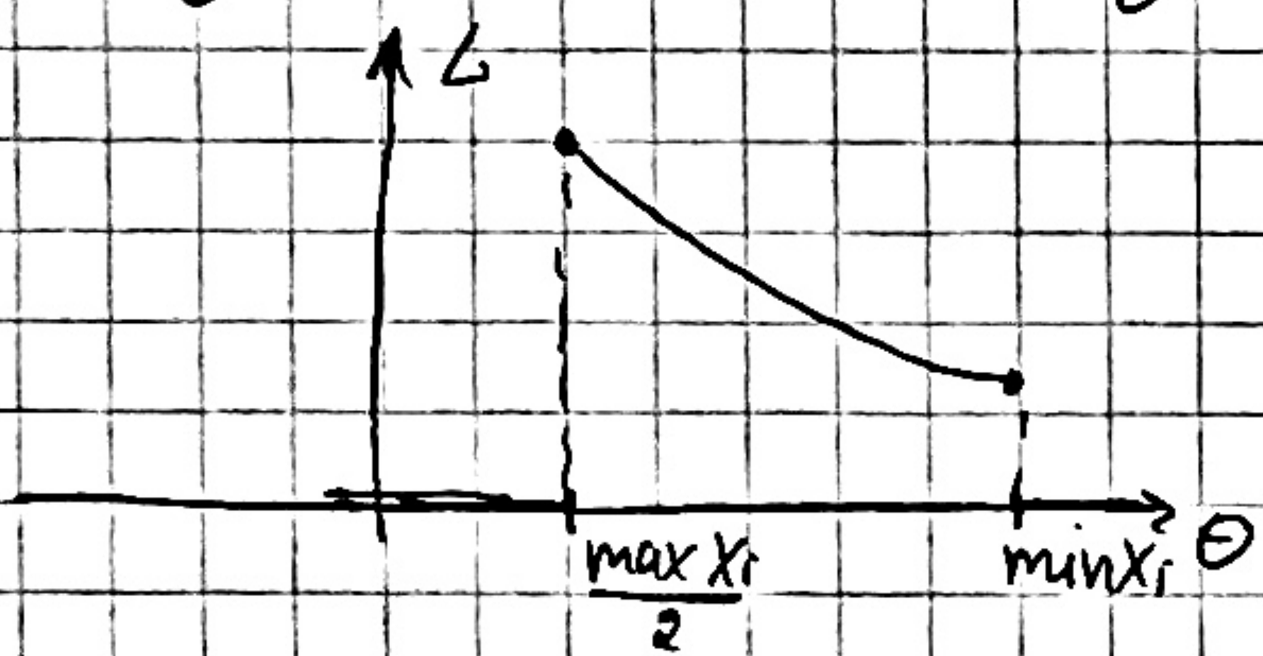
$\frac{3}{2}\tilde{\theta} = \overline{X} \quad \Rightarrow \quad \tilde{\theta}_1 = \frac{2}{3}\overline{X}$

• ОМП

$p(x_i, \theta) = \frac{1}{\theta} \; (\theta \le x_i \le 2\theta)$

$L = \frac{1}{\theta^n}$ - если все $x_i \in [\theta, 2\theta]$

$L = \frac{1}{\theta^n} \; (\theta \le x_i \le 2\theta \;\; \forall i) = \frac{1}{\theta^n} \; (\min x_i \geq \theta, \; \max x_i \le 2\theta)$



$\sup\limits_{\theta} L$ при $\theta = \frac{\max x_i}{2}$

$\tilde{\theta}_2 = \frac{\max x_i}{2}$

• $\theta_3 = \frac{1}{5}(X_{min} + 2X_{max})$

Имеем: $\tilde{\theta}_1 = \frac{2}{3}\overline{X}$, $\tilde{\theta}_2 = \frac{1}{2}X_{max}$, $\tilde{\theta}_3 = \frac{1}{5}(X_{min} + 2X_{max})$

б) • $\tilde{\theta}_1 = \frac{2}{3}\overline{X}$ (несмещена-✓ ; состоятельна-✓)

$M[\tilde{\theta}_1] = M[\frac{2}{3} \cdot \frac{1}{n} \cdot \sum_{i=1}^{n} x_i] = \frac{2}{3} \cdot \frac{1}{n} \cdot n \cdot \frac{3}{2}\theta = \theta \;\Rightarrow$ несмещена

$D[\tilde{\theta}_1] = D[\frac{2}{3} \cdot \frac{1}{n} \cdot \sum_{i=1}^{n} x_i] = \frac{4}{9n^2} \cdot n \cdot D\xi = \frac{4}{9n} \cdot \frac{\theta^2}{12} = \frac{\theta^2}{27n} \xrightarrow[n \to +\infty]{} 0$

$\Rightarrow$ состоятельна

(по дост. усл сост)

• $\tilde{\theta}_2 = \frac{1}{2}X_{max}$

$\max(\xi_1, \ldots, \xi_n) \sim (F(y))^n = \Phi(y)$

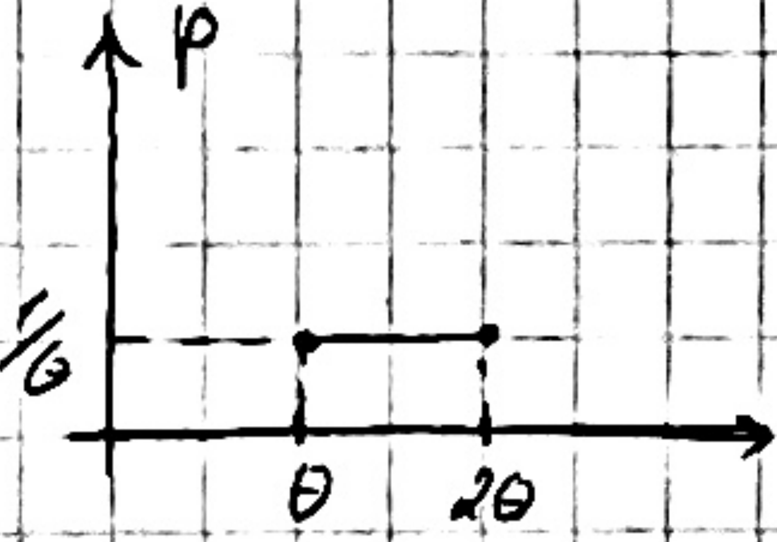
$\varphi(y) = n \cdot (F(y))^{n-1} \cdot p(y) = n \cdot \left(\frac{x}{\theta}\right)^{n-1} \cdot \frac{1}{\theta} \; (\theta, 2\theta)$

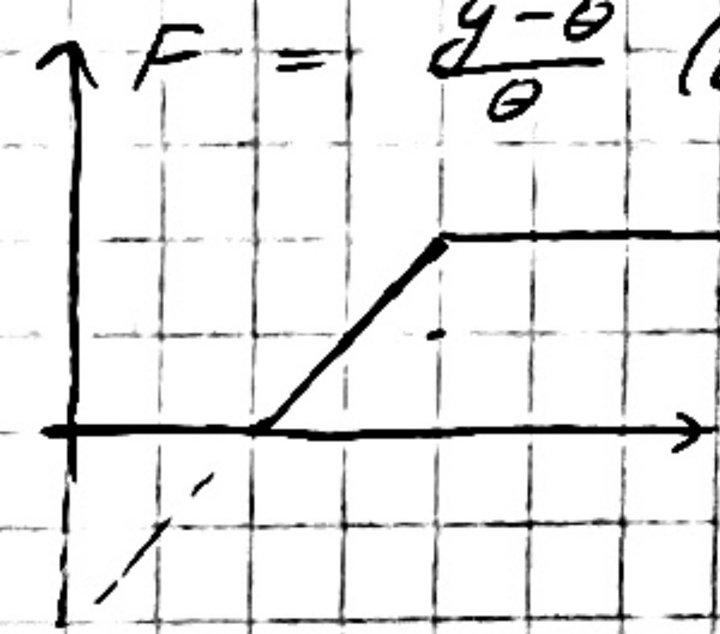
$M[\tilde{\theta}_2] = M[\frac{1}{2}X_{max}] = \int_{\theta}^{2\theta} \frac{x}{2} n \frac{x^{n-1}}{\theta^{n-1}} \cdot \frac{dx}{\theta} = \int_1^2 \frac{1}{2} t^{n-1} dt =$

$= \frac{1}{2_{2\theta}} \cdot \frac{t^n}{n}\Big|_1^2 = \frac{2^n - 1}{2n}$

$M[\tilde{\theta}_2] = M[\frac{1}{2}X_{max}] = \int_{\theta}^{2\theta} \frac{x}{2} \cdot n \cdot \left(\frac{x}{\theta}\right)^{n-1} \frac{1}{\theta} dx = \frac{n}{2}$

• $\tilde{\theta}_2 = \frac{1}{2} x_{max}$ (несм fix сост $\checkmark$)





$x_{max} \sim (F(x))^n$

$$\varphi(y) = \left(F^n(y)\right)'_y = \left(\left(\frac{y-\theta}{\theta}\right)^n\right)'_y =$$

$$= n\left(\frac{y-\theta}{\theta}\right)^{n-1}\cdot\frac{1}{\theta} \quad (\theta, 2\theta)$$

$$M[\tilde{\theta}_2] = M[\tfrac{1}{2} x_{max}] = \int_{\theta}^{2\theta} x\cdot\frac{1}{2}\cdot\frac{n}{\theta}\cdot\frac{(x-\theta)^{n-1}}{\theta^{n-1}}\,dx =$$

$$= \{t = x-\theta\} = \frac{n}{2}\int_0^{\theta}\frac{t+\theta}{\theta}\,\frac{(t^{n-1})}{\theta^{n-1}}\,dt = \frac{n}{2}\int_0^{\theta}\frac{t^n}{\theta^n}\,dt + \frac{n}{2}\int_0^{\theta}\frac{t^{n-1}}{\theta^{n-1}}\,dt =$$

$$= \frac{n}{2}\cdot\theta\cdot\frac{1}{n+1} + \frac{n}{2}\cdot\theta\cdot\frac{1}{n} = \frac{\theta}{2}\left(1+\frac{n}{n+1}\right) = \frac{2n+1}{2n+2}\,\theta \Rightarrow \text{смещена}$$

исправим: $\tilde{\theta}_2' = \frac{2n+2}{2n+1}\cdot\frac{1}{2}x_{max} = \frac{n+1}{2n+1}x_{max}$

$$M[\tilde{\theta}_2^2] = \frac{1}{4}\int_{\theta}^{2\theta} x^2\, n\left(\frac{x-\theta}{\theta}\right)^{n-1}\frac{1}{\theta}\,dx = \frac{n}{4}\int_{\theta}^{2\theta}\frac{x^2(x-\theta)^{n-1}}{\theta^n}\,dx = \{t = x-\theta\} =$$

$$= \frac{n}{4}\int_0^{\theta}\frac{(t+\theta)^2\,t^{n-1}}{\theta^n}\,dt = \frac{n}{4}\int_0^{\theta}\frac{t^{n+1}}{\theta^n}\,dt + \frac{n}{2}\int_0^{\theta}\frac{t^n}{\theta^{n-1}}\,dt + \frac{n}{4}\int_0^{\theta}\frac{t^{n-1}}{\theta^{n-2}}\,dt =$$

$$= \left\{\tau = \frac{t}{\theta}\quad dt = \theta\,d\tau\right\} = \frac{n}{4}\int_0^1\theta^2\tau^{n+1}\,d\tau + \frac{n}{2}\int_0^1\theta^2\tau^n\,d\tau + \frac{n}{4}\int_0^1\theta^2\tau^{n-1}\,d\tau =$$

$$\cancel{= \frac{n\theta^2}{4}\left(\frac{1}{n+2} + \frac{1}{2}\cdot\frac{2}{n+1} + \frac{1}{n}\right) = \frac{n\theta^2}{4}\left(\frac{2n^2+2n+n^2+2n+2n^2+6n+4}{2n(n+1)(n+2)}\right) =}$$

$$\cancel{= \frac{\theta^2}{8(n+1)(n+2)}(5n^2+8n+4)}$$

$$= \frac{n\theta^2}{4}\left(\frac{1}{n}+\frac{1}{n+2}+\frac{2}{n+1}\right) = \frac{\cancel{n}\theta^2(4n^2+8n+2)}{4\cancel{n}(n+1)(n+2)}$$

$$\cancel{D[\tilde{\theta}_2] = \theta^2\,\frac{5n^2+8n+2}{8(n+1)(n+2)} - \theta^2\frac{(2n+1)^2}{4(n+1)^2} = \frac{\theta^2}{8} =}$$

$$\cancel{= \frac{\theta^2}{8(n+1)}\left\{\frac{5n^2+8n+2}{n+2} - \frac{8n^2+8n+2}{n+1}\right\} = \frac{\theta^2}{8(n+1)^2(n+2)}\{(n+1)(5n^2+8n+2)-(n+2)(8n^2+8n+2)\}}$$

$$D[\tilde{\theta}_2] = \frac{\theta^2}{4}\cdot\frac{4n^2+8n+2}{(n+1)(n+2)} - \frac{\theta^2}{4}\cdot\frac{(2n+1)^2}{(n+1)^2} = \frac{\theta^2}{4(n+1)}\left\{\frac{4n^2+8n+2}{n+2} - \frac{4n^2+4n+1}{n+1}\right\} =$$

$$= \frac{\theta^2}{4(n+1)}\cdot\frac{\cancel{n(4n-3)}\,n}{(n+1)(n+2)} = \frac{\theta^2\cdot n\cancel{(4n-3)}}{4(n+1)^2(n+2)}$$

$$D[\tilde{\theta}_2'] = \left(\frac{2n+2}{2n+1}\right)^2\cdot\frac{n\cancel{(4n-3)}}{(n+1)^2(n+2)}\,\frac{\theta^2}{4} \xrightarrow[n\to+\infty]{} 0$$

$\to$ сост по дост усл

$\cdot\ \tilde{\theta}_3 = \frac{1}{5} X_{min} + \frac{2}{5} X_{max}$ (несм - fix ; сост ✓)

$M[\tilde{\theta}_3] = \frac{1}{5} M[X_{min}] + \frac{2}{5} M[X_{max}]$

$M[X_{min}] = \int_{\theta}^{2\theta} x \cdot n(1-F(x))^{n-1} p(x)\, dx =$

$= \int_{\theta}^{2\theta} x \cdot n \left(1 - \frac{x-\theta}{\theta}\right)^{n-1} \frac{1}{\theta}\, dx = \int_{\theta}^{2\theta} \frac{xn}{\theta} \left(\frac{2\theta - x}{\theta}\right)^{n-1} dx =$

$= \{ t = 2\theta - x \quad dx = -dt \} = -\int_{\theta}^{0} \frac{2\theta - t}{\theta} \frac{t^{n-1}}{\theta^{n-1}} dt =$

$= \int_0^{\theta} n \frac{2\theta - t}{\theta} \frac{t^{n-1}}{\theta^{n-1}} dt = 2n \int_0^{\theta} \left(\frac{t}{\theta}\right)^{n-1} dt - n \int_0^{\theta} \left(\frac{t}{\theta}\right)^{n} dt =$

$= 2n\theta \int_0^1 \tau^{n-1} d\tau - n\theta \int_0^1 \tau^n d\tau = 2n\theta \cdot \frac{1}{n} - n\theta \frac{1}{n+1} =$

$= 2\theta - \frac{n}{n+1}\theta = \theta \frac{2n+2-n}{n+1} = \theta \cdot \frac{n+2}{n+1}$

$M[\tilde{\theta}_3] = \frac{1}{5} \cdot \frac{n+2}{n+1}\theta + \frac{2}{5} \frac{2n+1}{n+1}\theta = \frac{\theta}{5(n+1)} \{ n+2+4n+2 \} = \frac{5n+4}{5n+5}\theta$

→ смещена → исправим $\tilde{\theta}_3' = \frac{5(n+1)}{5n+4}\tilde{\theta}_3$ - несм

$X_{min}, X_{max} \to K(y,z)$

$K(y,z) = \begin{cases} F^n(z) - (F(z)-F(y))^n, & z \geq y \\ F^n(z), & z < y \end{cases}$

$\varkappa(y,z) = \frac{\partial^2 K}{\partial y \partial z} = \frac{\partial}{\partial z}\left(n(F(z)-F(y))^{n-1}\varphi(y)\right) =$

$= n\varphi(y) \cdot (n-1)(F(z)-F(y))^{n-2} \cdot \varphi(z) =$

$= n(n-1)\varphi(y)\varphi(z)(F(z)-F(y))^{n-2} \quad (z \geq y)$

$M[X_{min} \cdot X_{max}] = \iint_{-\infty}^{+\infty} yz\, n(n-1)\varphi(y)\varphi(z)(F(z)-F(y))^{n-2} dy\, dz =$

$= \int_{\theta}^{2\theta} dy \int_{y}^{2\theta} n(n-1) \cdot \varphi(y)\, y \cdot z\varphi(z) \cdot (F(z)-F(y))^{n-2} dz = \left\{ \varphi(x) = \frac{1}{\theta},\ F(x) = \frac{x-\theta}{\theta} \right\} =$

$= n(n-1)\frac{1}{\theta^2} \int_{\theta}^{2\theta} y\, dy \cdot \int_{y}^{2\theta} \left(\frac{z-y}{\theta}\right)^{n-2} dz = | \tilde{z} = z - y | =$

$= \frac{n(n-1)}{\theta^2} \int_{\theta}^{2\theta} y\, dy \int_0^{2\theta - y} \left(\frac{\tilde{z}}{\theta}\right)^{n-2} d\tilde{z} = \frac{n(n-1)}{\theta^2} \int_{\theta}^{2\theta} \frac{\theta}{n-1} \left(\frac{2\theta - y}{\theta}\right)^{n-1} y\, dy =$

$= \frac{n}{\theta} \int_{\theta}^{2\theta} \left(\frac{2\theta - y}{\theta}\right)^{n-1} y\, dy = \frac{n}{\theta} \int_0^{\theta} \left(\frac{\tilde{y}}{\theta}\right)^{n-1} (2\theta - \tilde{y})\, d\tilde{y} =$

$= 2n \int_0^{\theta} \left(\frac{y}{\theta}\right)^{n-1} dy - n \int_0^{\theta} \left(\frac{y}{\theta}\right)^{n} dy = 2n\theta \int_0^1 y^{n-1} dy - n\theta \int_0^1 y^n dy =$

$= 2n\theta \cdot \frac{1}{n} - n\theta \frac{1}{n+1} = 2\theta - \frac{n}{n+1}\theta = \theta \frac{n+2}{n+1}$

$D[\tilde\theta_3] = M[\tilde\theta_3^2] - (M[\tilde\theta_3])^2 =$

$= M[(\frac{1}{5}X_{min} + \frac{2}{5}X_{max})^2] - (M[\tilde\theta_3])^2 =$

$= \frac{1}{25}M[X_{min}^2] + \frac{4}{25}M[X_{min}X_{max}] + \frac{4}{25}M[X_{max}^2] - \frac{1}{25}\frac{(5n+4)^2}{(n+1)^2}\theta^2$

$M[X_{min}^2] = \int_\theta^{2\theta} x^2\, n\left(1-\frac{x-\theta}{\theta}\right)^{n-1}\frac{1}{\theta}\,dx = \int_\theta^{2\theta}\frac{nx^2}{\theta}\left(\frac{2\theta-x}{\theta}\right)^{n-1}dx = \left\{\begin{matrix} t = 2\theta - x \\ dx = -dt\end{matrix}\right\} =$

$= \int_0^\theta \frac{n}{\theta}(2\theta-t)^2\cdot\left(\frac{t}{\theta}\right)^{n-1}dt = \frac{n}{\theta}4\theta^2\cdot\int_0^\theta\left(\frac{t}{\theta}\right)^{n-1}dt - \frac{n}{\theta}\cdot 4\theta\int_0^\theta\frac{t^n}{\theta^{n-1}}dt + \frac{n}{\theta}\int_0^\theta t\frac{t^{n+1}}{\theta^{n-1}}dt =$

$= 4n\theta^2\frac{1}{n} - 4n\theta^2\cdot\frac{1}{n+1} + n\theta^2\frac{1}{n+2} = n\theta^2\left\{\frac{4}{n} - \frac{4\cancel{n}}{n+1} + \frac{\cancel{n}1}{n+2}\right\}$

$M[X_{max}^2] = \int_\theta^{2\theta} x^2\cdot n\cdot\left(\frac{x-\theta}{\theta}\right)^{n-1}\frac{1}{\theta}\,dx = \int_\theta^{2\theta}\frac{nx^2}{\theta}\left(\frac{x-\theta}{\theta}\right)^{n-1}dx = \{t = x-\theta\} =$

$= \int_0^\theta\frac{n}{\theta}(t+\theta)^2\left(\frac{t}{\theta}\right)^{n-1}dt = \frac{n}{\theta}\int_0^\theta t^2\left(\frac{t}{\theta}\right)^{n-1}dt + \frac{n}{\theta}\cdot 2\theta\cdot\int_0^\theta t\left(\frac{t}{\theta}\right)^{n-1}dt + \frac{n}{\theta}\theta^2\int_0^\theta\left(\frac{t}{\theta}\right)^{n-1}dt$

$= n\theta\int_0^\theta\left(\frac{t}{\theta}\right)^{n+1}dt + 2n\theta\int_0^\theta\left(\frac{t}{\theta}\right)^{n}dt + n\theta\int_0^\theta\left(\frac{t}{\theta}\right)^{n-1}dt =$

$= n\theta^2\frac{1}{n+2} + 2n\theta^2\cdot\frac{1}{n+1} + n\theta^2\cdot\frac{1}{n} = n\theta^2\left\{\frac{1}{n+2} + \frac{2}{n+1} + \frac{1}{n}\right\}$

$M[X_{min}X_{max}] = 2\theta^2 + \theta^2\frac{1}{n+2}$

$D[\tilde\theta_3] = \ldots = \frac{\theta^2}{25}\cdot\frac{\cancel{5n+2}\ 5n+4}{n^3+4n^2+5n+2}$ ~~...~~ дост усл

$D[\tilde\theta_3'] = \frac{25(n+1)^2}{(5n+4)^2}\cdot\frac{\theta^2}{25}\cdot\frac{5n+4}{n^3+4n^2+5n+2} = \theta^2\frac{(n+1)^2}{(5n+4)(n^3+4n^2+5n+2)} \xrightarrow[n\to\infty]{} 0$

$\Rightarrow$ сост. по дост усл

е) какая более эффективна? асс. эфф?

$D[\tilde\theta_1] = \frac{\theta^2}{27n}$ ~ $\frac{1}{27n}$

$D[\tilde\theta_2'] = \theta^2\cdot\frac{n}{(2n+1)^2(n+2)}$ ~ $\frac{1}{4n^2}$

$D[\tilde\theta_3'] = \theta^2\frac{\cancel{(n+1)^2}}{(5n+4)(\cancel{n^3+4n^2+5n+2})\,(n+2)} = \theta^2\frac{1}{(5n+4)(n+2)}$ ~ $\frac{1}{5n^2}$

$\to$ $\tilde\theta_3'$ ~~самая более эффективная~~, асимптотически эффективнее $\tilde\theta_2'$, а $\tilde\theta_2'$ — $\tilde\theta_1$

Исследуя $n = 1, 2, 3, 4.$, получим, что

- $n=1$ — одинаковы по эф
- $n=2$ $\tilde\theta_3'$ эфф $\tilde\theta_1$, θ_1 эфф $\tilde\theta_2'$
- $n \geq 3$ $\tilde\theta_3'$ эфф $\tilde\theta_2'$, $\tilde\theta_2'$ эфф $\tilde\theta_1$

Итого: • $n=1$ – $\tilde{\theta}_1, \tilde{\theta}_2', \tilde{\theta}_3'$ – одинаковы по эф

• $n=2$ – $\tilde{\theta}_3'$ эфф $\tilde{\theta}_1$, $\tilde{\theta}_1$ эфф $\tilde{\theta}_2'$

• $n \geq 3$, $n \to +\infty$ – $\tilde{\theta}_3'$ эфф $\tilde{\theta}_2'$, $\tilde{\theta}_2'$ эфф $\tilde{\theta}_4$

d) Построить точный довер интервал для пар-ра θ.

$$h = \theta, \quad \tilde{h} = \tilde{\theta} = \frac{X_{max}}{2}$$

$$f(h, \tilde{h}) = \frac{\tilde{h}}{h} = \underbrace{\frac{X_{max}}{2\theta}}_{y}$$

~~$\Phi(y) = P(y < \xi) = P(X_{max} < 2\theta\xi) = (P(X_1 < 2\theta\xi)$~~

$$\Phi(y) = P(\xi < y) = P\left(\frac{X_{max}}{2\theta} < y\right) = (P(x < 2\theta y))^n = (F(2\theta y))^n =$$

$$= (2\cancel{\theta} y - 1)^n = (2y-1)^n$$

~~$\varphi(y) = n(2\theta y - 1)^{n-1} \cdot 2\theta$~~ $\quad \varphi(y) = 2n(2y-1)^{n-1}$

$$2\theta y \in [\theta; 2\theta] \qquad y \in [1/2; 1]$$

$$\int_{1/2}^{q_{\alpha/2}} 2n(2y-1)^{n-1} dy = 2n \cdot \int_{1/2}^{q_{\alpha/2}} (2y-1)^{n-1} dy = (2q_{\alpha/2} - 1)^n = \frac{\alpha}{2}$$

$$q_{\alpha/2} = \frac{1 + \sqrt[n]{\alpha/2}}{2}$$

$$q_{\alpha/2+\beta} = \frac{1 + \sqrt[n]{\beta + \alpha/2}}{2}$$

$$P\left(\frac{1 + \sqrt[n]{\alpha/2}}{2} < \frac{X_{max}}{2\theta} < \frac{1 + \sqrt[n]{\alpha/2 + \beta}}{2}\right) = \beta$$

$$P\left(\frac{X_{max}}{1 + \sqrt[n]{\alpha/2}} > \theta > \frac{X_{max}}{1 + \sqrt[n]{\alpha/2 + \beta}}\right) = \beta$$

Доверительный интервал: $I\left(\frac{X_{max}}{1 + \sqrt[n]{0{,}025}}; \frac{X_{max}}{1 + \sqrt[n]{0{,}975}}\right)$

$$I\left(\frac{X_{max}}{1 + \sqrt[n]{0{,}975}}; \frac{X_{max}}{1 + \sqrt[n]{0{,}025}}\right)$$

е) Построить асс. дов. инт.

$$\frac{f(\tilde{\alpha}) - f(\alpha)}{\sigma}\sqrt{n} \sim N(0,1)$$

$$\sigma = \sqrt{(\nabla f(\alpha))^T \, K \, \nabla f(\alpha)}$$

$$K_{ij} = \alpha_{i+j} - \alpha_i \alpha_j$$

$$f(\alpha) = \frac{2}{3}\alpha_1 \qquad f(\tilde{\alpha}_1) = \frac{2}{3}\tilde{\alpha}_1$$

$$\sigma = \sqrt{\frac{2}{3}\cdot\frac{2}{3}\,(\alpha_2 - \alpha_1^2)} = \frac{2}{3}\sqrt{\alpha_2 - \alpha_1^2}$$

$$\frac{\frac{2}{3}\tilde{\alpha}_1 - \frac{2}{3}\alpha_1}{\frac{2}{3}\sqrt{\tilde{\alpha}_2 - \tilde{\alpha}_1^2}}\sqrt{n} \sim N(0,1)$$

$$\frac{\tilde{\theta} - \theta}{2/3\sqrt{\tilde{\alpha}_2 - \tilde{\alpha}_1^2}}\sqrt{n} \sim N(0,1)$$

$$-1{,}96 \le \frac{\tilde{\theta} - \theta}{2/3\sqrt{\tilde{\alpha}_2 - \tilde{\alpha}_1^2}}\cdot\sqrt{n} \le 1{,}96$$

$$\tilde{\theta} - 1{,}96\cdot\frac{2}{3}\sqrt{\frac{\tilde{\alpha}_2 - \tilde{\alpha}_1^2}{n}} \le \theta \le \tilde{\theta} + 1{,}96\cdot\sqrt{\frac{\tilde{\alpha}_2 - \tilde{\alpha}_1^2}{n}}$$