Math Quiz - Brent Questions

February 20, 2017

2. The answer to each of the following is zero, one, or infinity. Choose wisely.

   (a) What does the following expression evaluate to?

$$\lim_{n\to\infty} \frac{n!e^n}{2\pi n^{n+\frac{1}{2}}} \int_{-\infty}^{\infty} \left[\sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{-x^2}{2} \right)^k \right] \mathrm{d}x$$

   (b) Consider the system of linear equations.

$$\begin{array}{rcrcrcrcl} w & + & 3x & + & 2y & + & 2z & = & 0 \\ w & + & 4x & + & y & + & & = & 0 \\ 3w & + & 5x & + & 10y & + & 14z & = & 0 \\ 2w & + & 5x & + & 5y & + & 6z & = & 0 \end{array}$$

   How many solutions (w, x, y, z) are there (where w, x, y, z are real)?

   (c) Let y is a real-valued function defined on the real line and satisfying the initial value problem below:

$$\begin{aligned} y' + xy &= x \\ y(0) &= -1 \end{aligned}$$

   Find

$$\lim_{x\to-\infty} y(x)$$

   (d) Let $G = (V, E)$ be the graph with $V = \mathbb{R}$, and $E = \{\{x, y\} : x, y \in \mathbb{R}\}$, i.e., G is the complete graph on the reals. Suppose each edge in E is colored either red or blue. Call a subgraph $H \subseteq G$ "monochromatic" if all of its edges are the same color. How many monochromatic, complete subgraphs $H \subseteq G$ must one find such that $|V(H)| = |\mathbb{R}|$?

   (e) The function $f : \mathbb{R} \to \mathbb{R}$ is defined as follows:

$$f(x) = \begin{cases} 3x^2 & \text{if } x \in \mathbb{Q} \\ -5x^2 & \text{if } x \notin \mathbb{Q} \end{cases}$$

   At how many places is f differentiable?

Solutions

(a) 1 (it's the integral of the normal distribution, cleverly disguised).

(b) ∞ (Math GRE practice 24)

(c) 1 (Math GRE practice 44)

(d) 0. Ramsey theory; trust me.

(e) 1 (Math GRE practice 47)

5. "Paradoxes". Match the following with their common names:

   (a) The expression "the smallest positive integer definable in under sixty letters"

   (b) An adjective is autological if and only if it describes itself, e.g., 'English', 'unhyphenated', 'pentasyllabic'.

   An adjective is heterological if and only if it does not describe itself, e.g., 'hyphenated' and 'monosyllabic'.

   Into which category does 'heterological' fall?

   (c) Let $R = \{x : x \notin x\}$. Then $R \in R \iff R \notin R$.

   (d) "All Cretans are liars." — a Cretan

   (e) If this sentence is true, then Germany borders China.

   (f) The graph of $x \mapsto 1/x$, rotated about the x-axis gives a finite volume, but the outer shell has infinite surface area.

   (g) When the Okies left Oklahoma and moved to California, they reaised the average intelligence level in both states.

   (h) "yields falsehood when preceded by its quotation." yields falsehood when preceded by its quotation.

   (i) Consider the following infinite set of sentences:

   (S_1) For each $i > 1$, S_i is not true.

   (S_2) For each $i > 2$, S_i is not true.

   (S_3) For each $i > 3$, S_i is not true.

   $\vdots$

   (j) "Some numbers are squares, while others are not; therefore, all the numbers, including both squares and non-squares must be more numerous than just the squares. And yet, for every square there is exactly one positive number that is its square root, and for every number there is exactly one square; hence there cannot be more of one than of the other."

Solution set:

(1) Painter's paradox

(2) The Epimenides paradox

(3) Berry paradox

(4) Russell's paradox

(5) Quine's paradox

(6) Curry's paradox

(7) Grelling–Nelson paradox

(8) Will Rogers phenomenon

(9) Galileo's paradox

(10) Yablo's paradox

Answers:

(a) 3

(b) 7

(c) 4

(d) 6

(e) 2

(f) 1

(g) 8

(h) 5

(i) 10

(j) 9