Homework Assignment 8

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Wavelets Filters

The 2-D wavelet transform of a continuous image $f(x)$ computes the set of inner products

$$d_j^k[n] = \langle f, \psi_{j,n}^k \rangle$$

for scales $j \in \mathbb{Z}$, position $n \in \mathbb{Z}^2$ and orientation $k \in \{H, V, D\}$.

The wavelet atoms are defined by scaling and translating three mother atoms $\{\psi^H, \psi^V, \psi^D\}$:

$$\psi_{j,n}^k(x) = \frac{1}{2^j}\psi^k\left(\frac{x - 2^j n}{2^j}\right)$$

These oriented wavelets are defined by a tensor product of a 1-D wavelet function $\psi(t)$ and a 1-D scaling function $\varphi(t)$

$$\psi^H(x) = \varphi(x_1)\psi(x_2), \quad \psi^V(x) = \psi(x_1)\varphi(x_2) \quad \text{and} \quad \psi^D(x) = \psi(x_1)\psi(x_2).$$

The fast wavelet transform algorithm does not make use of the wavelet and scaling functions, but of the filters h and g that caracterize their interaction:

$$g[n] = \frac{1}{\sqrt{2}}\langle \psi(t/2), \varphi(t-n) \rangle \quad \text{and} \quad h[n] = \frac{1}{\sqrt{2}}\langle \varphi(t/2), \varphi(t-n) \rangle.$$

The simplest filters are the Haar filters

$$h = [1, 1]/\sqrt{2} \quad \text{and} \quad g = [-1, 1]/\sqrt{2}.$$

Daubechies wavelets extends the haar wavelets by using longer filters, that produce smoother scaling functions and wavelets. Furthermore, the larger the size $p = 2k$ of the filter, the higher is the number k of vanishing moment.

A high number of vanishing moments allows to better compress regular parts of the signal. However, increasing the number of vanishing moments also inceases the size of the support of the wavelets, wich can be problematic in part where the signal is singular (for instance discontinuous).

Choosing the *best* wavelet, and thus choosing k, that is adapted to a given class of signals, thus corresponds to a tradeoff between efficiency in regular and singular parts.

- The filter with $k = 1$ vanishing moments corresponds to the Haar filter.
- The filter with $k = 2$ vanishing moments corresponds to the famous |D4| wavelet, which compresses perfectly linear signals.
- The filter with $k = 3$ vanishing moments compresses perfectly quadratic signals.

Set the support size. To begin, we select the D4 filter.

We first imporant some functions of the toolbox (that I implemented for Shearlab in julia already and I think are well documented) and some libraries that will help us that will let us import the picture correctly

In [1]:
```
using Autoreload
arequire("nt_toolbox/nt_signal.jl")
arequire("nt_toolbox/nt_general.jl")
using nt_signal
using nt_general
using PyPlot
using Wavelets
```

```
INFO: Loading help data...
```

As the wavelets.jl library is not yet understendable for me I will just use it to generate the filters automatically, we will use it just to generate Daubechies 4 filters

In [2]:
```
p = 4;
```

Create the low pass filter h and the high pass g. We add a zero to ensure that it has a odd length. Note that the central value of h corresponds to the 0 position.

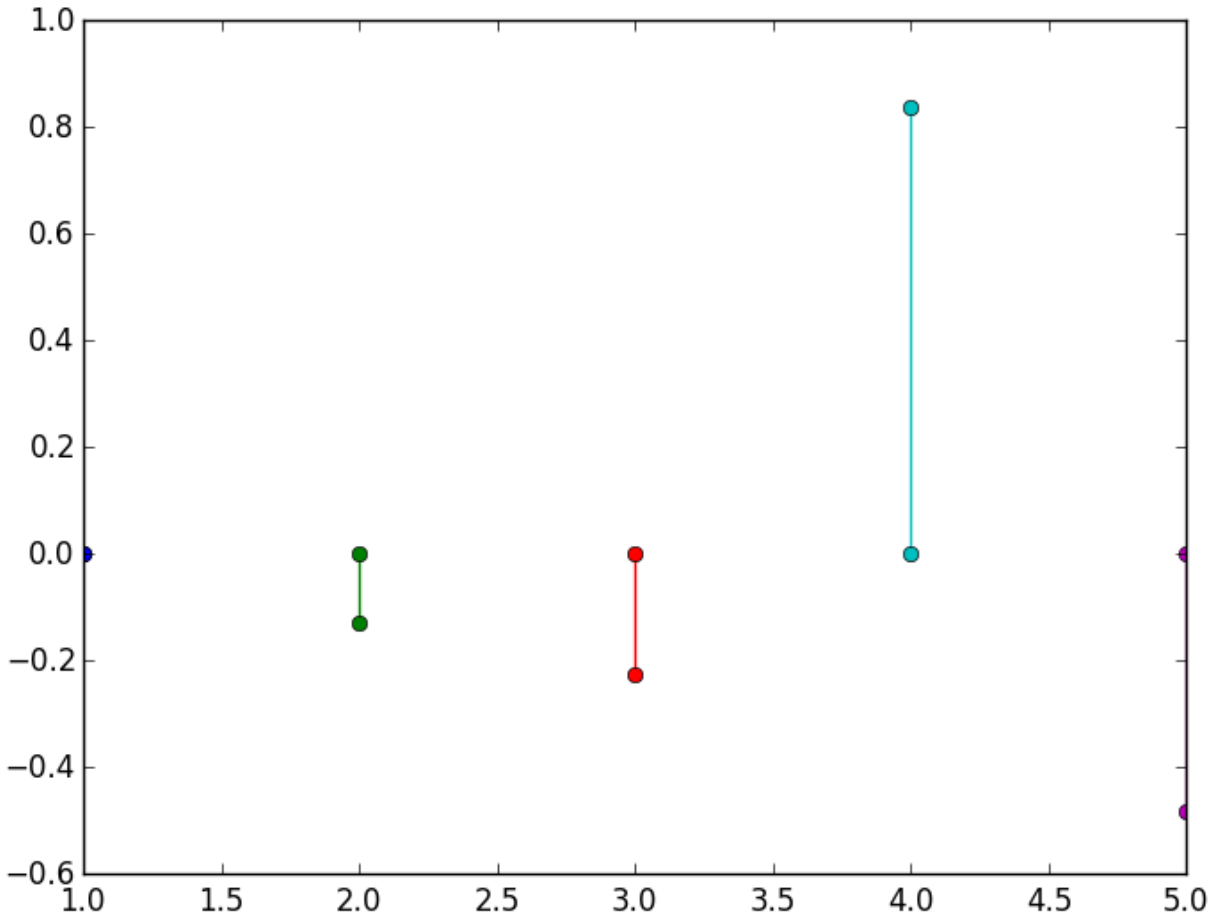
In [3]:
```
wavelet(WT.db2)
```

Out[3]:
```
OrthoFilter{PerBoundary}([0.482963,0.836516,0.224144,-0.12941],"db2")
```

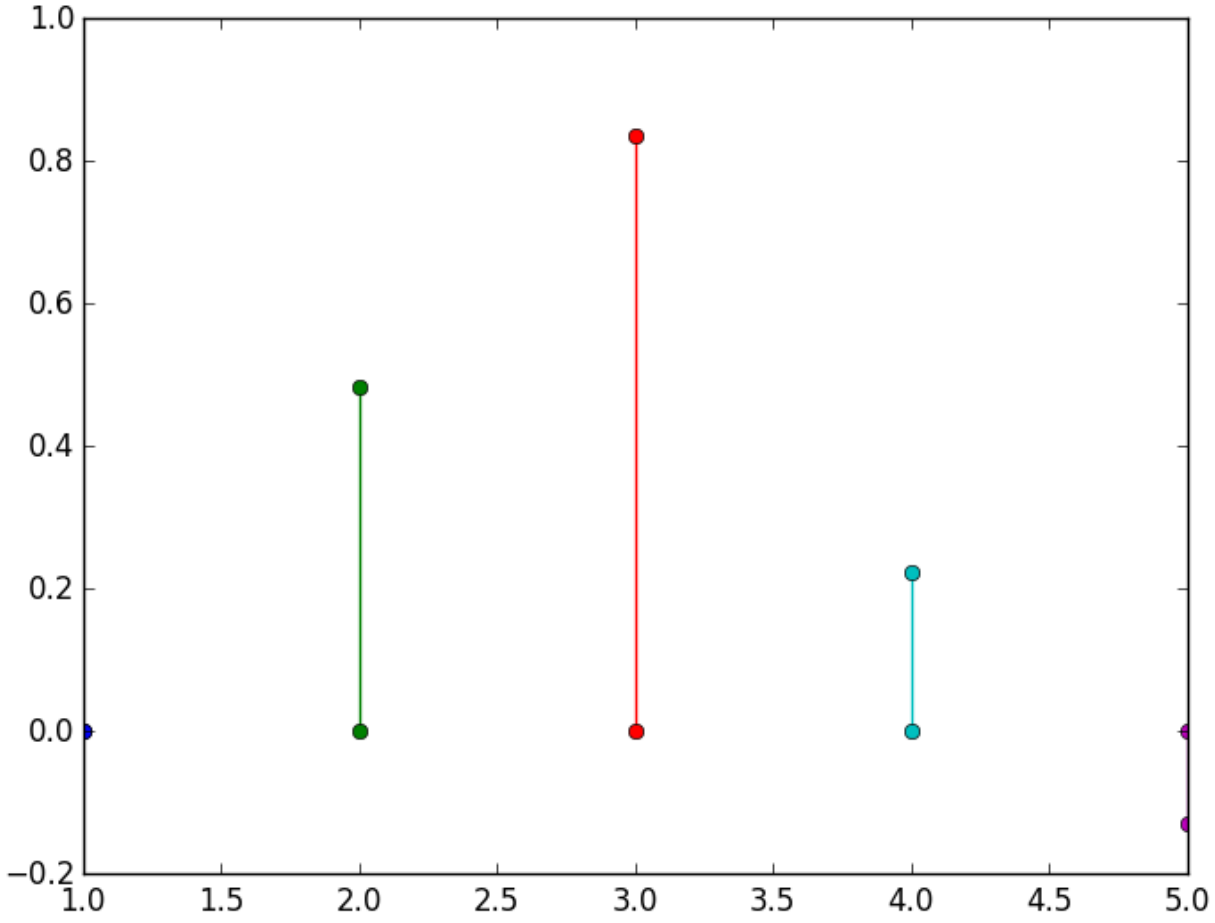
In [4]:
```
h = [0, .482962913145, .836516303738, .224143868042, -.129409522551];
h = h/norm(h);
g = cat(1, 0, h[length(h):-1:2]) .* ( (-1).^(1:length(h)) );
```

Lets visualize the high pass and low pass filter

In [5]:
```
for i in 2:(length(h)+1)
    plot([i-1,i-1],[0,g[i-1]],"-o")
end
```



In [6]:
```
for i in 2:(length(h)+1)
    plot([i-1,i-1],[0,h[i-1]],"-o")
end
```



Note that the high pass filter g is computed directly from the low pass filter as:

$$g[n] = (-1)^{1-n} h[1-n]$$

Display.

In [7]:
```
println("h filter = ", h);
println("g filter = ", g);
```

```
h filter = [0.0,0.48296291314483153,0.8365163037377082,0.2241438680419218
2,-0.12940952255095486]
g filter = [-0.0,-0.12940952255095486,-0.22414386804192182,0.836516303737
7082,-0.48296291314483153]
```

Up and Down Filtering

The basic wavelet operation is low/high filtering, followed by down sampling.

Starting from some 1-D signal $f \in \mathbb{R}^N$, one thus compute the low pass signal $a \in \mathbb{R}^{N/2}$ and the high pass signal $d \in \mathbb{R}^{N/2}$ as

$$a = (f \star h) \downarrow 2 \quad \text{and} \quad d = (f \star g) \downarrow 2$$

where the sub-sampling is defined as

$$(u \downarrow 2)[k] = u[2k].$$

Create a random signal $f \in \mathbb{R}^N$.

In [8]:
```
N = 1024;
f = rand(N,1);
```

Low/High pass filtering followed by sub-sampling.

In [9]:
```
a = subsampling( cconvol(f,h) );
d = subsampling( cconvol(f,g) );
```

For orthogonal filters, the reverse of this process is its dual (aka its transpose), which is upsampling followed by low/high pass filtering with the reversed filters and summing:

$$(a \uparrow 2) \star \tilde{h} + (d \uparrow 2) \star \tilde{g} = f$$

where $\tilde{h}[n] = h[-n]$ (computed modulo N) and $(u \uparrow 2)[2n] = u[n]$and $(u \uparrow 2)[2n+1] = 0$

Test for energy conservation.

In [10]:
```
e0 = norm(f[:])^2;
e1 = norm(a[:])^2 + norm(d[:])^2;
println("Energy before : $e0");
println("Energy before : $e1");
```

```
Energy before : 350.68268219927626
Energy before : 350.68268219943207
```

Up-sampling followed by filtering.

In [11]:
```
f1 =  cconvol(upsampling(a),reverse(h)) + cconvol(upsampling(d),reverse(g));
```

Check that we really recover the same signal.

In [12]:
```
println("Error |f-f1|/|f| = ", norm(f[:]-f1[:])/norm(f[:]) );
```

```
Error |f-f1|/|f| = 5.466285991302883e-13
```

Forward 2-D Wavelet transform

The set of wavelet coefficients are computed with a fast algorithm that exploit the embedding of the approximation spaces V_j spanned by the scaling function $\{\varphi_{j,n}\}_n$ defined as

$$\varphi_{j,n}(x) = \frac{1}{2^j}\varphi^0\left(\frac{x - 2^j n}{2^j}\right) \quad \text{where} \quad \varphi^0(x) = \varphi(x_1)\varphi(x_2).$$

The wavelet transform of f is computed by using intermediate discretized low resolution images obtained by projection on the spaces V_j:

$$a_j[n] = \langle f, \varphi_{j,n} \rangle.$$

First we load an image of $N = n \times n$ pixels, that for the first case is a picture of Ernst Reuter Haus

In [13]:
```
n = 1024;
name = "nt_toolbox/data/ernst_reuter_haus.bmp";
f = load_image(name, N);
f = rescale(sum(f,3));
f = f[:,:,1];
imageplot(f);
```

The algorithm starts at the coarsest scale $j = \log_2(n) - 1$

In [14]:
```
j = log2(n)-1;
```

The first step of the algorithm perform filtering/downsampling in the horizontal direction.

$$\tilde{a}_{j-1} = (a_j \star^H h) \downarrow^{2,H} \quad \text{and} \quad \tilde{d}_{j-1} = (a_j \star^H g) \downarrow^{2,H}$$

Here, the operator $\star^H$ and $\downarrow^{2,H}$ are defined by applying $\star$ and $\downarrow^2$ to each column of the matrix.

The second step computes the filtering/downsampling in the vertical direction.

$$\begin{aligned} a_{j-1} &= (\tilde{a}_j \star^V h) \downarrow^{2,V} \quad \text{and} \quad d^V_{j-1} = (\tilde{a}_j \star^V g) \downarrow^{2,V}, \\ d^H_{j-1} &= (\tilde{d}_j \star^V h) \downarrow^{2,V} \quad \text{and} \quad d^D_{j-1} = (\tilde{d}_j \star^V g) \downarrow^{2,V}. \end{aligned}$$

A wavelet transform is computed by iterating high pass and loss pass filterings with |h| and |g|, followed by sub-samplings. Since we are in 2-D, we need to compute these filterings+subsamplings in the horizontal and then in the vertical direction (or in the reverse order, it does not mind).

Initialize the transformed coefficients as the image itself and set the initial scale as the maximum one. |fW| will be iteratively transformated and will contains the coefficients.

In [15]:
```
fW = copy(f);
```

Select the sub-part of the image to transform.

In [16]:
```
A = fW[1:2^(j+1),1:2^(j+1)];
B = fW[1:2^(j+1),1:2^(j+1)];
```

Apply high and low filtering+subsampling in the vertical direction (1st coordinate), to get coarse and details.

Apply high and low filtering+subsampling in the horizontal direction (2nd coordinate), to get coarse and details.

In [17]:
```
Coarse = subsampling(cconvol(A,h,1),1);
Detail = subsampling(cconvol(A,g,1),1);
```

In [18]:
```
Coarse2 = subsampling(cconvol(B,h,2),2);
Detail2 = subsampling(cconvol(B,g,2),2);
```

Check for energy conservation.

In [19]:
```
norm(A[:])^2 - norm(Coarse[:])^2 - norm(Detail[:])^2
```

Out[19]:
```
-2.313968252565246e-7
```

In [20]:
```
norm(B[:])^2 - norm(Coarse[:])^2 - norm(Detail[:])^2
```

Out[20]:
```
-2.313968252565246e-7
```

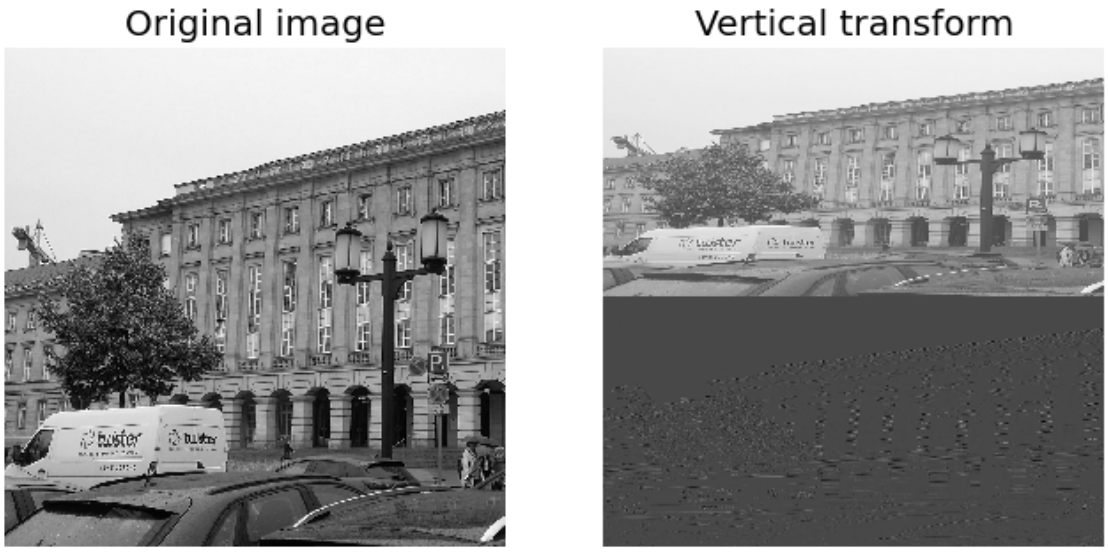
Note: |subsamplling(A,1)| is equivalent to |A(1:2:end,:)| and |subsamplling(A,2)| is equivalent to |A(:,1:2:end)|.

Concatenate them in the vertical direction to get the result.

In [21]:
```
A = cat(1, Coarse, Detail );
B= cat(1, Coarse2, Detail2 );
```

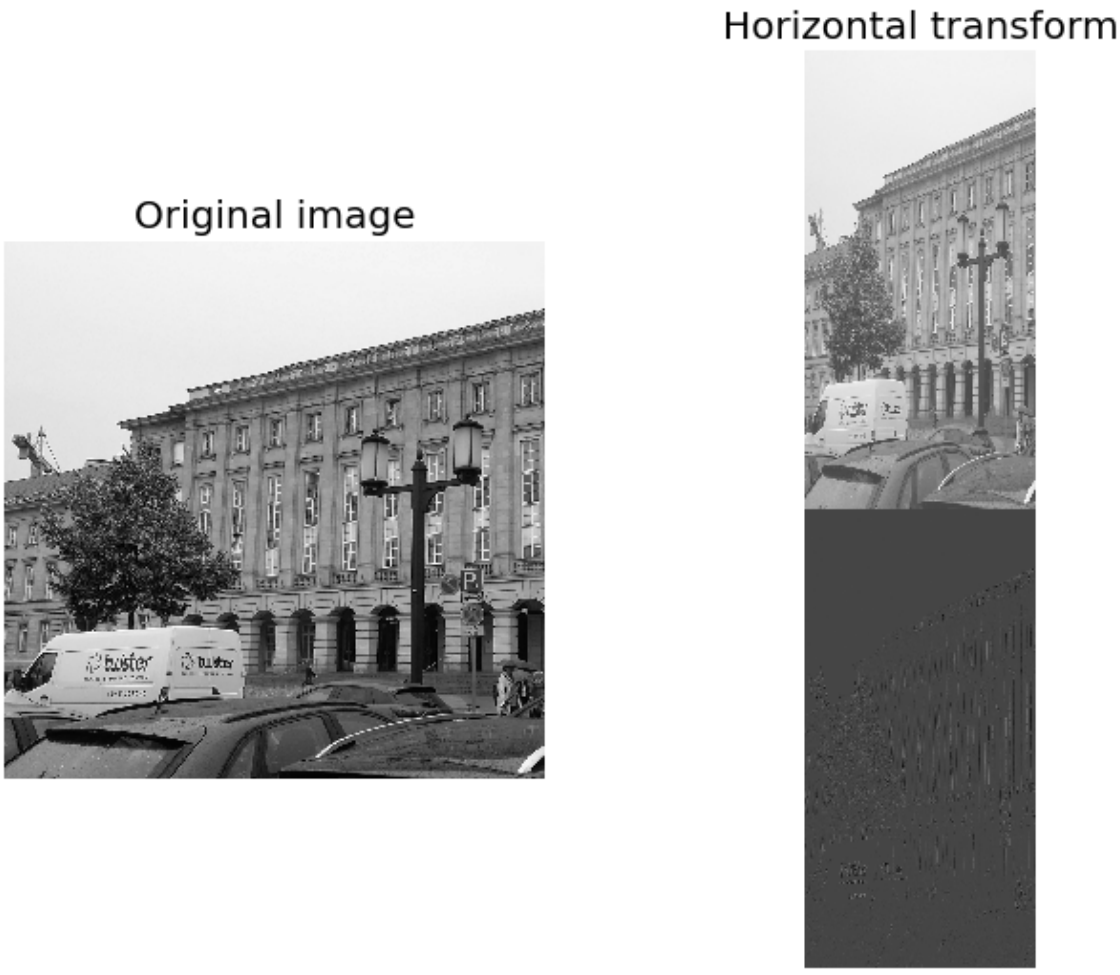
Display the result of the vertical transform.

In [22]:
```
clf;
imageplot(f,"Original image",1,2,1);
imageplot(A,"Vertical transform",1,2,2);
```



Display the result of the vertical transform.

In [23]:
```
clf;
imageplot(f,"Original image",1,2,1);
imageplot(B,"Horizontal transform",1,2,2);
```



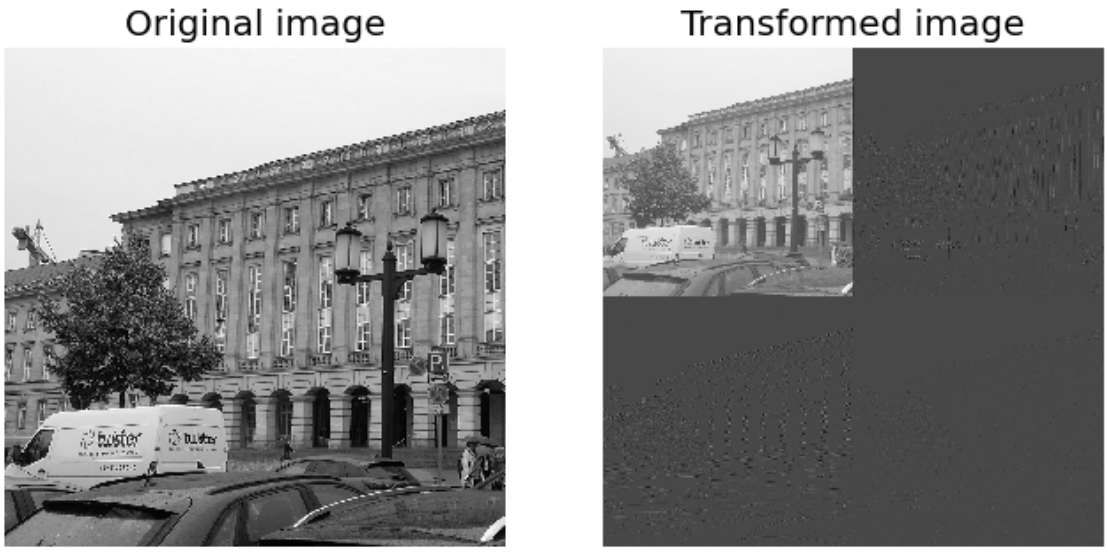
Apply high and low filtering+subsampling in the horizontal direction (2nd coordinate), to get coarse and details.

In [28]:
```
Coarse = subsampling(cconvol(A,h,2),2);
Detail = subsampling(cconvol(A,g,2),2);
```

Concatenate them in the horizontal direction to get the result.

In [29]:
```
A = cat(2, Coarse, Detail );
```

In [30]:
```
clf;
imageplot(f,"Original image",1,2,1);
imageplot(A,"Transformed image",1,2,2);
```

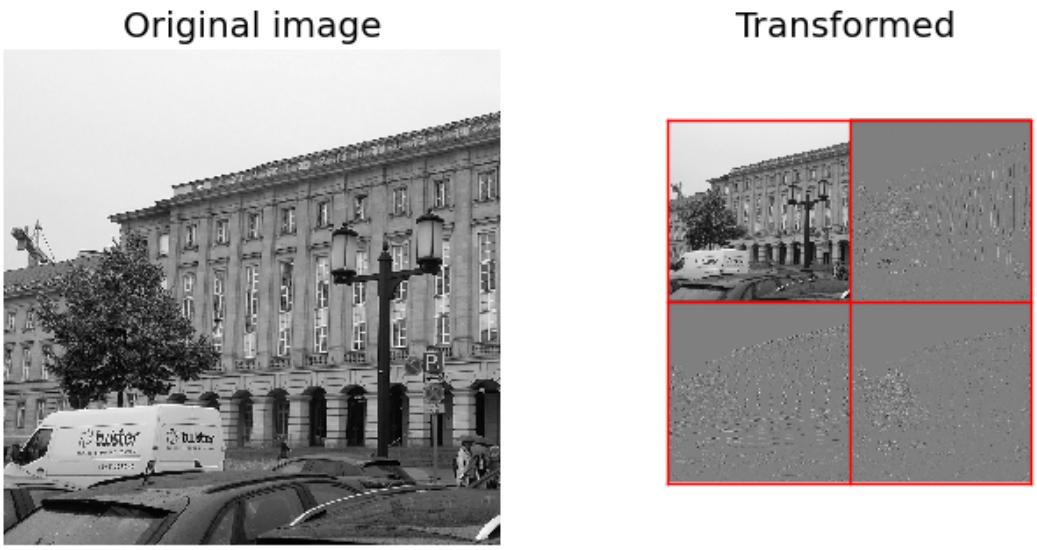


Assign the transformed data.

In [31]:
```
fW[1:2^(j+1),1:2^(j+1)] = A;
```

Display the result of the horizontal transform.

In [32]:
```
clf;
imageplot(f,"Original image",1,2,1);
subplot(1,2,2);
plot_wavelet(fW,log2(n)-1);
title("Transformed");
```

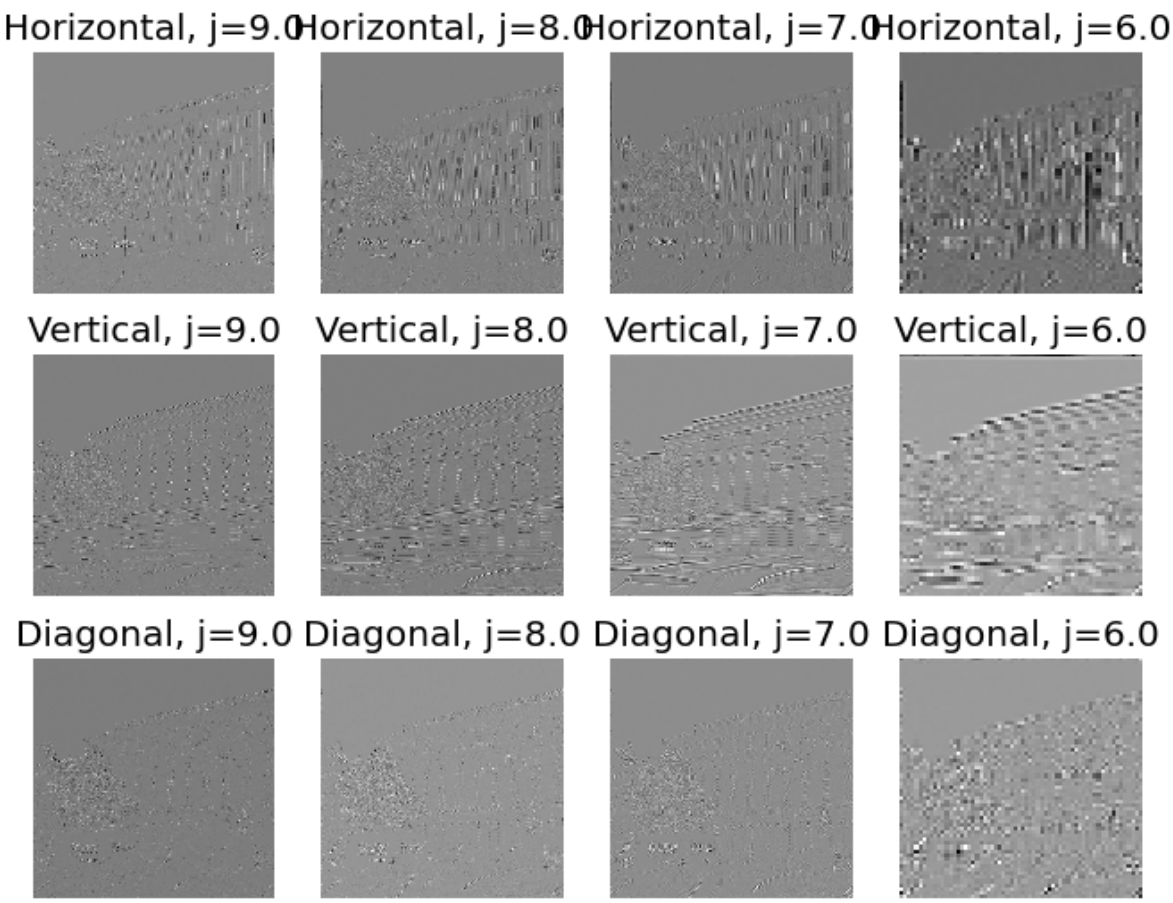


In [33]:
```
print( norm(f[:])-norm(fW[:]) )
```

```
-3.643663148977794e-10
```

Now lets Implement a full wavelet transform that extract iteratively wavelet coefficients, by repeating these steps. Take care of choosing the correct number of steps.

In [34]:
```
Jmax = log2(n)-1;
Jmin = 1;
fW = copy(f);
clf;
for j=Jmax:-1:Jmin
    A = fW[1:2^(j+1),1:2^(j+1)];
    for d=1:2
        Coarse = subsampling(cconvol(A,h,d),d);
        Detail = subsampling(cconvol(A,g,d),d);
        A = cat(d, Coarse, Detail );
    end
    fW[1:2^(j+1),1:2^(j+1)] = A;
    j1 = Jmax-j;
    if j1<4
        imageplot(A[1:2^j,2^j+1:2^(j+1)], "Horizontal, j=$j", 3,4, j1 + 1);
        imageplot(A[2^j+1:2^(j+1),1:2^j], "Vertical, j=$j", 3,4, j1 + 5);
        imageplot(A[2^j+1:2^(j+1),2^j+1:2^(j+1)], "Diagonal, j=$j", 3,4, j1
    end
end
```



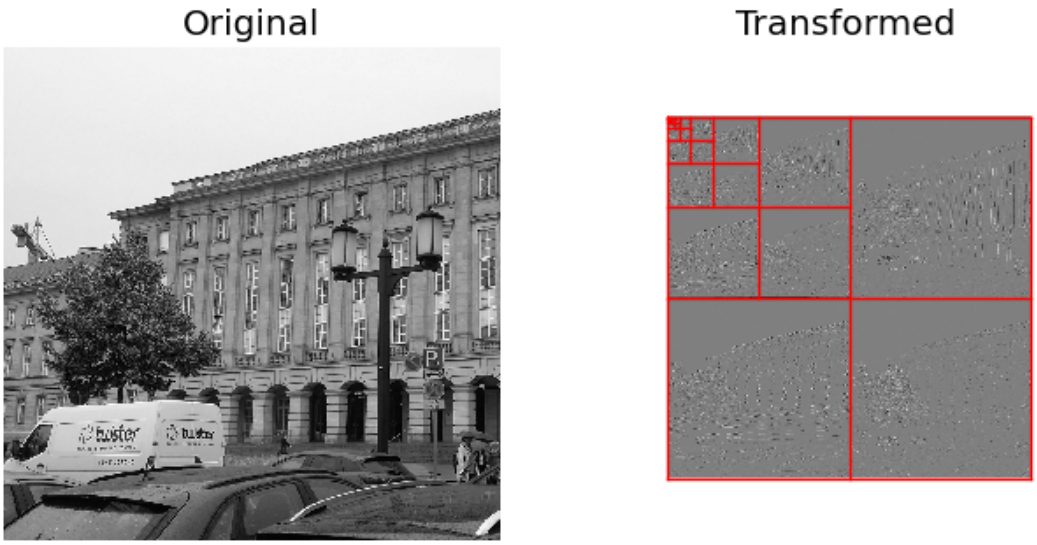
Check for orthogonality of the transform (conservation of energy).

In [35]:
```
e0 = norm(f[:])^2;
e1 = norm(fW[:])^2;
println("Energy of the signal       = $e0");
println("Energy of the coefficients = $e1");
```

```
Energy of the signal       = 405555.7888036768
Energy of the coefficients = 405555.7888075407
```

Display the wavelet coefficients.

In [36]:
```
clf;
imageplot(f, "Original", 1,2,1);
subplot(1,2,2);
plot_wavelet(fW, Jmin);
title("Transformed");
```



Inverse 2-D Wavelet transform.

Inversing the wavelet transform means retrieving a signal |f1| from the coefficients |fW|. If |fW| are exactly the coefficients of |f|, then |f=f1| up to machine precision.

Initialize the image to recover |f1| as the transformed coefficient, and select the smallest possible scale.

In [37]:
```
f1 = copy(fW);
j = 0;
```

Select the sub-coefficient to transform.

In [38]:
```
A = f1[1:2^(j+1),1:2^(j+1)];
```

Retrieve coarse and detail coefficients in the vertical direction (you can begin by the other direction, this has no importance).

In [39]:
```
Coarse = A[1:2^j,:];
Detail = A[2^j+1:2^(j+1),:];
```

Undo the transform by up-sampling and then dual filtering.

In [40]:
```
Coarse = cconvol(upsampling(Coarse,1),reverse(h),1);
Detail = cconvol(upsampling(Detail,1),reverse(g),1);
```

Recover the coefficient by summing.

In [41]:
```
A = Coarse + Detail;
```

Retrieve coarse and detail coefficients in the vertical direction (you can begin by the other direction, this has no importance).

In [42]:
```
Coarse = A[:,1:2^j];
Detail = A[:,2^j+1:2^(j+1)];
```

Undo the transform by up-sampling and then dual filtering.

In [43]:
```
Coarse = cconvol(upsampling(Coarse,2),reverse(h),2);
Detail = cconvol(upsampling(Detail,2),reverse(g),2);
```

Recover the coefficient by summing.

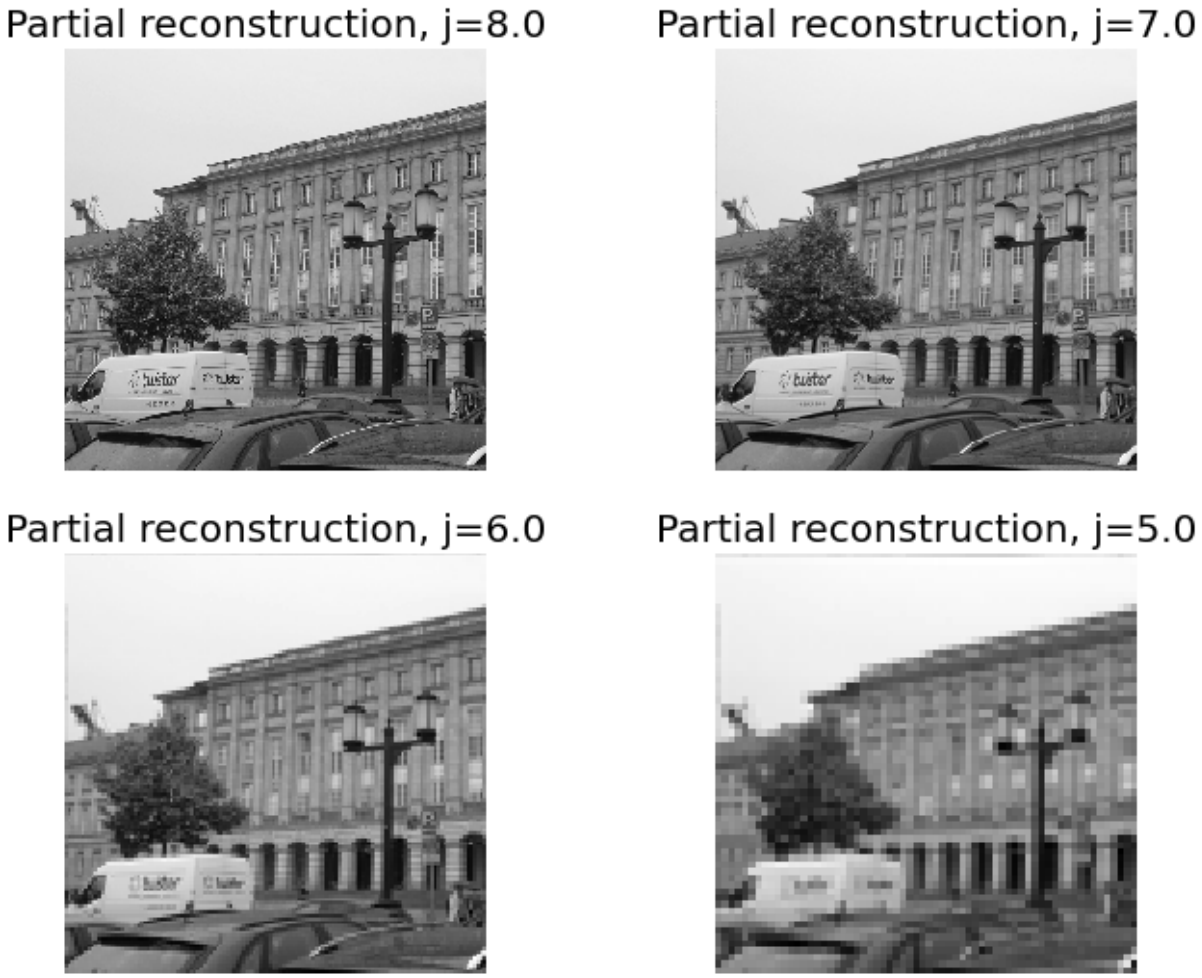
In [44]:
```
A = Coarse + Detail;
```

Assign the result.

In [45]:
```
f1[1:2^(j+1),1:2^(j+1)] = A;
```

Now lets write the inverse wavelet transform that computes |f1| from the coefficients |fW|. Compare |f1| with |f|.

In [46]:
```
f1 = copy(fW);
clf;
for j=Jmin:Jmax
    A = f1[1:2^(j+1),1:2^(j+1)];
    for d=1:2
        if d==1
            Coarse = A[1:2^j,:];
            Detail = A[2^j+1:2^(j+1),:];
        else
            Coarse = A[:,1:2^j];
            Detail = A[:,2^j+1:2^(j+1)];
        end
        Coarse = cconvol(upsampling(Coarse,d),reverse(h),d);
        Detail = cconvol(upsampling(Detail,d),reverse(g),d);
        A = Coarse + Detail;
        j1 = Jmax-j;
        if j1>0 && j1<5
            imageplot(A, "Partial reconstruction, j=$j", 2,2,j1);
        end
    end
    f1[1:2^(j+1),1:2^(j+1)] = A;
end
```



Check that we recover exactly the original image.

In [47]:
```
e = norm(f[:]-f1[:])/norm(f[:]);
print("Error |f-f1|/|f| = $e");
```

```
Error |f-f1|/|f| = 9.777840383145025e-12
```

Linear 2-D Wavelet Approximation

Linear approximation is performed by setting to zero the fine scale wawelets coefficients and then performing the inverse wavelet transform.

Here we keep only 1/16 of the wavelet coefficient, thus calculating an |m| term approximation with |m=n^2/16|.

In [48]:
```
eta = 4;
fWLin = zeros(n,n);
fWLin[1:n/eta,1:n/eta] = fW[1:n/eta,1:n/eta];
```

Now, lets compute and display the linear approximation |fLin| obtained from the coefficients |fWLin| by inverse wavelet transform.

In [50]:
```
min = 1;
# forward transform
fW = perform_wavortho_transf(f,Jmin,+1,h);
# linear approximation
eta = 4;
fWLin = zeros(n,n);
fWLin[1:n/eta,1:n/eta] = fW[1:n/eta,1:n/eta];
# backward transform
fLin = perform_wavortho_transf(fWLin,Jmin,-1,h);
elin = snr(f,fLin);
# display
clf;
imageplot(f, "Original", 1,2,1);
u = @sprintf("Linear, SNR=%.3f", elin);
imageplot(clamp(fLin), u, 1,2,2);
```

Non-Linear 2-D Wavelet Approximation

A non-linear |m|-term approximation is obtained by keeping only the |m| largest coefficients, which creates the smallest possible error.

Removing the smallest coefficient, to keep the |m|-largest, is equivalently obtainedby thresholding the coefficients to set to 0 the smallest coefficients.

First select a threshold value (the largest the threshold, the more agressive the approximation).

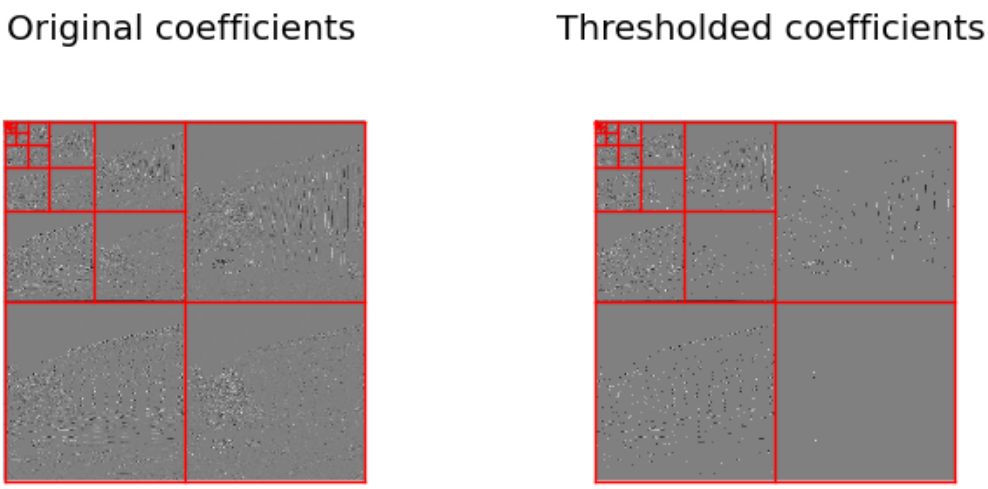
In [51]:
```
T = .2;
```

Then set to 0 coefficients with magnitude below the threshold.

In [52]:
```
fWT = fW .* float(abs(fW).>T);
```

Display thresholded coefficients.

In [53]:
```
clf;
subplot(1,2,1);
plot_wavelet(fW,Jmin);
title("Original coefficients");
subplot(1,2,2);
plot_wavelet(fWT,Jmin);
title("Thresholded coefficients");
```



Now lets find the thresholds |T| so that the number |m| of remaining coefficients in |fWT| are |m=n^2/16|. Use this threshold to compute |fWT| and then display the corresponding approximation |Mnlin| of |f|. Compare this result with the linear approximation. umber of kept coefficients ompute the threshold T elect threshold nverse transform nverse isplay

In [54]:
```
# number of kept coefficients
m = round(n^2/16);
# compute the threshold T
Jmin = 1;
fW = perform_wavortho_transf(f,Jmin,+1, h);
# select threshold
v = sort(abs(fW[:]));
if v[1]<v[n^2]
    v = reverse(v);
end
# inverse transform
T = v[m];
fWT = fW .* (abs(fW).>T);
# inverse
fnlin = perform_wavortho_transf(fWT,Jmin,-1, h);
# display
clf;
u1 = @sprintf("Linear, SNR=%.3fdB", snr(f,fLin));
u2 = @sprintf("Non-linear, SNR=%.3fdB", snr(f,fnlin));
imageplot(clamp(fLin),u1, 1,2,1 );
imageplot(clamp(fnlin),u2, 1,2,2 );
```