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Master's Thesis

Interference in ad-hoc telecommunication systems in the high-density limit

Author:
Adrián Hinojosa Calleja

Supervisor: Prof. Dr. Wolfgang König
Second reader: Dr. Christian Hirsch

Fakultät II
Institut für Mathematik
AG Stochastik und Finanzmatematik
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Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbstständig und eigenhändig sowie ohne unerlaubte fremde Hilfe und ausschließlich unter Verwendung der aufgeführten Quellen und Hilfsmittel angefertigt habe.

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Berlin, den 4. August 2016

Adrián Hinojosa Calleja

Zusammenfassung in deutscher Sprache

Störungen in Ad-hoc-Telekommunikationssystemen in hoher Dichte Grenze

Diese These ist angewendete Punktprozesse für Telekommunikations Modelle. Es ist in drei verschiedenen Hauptteile geteilt. Der erste Teil (Kapitel 1) ist eine kurze Einfährung in die Theorie und die wichtigsten Ergebnisse der Punktprozesse, die im Rest der Arbeit verwendet werden. Der zweite Teil (Kapitel 2 und 3) geht um die Untersuchung der Verkehrsflussdichten und es ist hauptsächlich basierend auf [4]. Schliesslich im dritten Teil (Kapitel 4) basierend auf [5] wir haben das Konzept der Informationsgeschwindigkeit ausgesetzte.

A Trini, Rai y él/la que viene
A Paqui, Nati y Elsa
A Ma, Pa y Gon

There must be some kind of way outta here
Said the joker to the thief
There's too much confusion
I can't get no relief

Business men, they drink my wine
Plowman dig my earth
None were level on the mind
Nobody up at his word

No reason to get excited
The thief he kindly spoke
There are many here among us
Who feel that life is but a joke
But you and I, we've been through that
And this is not our fate
So let us stop talkin' falsely now
The hour's getting late

All along the watchtower
Princes kept the view
While all the women came and went
Barefoot servants, too
Outside in the cold distance
A wildcat did growl
Two riders were approaching
And the wind began to howl

All along the watchtower, Bob Dylan

FIGURE 1: Department of Mathematics, University of Coimbra, Portugal. Picture by the author.

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To all of you, thank you very much.

Berlin, August 2016

Contents

Introduction

This thesis is about point processes applied to telecommunication models. It is divided in three different main parts. The first part (Chapter 1) is a brief introduction to the theory and main results of point processes that will be used in the rest of the work. The second part (Chapters 2 and 3) is about the study of the traffic flow densities and its mainly based on [4]. Finally in the third part (Chapter 4) based in [5] we expose the concept of information velocity.

In chapter one first we define what is a point process, we explain some details about its construction and we proof one of the main results related to them, which allow us to compute expected value of a random sum evaluated on the points of a point process: the Campbell's theorem. We also define what is a stationary point process an the related concept of nearest neighbor density. Also we state an ergodic theorem for stationary point processes. Second we focus on the most classical example of point processes: the Poisson point process, which independence properties makes simply to characterize and study, we also mention a more complete version of Campbell's theorem and even a generalization of it: the Slyvniak-Mecke's theorem. We also compute the nearest neighbor density of a Poisson point process. Due to its characteristics the Poisson point process is quite often used in telecommunication models. Third we give a brief introduction to the concept of Hausdorff measure and its relation with the coarea formula.

Second chapter is about directed navigations and traffic flows. A directed navigation is a function $\mathcal{A}$ that assigns to each point of a point process a successor in the direction e_1. A traffic flow is a function that designates some weight to each node according to the location of the other nodes that are on its tail. Under certain hypothesis for $\mathcal{A}$ a limiting result about the average of the traffic flow on an orthogonal window to e_1 which collapses to a point is proved in [4]. Using similar techniques we prove an analogous result when the window not collapse. Finally also inspired in [4] we provide an example of directed navigation which satisfies the hypothesis of our result: the spanning three navigation.

Third chapter is about radial navigations and traffic flows. A radial navigation is a function $\mathcal{A}$ that assigns to each point of a point process a successor that is approaching to the origin. As in second chapter we proof a limiting result about the average traffic of nodes that pass through a spherical environment. In [4] is analyzed the case when the environment collapse to a point, here we proof an analogous result when the environment does not collapse.

Finally the fourth chapter is centered in the concept of Signal Interference Noise Ratio (SINR). If we pick two points x and y of a point process X the $SINR_{x,y}(t)$ measures how good or bad is the quality of communication between them at time t. If the SINR is bigger that a threshold then the message from x to y is sent if not we try again at time $t+1$. Using a conic strategy which assigns to x to point to which the message will be send we proof two results. The first is a generalization from a result provided in [5] to non stationary Poisson point process and it states that the expected value of the time that takes to a message from being send from x to y is finite. The second result was taken from [5] and it states that if we continue sending the message between points, the asymptotic limit of the velocity is positive.

For reading this work is not necessary more than the basic mathematics undergraduate knowledge in probability theory. In the first chapter we do not provide proofs of most of the results but this is not relevant for the correct understanding of the following pages.

Chapter 1

Preliminaries

The main results of this work are about point processes which are used to model users of a telecommunications network. On this chapter we will define what is a point process and we will enunciate some classic results that will be used in the following chapters. We will also refer to one of the most elementary example of point process the Poisson point process which is quite used in modeling. Finally we will briefly discuss the concept of Hausdorff measure its relation with Lebesgue measure and we will enunciate the coarea formula. We will not offer many proofs in this chapter but we will provide the corresponding bibliography to the interested reader for consulting them.

1.1 Point process

A point process is a set of points that are randomly distributed on the space and which does not densely accumulate in any part of it. For $A \subset \mathbb{R}^d$ we say that A is *locally finite* if for any bounded set $B \subset \mathbb{R}^d$ the cardinality of $A \cap B$ is finite. A point process X will take values on the space

$$\mathbf{N} = \{A \subset \mathbb{R}^d : A \text{ is locally finite } \},$$

where we will equip $\mathbf{N}$ with $\mathcal{N}$ the minimum σ-algebra that contains all the locally finite sets of $\mathbb{R}^d$. We will denote by $\mathcal{B}_0(\mathbb{R}^d)$ the bounded Borel sets.

Definition 1.1. A point process X on $\mathbb{R}^d$ is a measurable mapping defined on some probability space $(\Omega, \mathcal{F}, P)$ taking values in $(\mathbf{N}, \mathcal{N})$. This mapping induces a distribution P_X of X given by

$$P_X(B) = P(\omega \in \Omega : X(\omega) \in B), \text{ for } B \in \mathcal{N}.$$

In general is hard to study the distribution of a point process, it is more easy to answer questions such as: How many points has a point process in some borelian bounded region of the space? or Which is the sum of a function evaluated in each of the points of the process? The intensity measure will be our big allied for answering that kind of questions .

Definition 1.2. Let X be a pointed process, for $B_1, ..., B_n \in \mathcal{B}_0(\mathbb{R}^d)$, we have that $\#(X \cap B_1), ..., \#(X \cap B_1)$ where $\#(A)$ denotes the cardinality of a set $A \in \mathbb{R}^d$ are random variables taking values on $\mathbb{N}$. We say that the measure

$$\Lambda(B) = E\#(X \cap B), B \in \mathcal{B}_0(\mathbb{R}^d),$$

is the intensity measure of X. If it exists we will denote by λ the density of Λ with respect to the Lebesgue measure. We say that λ is the intensity function of X.

It is much easier to define a point process by its intensity measure than by its distribution in fact the joint

$$(\#(X \cap B_1), ..., \#(X \cap B_n))$$

characterizes it.

Proposition 1.1. The distribution of a point process X is determined by the family of joint distributions

$$(\#(X \cap B_1), ..., \#(X \cap B_n)), B_1, ..., B_n \in \mathcal{B}_0(\mathbb{R}^d).$$

Proof. [8] Lemma B. 2. □

The next theorem could be called the Fundamental Theorem of point processes and it is key in the development of this work. It relates the expected value of random sums evaluated in the points of the process with its intensity measure.

Theorem 1.1 (Campbell)**.** Let X be a point process and let $f : \mathbb{R}^d \to \mathbb{R}$ a non negative measurable function. Then the random sum

$$\Sigma = \sum_{x \in X} f(x)$$

is a random variable, with expected value

$$E(\Sigma) = \int_{\mathbb{R}^d} f(x)\Lambda(dx).$$

Proof. Lets assume that f is a step function, then we can write $f = \sum_{i=1}^{n} c_i \mathbb{1}_{B_i}$, $c_i \geq 0$, $B_i \in \mathcal{B}_0(\mathbb{R}^d)$ disjoint. Then we have that

$$\Sigma = \sum_{x \in X} \sum_{i=1}^{n} c_i \mathbb{1}_{B_i}(x) = \sum_{i=1}^{n} c_i \#(X \cap B_i)$$

so

$$E(\Sigma) = \sum_{i=1}^{n} c_i E\#(X \cap B_i) = \sum_{i=1}^{n} c_i \Lambda(B_i) = \int_{\mathbb{R}^d} f(x)\Lambda(dx).$$

The result follows by monotone approximation. □

We are interested in simple point process since in our models each point will represent an user of a telecommunication network and it does not make sense to have two users in the same spatial location.

Definition 1.3. We say that a point process X is simple if for all $x \in \mathbb{R}^d$ satisfies that $P(\#(X \cap \{x\}) > 1) = 0$.

For finishing our review of basic results we talk about stationary point process which has been inspired in the classic definition for random variables.

Definition 1.4. A point process X is called stationary if for all $x \in \mathbb{R}^d$, $X + x$ and X have the same distribution where

$$X + x = \{Y_i : Y_i = X_i + x, X_i \in X\}.$$

Additionally we say that X is ergodic if for all $B, C \in \mathcal{N}$,

$$\lim_{t\to\infty} \frac{1}{(2t)^d} \int_{[-t,t]^d} \mathbb{1}\{X \in B, X + x \in C\} dx = P(X \in B)P(X \in C).$$

The homogeneity of a stationary point process aloud us to define the concept of nearest neighbor distribution.

Definition 1.5. Let X be a stationary point process we define the nearest neighbor distribution $G(r)$ as the probability that given that x is in X, its nearest neighbor on X is at distance r, more precisely

$$G(r) = \frac{1}{\Lambda(B)} E \sum_{X_i \in X \cap B} \mathbb{1}\{(X \setminus \{x\}) \cap B_r^d(x) \neq \emptyset\}$$

for $r > 0$ and $B \in \mathcal{B}_0(\mathbb{R}^d)$.

And of course there is also an ergodic theorem for pointed process, for stating it we need the concept of sequence of convex averaging windows.

Definition 1.6. $\{W_n\}$ a sequence of subsets of $\mathbb{R}^d$ is a sequence of convex averaging windows if

1. each W_n is convex and compact,
2. $W_n \subset W_{n+1}$,
3. $\sup_{r\geq 0}\{B(x,r) \subset W_n$ for some $x\} \to \infty$ as $n \to \infty$.

Theorem 1.2. Let X be a stationary ergodic pointed process, $\{W_n\}$ a sequence of convex averaging windows and f a non negative measurable function on $\mathbb{R}$. Then almost surely

$$\lim_{n\to\infty} \frac{1}{\nu_d(W_n)} \int_{W_n} f(X + x) dx = E(f(X)),$$

where ν_d is the d-dimensional Lebesgue measure.

[2] Section 12.2.

1.2 Poisson point process

The Poisson point processes are the most common and basic example of point processes. Due to its independence characteristics it becomes very easy to work with them. Quite often it is used in telecommunications models.

Definition 1.7. Let Λ a non atomic measure on $\mathbb{R}^d$ such that for all $B \in \mathcal{B}_0(\mathbb{R}^d)$, $\Lambda(B) < \infty$. We say that X is a Poisson pointed process with intensity measure Λ if

1. for $B \in \mathcal{B}_0(\mathbb{R}^d)$, $\#(X \cap B)$ has Poisson distribution with mean $\Lambda(B)$.
2. for $B_1, ..., B_k \in \mathcal{B}_0(\mathbb{R}^d)$ disjoint sets $\#(X \cap B_1), ..., \#(X \cap B_k)$ are independent.

A good resource for studying the construction of this point process is [6] Section 5.5. The Campbell's theorem can be translated in to a more simple version and also extended to compute the moment generator function of a random sum.

Theorem 1.3 (Campbell)**.** Let X be a Poisson point process with intensity function λ and f a measurable real valued function on $\mathbb{R}^d$. Then the random sum

$$\Sigma = \sum_{x \in X} f(x)$$

is almost surely absolutely convergent if and only if

$$\int_{\mathbb{R}^d} 1 \wedge |f(x)| \Lambda(dx) < \infty.$$

If this condition holds, then

$$E(e^{\theta \Sigma}) = \exp(\int_{\mathbb{R}^d} (e^{\theta f(x)} - 1)\Lambda(dx))$$

for any complex θ for which the integral on the right converges and in particular whenever θ is imaginary. Moreover

$$E(\Sigma) = \int_{\mathbb{R}^d} f(x)\Lambda(dx)$$

in the sense that the expectation exists if and only if the integral converges. If the integral converges then

$$Var(\Sigma) = \int_{\mathbb{R}^d} f(x)^2 \Lambda(dx),$$

finite or infinite.

Proof. See [6] Section 3.2. □

The next two theorems are a generalization of the Campbell's theorem.

Theorem 1.4 (Slyvniak-Mecke)**.** Let X be a Poisson point process with intensity measure Λ then

$$E \sum_{x \in X} f(x, X) = \int_{\mathbb{R}^d} E(f(x, X \cup \{x\}))\Lambda(dx)$$

$$E \sum_{x \in X} f(x, X \setminus \{x\})) = \int_{\mathbb{R}^d} E(f(x, X)\Lambda(dx)$$

for all non negative measurable functions f on $\mathbb{R}^d \times \mathbf{N}$.

Proof. [10] Theorem 3.2.5,[8] Theorem 3.2. □

As an example of an application of this result we will compute the nearest neighbor density of a stationary Poisson point process.

Proposition 1.2. Let X be a stationary Poisson point process of constant intensity $\lambda(x) = \lambda$ then is nearest neighbor density is

$$G(r) = 1 - e^{-\Lambda(B_r^d(o))},$$

Proof. By the Slyvniak-Mecke's theorem we have that

$$G(r) = \frac{1}{\Lambda(B)} \int_{\mathbb{R}^d} P(X \cap B \cap B_r^d(x) \neq \emptyset)\Lambda(dx) = 1 - e^{-\Lambda(B_r^d(x))}.$$

□

It is very useful to have a way to compare Poisson point processes with different intensities on a same space this can be done using coupling techniques. For that let X be an stationary Poisson point process in $\mathbb{R}^d \times [0, \lambda_M]$ with constant intensity function $\mathbb{1}$. If $\lambda^* : \mathbb{R}^d \to [0, \lambda_M]$ is a measurable function, then we define

$$X^{[\lambda^*]} = \{X_i : (X_i, U_i) \in X \text{ and } U_i \leq \lambda^*(X_i)\}.$$

Proposition 1.3. $X^{[\lambda^*]}$ as before is a Poisson point process in $\mathbb{R}^d$ with intensity function λ^*.

Proof. Since X is a Poisson point process with intensity function $\mathbb{1}$ we have that for $B \in \mathcal{B}_0(\mathbb{R}^d)$,

$$E\#(X^{[\lambda^*]} \cap B) = \int_{B \times [0 \times \lambda_M]} \mathbb{1}_{x_{d+1} \leq \lambda^*(x_1, \ldots, x_d)} dx = \int_B \lambda^*(\hat{x}) d\hat{x}$$

for $x \in \mathbb{R}^{d+1}$ and $\hat{x} = (x_1, ..., x_d)$ the projection on the first d coordinates. Hence by the Marking theorem on [6], $X^{[\lambda^*]}$ is a Poisson point process with intensity function λ^*. □

1.3 The Hausdorff measure and the coarea formula

In this section we mention a result that we will use in the third chapter of this work. This can be interpreted as a generalized "curvilinear" version of Fubini's theorem: the coarea formula. This is related with the Hausdorff measure which generalizes the Lebesgue measure.

Definition 1.8 (Hausdorff measure)**.** Let $A \subset \mathbb{R}^n$, $0 \leq s < \infty$, $0 < \delta \leq \infty$ we write

$$\eta^s_\delta(A) = \inf\{\sum_{j=1}^{\infty} \alpha(s)(\frac{\text{diam} C_j}{2})^s | A \subset \bigcup_{j=1}^{\infty} C_j, \text{diam} C_j \leq \delta\},$$

where

$$\alpha(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2}) + 1}.$$

For A and s as above, define

$$\eta^s(A) = \lim_{\delta \to 0} \eta^s_\delta(A) = \sup_{d > 0} \eta^s_\delta(A).$$

We call η^s the s-dimensional Hausdorff measure on $\mathbb{R}^d$.

The d-dimensional Lebesgue measure just allows us to measure the d-dimensional volume of a measurable subset of $\mathbb{R}^d$. More generally the Hausdorff measure can measure the d-dimensional (or even s-dimensional) volume of any object in any dimension. The d-dimensional Lebesgue and Hausdorff measure coincide on $\mathbb{R}^d$.

Theorem 1.5. $\eta^d = \nu^d$ on $\mathbb{R}^d$ with ν^d the d-dimensional Lebesgue measure.

Proof. [3] Section 2.2 □

Finally we enunciate the coarea formula and its relation with the Hausdorff measure.

Theorem 1.6 (Coarea formula)**.** Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz continuous, $n \geq m$. Then for each ν^n-measurable function $g : \mathbb{R}^n \to \mathbb{R}$,

$$\int_{\mathbb{R}^n} gJf dx = \int_{\mathbb{R}^m} \int_{f^{-1}(y)} g d\eta^{n-m} dy,$$

where Jf is the Jacobian of f.

Proof. [3] Section 3.4. $\square$

Chapter 2

Directed networks

In this chapter we will analize the traffic flow generated by a directed navigation on a point process defined on a d-dimensional hypercube. A directed navigation assigns a successor to each node of a point process who lies near a specific direction, iterating that process is how the traffic flow is generated.

We will proof two main results. The first one measures how is the average behavior of the traffic flow on an $\mathbb{R}^{d-1}$ micro window orthogonal to the direction of the flow. On the second one we construct a directed navigation that satisfies the hypothesis of our first result. Both results follows a similar construction to Section 1.1 of [4] where the case of a $\mathbb{R}^{d-1}$ micro window which collapses in to a point is studied. Now we consider the case when the window does not collapse.

2.1 Directed navigation

Let $X^{(s)}$ a simple point process with intensity $\lambda^{(s)}$ on sC where C is the interior of the d-dimensional hypercube $[-2, 2]^d$, $\lambda^{(s)}(x) = \lambda(\frac{x}{s})$ with $\lambda : C \to [0, \infty)$ a continuous and bounded mapping with positive maximum and minimum λ_M, λ_m respectively. We assume that each node $X_i \in X^{(s)}$ generates traffic at rate $\mu^{(s)}(x) = \mu(\frac{x}{s})$ with $\mu : C \to [0, \infty)$ also a continuous and bounded mapping with maximum and minimum μ_M, μ_m respectively.

We want to analyze the traffic flow who moves on a certain direction, for simplicity we will investigate a navigation on the direction e_1. For that let be $\pi_1 : \mathbb{R}^d \to \mathbb{R}$ the projection to the first coordinate and $\mathbf{N}$ the set of locally finite sets of points in $\mathbb{R}^d$.

Definition 2.1. A measurable function $\mathcal{A} : \mathbb{R}^d \times \mathbf{N} \to \mathbb{R}^d$ is called *directed navigation* (on the direction e_1) if for all $\varphi \in \mathbf{N}$, $x \in \varphi$,

- $\mathcal{A}(x, \varphi) \in \varphi$,
- $\varphi(\mathcal{A}(x, \varphi)) \geq \pi_1(x)$.

On the case of a point process X for $X_i \in X$, we have that $\mathcal{A}(X_i, X) \in X$, so we will simply write $\mathcal{A}(X_i)$.

In Figure 2.1 we present two examples of directed navigations, where the successor of a node is the nearest neighbor on a cone with fixed angle pointing on the e_1 direction and whose vertex is X_i. Each picture represents a different angle.

Now we want to define the traffic flow at a node $X_i \in X^{(s)}$ with respect to $\mathcal{A}$, for that let $\mathcal{A}^k$ be the k-fold iteration of $\mathcal{A}$ with $\mathcal{A}^0(X_i) = X_i$ and

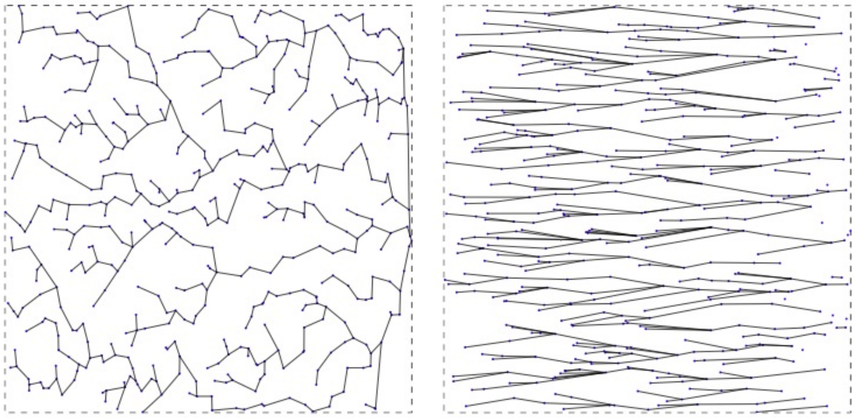

Figure 2.1: Directed navigation based on a Poisson pointed process on the unit square. Each node connects to its nearest neighbor which is also contained in a horizontal cone starting at the node. Figure took from [4].

$\mathcal{A}^k(X_i) = \mathcal{A}(\mathcal{A}^{k-1}(X_i))$, $k \geq 1$. So the *trajectory* of a node $X_i \in X^{(s)}$ is

$$\Gamma(X_i) = \{\mathcal{A}^k(X_i) : k \geq 0\}.$$

In other words $\Gamma(X_i)$ consists all the iterated successors of X_i. Also the *interpolated trajectory* is

$$\bar{\Gamma}(X_i) = \bigcup_{k\geq 0} [\mathcal{A}^k(X_i), \mathcal{A}^{k+1}(X_i)]$$

where $[x, y]$ is the line segment between x and y. Finally we define the *traffic flow* as

$$\Delta(X_i) = \sum_{X_j : X_i \in \Gamma(X_j)} \mu^{(s)}(X_j).$$

Then the traffic flow over a node is generated by the nodes that after some iterations will arrive to it.

Now we want to specify the notion of a density that measures the number of links induced by the navigation $\mathcal{A}$ on a neighborhood of sx. For that let

$$I_s^D(x) = B_{g(s)}(sx) \cap \{y \in \mathbb{R}^d : \pi(y) = \pi_1(sx)\}$$

where $g(s)$ is a positive real valued function, denote the environment of x inside the hyperplane through x perpendicular to e_1. Let

$$\Xi_s^D(x) = \{X_i \in X^{(s)} : [X_i, \mathcal{A}(X_i)] \cap I_s^D(x) \neq \emptyset\}$$

be the set of points such that the segment $[X_i, \mathcal{A}(X_i)]$ crosses $I_s^D(x)$.

We are ready to define the *link density condition*

Condition 2.1. There exists a function $\lambda_{\mathcal{A}}(x) : C \to (0, \infty)$ such that for every $x \in C$,

$$\lambda_{\mathcal{A}}(x) = \lim_{s\to\infty} \frac{E\#\Xi_s^D(x)}{s\eta_{d-1}(I_s^D(x))}$$

where η_{d-1} denotes the $(d-1)$-dimensional Hausdorff measure in $\mathbb{R}^d$.

We write

$$C_\varepsilon = \{x \in C : B_\varepsilon(x) \subset C\}$$

for the points of distance at least ε to the boundary of C. Moreover, we denote by

$$Z_r^D(x) = \mathbb{R} \times B_r^{d-1}(o) + x$$

the horizontal cylinder with radius r shifted to the point $x \in \mathbb{R}^d$. We define

$$E_{s,\varepsilon}^D = E_{s,\varepsilon,1}^D \cap E_{s,\varepsilon,2}^D$$

where

$$E_{s,\varepsilon,1}^D = \{\mathcal{A}(X_i) \neq X_i \text{ for all } X_i \in X^{(s)} \cap (sC)_{\varepsilon s}\}$$

is the event that away from the boundary of sC, there are no *dead ends*, that is no points $X_i \in X^{(s)}$ satisfying $\mathcal{A}(X_i) = X_i$ and

$$E_{s,\varepsilon,2}^D = \{(\bar{\Gamma}(X_i)) \subset Z_{h(s)}^D(X_i) \text{ for all } X_i \in X^{(s)} \cap (sC)_{\varepsilon s}\}$$

for $h(s)$ a positive real valued function, is the event that, away from the boundary of sC, all interpolated trajectories remain in the $h(s)$-cylinder at their starting points. Let us now define the *sub-ballisticity condition.*

Condition 2.2. For all $\varepsilon > 0$, we have $P((E_{s,\varepsilon}^D)^c) \in O(s^{-2d})$.

2.2 Convergence of the traffic flow average in the directed case

Now we are ready to enunciate and prove the main result of this chapter which states that under the link density and sub-ballisticity conditions we will have an almost surely convergence of the traffic flow average on a orthogonal micro environment to the flow. The case when $g(s)/s$ tends monotonically to zero is studied in [4] Theorem 2, now we will assume that $g(s)/s$ is constant getting a slightly different result.

Theorem 2.1. Let $x \in C$, assume that Conditions 2.1, 2.2 are satisfied for $g(s) = s$, $h(s) = s^\xi$, $0 < \xi < 1$ and that $E[(\#X^{(s)})^2] \in O(s^{2d})$.

1.Then,

$$\lim_{s\to\infty} \frac{E\sum_{X_i \in \Xi_s^D(x)} \Delta(X_i)}{E\#\Xi_s^D(x)} = \frac{1}{\kappa_{d-1}\lambda_{\mathcal{A}}(x)} \int_{-\infty}^0 \int_{B_1^{d-1}(o)} \mu(x+re_1+\hat{y})\lambda(x+re_1+\hat{y})d\bar{y}dr,$$

where $\hat{y} = (0, y_2, ..., y_d)$, $\bar{y} = (y_2, ..., y_d)$ and κ_{d-1} denotes the volume of the unit ball in $\mathbb{R}^{d-1}$.

2. If additionally $X^{(s)}$ is either a Poisson point process or μ is constant and $X^{(s)} = X \cap sC$ for some ergodic point process X, then

$$\lim_{s\to\infty} \frac{\sum_{X_i \in \Xi_s^D(x)} \Delta(X_i)}{E\#\Xi_s^D(x)} = \frac{1}{\kappa_{d-1}\lambda_{\mathcal{A}}(x)} \int_{-\infty}^0 \int_{B_1^{d-1}(o)} \mu(x+re_1+\hat{y})\lambda(x+re_1+\hat{y})d\bar{y}dr$$

in probability.

Proof. Let us start with two lemmas giving estimates of the accumulated traffic flow under the events $E_{s,\varepsilon}^D$ and $(E_{s,\varepsilon}^D)^c$. For this we need the following

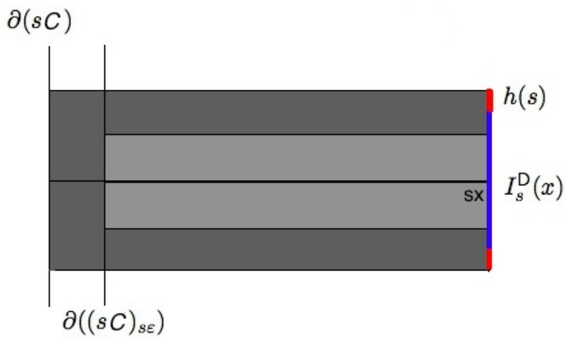

FIGURE 2.2: Construction of the cylinders $R^{l,-}_{s,\varepsilon}(x)$ (light gray) and $R^{l,+}_s(x)$ (union of light and dark gray). After re-scaling $\zeta^-(\varepsilon)$ does not depend on s.

definitions. We write $R^+_s(x) = Z^D_{g(s)+h(s)}(sx)$ and

$$R^{l,+}_s(x) = \{sy \in sC \cap R^+_s(x) : \pi_1(y) \leq \pi_1(x)\}$$

for the set of points in sC which lie in the microscopic cylinder $R^+_s(x)$ to the left of sx. Further we write $R^-_s(x) = Z^D_{g(s)-h(s)}(sx)$ and

$$R^{l,-}_{s,\varepsilon}(x) = \{sy \in sC \cap R^-_s(x) : \zeta^-(\varepsilon) \leq \pi_1(y) \leq \pi_1(x)\}$$

where

$$\zeta^-(\varepsilon) = \inf\{\pi(z) : z \in \partial C_\varepsilon \cap (R^-_s(x)/s)\}$$

denotes the infimum of the projection of the intersection of ∂C_ε with the microscopic cylinder $R^-_s(x)/s$ that is orthogonal to the horizontal axis, it is important to remark that since C is an hypercube this value does not depend on s, see also Figure 2.2.

Lemma 2.1. Let $\varepsilon > 0$ and $x \in C_{2\varepsilon}$. Furthermore let $X^{(s)}$ on the event $E^D_{s,\varepsilon}$ then

$$\sum_{X_j \in X^{(s)} \cap R^{l,-}_{s,\varepsilon}(x)} \mu^{(s)}(X_j) \leq \sum_{X_j \in \Xi^D_s(x)} \Delta^{(s)}(X_j) \leq \sum_{X_j \in X^{(s)} \cap R^{l,+}_s(x)} \mu^{(s)}(X_j).$$

Proof. For the upper bound, the cylinder condition given by $E^D_{s,\varepsilon}$ ensures that we can estimate

$$\begin{aligned}\sum_{X_i \in X^{(s)} \cap \Xi^D_s(x)} \Delta(x)(X_i) &= \sum_{X_j \in X^{(s)}} \mu^{(s)}(X_j) \sum_{X_i \in \Xi^D_s(x)} \mathbb{1}_{\Gamma(X_j)}(X_i) \\ = \sum_{X_i \in X^{(s)} \cap R^{l,+}_s(x)} \mu^{(s)}(X_j) &\sum_{X_i \in \Xi^D_s(x)} \mathbb{1}_{\Gamma(X_j)}(X_i) \leq \sum_{X_i \in X^{(s)} \cap R^{l,+}_s(x)} \mu^{(s)}(X_j).\end{aligned}$$

For the lower bound, since $E^D_{s,\varepsilon}$ does not give control over nodes in $sC \backslash (sC)_{s\varepsilon}$, those nodes have to be excluded.

□

Lemma 2.2.

$$E(\mathbb{1}_{(E^D_{s,\varepsilon})^c} \sum_{X_i \in \Xi^D_s(x)} \Delta(X_i)) \leq E(\mathbb{1}_{(E^D_{s,\varepsilon})^c} \sum_{X_i \in X^{(s)}} \mu(X_i)) \in O(1).$$

Proof. Since $\mathcal{A}$ is a directed navigation we have that for distinct $X_i, X_k \in \Xi_s^D(x)$, the intersection of the sets

$$\{X_j : X_i \in \Gamma(X_j)\}, \{X_j : X_k \in \Gamma(X_j)\}$$

is empty. This implies that under $(E_{s,\varepsilon}^D)^c$,

$$\sum_{X_i \in \Xi_s^D(x)} \Delta^{(s)}(X_i) \le \sum_{X_i \in X^{(s)}} \mu^{(s)}(X_i) \le \mu_M \# X^{(s)}$$

and hence by Cauchy-Schwartz inequality, since $E[(\#X^{(s)})^2] \in O(s^{2d})$ and Condition 2.2,

$$\begin{aligned} E(\mathbb{1}_{(E_{s,\varepsilon}^D)^c} \sum_{X_i \in \Xi_s^D(x)} \Delta^{(s)}(X_i)) &\le E(\mathbb{1}_{(E_{s,\varepsilon}^D)^c} \sum_{X_i \in X^{(s)}} \mu^{(s)}(X_i)) \\ &\le \mu_M \sqrt{(P(E_{s,\varepsilon}^D))^c} \sqrt{E[(\#X^{(s)})^2]} \in O(1). \end{aligned}$$

□

We are ready to proof the theorem. Let us abbreviate $i_s = s\eta_{d-1}(I_s^D(x))$,

$$N_s = i_s^{-1} \sum_{X_i \in \Xi_s^D(x)} \Delta(X_i), S = \frac{1}{\kappa_{d-1}} \int_{-\infty}^{0} \int_{B_1^{d-1}(o)} \mu(x+re_1+\hat{y})\lambda(x+re_1+\hat{y}) d\bar{y} dr.$$

Proof of Theorem. **1.** By Condition 2.1 it is enough to proof that

$$\lim_{s\to\infty} EN_s = S.$$

We have that

$$EN_s = E\mathbb{1}_{E_{s,\varepsilon}^D} N_s + E\mathbb{1}_{(E_{s,\varepsilon}^D)^c} N_s.$$

On the one hand since

$$i_s = s\eta_{d-1}((I_s^D(x)) = sg(s)^{d-1}\kappa_{d-1} = s^d \kappa_{d-1},$$

then $i_s^{-1} \in o(1)$. So by Lemma 2.2 $E\mathbb{1}_{(E_{s,\varepsilon}^D)^c} N_s \in o(1)$.

On the other hand by Lemma 2.1, Campbell's theorem and the coordinate transformation $y = sz$,

$$\begin{aligned} E\mathbb{1}_{E_{s,\varepsilon}^D} N_s &\le i_s^{-1} E \sum_{X_j \in X^{(s)} \cap R_s^{l,+}(x)} \mu^{(s)}(X_j) \\ &= i_s^{-1} \int_{R_s^{l,+}(x)} \mu^{(s)}(y)\lambda^{(s)}(y) dy \\ &= s^d i_s^{-1} \int_{R^+(x)/s} \mathbb{1}_{\pi_1(y) \le \pi_1(x)} \mu(y)\lambda(y) dy, \end{aligned}$$

Hence, by Fubini's theorem, we have for $\hat{y} = (0, y_2, ..., y_d)$, $\bar{y} = (y_2, ..., y_d)$

$$
\begin{aligned}
E\mathbb{1}_{E^D_{s,\varepsilon}} N_s &\leq \frac{s^{d-1}}{g(s)^{d-1}\kappa_{d-1}} \int_{-\infty}^{0} \int_{B^{d-1}_{\frac{(h(s)+g(s))}{s}}(o)} \mu(x+re_1+\hat{y})\lambda(x+re_1+\hat{y})d\bar{y}dr \\
&= \frac{1}{\kappa_{d-1}} \int_{-\infty}^{0} \int_{B^{d-1}_{1+s^{\xi-1}}(o)} \mu(x+re_1+\hat{y})\lambda(x+re_1+\hat{y})d\bar{y}dr.
\end{aligned}
$$

And by the Dominated Convergence theorem we get the upper bound.

For the lower bound lets pick s enough large that $g(s)-h(s)>0$. By Lemma 2.1 we can estimate

$$
i_s^{-1} E \sum_{X_j \in X^{(s)} \cap R^{l,-}_{s,\varepsilon}(x)} \mu^{(s)}(X_j) - i_s^{-1} E\mathbb{1}_{(E^D_{s,\varepsilon})^c} \sum_{X_j \in X^{(s)}} \mu^{(s)}(X_j) \leq E\mathbb{1}_{E^D_{s,\varepsilon}} N_s.
$$

The second summand tends to zero when s tends to infinity since $i_s^{-1} \in o(1)$ and by Lemma 2.2.

Lets proceed to estimate the first one. Again by Campbell's theorem and the coordinate transformation $y = sz$,

$$
\begin{aligned}
i_s^{-1} E \sum_{X_j \in X^{(s)} \cap R^{l,+}_{s,\varepsilon}(x)} \mu^{(s)}(X_j) &= i_s^{-1} \int_{R^{l,-}_{s,\varepsilon}(x)} \mu^{(s)}(y)\lambda^{(s)}(y)dy \\
&= s^d i_s^{-1} \int_{R^-(x)/s} \mathbb{1}_{\pi_1(y)\leq\pi_1(x)}\mu(y) - s^d i_s^{-1} \int_{R^-(x)/s} \mathbb{1}_{\pi_1(y)<\zeta_s^-(\varepsilon)}\mu(y)\lambda(y)dy,
\end{aligned}
$$

By an analogous reasoning to the upper bound the first summand converges to S.

For the second summand lets observe that

$$
s^d i_s^{-1} \int_{R^-(x)/s} \mathbb{1}_{\pi_1(y)<\zeta_s^-(\varepsilon)}\mu(y)\lambda(y)dy \leq \mu_M \lambda_M \frac{\eta_{d-1}(B^{d-1}_{g(s)-h(s)}(o))}{\kappa_{d-1}}(\zeta^-(\varepsilon)-\zeta^-)
$$

where $\zeta^- = \inf\{\pi_1(z), z \in \partial C \cap (R_s^-(x)/s)\}$ denotes the smallest first coordinate of points on the boundary of S intersected with the cylinder $R_s^-(x)/s$ and also does not depend on s since C is an hypercube. To conclude that the upper bound of the inequality tends to zero, note that

$$
\lim_{s\to\infty} \frac{\eta_{d-1}(B^{d-1}_{g(s)-h(s)}(o))}{\kappa_{d-1}} = \lim_{s\to\infty}(g(s)-h(s))^{d-1} = 1.
$$

And finally when ε tends to zero, $\zeta^-(\varepsilon)$ tends to ζ^-.

2. Again by Condition 2.1 it is enough to proof that

$$
\lim_{s\to\infty} N_s = S \text{ in probability.}
$$

By Markov's inequality

$$
P(|N_s - S| > \varepsilon) \leq \varepsilon^{-1} E(|N_s - S|),
$$

so we will prove that the upper bound tends to zero when s tends to infinity. We start by estimating

$$E(|N_s - S|) \leq E(|N_s - EN_s|) + E(|EN_s - S|)$$

where by the Dominated Convergence theorem and 1 the second summand tends to zero. Furthermore we can write

$$E(|N_s - EN_s|) = E(\mathbb{1}_{E^D_{s,\varepsilon}} |N_s - EN_s|) + E(\mathbb{1}_{(E^D_{s,\varepsilon})^c} |N_s - EN_s|).$$

On the one hand the second summand can be bounded

$$E(\mathbb{1}_{(E^D_{s,\varepsilon})^c} |N_s - EN_s|) \leq E(\mathbb{1}_{(E^D_{s,\varepsilon})^c} N_s) + E\mathbb{1}_{(E^D_{s,\varepsilon})^c} EN_s.$$

By Lemma 2.2 the first term is in $o(1)$, by Condition 2.2 and 1.

$$\lim_{s\to\infty} E\mathbb{1}_{(E^D_{s,\varepsilon})^c} EN_s = 0 \cdot S = 0.$$

On the other hand by Lemma 2.1 the first summand can be bounded

$$\begin{aligned} E(\mathbb{1}_{E^D_{s,\varepsilon} \cap \{N_s \geq EN_s\}} (N_s - EN_s)) + E(\mathbb{1}_{E^D_{s,\varepsilon} \cap \{EN_s \geq N_s\}} (EN_s - N_s)) \\ \leq E(|N_s^+ - EN_s|) + E(|N_s^{-,\varepsilon} - EN_s|), \end{aligned}$$

where we write

$$N_s^+ = i_s^{-1} \sum_{X_j \in X^{(s)} \cap R_s^{l,+}(x)} \mu^{(s)}(X_j) \text{ and } N_s^{-,\varepsilon} = i_s^{-1} \sum_{X_j \in X^{(s)} \cap R_{s,\varepsilon}^{l,-}(x)} \mu^{(s)}(X_j).$$

Finally we have that

$$\begin{aligned} E(|N_s^+ - EN_s|) &\leq E(|N_s^+ - EN_s^+|) + E(|EN_s^+ - EN_s|) \\ E(|N_s^{-,\varepsilon} - EN_s|) &\leq E(|N_s^{-,\varepsilon} - EN_s^{-,\varepsilon}|) + E(|EN_s^{-,\varepsilon} - EN_s|). \end{aligned}$$

On 1 we showed that

$$\lim_{s\to\infty} |EN_s^+ - EN_s| = \lim_{\varepsilon\to 0} \lim_{s\to\infty} |EN_s^{-,\varepsilon} - EN_s| = 0.$$

So we have to find estimates for $E(|N_s^+ - EN_s^+|)$ and $E(|N_s^{-,\varepsilon} - EN_s^{-,\varepsilon}|)$. This will be done distinguishing between cases.

Case 1) Let us start with the case of a Poisson point process $X^{(s)}$. Then by Jensen's inequality, Campbell's theorem for the variance and Fubini's theorem we have

$$\begin{aligned} E(|N_s^+ - EN_s^+|) &\leq (VN_s^+)^{1/2} = i_s^{-1} V(\sum_{X_j \in X^{(s)} \cap R_s^{l,+}(x)} \mu^{(s)}(X_j))^{1/2} \\ &= i_s^{-1}(s^d \int_{R_s^+(x)/s} \mathbb{1}_{\pi_1(y) \leq \pi_1(x)} \mu(y)^2 \lambda(y) dy)^{1/2} \\ &= ((i_s^{-1})^2 s^d \int_{-\infty}^0 \int_{B^{d-1}_{\frac{(h(s)+g(s))}{s}}(o)} \mu(x + re_1 + \hat{y})^2 \lambda(x + re_1 + \hat{y}) d\bar{y} dr)^{1/2}. \end{aligned}$$

For finishing lets observe that

$$
\begin{aligned}
(i_s^{-1})^2 s^d \int_{B^{d-1}_{\frac{(h(s)+g(s))}{s}}(o)} & \mu(x + re_1 + \hat{y})^2 \lambda(x + re_1 + \hat{y}) d\bar{y} \\
&\le \frac{s^d \eta_{d-1}(B^{d-1}_{\frac{(h(s)+g(s))}{s}}(o))}{(s^d \kappa_{d-1})^2} \mu_M^2 \lambda_M = \frac{\mu_M^2 \lambda_M}{\kappa_{d-1} s^d} (\frac{s + s^\xi}{s})^{d-1},
\end{aligned}
$$

which tends to zero when s tends to infinity. The argument for $N^{-,\varepsilon}$ is analogous.

Case 2) Since X is ergodic λ is constant, by hypothesis μ is also constant, let us abbreviate $r_s^+(x) = \eta_d(R_s^{l,+}(x))$. Then by Campbell's theorem

$$
\begin{aligned}
E(|N_s^+ - EN_s^+|) &= i_s^{-1} E(|\#(X \cap R_s^{l,+}(x))\mu - r_s^+(x)\lambda\mu|) \\
&= \mu r_s^+(x) i_s^{-1} E(|\frac{\#(X \cap R_s^{l,+}(x))}{r_s^+(x)} - \lambda|),
\end{aligned}
$$

where by analogous computations to 1.

$$
\lim_{s \to \infty} r_s^+(x) i_s^{-1} = (\pi_1(x) - \zeta^-).
$$

Since X is translation invariant, $R_s^{l,+}$ can be replaced by

$$
R_{s,x}^{l,+} = \{sy \in \partial(C - x) \cap R_s^+(o) : \pi_1(y) \le 0\},
$$

which since C is an hypercube is a is a sequence of convex averaging windows. Hence

$$
E(|\frac{\#(X \cap R_s^{l,+})}{r_s^+(x)} - \lambda|) = E(|\frac{\#(X \cap R_{s,x}^{l,+})}{r_s^+(x)} - \lambda|,
$$

which by the ergodic theorem tends to zero when s tends to infinity. The argument for $N^{-,\varepsilon}$ is analogous. $\square$

2.3 Directed spanning tree navigation

We will give an example of a directed navigation that satisfies the Conditions 2.1 and 2.2 on a Poisson point process.

Definition 2.2 (Spanning tree navigation)**.** Let $\mathcal{A} : \mathbb{R}^d \times \mathbf{N} \to \mathbf{N}$, $\varphi \in \mathbf{N}$ and $x \in \mathbb{R}^d$. $\mathcal{A}(x, \varphi)$ is defined to be the point in $\varphi \cap ((\pi_1(x), \infty) \times \mathbb{R}^{d-1})$ that is of minimal Euclidean distance to x. If the minimum is realized in several points, we choose the lexicographic smallest of them. If there is no point of φ to the right of x, then we put $\mathcal{A}(x, \varphi) = x$.

The spanning tree is an example of a directed navigation, from now until the end of this chapter we will use $\mathcal{A}$ to refer it. Firs we will proof that $\mathcal{A}$ satisfies Condition 2.1 in [4] Proposition 5 is considered the case where $g(s) = s^\xi$, $0 < \xi < 1$. Here we follow similar techniques to prove the next result.

Proposition 2.1. Let $\lambda : C \to [0, \infty)$ a locally Lipschitz and bounded mapping. Further, let $X^{(s)}$ be a Poisson point process on sC with intensity function $\lambda^{(s)}$. Then Condition 2.1 is satisfied for the directed spanning tree on $X^{(s)}$ together with the fluctuation function $g(s) = s$.

Proof. First we will show three auxiliary lemmas. We will simplify the notation writing $I_s(x) = I_s^D(x)$ and we will use $X^{[\lambda(x)]}$ for the marked process of constant intensity function $\lambda(x)$ which construction was explained in Proposition 1.3. Also for $\xi' \in (1,2)$ we write $I_s^+(x) = I_s(x) \oplus B^d_{s^{2-\xi'}}(o)$ where $\oplus$ denotes the Minkowski's sum. Our first result analyze the crossings out of $I_s^+(x)$ when s tends to infinity.

Lemma 2.3. Let $x \in C$ be arbitrary. Then,

1. $\int_{sC\setminus I_s^+(x)} P([y, \mathcal{A}(y, X^{(s)} \cup \{y\})] \cap I_s(x) \neq \emptyset)\lambda^{(s)}(y)dy \in o(s\eta_{d-1}(I_s(x)))$,

2. $\int_{sC\setminus I_s^+(x)} P([y, \mathcal{A}(y, X^{[\lambda(x)]} \cup \{y\})] \cap I_s(x) \neq \emptyset)dy \in o(s\eta_{d-1}(I_s(x)))$.

Proof. **1.** Let $\varepsilon > 0$ be such that $B_{3\varepsilon}(x) \subset C$. If $y \in sC \setminus B_{2\varepsilon s}(sx)$ is such that $[y, \mathcal{A}(y, X^{(s)} \cup \{y\})] \cap I_s(x) \neq \emptyset$ then $X^{(s)}$ does not contain any points in the set

$$B_{|y-sx|-\varepsilon s}(y) \cap B_{2\varepsilon s}(sx) \cap ([\pi_1(y), \infty) \times \mathbb{R}^{d-1})$$

which contains half ball of radius $2^{-1}\varepsilon s$. Hence since $X^{(s)}$ is a Poisson point process of intensity $\lambda^{(s)}$ we can estimate,

$$P([y, \mathcal{A}(y, X^{(s)} \cup \{y\})] \cap I_s(x) \neq \emptyset) \leq \exp(-\lambda_m \kappa_d 2^{-1}(2^{-1}\varepsilon s)^d),$$

where κ_d denotes the volume of the unit ball in $\mathbb{R}^d$. And we can bound

$$\begin{aligned}\int_{sC\setminus B_{2\varepsilon s}(sx)} & P([y, \mathcal{A}(y, X^{(s)} \cup \{y\})] \cap I_s(x) \neq \emptyset)\lambda^{(s)}(y)dy \\ &\leq \exp(-\lambda_m \kappa_d 2^{-1}(2^{-1}\varepsilon s)^d)\eta_d(sC) \\ &= \eta_d(C)\exp(-\lambda_m \kappa_d 2^{-1}(2^{-1}\varepsilon s)^d)s^d \in o(1).\end{aligned}$$

Which implies that

$$\int_{sC\setminus B_{2\varepsilon s}(sx)} P([y, \mathcal{A}(y, X^{(s)} \cup \{y\})] \cap I_s(x) \neq \emptyset)\lambda^{(s)}(y)dy \in o(s\eta_{d-1}(I_s(x))),$$

since for s big enough $s\eta_{d-1}(I_s(x)) \geq 1$.

For finishing the claim, we have to proof that

$$\int_{B_{2\varepsilon s}(sx)\setminus I_s^+(x)} P([y, \mathcal{A}(y, X^{(s)} \cup \{y\})] \cap I_s(x) \neq \emptyset)\lambda^{(s)}(y)dy \in o(s\eta_{d-1}(I_s(x))).$$

Let $y \in B_{2\varepsilon s}(sx) \setminus I_s^+(x)$ such that $[y, \mathcal{A}(y, X^{(s)} \cup \{y\})] \cap I_s(x) \neq \emptyset$ then $|y - \mathcal{A}(y, X^{(s)} \cup \{y\})| \geq s^{2-\xi'}$ and we can estimate

$$\begin{aligned}\int_{B_{2\varepsilon s}(sx)\setminus I_s^+(x)} & P([y, \mathcal{A}(y, X^{(s)} \cup \{y\})] \cap I_s(x) \neq \emptyset)\lambda^{(s)}(y)dy \\ &\leq \int_{B_{2\varepsilon s}(sx)} P(|y - \mathcal{A}(y, X^{(s)} \cup \{y\})| \geq s^{2-\xi'})\lambda^{(s)}(y)dy.\end{aligned}$$

The event $|y - \mathcal{A}(y, X^{(s)} \cup \{y\})| \geq s^{2-\xi'}$ implies that half a ball of radius $s^{2-\xi'}$ contains no points so we can bound

$$\begin{aligned}\int_{B_{2\varepsilon s}(sx)} & P(|y - \mathcal{A}(y, X^{(s)} \cup \{y\})| \geq s^{2-\xi'})\lambda^{(s)}(y)dy \\ &\leq \exp(-\lambda_m \kappa_d 2^{-1}(s^{2-\xi'})^d)\eta_d(B_{2\varepsilon s}(sx)) \\ &= \kappa_d \exp(-\lambda_m \kappa_d 2^{-1}(s^{2-\xi'})^d)(2\varepsilon s)^d \in o(1).\end{aligned}$$

Which implies that

$$\int_{B_{2\varepsilon s}(sx)} P(|y - \mathcal{A}(y, X^{(s)} \cup \{y\})| \geq s^{2-\xi'})\lambda^{(s)}(y)dy \in o(s\eta_{d-1}(I_s(x)))$$

since for s big enough $\eta_{d-1}(sI_s(x)) \geq 1$.

2. It is implied from 1 since $X^{[\lambda(x)]}$ has constant intensity function $\lambda^{(s)}(y) = \lambda(x)$. □

We show that if we replace integrals over $I_s^+(x)$ with respect to the intensity $\lambda^{(s)}$ by integrals with respect to the constant intensity $\lambda(x)$, then the error is of order $o(s\eta_{d-1}(I_s(x)))$.

Lemma 2.4. Let $x \in C$ be arbitrary and let $f : I_s^+(x) \to [0,1]$ be a measurable function. Then,

$$|\int_{I_s^+(x)} f(y)\lambda^{(s)}(y)dy - \lambda(x)\int_{I_s^+(x)} f(y)dy| \in o(s\eta_{d-1}(I_s(x))$$

Proof. Let L be the Lipschitz constant of λ in C. Since the biggest distance from between two points in $I_s^+(x)$ is $2(g(s) + s^{2-\xi'}) = 2(s + s^{2-\xi'})$, for $y \in I_s^+(x)$ we can estimate

$$|\lambda^{(s)}(y) - \lambda(x)| = |\lambda(\frac{y}{s}) - \lambda(x)| \leq 2L(s + s^{2-\xi'})s^{-1}.$$

Hence

$$\int_{I_s^+(x)} f(y)|\lambda^{(s)}(y) - \lambda(x)|dy \leq 2L(s + s^{2-\xi'})s^{-1}\eta_d(I_s^+(x)).$$

Finally lets observe that since $o(\eta_d(I_s^+(x))) = o(s^{2-\xi'}\eta_{d-1}(I_s(x)))$, after dividing by $s\eta_{d-1}(I_s(x))$ we get the bound

$$2L(s^{1-\xi'} + s^{2(1-\xi')})$$

which tends to zero when s tends to infinity. □

Finally we show that replacing the Poisson point process $X^{(s)}$ by the stationary Poisson point process $X^{[\lambda(x)]}$ in

$$P([y, \mathcal{A}(y, X^{(s)} \cup \{y\})] \cap I_s(x) \neq \emptyset)$$

leads to a negligible error.

Lemma 2.5. Let $x \in C$ be arbitrary. Then

$$\int_{I_s^+(x)} |P([y, \mathcal{A}(y, X^{(s)} \cup \{y\})] \cap I_s(x) \neq \emptyset) - P([y, \mathcal{A}(y, X^{[\lambda(x)]} \cup \{y\})] \cap I_s(x) \neq \emptyset)|dy$$

is of order $o(s\eta_{d-1}(I_s(x))$.

Proof. Let $I_s^{++}(x) = I_s^+ \oplus B_{s^{2-\xi'}}$ and $\alpha d < -1$. Let $y \in I_s^+(x)$ and lets assume that $[y, \mathcal{A}(y, X^{(s)} \cup \{y\})] \cap I_s(x) \neq \emptyset$. Then we have two possibilities

1. $[y, \mathcal{A}(y, X^{(s)} \cup \{y\})] \neq [y, \mathcal{A}(y, X^{[\lambda(x)]} \cup \{y\})]$,
2. $[y, \mathcal{A}(y, X^{(s)} \cup \{y\})] = [y, \mathcal{A}(y, X^{[\lambda(x)]} \cup \{y\})]$ which implies that $[y, \mathcal{A}(y, X^{[\lambda(x)]} \cup \{y\})] \cap I_s(x) \neq \emptyset$.

We can follow an analogous reasoning exchanging $X^{(s)}$ by $X^{[\lambda(x)]}$. As a consequence we estimate

$$
\begin{aligned}
&|P([y, \mathcal{A}(y, X^{(s)} \cup \{y\})] \cap I_s(x) \neq \emptyset) - P([y, \mathcal{A}(y, X^{\lambda(x)} \cup \{y\})] \cap I_s(x) \neq \emptyset)| \\
&\quad \leq P([y, \mathcal{A}(y, X^{(s)} \cup \{y\})] \neq [y, \mathcal{A}(y, X^{[\lambda(x)]} \cup \{y\})]) \\
&\quad \leq P(X^{[\lambda(x)]} \cap B_{s^\alpha}(y) \neq X^{(s)} \cap B_{s^\alpha}(y)) + P(|y - \mathcal{A}(y, X^{[\lambda(x)]} \cup \{y\})| \geq s^\alpha).
\end{aligned}
$$

On the one hand, lets estimate the first summand, hence lets remember that we can write

$$X^{(s)} = \{X_i : (X_i, U_i) \in X \text{ and } U_i \leq \lambda^{(s)}(X_i)\},$$

$$X^{[\lambda(x)]} = \{X_i : (X_i, U_i) \in X \text{ and } U_i \leq \lambda(x)\},$$

which by elemental computations implies that

$$X^{(s)} \triangle X^{[\lambda(x)]} \subset \{X_i : (X_i, U_i) \in X \text{ and } U_i \leq |\lambda^{(s)}(X_i) - \lambda(x)|\}$$

where $\triangle$ denotes the symmetrical difference.

The Lipschitz continuity implies that

$$\sup_{y \in I_s^{++}(x)} |\lambda^{(s)}(y) - \lambda(x)| \leq 2L(s + 2s^{2-\xi'})s^{-1}.$$

Hence by Campbell's theorem,

$$
\begin{aligned}
\int_{I_s^+(x)} &P(X^{[\lambda(x)]} \cap B_{s^\alpha}(y) \neq X^{(s)} \cap B_{s^\alpha}(y))dy \\
&\leq \int_{I_s^+(x)} E\#(X^{[|\lambda(\cdot) - \lambda(x)|]} \cap B_{s^\alpha}(y))dy \\
&\leq \int_{I_s^+(x)} 2L(s + 2s^{2-\xi'})s^{-1}\eta_d(B_{s^\alpha}(y))dy \\
&= 2L\kappa_d\eta_d(I_s^+(x))(s + 2s^{2-\xi'})s^{\alpha d - 1}.
\end{aligned}
$$

Since $o(\eta_d(I_s^+(x))) = o(s^{2-\xi'}\eta_{d-1}(I_s(x)))$, after dividing by $\eta_{d-1}(I_s(x))$ we get the bound

$$2L\kappa_d(s^{\alpha d + 2 - \xi'} + 2s^{\alpha d + 3 - 2\xi'})$$

which due to the conditions imposed to α tends to zero as s tends to infinity.

On the other hand, by since $X^{[\lambda(x)]}$ is stationary

$$\int_{I_s^+(x)} P(|y - \mathcal{A}(y, X^{[\lambda(x)]} \cup \{y\})| \geq s^\alpha)dy = P(|\mathcal{A}(o, X^{[\lambda(x)]} \cup \{o\})| \geq s^\alpha)\eta_d(I_s^+(x)).$$

And as in the proof of Lemma 2.3 we can estimate

$$\int_{I_s^+(x)} P(|\mathcal{A}(o, X^{[\lambda(x)]} \cup \{o\})| \geq s^\alpha)dy \leq \exp(-\lambda_m \kappa_d 2^{-1} s^{d\alpha})\eta_d(I_s^+(x)).$$

Hence after dividing by $o(s\eta_{d-1}(I_s(x))$ we get the bound

$$\exp(-\lambda_m \kappa_d 2^{-1}(s^{d\alpha}))s^{1-\xi'}$$

which tends to zero when s tends to infinity. □

Proof of Proposition. Let $x \in C$, we claim that

$$\lambda_{\mathcal{A}}(x) = \frac{\lambda(x) \int_C P([y, \mathcal{A}(y, X^{[\lambda(x)]} \cup \{y\})] \cap I_1(x) \neq \emptyset)dy}{\eta_{d-1}(I_1(x))}.$$

On the one hand by Slyvniak-Mecke's theorem we get that

$$\begin{aligned} &E\#\{X_i \in X^{[\lambda(x)]} \cap sC : [X_i, \mathcal{A}(X_i, X^{(s)})] \cap I_s(x) \neq \emptyset\} \\ &= \lambda(x) \int_{sC} P([y, \mathcal{A}(y, X^{[\lambda(x)]} \cup \{y\})] \cap I_s(x) \neq \emptyset)dy. \end{aligned}$$

Since $X^{[\lambda(x)]}$ is stationary we have that

$$\frac{\lambda(x) \int_{sC} P([y, \mathcal{A}(y, X^{[\lambda(x)]} \cup \{y\})] \cap I_s(x) \neq \emptyset)dy}{s\eta_{d-1}(I_s(x))}$$

is constant for all s. On the other hand, also by Slyvniak-Mecke's theorem

$$\begin{aligned} E\#\Xi_s^D(x) &= E\#\{X_i \in X^{(s)} : [X_i, \mathcal{A}(X_i, X^{(s)})] \cap I_s(x) \neq \emptyset\} \\ &= \int_{sC} P([y, \mathcal{A}(y, X^{(s)} \cup \{y\})] \cap I_s(x) \neq \emptyset)\lambda^{(s)}(y)dy. \end{aligned}$$

Therefore, by Lemma 2.3, it suffices to show that the difference

$$\begin{aligned} &\int_{I_s^+(x)} P([y, \mathcal{A}(y, X^{(s)} \cup \{y\})] \cap I_s(x) \neq \emptyset)\lambda^{(s)}(y)dy \\ &- \int_{I_s^+(x)} P([y, \mathcal{A}(y, X^{[\lambda(x)]} \cup \{y\})] \cap I_s(x) \neq \emptyset)\lambda(x)dy \end{aligned}$$

is of order $o(s\eta_{d-1}(I_s(x)))$. This follows by the triangle inequality and Lemmas 2.4, 2.5. □

Now we show that the second condition is also satisfied, this strictly depends on the value of $h(s)$ which in our model satisfies the same hypothesis that in [4]. This proof has been took from [4] Proposition 6.

Proposition 2.2. Let $\lambda : C \to [0, \infty)$ a locally Lipschitz and bounded mapping. Furthermore, let $X^{(s)}$ be a Poisson point process on sC with intensity function $\lambda^{(s)}$. Then the Condition 2.2 is satisfied for the directed spanning tree on $X^{(s)}$ together with the fluctuation function $h(s) = s^{1-1/(64d)}$.

Proof. The proof will take several steps. Lets define the maximal radius of stabilization as

$$R_{-,s} = \max_{X_i \in X^{(s)} : \pi_1(X_i) \leq 2s - s^{1/2d}} |X_i - \mathcal{A}(X_i)|$$

and let $E_s^{(1)}$ denote the event $\{R_{-,s} \geq \frac{1}{6}s^{1/(2d)}\}$.

Lemma 2.6. As $s \to \infty$, $P(E_s^{(1)}) \in O(s^{-2d})$.

Proof. Let $x \in sC$ with $2s - s^{1/(2d)} \leq \pi_1(x)$, since $X^{(s)}$ is a Poisson point process

$$\begin{aligned}
&P(|x - \mathcal{A}(x, X^{(s)} \cup \{x\})| > \frac{1}{6}s^{1/(2d)}) \\
&\quad \leq P(X^{(s)} \cap (B_{\frac{1}{6}s^{1/(2d)}}(x) \cap ([\pi_1(x), \infty) \times \mathbb{R}^{d-1}) = \emptyset) \\
&\quad \leq \exp(-\lambda_m \kappa_d 2^{-1} 6^{-d} s^{1/2})
\end{aligned}$$

And then by the Slyvniak-Mecke's theorem

$$\begin{aligned}
P(E_s^{(1)}) &= E(\mathbb{1}\{\exists X_i \in X^{(s)} : |X_i - \mathcal{A}(X_i, X^{(s)})| > \frac{1}{6}s^{1/(2d)}\}) \\
&\leq E(\sum_{X_i \in X^{(s)}} \mathbb{1}\{|X_i - \mathcal{A}(X_i, X^{(s)})| > \frac{1}{6}s^{1/(2d)}\}) \\
&\leq \lambda_M s^d \eta_d(C) \exp(-\lambda_m \kappa_d 2^{-1} 6^{-d} s^{1/2}) \in O(s^{-2d}).
\end{aligned}$$

□

We write

$$C_{-,s} = [-2s, 2s - 2s^{1/(2d)}] \times [-2s + 2s^{1/(2d)}, 2s - 2s^{1/(2d)}]^{d-1}.$$

Furthermore, we replace $s^{1/(2d)}$ by s' and assume that $4s(s')^{-1}$ is an odd integer. This is no a restriction, since otherwise s' can be adjusted in such a way that it is of the same order as $s^{1/(2d)}$, hence we can subdivide sC on a cubic grid where each cube has center on $s'z$ and side s'.

Next, for $s \geq 1$ and $z \in \mathbb{Z}^d$ we let $H_{s,z}$ denote the event that the point processes $X^{(s)}$ and $X^{[\lambda_{s,z}]}$ agree on $Q_{3s'}(s'z) \cap sC$, where

$$\lambda_{s,z} = \max\{\lambda^{(s)}(x) : x \in Q_{3s'}(s'z)\}$$

denotes the maximum of the density $\lambda^{(s)}(\cdot)$ in the cube $Q_{3s'}(s'z)$ of side length $3s'$ centered at $s'z$. We say that the site z (or the associated cube $Q_{3s'}(s'z)$) is *s-good* if the event $H_{s,z}$ occurs. Sites that are not s-good are called s-bad.

We want to use that locally, trajectories do not deviate substantially from the horizontal line. More precisely, let $H'_{s,z}$ be the event that for every path $\Gamma(X_0)$ in the directed spanning tree on $X^{[\lambda_{s,z}]} \cap Q_{3s'}(s'z)$ such that X_0 is contained in $Q_{s'}(s'z)$ and whose endpoint X_e satisfies $\pi_1(X_e) \leq \pi_1(s'z) + s'$ we have that $\Gamma(X_0) \subset Z^D_{(s')^{5/8}}$. We say that $E_s^{(2)}$ occurs if there exists $z \in \mathbb{Z}^d$ such that $s'z \in sC_{-,s}$ and $H'_{s,z}$ fails to occur. By [1] Theorem 4.10 the next result follows.

Lemma 2.7. $P(E_s^{(2)}) \in O(s^{-2d})$ when s tends to infinity.

For any path Γ in the directed spanning tree on $X^{(s)}$ we let

$$\#_s\Gamma = \#\{z \in \mathbb{Z}^d : X_i \in Q_{s'}(s'z) \text{ for some } X_i \in \Gamma\}$$

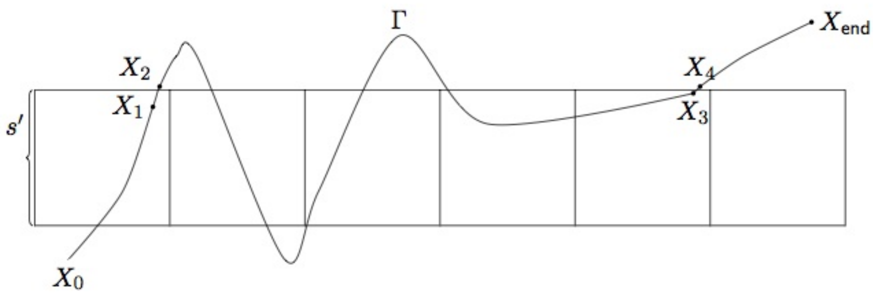

FIGURE 2.3: Illustration of the induction step in the proof of Lemma 2.8. Figure took from [4].

denote the number of s'-cubes intersected by Γ. Similarly, we let $\#_s\Gamma_{s,g}$ and $\#_s\Gamma_{s,b}$ denote the number of good and bad cubes that are intersected by Γ. We want to find an upper bound for the vertical displacement of a path Γ. If Γ is a path in the directed spanning tree on $X^{(s)}$ starting from X_0 we let

$$V(\Gamma) = \max_{X_i \in \Gamma} d(X_0, X_i)$$

where we write $d(X_0, X_i) = d_\infty(X_0 + \mathbb{R}e_1, X_i)$, for the d_∞ distance of X_i to the horizontal line $X_0 + \mathbb{R}e_1$.

Lemma 2.8. Let $\Gamma \subset C_{-,s}$ be an arbitrary path in the directed spanning tree on $X^{(s)}$. Then under the complement of the event $E_s^{(1)} \cup E_s^{(2)}$ almost surely,

$$V(\Gamma) \leq 2s' + 3(s')^{5/8}\#_{s,g}\Gamma + 3s'\#_{s,b}\Gamma.$$

Proof. We will proceed by induction under the number of vertex. The assertion is trivial if $V(\Gamma) \leq 2s'$, so we will assume that $V(\Gamma) > 2s'$. Now lets proof the induction step. By shortening the path if necessary we may assume that the maximal displacement $V(\Gamma)$ is achieved at the endpoint X_e of Γ.

Let $z_0 \in \mathbb{Z}^d$ such that $Q_{s'}(s'z_0)$ contains the starting point X_0 of Γ. Then, we let $X_2 \in \Gamma$ denote the first vertex of Γ such that $X_2 \in Q_{s'}(s'z_2)$ for some $z_2 \in \mathbb{Z}^d$ with $d(s'z_0, s'z_2) > s'$. We also let $X_1 \in \Gamma$ be such that $X_2 = \mathcal{A}(X_1, X^{(s)})$. Similarly, we let $X_3 \in \Gamma$ denote the last vertex of Γ such that $X_3 \in Q_{s'}(s'z_3)$ for some $z_3 \in \mathbb{Z}^d$ with $d(s'z_0, s'z_3) \leq s'$. Finally, we put $X_4 = \mathcal{A}(X_3, X^{(s)})$. This points can be constructed since we are under $(E_s^{(1)})^c$. Saw Figure 2.3 for an illustration of the construction.

We will denote the subpaths of Γ from X_0 to X_1 and X_4 to X_e as $\Gamma[X_0, X_1]$ and $\Gamma[X_4, X_e]$ respectively. Since $V(\Gamma) > 2s'$ no s'-subcube is hit both $\Gamma[X_0, X_1]$ and $\Gamma[X_4, X_e]$, using this fact, Pythagoras' theorem and $(E_s^{(1)})^c$,

$$\begin{aligned} V(\Gamma) &\leq V(\Gamma[X_0, X_2]) + V(\Gamma[X_3, X_4]) + V(\Gamma[X_4, X_e]) \\ &\leq 2s' + |X_3 - X_4| + V(\Gamma[X_4, X_e]) \leq \frac{5}{2}s' + V(\Gamma[X_4, X_e]) \end{aligned}$$

Hence using the induction step, we arrive that

$$V(\Gamma) \leq \frac{5}{2}s' + 2s' + 3(s')^{5/8}\#_{s,g}\Gamma[X_4, X_e] + 3s'\#_{s,b}\Gamma[X_4, X_e].$$

The assertion follows if $\Gamma[X_0, X_1]$ hits at least one s-bad cube. lets assume that $\Gamma[X_0, X_1]$ just intersects s-good cubes. Since $d(s'z_0, s'z_2) > s'$ and $(E_s^{(1)} \cup E_s^{(2)})^c$ occurs, it follows that for s' big enough

$$\frac{5}{6}s' \leq d(X_0, X_1) \leq V(\Gamma[X_0, X_1]) \leq (s')^{5/8}\#_{s,g}\Gamma[X_0, X_1].$$

And we complete the induction step by noting that

$$\begin{aligned}
V(\Gamma) &\leq \frac{5}{2}s' + 2s' + 3(s')^{5/8}\#_{s,g}\Gamma[X_4, X_e] + 3s'\#_{s,b}\Gamma[X_4, X_e] \\
&\leq 2s' + 3(s')^{5/8}(\#_{s,g}\Gamma[X_0, X_1] + \#_{s,g}\Gamma[X_4, X_e]) + 3s'\#_{s,b}\Gamma[X_4, X_e] \\
&\leq 2s' + 3(s')^{5/8}\#_{s,g}\Gamma + 3s'\#_{s,b}\Gamma.
\end{aligned}$$

□

We need to provide appropriate upper bounds on $\#_{s,g}\Gamma(X_i)$ and $\#_{s,b}\Gamma(X_i)$ for any $X_i \in X^{(s)}$.

Lemma 2.9. 1. For every $z \in \mathbb{Z}^d$ with $s'z \in C_{-,s}$ it holds that almost surely

$$P(H_{s,z}^c | X^{(s)}) \leq L\sqrt{d}3^{d+1}s^{-1/4}$$

where L is the global Lipschitz constant of λ.

2. Every path $\Gamma \subset C_{-,s}$ in the directed spanning three on $X^{(s)}$ almost surely satisfies that

$$\mathbb{1}\{\#_s\Gamma \leq 3^{d+3}s(s')^{-1}\}P(\#_{s,b}\Gamma \geq s^{1-5/(8d)}|X^{(s)}) \leq \exp(-\sqrt{s}).$$

Proof. **1.** The event $H_{s,z}$ is expressed as

$$X \cap \{(X_i, U_i) : X_i \in Q_{3s'}(s'z) \text{ and } \lambda^{(s)}(X_i) \leq U_i \leq \lambda_{s,z}\} = \emptyset.$$

Hence

$$H_{s,z}^c \subset \{(X_i, U_i) : X_i \in Q_{3s'}(s'z) \text{ and } \lambda_{z,m}^{(s)} \leq U_i \leq \lambda_{s,z}\},$$

with $\lambda_{s,z,m} = \min\{\lambda^{(s)}(x) : x \in Q_{3s'}(s'z)\}$, i.e. a Poisson point process with intensity $\lambda_{s,z} - \lambda_{s,z,m}$. As a consequence if we condition $H_{s,z}^c$ on $X^{(s)}$ the number of points in $P(H_{s,z}^c|X^{(s)})$ are bounded by a Poisson random variable of parameter

$$(\lambda_{s,z} - \lambda_{s,z,m})\eta_d(Q_{3s'}(s'z)) = (\lambda_{s,z} - \lambda_{s,z,m})3^d s^{1/2}.$$

Since λ is locally Lipschitz, and the longest diagonal of $Q_{3s'}(s'z)$ has length $\sqrt{d}3s^{-1/2}$, we obtain that almost surely

$$P(H_{s,z}^c|X^{(s)}) \leq L\sqrt{d}3^{d+1}s^{-1/2+1/(2d)}.$$

We finish the first inequality observing that $-1/2 + 1/(2d) \leq -1/4$.

2. Let $\gamma = \{z \in \mathbb{Z}^d : Q_{s'}(s'z) \cap \Gamma \neq \emptyset\}$ be the discretization of Γ and define $M_i = z_i + 3\mathbb{Z}^d$, where $z_i \in \mathbb{Z}^d$ can be chosen such that $\{M_1, ..., M_K\}$ is a partition of $\mathbb{Z}^d$ of $K = 3^d$.

For every i conditioned on $X^{(s)}$ the process of s-good sites is an independent site process on $\gamma_i = \gamma \cap M_i$. By 1. conditioned on $X^{(s)}$ the probability for a site to be s-bad is of order $O(s^{-\frac{1}{4}})$. Let $\#_b\gamma$ denote the number of bad sites in γ then by the Binomial concentration inequality ([9] Lemma 1.1) under the event $\{\#\Gamma \leq 3^{d+1}s(s')^{-1}\}$ we have that

$$P(\#_{s,b}\Gamma \geq s^{1-5/(8d)}|X^{(s)}) \leq \sum_{i=1}^{K} P(\#_b\gamma_i \geq K^{-1}s^{1-5/(8d)}) \leq \exp(-\sqrt{s}).$$

□

We proceed to proof our last auxiliary lemma for that let $E_s^{(3)}$ denote the event that there exists a point $X_i \in X^{(s)}$ such that

$$\#_s\Gamma_{-,s}(X_i) \geq 3^{d+3}s(s')^{-1},$$

where $\Gamma_{-,s}(X_i)$ denote the longest subpath of $\Gamma(X_i)$ that starts at X_i and is contained in $C_{-,s}$.

Lemma 2.10. **1.** Let $E_s^{(4)}$ the event that exists a finite connected set $\gamma \in \mathbb{Z}^d$ such that

- $\#\gamma \geq \sqrt{s}$,
- $s'z \in D_{-,s}$ holds for every $z \in \gamma$,
- the number of s-good sites intersected by γ is at most $\#\gamma/2$.

Then $\lim_{s\to\infty} P(E_s^{(4)}) = 0$.

2.Suppose that $(E_s^{(1)} \cup E_s^{(2)})^c$ occurs and let $X_i \in X^{(s)}$ be arbitrary. Furthermore, let $X_e \in X^{(s)}$ be the end point of $\Gamma_{-,s}(X_i)$. Then

$$\pi_1(x)(X_e - X_i) \geq 3^{-d-2}s'\#_{s,g}\Gamma_{-,s}(X_i).$$

3. $P(E_s^{(3)}) \in O(s^{-2d})$ when s tends to infinity.

Proof. **1.** The process of s- good sites is dominated by a Bernoulli site percolation process with probability $p \in (0, 1)$ due two [7] Theorem 0.0 which hypothesis are satisfied by Lemma 2.9 1. For s sufficiently large p can be chosen arbitrarily close to 1.

We have that for a fixed connected set the probability that γ contains at least $\#/2$ bad sites is at most $2^{\#\gamma/2}(1-p)^{\#\gamma/2}$. Also by [9] Lemma 9.3 the number of connected sets containing $k \geq 1$ sites is bounded above by $s^d 2^{3^d k}$. Therefore

$$P(E_s^{(4)}) \leq s^d \sum_{k\geq\sqrt{s}} 2^{3^d k} 2^k (1-p)^{k/2},$$

which is of order $O(s^{-2d})$, since p can be chosen sufficiently close to 1.

2. Let γ be a subset of s- good sites whose cubes are intersected by $\Gamma_{-,s}(X_i)$ such that every pair of distinct sites in γ is at least a three cubes of distance and $\#\gamma \geq 3^{-d}\#_{s,g}\Gamma_{-,s}(X_i)$. Writing $\gamma = \{z_1, ..., z_k\}$ we define

$\Gamma_1, ..., \Gamma_k$ sub paths of Γ, where the starting point $X_{j,0}$ of Γ_j is the first point of Γ that is contained in $Q_{s'}(s'z_j)$. Starting from $X_{j,0}$, Γ_j is the longest subpath of Γ that is contained in the left half-space $(-\infty, \pi_1(s'z_j)+s') \times \mathbb{R}^d$.

Since the events $E_s^{(1)}$ and $E_s^{(2)}$ do not occur these sub paths are disjoint and $\pi_1(X_{j,e}-X_{j,0}) \geq s'/6$ where $X_{j,e}$ denotes the endpoint of Γ_j. Combining these lower bounds shows that

$$\pi_1(X_e - X_i) \geq \frac{s'}{6}\#\gamma \geq 3^{-d-2}s'\#_{s,g}\Gamma_{-,s}(X_i).$$

3. If we show that $E_s^{(3)} \subset E_s^{(1)} \cup E_s^{(2)} \cup E_s^{(4)}$ the result follows by Lemmas 2.6, 2.7 and 1. on this lemma. Assume that $E_s^{(3)}$ and $(E_s^{(1)} \cup E_s^{(2)} \cup E_s^{(4)})^c$ occur simultaneously, we will derive a contradiction. Then there exists $X_i \in X^{(s)}$ such that $\#_s\Gamma_{-,s}(X_i) \geq 3^{d+3}2s(s')^{-1}$. Since $(E_s^{(4)})^c$ occurs, we obtain that

$$\#_{s,g}\Gamma_{-,s}(X_i) \geq \frac{1}{2}\#_s\Gamma_{-.s}(X_i) \geq 3^{d+2}s(s')^{-1}.$$

In particular 2. on this lemma would imply that $\pi_1(X_e - X_i) \geq 2s$. But this is impossible since both X_1 and X_e are contained in sC an open hypercube of side length $2s$.

And we are ready to proof the proposition. □

Proof of Proposition. We can bound

$$P((E_s^D)^c) \leq P((E_{s,\varepsilon,1}^D)^c) + P((E_{s,\varepsilon,2}^D)^c).$$

For the first summand lets observe that we can estimate

$$(E_{s,\varepsilon,1}^D)^c \subset \{\#(X^{(s)} \cap (sC)_{\varepsilon s}) \leq 1\}$$

As since $X^{(s)}$ is a Poisson point process

$$P(\#(X^{(s)} \cap (sC)_{\varepsilon s}) \leq 1) = e^{-\lambda_m s^d \eta_d(C_\varepsilon)}(1 + s^d\lambda_M\eta_d(C_\varepsilon)).$$

Which implies that $P((E_{s,\varepsilon,1}^D)^c) \in O(s^{-2d})$.

Lets proceed to bound the second summand. First we write the event

$$E_{s,\varepsilon,2}^{D-} = \{\Gamma_{-,s}(X_i) \subset Z_{h(s)}^D(X_i) \text{ for all } X_i \in X^{(s)}\},$$

we claim that

$$(E_{s,\varepsilon,2}^D)^c \subset (E_{s,\varepsilon,2}^{D,-})^c \cup E_s^{(1)},$$

which is equivalent to show that $E_{s,\varepsilon,2}^{D-} \cap (E_s^{(1)})^c \subset E_{s,\varepsilon,2}^D$. Then lets assume that we are under the event $E_{s,\varepsilon,2}^{D-} \cap (E_s^{(1)})^c$. Let $X_i \in X^{(s)}$ and lets assume that s is big enough that $(sC)_{\varepsilon s} \subset C_{-,s}$. We have two cases

Case 1) $\Gamma(X_i) \subset C_{-,s}$. Hence $\Gamma(X_i) = \Gamma_{-,s}(X_i) \subset Z_{h(s)}^D(X_i)$ and since $(sC)_{\varepsilon s} \subset C_{-,s}$ then $\bar{\Gamma}(X_i) \cap (sC)_{\varepsilon s} \subset Z_{h(s)}^D(X_i)$.

Case 2) $\Gamma(X_i) \not\subseteq C_{-,s}$. We will proof that $\Gamma(X_i) \cap (sC)_{\varepsilon s} = \emptyset$ which implies that $\bar{\Gamma}(X_i) \cap (sD)_{\varepsilon s} \subset Z_{h(s)}^D(X_i)$ then without lost of generality we can assume that $X_i \in sC \setminus C_{-,s}$. Lets assume that $\Gamma(X_i) \cap (sC)_{\varepsilon s} \neq \emptyset$ and let $X_{i_2} \in \Gamma(X_i) \cap (sC)_{\varepsilon s}$. Since we are on the event $(E_s^{(1)})^c$ for s big

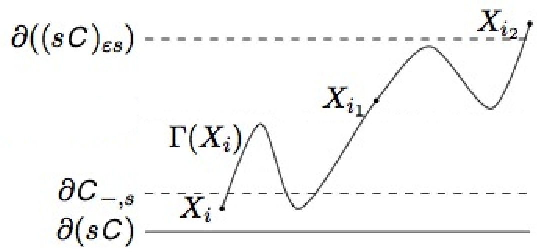

FIGURE 2.4: A not possible path under $E^{D-}_{s,\varepsilon,2} \cap (E^{(1)}_s)^c$. Figure took from [4].

enough it implies that exists $X_{i_1} \in \Gamma(X_i)$ such that $d(X_{i_1}, X_{i_2}) \geq h(s)$ which contradicts the occurrence of $E^{D-}_{s,\varepsilon,2}$ and the claim is finished. The construction is illustrated in Figure 2.4.

Now that we proved that $(E^{D}_{s,\varepsilon,2})^c \subset (E^{D-}_{s,\varepsilon,2})^c \cup (E^{(1)}_s)$ by Lemma 2.6 it remains to show that $P((E^{D}_{s,\varepsilon,2})^c \in O(s^{-2d})$. For $X_i \in X^{(s)} \cap C_{-,s}$ such that $\Gamma_{-,s}(X_i) \nsubseteq Z^{D}_{s^{1-1/(64d)}}(X_i)$ then $V(\Gamma_{-,s}(X_i)) \geq s^{1-1/(32d)}$. Therefore by Lemma 2.8 after some technical computations yields that

$$
\begin{aligned}
P((E^{D}_{s,\varepsilon,2})^c) &\leq P(\sup_{X_i \in X^{(s)} \cap C_{-,s}} \#_{s,b}\Gamma_{-,s}(X_i) \geq s^{1-1/(32d)}) \\
&\leq P(E^{(1)}_s \cup E^{(2)}_s \cup E^{(3)}_s) + P((E^{(3)}_s)^c \cap \{\sup_{X_i \in X^{(s)} \cap C_{-,s}} \#_{s,b}\Gamma_{-,s}(X_i) \geq s^{1-1/(32d)}\}).
\end{aligned}
$$

By Lemmas 2.6, 2.7 and 2.10 the first summand is of order $O(s^{-2d})$. For the second we may condition on $X^{(s)}$ so by Lemma 2.9 and Campbell's theorem

$$
\begin{aligned}
&P((E^{(3)}_s)^c \cap \{\#_{s,b}\Gamma_{-,s}(X_i) \geq s^{1-9/(16d)} \text{ for some } X_i \in X^{(s)} \cap C_{-,s}\}) \\
&\quad \leq E(\mathbb{1}\{(E^{(3)}_s)^c\} \sum_{X_i \in X^{(s)} \cap D_{-,s}} P(\#_{s,b}\Gamma_{-,s}(X_i) \geq s^{1-5/(8d)} | X^{(s)})) \\
&\quad \leq \exp(-\sqrt{s}) E\#X^{(s)} \leq \exp(-\sqrt{s})\lambda_M s^d \eta_d(C),
\end{aligned}
$$

which is of order $O(s^{-2d})$. $\square$

Chapter 3

Radial networks

In this chapter we will analyze the traffic flow generated by a radial navigation on a point process defined on a convex bounded subset of $\mathbb{R}^d$. A radial navigation assigns a successor to each node of a point process who lies near a specific target point in space, iterating that process is how the traffic flow is generated.

We will analyze how is the average behavior of the traffic flow on an $\mathbb{R}^d$ spherical environment pointing to the target point of the flow. We follow a similar construction to [4] Section 1.2 where the case of a $\mathbb{R}^d$ spherical environment which collapses in to a point is studied. Now we consider the case when it does not collapse.

3.1 Radial Navigation

Let $X^{(s)}$ be a simple point process with intensity $\lambda^{(s)}$ on sD where D is an open, bounded, convex subset of $\mathbb{R}^d$; $\lambda^{(s)}(x) = \lambda(\frac{x}{s})$ with $\lambda : D \to [0, \infty)$ a continuous and bounded mapping with positive maximum and minimum λ_M, λ_m respectively. We assume that each node $X_i \in X^{(s)}$ generates traffic at rate $\mu^{(s)}(x) = \mu(\frac{x}{s})$ with $\mu : D \to [0, \infty)$ also a continuous and bounded mapping with maximum and minimum μ_M, μ_m respectively.

We want to propose a model where the traffic flow approaches to a target point, for simplicity we will investigate a navigation that approaches to the origin.

Definition 3.1. A measurable function $\mathcal{A} : \mathbb{R}^d \times \mathbf{N} \to \mathbb{R}^d$ is called *radial navigation* (approaching to o) if for all $\varphi \in \mathbf{N}$

- $\mathcal{A}(o, \varphi \cup \{o\}) = o$,
- $\mathcal{A}(x, \varphi) \in \varphi \cup \{o\}$ for all $x \in \varphi$,
- $|\mathcal{A}(x, \varphi)| < |x|$ for all $x \in \varphi \setminus \{o\}$.

If X is a pointed process for $X_i \in X \cup \{o\}$, we have that $\mathcal{A}(X_i, X) \in X \cup \{o\}$, so we will simply write $\mathcal{A}(X_i)$.

In Figure 3.1 we illustrate two examples of radial navigations, where the successor of a node is the nearest neighbor on a cone with fixed angle pointing to the origin and whose vertex is X_i. Each example presents a different angle.

For the link density condition we need to adapt the definitions of I^D and Ξ^D, then for $g(s)$ a positive real valued function, we write

$$I_s^R(x) = B^d_{g(s)}(sx) \cap \partial B^d_{s|x|}(o),$$

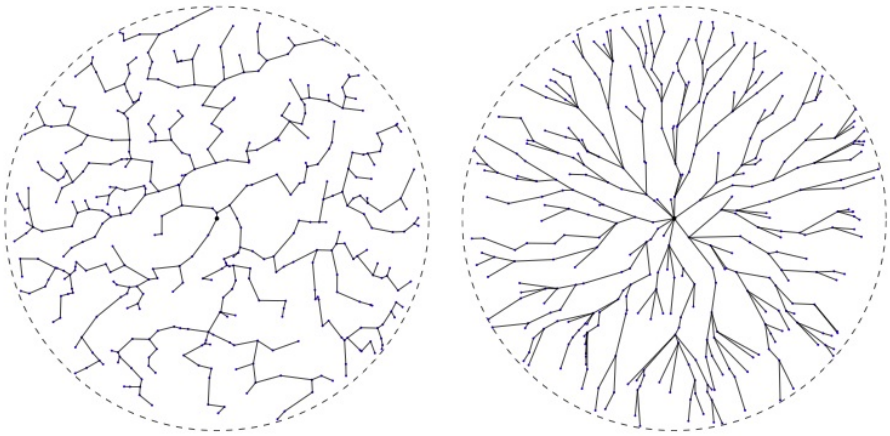

FIGURE 3.1: Radial navigation based on a Poisson point process on the disc. Each node connects to its nearest neighbor which is also closer to the origin, respectively additionally is contained in a cone starting at the node and opening towards the origin. Figure taken from [4].

which instead of the hyperplane surface in the case of directed networks defines a spherical cap. Moreover, let

$$\Xi_s^R(x) = \{X_i \in X^{(s)} : [X_i, \mathcal{A}(X_i)] \cap I_s^R(x) \neq \emptyset\}$$

be the points X_i such that the segment $[X_i, \mathcal{A}(X_i)]$ crosses $I_s^R(x)$. We get the analogous condition

Condition 3.1. There exists a function $\lambda_{\mathcal{A}}(x) : D \to (0, \infty)$ such that for every $x \in D \setminus \{o\}$,

$$\lambda_{\mathcal{A}}(x) = \lim_{s\to\infty} \frac{E\#\Xi_s^R(x)}{s\eta_{d-1}(I_s^R(x))},$$

where η_{d-1} denotes the $(d-1)$-dimensional Hausdorff measure in $\mathbb{R}^d$.

Continuing the analogy, we change the sub-ballisticity condition in such a way that the cylinders point towards the origin. More precisely, we write $\hat{v} = v/|v|$ for $v \in \mathbb{R}^d \setminus \{o\}$ and define

$$Z_r^R(v) = \{y \in \mathbb{R}^d : |y - \langle y, \hat{v}\rangle\hat{v}| \leq r\}$$

for the cylinder consisting of all points in $\mathbb{R}^d$ whose projection onto the orthogonal complement of the direction $\hat{v} \in \partial B_1^d(o)$ is of length at most $r \geq 0$. Moreover for $h(s)$ a positive real valued function we write

$$E_s^R = \{(\Gamma(X_i) \subset Z_{h(s)}^R(X_i) \text{ for all } X_i \in X(s)\},$$

the event that the trajectory is contained in a narrow cylinder towards the origin. An we get the condition

Condition 3.2. $P((E_s^R)^c) \in O(s^{-2d})$.

3.2 Convergence of the traffic flow average in the radial case

We are ready to enunciate and prove the main result of this chapter which states that under the link density and sub-ballisticity conditions we will have an almost surely convergence of the traffic flow average on a micro spherical cap orthogonal to the flow. The case when $g(s)/s$ tends monotonically to zero is studied in [4] Theorem 4, we will assume that $g(s)/s$ is constant getting a slightly different result.

Theorem 3.1. Let $x \in D \setminus \{o\}$. Assume that Conditions 3.1, 3.2 are satisfied assume that Conditions 2.1, 2.2 are satisfied for $g(s) = s$, $h(s) = s^{\xi}$, $0 < \xi < 1$ and that $E[(\#X^{(s)})^2] \in O(s^{2d})$.

1.Then,

$$\lim_{s\to\infty} \frac{E\sum_{X_i\in\Xi_s^R(x)} \Delta(X_i)}{E\#\Xi_s^R(x)} =$$

$$\frac{1}{\rho_x |x|^{d-1}\lambda_{\mathcal{A}}(x)} \int_{|x|}^{\infty} r^{d-1} \int_{\partial B_1^d(o)} \mathbb{1}_{|\hat{y}-\hat{x}|\le 1/|x|} \mu(ry)\lambda(ry)\eta_{d-1}(dy)dr,$$

where $\rho_x = \int_{\partial B_1(o)} \mathbb{1}_{|\hat{y}-\hat{x}|\le 1/|x|}\eta_{d-1}(dy)$.

2. If additionally $X^{(s)}$ is either a Poisson point process or μ is constant on D and $X^{(s)} = X \cap sD$ for some ergodic point process X, then

$$\lim_{s\to\infty} \frac{\sum_{X_i\in\Xi_s^R(x)} \Delta(X_i)}{E\#\Xi_s^R(x)} =$$

$$\frac{1}{\rho_x |x|^{d-1}\lambda_{\mathcal{A}}(x)} \int_{|x|}^{\infty} r^{d-1} \int_{\partial B_1^d(o)} \mathbb{1}_{|\hat{y}-\hat{x}|\le 1/|x|} \mu(ry)\lambda(ry)\eta_{d-1}(dy)dr,$$

in probability.

Proof. By a similar procedure that in the directed case first we have to give estimations of the accumulated traffic flow under E_s^R and $(E_s^R)^c$. Let us define the analogous of the cylinders $R_s^{l,+}(x)$ and $R^{l,-}(x)$. First,

$$C_s^+(x) = \{sy \in sD \setminus B_{|sx|}^d(o) : |\hat{y} - \hat{x}| \le (g(s) + 2h(s))/s|x|\}$$

is the set of points which lie in an extended cone around sx facing towards the boundary of sD. Second for s such that $g(s) \ge 2h(s)$

$$C_s^-(x) = \{sy \in sD \setminus B_{|sx|}^d(o) : |\hat{y} - \hat{x}| \le (g(s) - 2h(s))/s|x|\}$$

the set of points which lie in a diminished cone around sx facing towards the boundary of sD, see also Figure 3.2.

Lemma 3.1. Let $x \in D$ and $X^{(s)}$ on the event E_s^R then

$$\sum_{X_j\in X^{(s)}\cap C_s^-(x)} \mu^{(s)}(X_j) \le \sum_{X_j\in\Xi_s^R(x)} \Delta^{(s)}(X_j) \le \sum_{X_j\in X^{(s)}\cap C_s^+(x)} \mu^{(s)}(X_j).$$

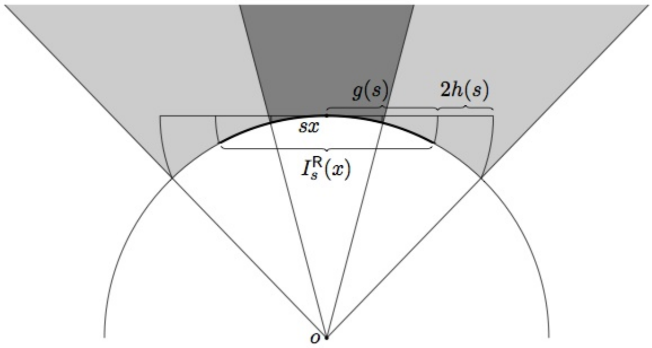

FIGURE 3.2: Construction of the cones $C_s^-(x)$ (dark gray) and $C_s^+(x)$ (union of light and dark gray). Figure took from [4].

Proof. For the upper bound, lets observe that if we pick $X_i \in \Xi_s^R(x)$ on the event E_s^R we have that $\Gamma(X_i) \subset Z_{h(s)}^R(X_i) \subset C_s^+(x)$ and in consequence

$$\sum_{X_j \in \Xi_s^R(x)} \Delta^{(s)}(X_j) \leq \sum_{X_j \in X^{(s)} \cap C_s^+(x)} \mu^{(s)}(X_j).$$

For the lower bound, lets pick $X_j \in X^{(s)} \cap C_s^-(x)$. On the one hand since $\mathcal{A}$ is a radial navigation there exists $X_j \in X^{(s)}$ s.t. $[X_j, \mathcal{A}(X_j)] \cap B_{|sx|}^d(o) \neq \emptyset$ and $X_i \in \Gamma(X_j)$. On the other hand $\Gamma(X_j) \subset Z_{h(s)}^R(X_j)$. Which implies that $X_i \in \Xi_s^R$ and we conclude

$$\sum_{X_j \in X^{(s)} \cap C_s^-(x)} \mu^{(s)}(X_j) \leq \sum_{X_j \in \Xi_s^R(x)} \Delta^{(s)}(X_j).$$

□

The following lemma is analogous to Lemma 2.2

Lemma 3.2. $E(\mathbb{1}_{(E_s^R)^c} \sum_{X_i \in \Xi_s^D(x)} \Delta(X_i)) \in O(1)$.

We proceed to proof the theorem. For that let

$$i_{s,x} = s\eta_{d-1}(I_s^R(x)), N_s = (i_{s,x})^{-1} \sum_{X_j \in \Xi_s^R(x)} \Delta^{(s)}(X_j),$$

$$S = \frac{1}{\rho_x |x|^{d-1}} \int_{|x|}^{\infty} r^{d-1} \int_{\partial B_1^d(o)} \mathbb{1}_{|\hat{y}-\hat{x}| \leq 1} \mu(ry)\lambda(ry)\eta_{d-1}(dy)dr.$$

*Proof of Theorem.***1.** By Condition 3.1 it is enough to proof that $\lim_{s\to\infty} EN_s = S$. We have that

$$EN_s = E\mathbb{1}_{E_s^R} N_s + E\mathbb{1}_{(E_s^R)^c} N_s.$$

First lets estimate

$$i_{s,x} = s\eta_{d-1}((I_s^D(x)) = s\int_{\partial B_{|sx|}(o)} \mathbb{1}_{|\hat{y}-\hat{x}| \leq g(s)/|sx|} \eta_{d-1}(dy)$$
$$= s^d \rho_x |x|^{d-1},$$

and then $i_{s,x}^{-1} \in o(1)$. So by Lemma 3.2 $E\mathbb{1}_{(E_s^R)^c} N_s \in o(1)$.

Lets find an upper found for $E\mathbb{1}_{E_s^R}N_s$. By Lemma 3.1 and Campbell's theorem

$$E\mathbb{1}_{E_s^R}N_s \leq i_{s,x}^{-1}E\sum_{X_j\in X^{(s)}\cap C_s^+(x)}\mu^{(s)}(X_j)$$
$$= i_{s,x}^{-1}\int_{C_s^+(x)}\mu^{(s)}(y)\lambda^{(s)}(y)dy.$$

By the coordinate transformation $y = sz$, the co-area formula applied to the function $f(x) = \|x\|$ and the transformation $y = rz$ we rewrite

$$\int_{C_s^+(x)}\mu^{(s)}(y)\lambda^{(s)}(y)dy = \int \mathbb{1}_{|y|>|sx|}\mathbb{1}_{|sx||\hat{y}-\hat{x}|\leq g(s)+2h(s))}\mu^{(s)}(y)\lambda^{(s)}(y)dy$$
$$= s^d\int \mathbb{1}_{|y|>|x|}\mathbb{1}_{|\hat{y}-\hat{x}|\leq(g(s)+2h(s))/|sx|}\mu(y)\lambda(y)dy$$
$$= s^d\int_{|x|}^{\infty}\int_{\partial B_r^d(o)}\mathbb{1}_{|\hat{y}-\hat{x}|\leq(g(s)+2h(s))/|sx|}\mu(y)\lambda(y)\eta_{d-1}(dy)dr$$
$$= s^d\int_{|x|}^{\infty}r^{d-1}\int_{\partial B_1^d(o)}\mathbb{1}_{|\hat{y}-\hat{x}|\leq(g(s)+2h(s))/|sx|}\mu(ry)\lambda(ry)\eta_{d-1}(dy)dr$$

And multiplying by $i_{s,x}^{-1}$ we get the final bound

$$\frac{1}{\rho_x|x|^{d-1}}\int_{|x|}^{\infty}r^{d-1}\int_{\partial B_1^d(o)}\mathbb{1}_{|\hat{y}-\hat{x}|\leq(g(s)+2h(s))/|sx|}\mu(ry)\lambda(ry)\eta_{d-1}(dy)dr.$$

Which by the Dominated Convergence theorem converges to S.

For the lower bound, using the same arguments as above we can estimate

$$E\mathbb{1}_{E_s^R}N_s \geq s^d i_{s,x}^{-1}\int_{|x|}^{\infty}r^{d-1}\int_{\partial B_1^d(o)}\mathbb{1}_{|\hat{y}-\hat{x}|\leq(g(s)-2h(s))/|sx|}\mu(ry)\lambda(ry)\eta_{d-1}(dy)dr.$$

And using an analogous reasoning that in the upper bound the result follows.

2. Again by Condition 3.1 it suffices to show that

$$\lim_{s\to\infty}N_s = S$$

in probability. Following the same arguments as in Theorem 2.1 2. replacing by the following definition

$$N_s^{\pm} = i_{s,x}^{-1}\sum_{X_j\in X^{(s)}\cap C_s^{\pm}(x)}\mu^{(s)}(X_j)$$

it suffices to show that $E(|N_s^+ - EN_s^+|)$ and $E(|N_s^- - EN_s^-|)$ tends to zero as s tends to infinity for a Poisson and an ergodic point process.

Case 1) Let $X^{(s)}$ be a Poisson point process, then by Jensen's inequality, Campbell's theorem for the variance

$$E(|N_s^+ - EN_s^+|) \leq (VN_s^+)^{1/2} = i_{s,x}^{-1}V(\sum_{X_j\in X^{(s)}\cap C_s^+(x)}\mu^{(s)}(X_j))^{1/2}$$
$$= i_{s,x}^{-1/2}(i_{s,x}^{-1}\int_{C_s^+(x)}(\mu^{(s)}(y))^2\lambda^{(s)}(y)dy)^{1/2}.$$

And by an analogous reasoning to 1. of we get that when s tends to infinity

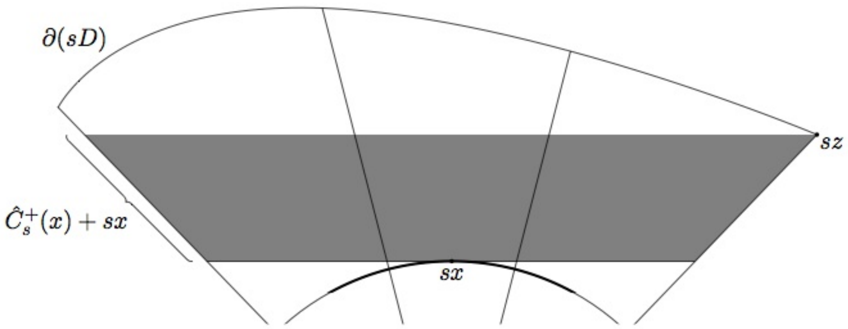

FIGURE 3.3: Construction of the cone $\hat{C}^+_{s,x}$ (gray area), where $\langle z, \hat{x}\rangle = \zeta_{s,x}$. Figure took from [4].

the second term converges to a constant and hence the complete expression goes to zero. The reasoning for $E(|N_s^- - EN_s^-|)$ is similar.

Case 2) Lets assume that X is ergodic on $\mathbb{R}^d$ and $X^{(s)} = X \cap sD$. Let $r_s^+(x) = \nu_d(C_s^+(x))$. Then by ergodicity X has constant density λ. Since by hypothesis also λ is constant by Campbell's theorem we compute

$$E(|N_s^+ - EN_s^+|) = \mu \frac{r_s^+(x)}{i_{s,x}} E(|\frac{\#(X \cap C_s^+(x))}{r_s^+(x)} - \lambda|),$$

where analogously to computations done in 1. we have that $\frac{r_s^+(x)}{i_{s,x}}$ converges to a constant when s tends to infinity We have to modify $C_s^+(x)$ to become a sequence of convex averaging windows. Let

$$\zeta_{s,x} = \inf\{\langle z, \hat{x}\rangle : z \in \partial D \cap (C_s^+(x)/s)\}$$

the smallest component in the direction $\hat{x}$ of points in the boundary sD intersected with the cone $C_s^+(x)$. Let

$$\dot{C}_s^+(x) = \{sy \in sD : sy \in C_s^+(x) \text{ and } |x| \leq \langle y, \hat{x}\rangle \leq \zeta_{s,x}\}, \hat{C}_s^+(x) = \dot{C}_s^+(x) - sx.$$

$\hat{C}_s^+(x)$ is a sequence of convex averaging windows since $g(s)$ and $h(s)$ are monotonically increasing, see also Figure 3.3 .

Let $r_s^+(x) = \eta_d(\hat{C}_s^+(x))$, by stationarity we can estimate

$$E(|\frac{\#(X \cap C_s^+(x))}{r_s^+(x)} - \lambda|) \leq E(|\frac{\#(X \cap C_s^+(x))}{r_s^+(x)} - \frac{\#(X \cap (\dot{C}_s^+(x))}{\hat{r}_s^+(x)}|) + E(|\frac{\#(X \cap \hat{C}_s^+(x)}{\hat{r}_s^+(x)} - \lambda|).$$

Due to the ergodic theorem second summand tends to zero when s tends to infinity. Lets proceed to estimate the first one.

Since $\hat{r}_s^+(x) \leq r_s^+(x)$ and $\#(X \cap \dot{C}_s^+(x)) \leq \#(X \cap C_s^+(x))$ we have that

$$\begin{aligned}\#(X \cap \dot{C}_s^+(x))(\frac{1}{r_s^+(x)} - \frac{1}{\hat{r}_s^+(x)}) &\leq \frac{\#(X \cap C_s^+(x))}{r_s^+(x)} - \frac{\#(X \cap \dot{C}_s^+(x))}{\hat{r}_s^+(x)} \\ &\leq \frac{\#(X \cap C_s^+(x)) - \#(X \cap \dot{C}_s^+(x))}{\hat{r}_s^+(x)},\end{aligned}$$

and in consequence by Campbell's theorem

$$
\begin{aligned}
&E(|\frac{\#(X \cap C_s^+(x))}{r_s^+(x)} - \frac{\#(X \cap (\dot{C}_s^+(x))}{\hat{r}_s^+(x)}|) \\
&\leq E(|\frac{\#(X \cap C_s^+(x)) - \#(X \cap \dot{C}_s^+(x))}{\hat{r}_s^+(x)}|) + E(|\#(X \cap \dot{C}_s^+(x))(\frac{1}{r_s^+(x)} - \frac{1}{\hat{r}_s^+(x)})|) \\
&\leq 2\lambda \frac{r_s^+(x) - \hat{r}_s^+(x)}{\hat{r}_s^+(x)}.
\end{aligned}
$$

Which tends to zero when s tends to infinity. Similar arguments apply for the case where N^+ is replaced by N^-. □

Chapter 4

Information velocity in the SINR graph

For finishing this work we present a telecommunication model. Every point of our process will represent an user that want to sent or receive a message. We want to present a sending strategy that satisfies two goals

1. Every message is send from one user to another in finite time,
2. The message travels around the network with positive velocity.

For achieving it we will use a power control policy and a conic forwarding strategy. The two main results of this chapter are taken from [5] where is considered the case of a stationary Poisson point process. Using coupling techniques we will extend the first result to the non stationary case.

4.1 SINR graph, power control policy and conic forwarding strategy

In this chapter X will be a point process and every $x \in X$ will represent a user of a network that want to transmit and receive information at time t. We will denote by $X_T(t)$, $X_R(t)$ the users that are transmitting and receiving messages at time t respectively, this sets are disjoint and satisfy $X_T(t) \cup X_R(t) = X$. The Signal Interference Noise plus Radio measures the quality of the communication from a user x to a user y at time t.

Definition 4.1. We define the SINR from $x \in X_T(t)$ to a $y \in X_R(t)$ as

$$SINR_{xy}(t) = \frac{P_x(t)h_t(x,y)\ell(x,y)}{\gamma I(t) + N}.$$

Where at time $t \in \mathbb{Z}^+$,

- $P_x(t)$ is the power of transmission of x,
- $h_t(x,y)$ are the space-time fading coefficients from x to y,
- $\ell(x,y)$ is the path-loss function which measures the decaying of the quality of the transmission according to the distance between x and y,
- $I(t) = \sum_{z \in X_T(t) \setminus \{x,y\}} P_z(t)h_t(z,y)\ell(z,y)$ is the interference coming from the other nodes,
- γ is a constant and N a Gaussian noise,

and all are positive functions.

We say that the transmission from $x \in X_T(t)$ to $y \in X_R(t)$ has been successful if $SINR_{xy} > \beta$ a positive constant.

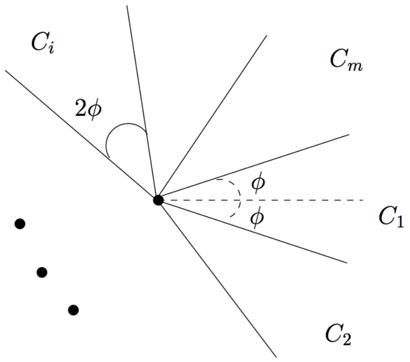

FIGURE 4.1: Definition of cones with angles 2ϕ. Figure taken from [5].

At all times the location of the users of the network will be fixed. If at time zero the transmission from x to y is not successful, at time one a new independent trial is done, and the process continue until the message is sent. Then once it arrives to y a new node z is picked and the process restart again from y to z. We will propose a model where the expected value of the delay time of successfully transmitting one message from a user to another is finite and the average velocity in which the message travels trough the network is strictly positive. For that we will assume that

- X is a Poisson point process in $\mathbb{R}^2$,
- $h_t(x,y), x, y \in X, t = 0, 1, ...$ are independent exponential random variables of parameter μ,
- $\ell(x,y) = 1 \wedge |x-y|^{-\alpha}$ for $\alpha > 2$,
- every node is a transmitter or a receiver at time t following a Bernoulli random variable $\mathbf{1}_x(t)$ with success probability $p_x(t)$,
- $0 < \gamma < 1$,
- $P_x(t) = c\ell(x,y)^{-1}$ where $c = M(1-\varepsilon)^{-1}$, $0 < \varepsilon < 1$,
- $M = P_x(t)p_x(t)$.

The last two assumptions is what is called *power control policy.* Then we can rewrite

$$SINR_{xy}(t) = \frac{P_x(t)h_t(x,y)\ell(x,y)\mathbf{1}_x(t)(1-\mathbf{1}_y(t))}{\gamma I(t) + N}.$$

The next random variable tell us if the transmission at time t has been successful

$$e_{xy}(t) = \begin{cases} 1 & 1 \text{ if } SINR_{xy} > \beta \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4.2. The space-time SINR graph is a graph with vertex on $X \times \mathbb{Z}^+$ where a directed edge exists from (x,t) to $(y,t+1)$ if $e_{xy}(t) = 1$. Since given X the location of the nodes does not change on time, the evolution of the graph is due to the changes in the fading variables $h_t(x,y)$ and X_T.

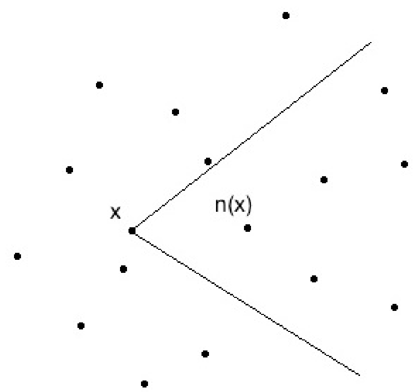

FIGURE 4.2: Each node transmits to its nearest neighbor in the destination cone.

For finishing our model we have to specify how does a user x choose a user y to transmit the message. The network will use a conic forwarding strategy. For that let $C_1, ..., C_m$ be cones centered in the origin with angle $2\phi < \frac{\pi}{2}$ such that. $\cup_{i=1}^{m} C_i = \mathbb{R}^2$ and are disjoint, also lets assume that C_1 is symmetric with respect to the x axis (see Figure 4.1).

At time t the user x will transmit through the cone $x + C_d(x,t)$ that contains the final destination of the package. It will choose the nearest user inside that cone which we will denote by $n_t(x)$ (see Figure 4.2).

4.2 Exiting time from the destination cone

Using similar techniques to [5] Theorem 3.2 where it is analyzed the case of a stationary Poisson point process we will proof that following the conic forwarding strategy a message can be transmitted in finite time from one user to another in the non stationary case.. Let the minimum exit time taken by any packet to be successfully transmitted from node x_o to its nearest neighbor $n(x_o)$ in the destination cone of the packet be

$$T_{x_o} = \min\{t > 0 : e_{x_o, n_t(x_o)}(t) = 1\}.$$

Theorem 4.1. Let X be a Poisson point process with bounded intensity function $0 < \lambda_m \leq \lambda(x) \leq \lambda_M$. If $\beta\gamma < 1$ then the SINR graph with power control policy and conic following strategy satisfies that $E(T_{x_o}) < \infty$ for any $x_o \in X$.

Proof. Lets assume that the package is sent by a fix user $x_o \in X$, let C_d the destination cone of this package and $n(x_o)$ the nearest neighbor of x_o in C_d.

We have that

$$P(T_{x_o} > k \mid X) = E\{\prod_{t=1}^{k} P(A(t) \cup B(t) \cup C(t) \mid \mathcal{G}_k) \mathbf{1}_F | X\}.$$

Where

- $F = \cap_{j=2}^{k} \{p_{x_o}(j) = p_{x_o}(1)\}$,

- $\mathcal{G}_k$ is the σ-algebra generated by X and the choice of the cones made at all nodes of X up to time k,
- $A(t) = \{x_o \in X_R(t)\}$,
- $B(t) = \{x_o \in X_T(t), n(x_o) \in x_R(t), SINR_{x_o,n(x_o)}(t) \le \beta\}$,
- $C(t) = \{x_o \in X_T(t), n(x_o) \in X_T(t)\}$.

On the event F since $c = P_{x_o}(t)\ell(x_o, n(x_o))$ and $h_t(x_o, n(x_o))$ has exponential distribution of parameter μ,

$$\begin{aligned}
P(A(t)|\mathcal{G}_k) &= 1 - p_{x_o}(1),\\
P(B(t) \mid \mathcal{G}_k) &= p_{x_o}(1)q_{n(x_o)}(t)P(SINR_{x_o,n(x_o)} \le \beta|\mathcal{G}_k)\\
&= p_{x_o}(1)q_{n(x_o)}(1)P(h_t(x_o, n(x_o)) \le \frac{\beta(N+\gamma I(t))}{c}|\mathcal{G}_k\})\\
&= p_{x_o}(1)q_{n(x_o)}(1)(1 - E\{e^{-\frac{\mu\beta}{c}(N+\gamma I(t))}|\mathcal{G}_k\})\\
P(C(t)|\mathcal{G}_k) &= p_{x_o}(1)(1 - q_{n(x_o)}(1)).
\end{aligned}$$

Hence since $q_{n(x_o)}(1) = 1 - \ell(x_o, n(x_o))(1-\varepsilon) \ge \varepsilon$, we can estimate

$$P(A(t) \cup B(t) \cup C(t)|\mathcal{G}_k) \le 1 - p_{x_o}(1)\varepsilon e^{-\frac{\mu\beta N}{c}} E(e^{-\frac{\mu\beta\gamma}{c}I(t)}|\mathcal{G}_k). \tag{4.1}$$

Suppose that $x \in X \setminus \{x_o, n(x_o)\}$ transmits using the cone $x + C_i$ at time t with transmission probability $p_x^{(i)}(t)$ and power $P_x^{(i)}(t)$. Then we have that for $a = \frac{\mu\beta\gamma}{c}$

$$E\{e^{-\frac{\mu\beta\gamma}{c}I(t)}|\mathcal{G}_k\} = \prod_{x \in X\setminus\{x_o,n(x_o)\}} E\{e^{-a\mathbf{1}_x P_x^{(i)}(t)h_t(x,n(x_o))\ell(x,n(x_o))}|\mathcal{G}_k\}.$$

And again from the fact that $h_t(x, n(x_o))$ has exponential distribution of parameter μ and by the power control policy,

$$\begin{aligned}
&E\{e^{-a\mathbf{1}_x P_x^{(i)}(t)h_t(x,n(x_o))\ell(x,n(x_o))}|\mathcal{G}_k\}\\
&\quad= (1 - p_x^{(i)}) + p_x^{(i)} E\{e^{-aP_x^{(i)}(t)h_t(x,n(x_o))\ell(x,n(x_o))}|X\}\\
&\quad= (1 - p_x^{(i)}) + p_x^{(i)} \frac{c}{c + \beta\gamma\ell(x, n(x_o))P_x^{(i)}(t)}.
\end{aligned} \tag{4.2}$$

For $x \in X \setminus \{x_o, n(x_o)\}$ let $C_x^* = C_x^*(X)$ the cone which minimizes (4.2) and in consequence maximizes (4.1), p_x^*, P_x^*, $\mathbf{1}_x^*$ the corresponding transmission probability, power and Bernoulli random variable. We write

$$I^*(t) = \sum_{x \in X\setminus\{x_o,n(x_o)\}} \mathbf{1}_x^* P_x^* h_t(x, n(x_o))\ell(x, n(x_o)).$$

Since on the event F, $I^*(t)$ have the same distribution for all $t \in \mathbb{Z}^+$ then

$$P(A(t) \cup B(t)|\mathcal{G}_k) \le 1 - p_{x_o}(1)\varepsilon e^{-\frac{\mu\beta N}{c}} E\{e^{-aI^*(1)}|X\}.$$

For $J = p_{x_o}(1)\varepsilon e^{-\frac{\mu\beta N}{c}} E\{e^{-aI^*(1)}|X\}$, we get that

$$P(T(x_o) > k|X) \le (1 - J)^k.$$

Since $0 < 1 - J < 1$ the corresponding geometric series converge and by the Cauchy-Schwartz inequality on J^{-1}

$$\begin{aligned} E(T_{x_o}) &= \sum_{k\geq 0} P(T_{x_o} > k) \\ &= E(\sum_{k\geq 0} P(T_{x_o} > k|X)) \leq E(J^{-1}) \\ &=\leq \frac{e^{\frac{\mu\beta N}{c}}}{\varepsilon}(E\{p_{x_o}(1)^{-2}\}E\{\frac{1}{(E\{e^{-aI^*(1)}|X\})^2}\})^{\frac{1}{2}}. \end{aligned}$$

For finishing the proof we will use a coupling argument, due to Proposition 1.3 we can construct an homogeneous Poisson point process $X^{[\lambda_m]}$ with constant intensity λ_m, such that $X^{[\lambda_m]} \subset X$.

On the one hand let us denote by $n'(x_o)$ the nearest neighbor of x_o at $X^{[\lambda_m]}$. Since $X^{[\lambda_m]} \subset X$ we have that $|n(x_o) - x_o| \leq |n'(x_o) - x_o|$. Hence from the definition of the transmission probability $p_{x_o}(t)$,

$$\begin{aligned} E\{p_{x_o}(1)^{-2}\} &= E\{(\frac{c}{M\ell(x_o, n(x_o))})^2\} = (\frac{c}{M})^2 E(1 \vee |n(x_o) - x_o|^{2\alpha}) \\ &\leq (\frac{c}{M})^2 E(1 \vee |n'(x_o) - x_o|^{2\alpha}) \\ &= \frac{\lambda_m 2\pi}{m}(\frac{c}{M})^2 \int_0^\infty (1 \vee |r|^{2\alpha}) r e^{-\frac{\lambda_m \pi}{m} r^2} dr < \infty, \end{aligned}$$

since by Proposition 1.2 $X^{[\lambda_m]}$ has nearest neighbor density

$$f(r) = \frac{\lambda_m 2\pi r}{m} e^{-\frac{\lambda_m \pi}{m} r^2}$$

in the cone C_d.

On the other hand, by the power control strategy and analogous computations to the already done

$$\begin{aligned} E\{e^{-a\mathbf{1}_x^* P_x^* h_1(x,n(o))\ell(z,n(o))}|X\} &= 1 - \frac{\beta\gamma p_x^* P_x^* \ell(x, n(o))}{c + \beta\gamma P_z^* \ell(x, n(o))} \\ &\geq 1 - \frac{\beta\gamma M \ell(x, n(o))}{c} = 1 - \beta\gamma(1-\varepsilon)\ell(x, n(o)). \end{aligned}$$

Let $c_1 = \beta\gamma(1-\varepsilon)$, note that since $\beta\gamma < 1$ then $0 < 1 - c_1\ell(x, n(x_o))$. Hence we can estimate

$$\frac{1}{(E\{e^{-aI^*(1)}|X\})^2} \leq \prod_{x\in X\setminus\{x_o, n(x_o)\}} \frac{1}{(1 - c_1\ell(x, n(x_o)))^2}.$$

Let Y a Poisson point process with intensity function $\lambda \mathbb{1}_{(x_o + C_d)\cap B_{|n(x_o)|}(x_o)}$ independent of X then $(X \setminus \{x_o, n(x_o)\}) \cup Y$ is a Poisson point process of intensity λ. By Campbell's theorem for Poisson point processes

$$\begin{aligned}
E(\frac{1}{(E\{e^{-aI^*(1)}|X\})^2}) &\le E \prod_{(X\setminus\{x_o,n(x_o)\})\cup Y} \frac{1}{(1-c_1\ell(x,n(x_o)))^2} \\
&= \exp \int_{\mathbb{R}^2} (e^{-2\log(1-c_1\ell(x,n(x_o)))}-1)\lambda(x)dx \\
&\le \exp(\frac{2\lambda_M c_1}{(1-c_1)^2}\int_{\mathbb{R}^2} \ell(x,n(x_o))dx) < \infty.
\end{aligned}$$

□

4.3 Positive information velocity

For finishing this work we offer a result from [5] Theorem 3.5 where it is analyzed the asymptotic behavior of the velocity of a message traveling through the network. Before stating it lets first define some key concepts.

Definition 4.3. Let T_0 be the time taken by a tagged message starting at $x_o \in X$ to successfully reach its nearest neighbor $x_1 = n(x_o)$ in the destination cone C_1. More generally let T_{i-1} be the time taken for the message to successfully reach the nearest neighbor x_i of x_{i-1} in the destination cone $x_{i-1}+C_1$. We define the distance $d(t)$ for $t \in \mathbb{Z}^+$ as the random variable that satisfies

$$\mathbb{1}_{\sum_{i=0}^{k-1} T_i \le t < \sum_{i=0}^{k} T_i} d(t) = |x_k - x_o|.$$

We say that the information velocity of SINR network is

$$v = \liminf_{t\to\infty} \frac{d(t)}{t}.$$

Theorem 4.2. Under the conditions of Theorem 4.1 if X is a stationary Poisson point process with constant intensity function $\lambda > 0$ then the information velocity is almost surely positive.

Proof. For $i \ge 0$ let $r_i = |x_{i+1} - x_i|$, $\theta_i = \arcsin(\frac{x_{i+1,2}-x_{i,2}}{r_i})$ where $x_i = (x_{i,1}, x_{i,2})$ in $\mathbb{R}^2$ coordinates. The cones $\{(x_i + C_1) \cap B_{r_i}(x_i), i \ge 0\}$ are non-overlapping since $2\phi < \frac{\pi}{2}$.

Since X is a stationary Poisson point process with intensity λ, we have that $\{(r_i,\theta_i), i \ge 0\}$ is an i.i.d. sequence of random vectors where θ_i is uniformly distributed on $(-\phi,\phi)$ and by Proposition 1.2 r_i has density

$$f(r) = \frac{2\lambda\pi r}{m} e^{-\frac{-\lambda\pi}{m}r^2}.$$

Our goal is to construct a stationary sequence of stopping times such that for all $i \ge 0$, $T_i' \ge T_i$.

To nullify the effect of moving to the nearest neighbor we progressively fill the voids with independent Poisson points as the packet traverses the network. This however leaves an increasing sequence of special points. The following construction is intended to take care of this issue and deliver an stationary sequence.

Let $\{(r_{-i},\theta_{-i}), i \ge 1)\}$ an i.i.d. sequence of random vectors with distribution (r_0,θ_0). Let $\widetilde{X} = \{x_{-i}, i \ge 1\}$ starting from x_{-1} such that $r_{-i} = |x_{-i} - x_{-i+1}|$, $\theta_{-i} = \arcsin(\frac{x_{-i+1,2}-x_{-i,2}}{r_{-i}})$. See Figure 4.3.

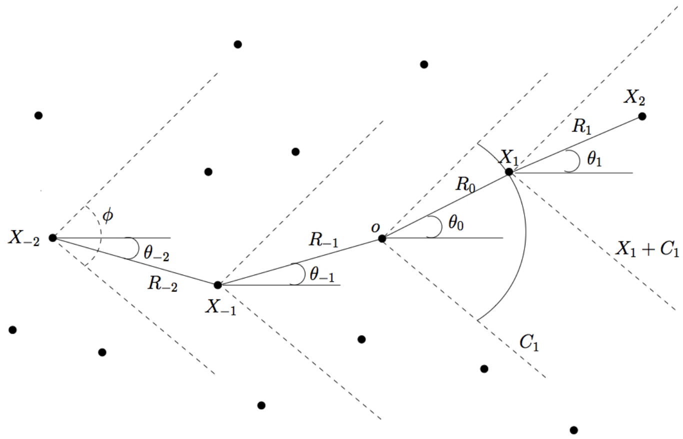

FIGURE 4.3: Illustration of the addition of an infinite sequence of points for constructing a stationary sequence of stopping times. Figure taken from [5].

Let X_i be a Poisson point process of intensity $\lambda \mathbf{1}_{\{(x_i+C_1)\cap B_{r_i}(x_i)\}}$ independent of everything else, $T_i^{'}$ be the delay experienced by the packet in going from X_i to X_{i+1} when the interference is coming from the nodes in

$$(X \setminus \{x_i, x_{i+1}\}) \cup \widetilde{X} \cup \bigcup_{j=0}^{i-1} X_j.$$

Then $(T_i^{'}, i \geq 0)$ is a stationary sequence with $T_i^{'} \geq T_i$. We want to proof that $E(T_0^{'}) < \infty$ and then use the Birkoff's ergodic theorem.

Following an analogous reasoning to Theorem 4.1 we have that

$$E(T_0^{'}) \leq \frac{e^{\frac{\mu\beta N}{c}}}{\varepsilon}(E\{p_o(1)^{-2}\}E\{\frac{1}{(E\{e^{-a(I^*(1)+\widetilde{I}^*(1))}|X \cup \widetilde{X}\})^2}\})^{\frac{1}{2}}$$

for $\widetilde{I}(t) = \sum_{x\in\widetilde{X}} \mathbf{1}_z P_z(t) h_t(x, n(x_o))\ell(x, n(x_o))$. We already proved that the first term is finite, lets focus on the second.

$I^*(1)$ and $\widetilde{I}^*(1)$ are independent since we conditioned on $X \cup \widetilde{X}$. Then we have that

$$E\{e^{-a(I^*(1)+\widetilde{I}^*(1))}|X \cup \widetilde{X}\} = E\{e^{-aI^*(1))}|X\}E\{e^{-a\widetilde{I}^*(1)}|\widetilde{X} \cup \{n(o)\}\}$$

By Cauchy-Schwartz's inequality the result follows if we show that

$$E\{\frac{1}{(E\{e^{-aI^*(1))}|X\})^4}\}E\{\frac{1}{(E\{e^{-a\widetilde{I}^*(1)}|\widetilde{X} \cup \{n(o)\}\})^4}\} < \infty.$$

Analogously to Theorem 4.1 by Campbell's theorem the first term can be bounded

$$E\{\frac{1}{(E\{e^{-aI^*(1))}|X\})^4}\} \leq \exp(\frac{\lambda}{(1-c_1)^4}\int_{\mathbb{R}^2}(1-(1-c_1\ell(|z|))^4)dz) < \infty.$$

Lets estimate the second term. For all $i \in \mathbb{N}$, by Pythagoras' theorem,

$$\sum_{j=0}^{i} r_{-j} \cos(\theta_{-j}) \leq |x_{-i} - n(x_o)|.$$

Hence by the last bound since ℓ is decreasing

$$\begin{aligned} E\{\frac{1}{(E\{e^{-a\widetilde{I}^*(1)}|\widetilde{X} \cup \{n(x_o)\}\})^4}\} &\leq E\{\prod_{i=1}^{\infty} e^{-4\log(1-c_1\ell(x_{-i},n(x_o)))}\} \\ &\leq E\{\prod_{i=1}^{\infty} e^{-4\log(1-c_1\ell(\sum_{j=0}^{i} r_{-j}\cos(\theta_{-j}))}\} \\ &= E\{e^{\sum_{n=1}^{\infty} g(S_{n+1})}\} \end{aligned}$$

where $S_n = \sum_{j=0}^{n-1} r_{-j} \cos(\theta_{-j})$ and $g(x) = -4\log(1 - c_1\ell(x))$.

Let $0 < \delta < E\{r_0 \cos(\theta_0)\}$, by the Chernoff's bound

$$P(\frac{S_n}{n} < \delta) \leq \min_{t>0} \exp(-n(\delta t + \log E(e^{tS_1}))) \leq e^{-n\delta t}.$$

Then by the Borel Cantelli lemma it exists N a random variable such that

$$\begin{aligned} P(N \geq m) &= P(S_n < n\delta \text{ for some } n \geq m) \\ &\leq \sum_{n=m}^{\infty} e^{-\delta t n} = \frac{1}{1-e^{-t\delta}} e^{-t\delta m}. \end{aligned}$$

Since g is non-increasing we can estimate

$$\begin{aligned} E\{e^{\sum_{n=1}^{\infty} g(S_n)}\} &= E\{e^{\sum_{n=1}^{N} g(S_n) + \sum_{n=N+1}^{\infty} g(S_n)}\} \\ &\leq E\{e^{\sum_{n=1}^{N} g(0) + \sum_{n=N+1}^{\infty} g(n\delta)}\} \leq e^{\sum_{n=1}^{\infty} g(n\delta)} E\{e^{g(0)N}\}. \end{aligned}$$

On the one hand by the series comparison test $\sum_{n=1}^{\infty} g(n\delta) < \infty$. On the other hand

$$Ee^{g(0)N} \leq \frac{1}{1-e^{-t\delta}} \sum_{n=0}^{\infty} e^{n(g(0)-t\delta)}$$

and choosing t big enough the last series converges.

Now that we proved that $E(T_0')$ is finite then by Birkoff's ergodic theorem there exists a random variable T' such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} T_k' = T'.$$

Also since $T_i' \geq 1$ then $T' \geq 1$.

Finally from the fact that $\mathbb{1}_{\sum_{i=0}^{n-1} T_i \leq t < \sum_{i=0}^{n} T_i} d(t) \geq \sum_{k=1}^{n-1} R_k \cos(\theta_k)$, we conclude that

$$\liminf_{t\to\infty} \frac{d(t)}{t} \geq \lim_{n\to\infty} \frac{\sum_{k=1}^{n-1} R_k \cos(\theta_k)}{\sum_{k=1}^{n-1} T_k'} = \frac{E(R\cos(\theta))}{T'} > 0,$$

as we wanted.

□

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