Fast Multidimensional Signal Processing with Shearlab.jl

Héctor Andrade Loarca

TU Berlin, BMS

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What is a signal?

Our definition

Function (or something that can be represented as) that contains information about the behavior or attributes of some phenomenon. It can be digital (discrete) or analog (continuous).

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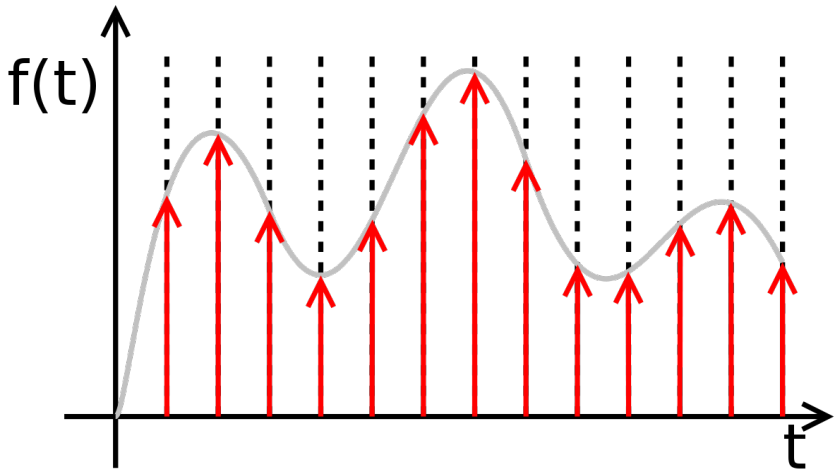

Figure: Digital and continuous one-dimensional signals

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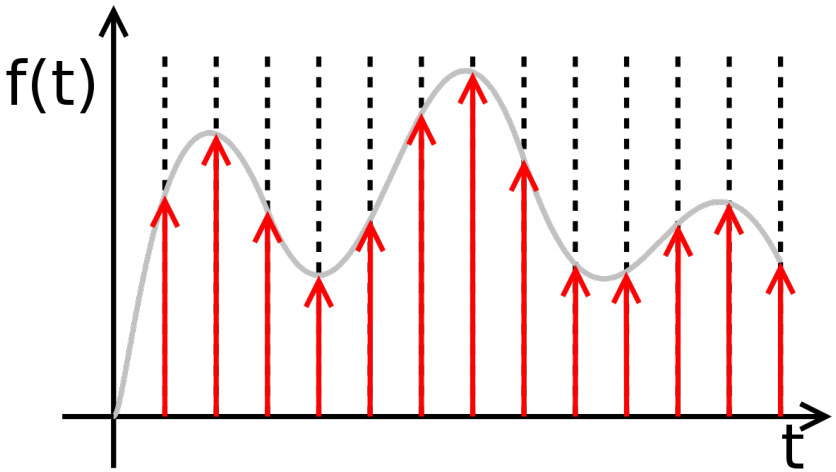

Figure: Digital and continuous one-dimensional signals

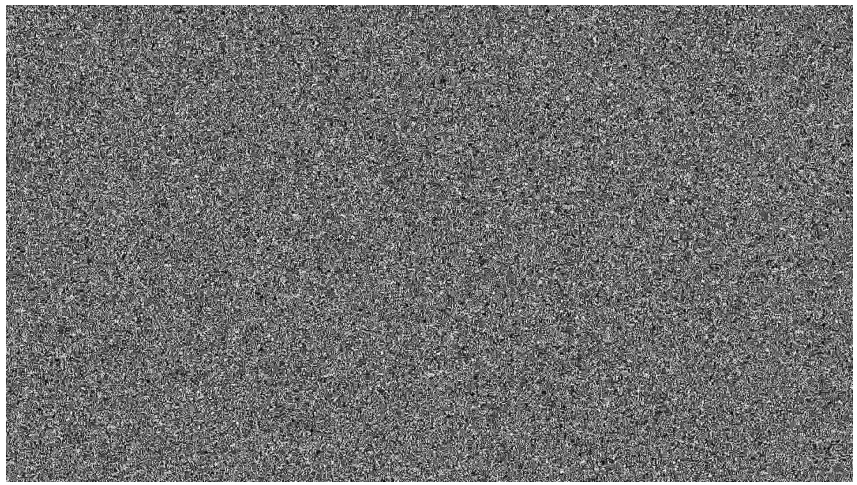

Figure: White noise, not a signal

Sparse representations of signals

- Relevant information in structured data is sparse, due the high correlation of its elements.

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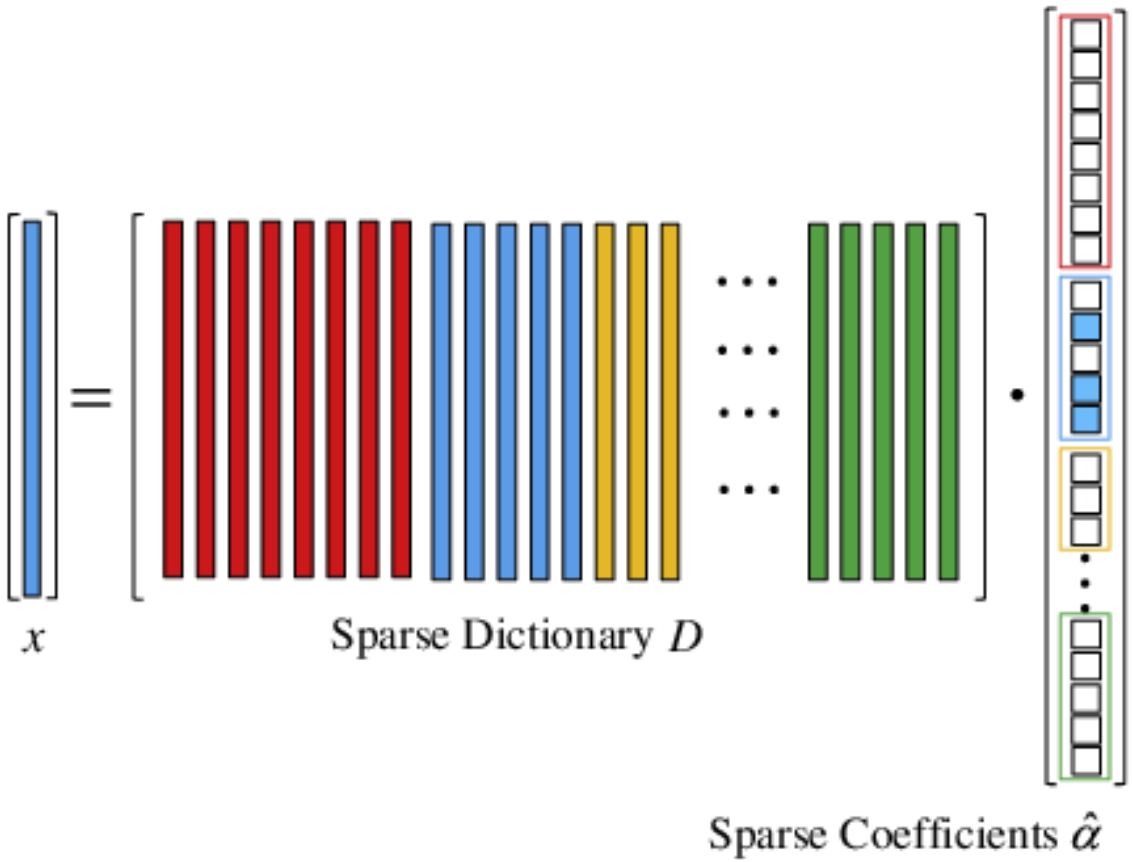
- Goal: Find the right dictionary to represent optimally our data.



Fourier Transform (Fourier, 1822)

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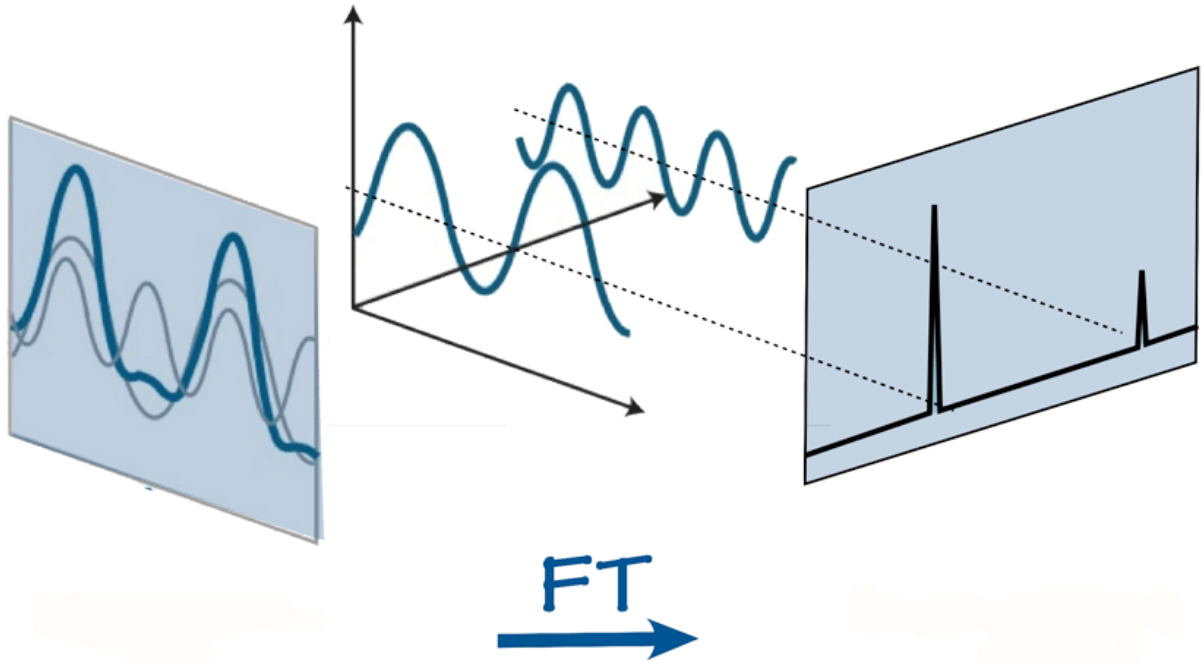

Short Time Fourier Transform (Gabor,1946)

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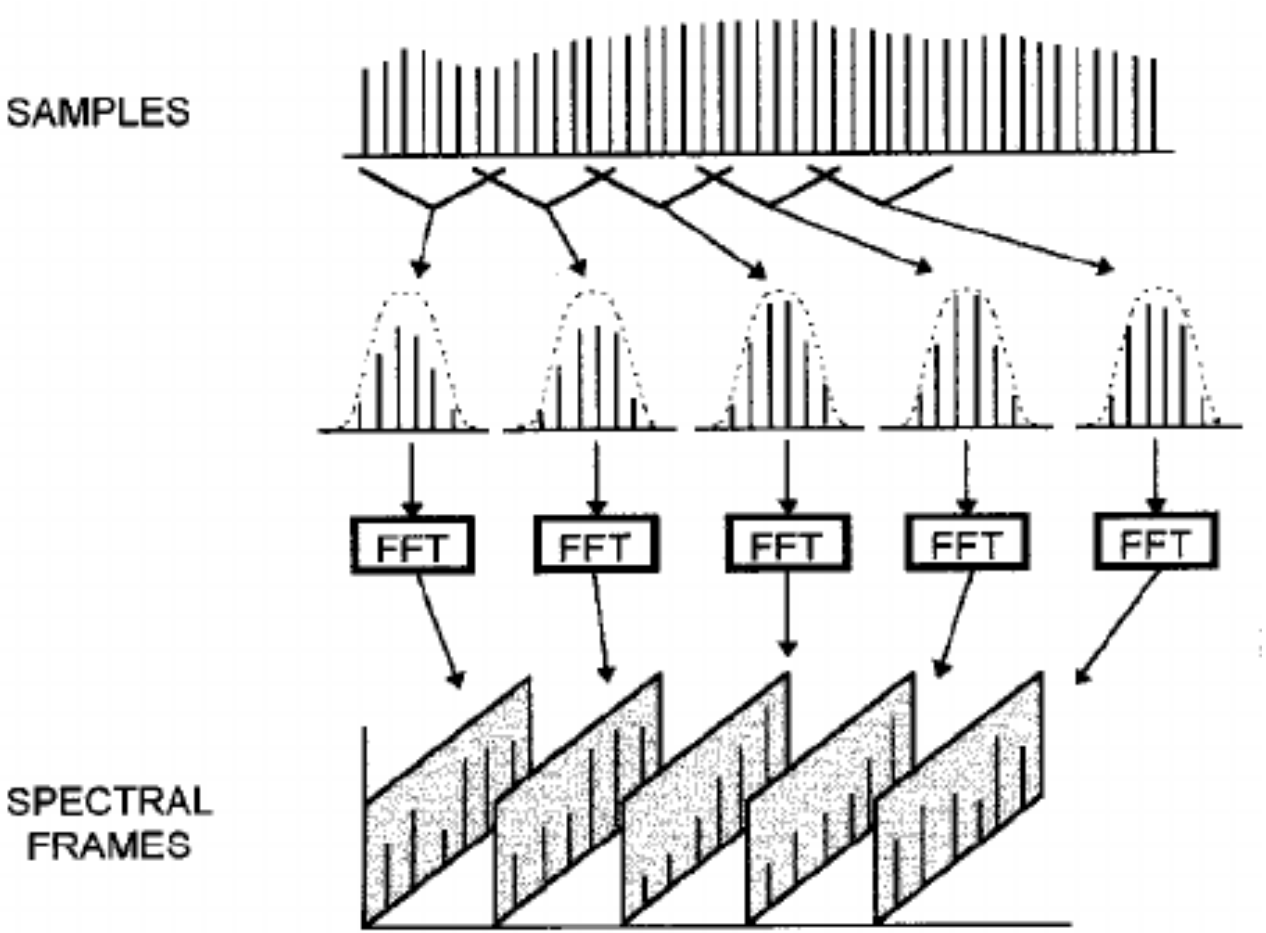

Wavelet Transform (Morlet and Grossman, 1984)

$$\mathcal{W}_\psi f(a,b) = \int_{\mathbb{R}} f(t) a^{-\frac{1}{2}} \overline{\psi\left(\frac{t-b}{a}\right)} dt$$
$$= (f * D_a \overline{\psi}^*)(b), \quad (a,b) \in \mathbb{R}^+ \times \mathbb{R}$$

where

$$\int_0^\infty \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega < \infty$$

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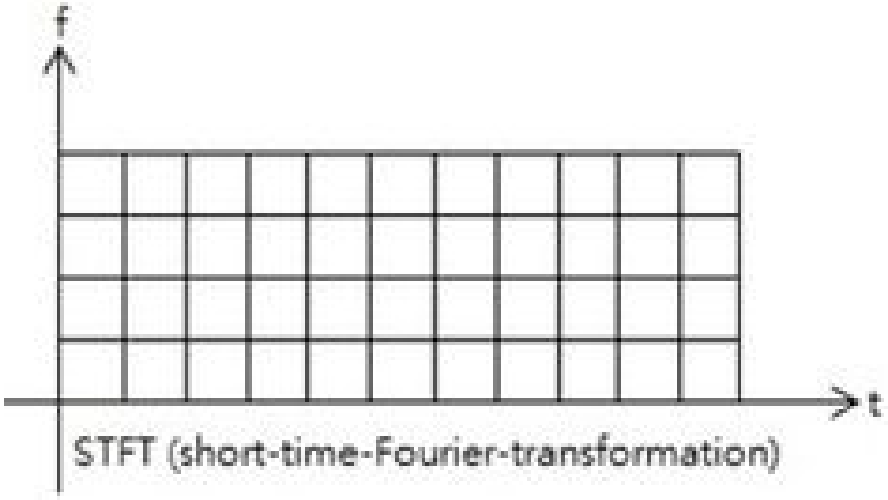

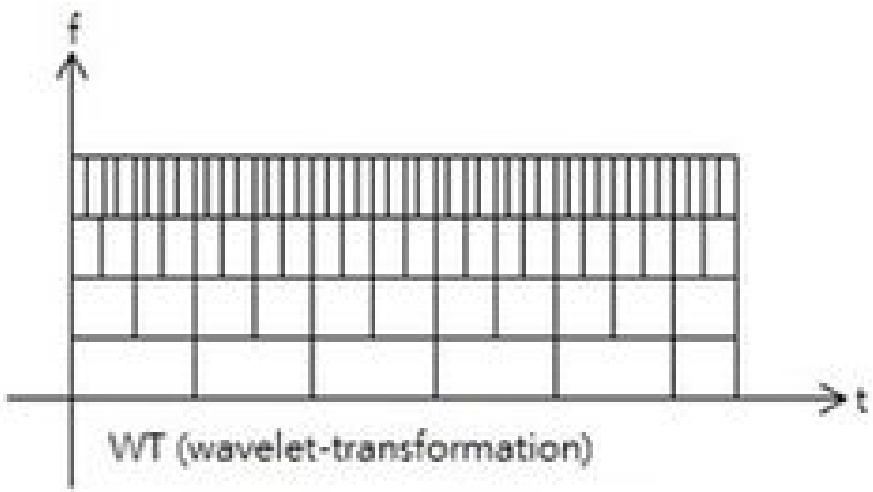

Cartoon-like functions

Definition

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$, $f \in \mathcal{E}^2(\mathbb{R}^2)$ if $f = f_0 + \chi_B f_1$, with $B \subset [0,1]^2$, $\partial B \in C^2$ and with bounded curvature. Moreover, $f_i \in C^2(\mathbb{R}^2)$ with $||f_i||_{C^2} \leq 1$ and $\text{supp} f_i \subset [0,1]^2$ for $i = 0, 1$.

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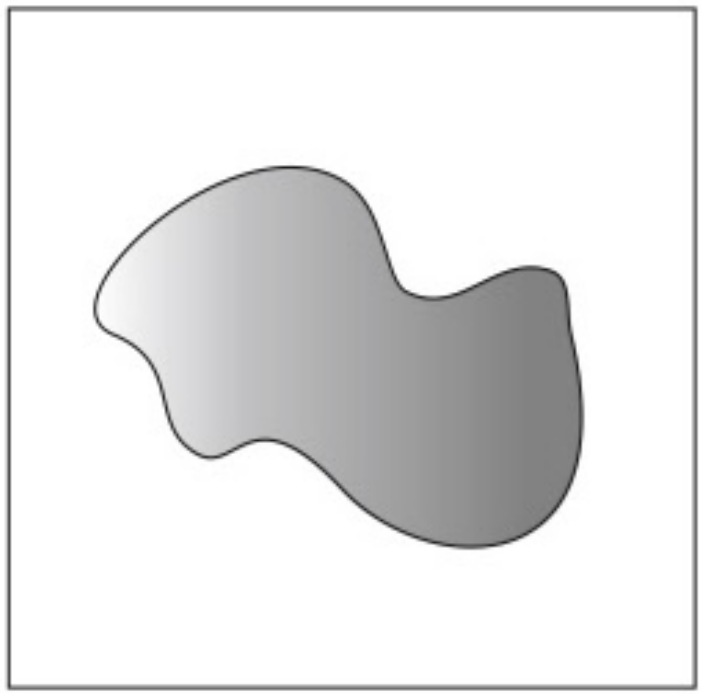

Optimal error for 2D signals

Best N-term approx. error (Donoho, 2001)

Let $\{\psi_\lambda\}_{\lambda\in\Lambda} \subset L^2(\mathbb{R}^2)$ a frame. The optimal best N-Term approximation error for any $f \in \mathcal{R}^2(\mathbb{R}^2)$ is

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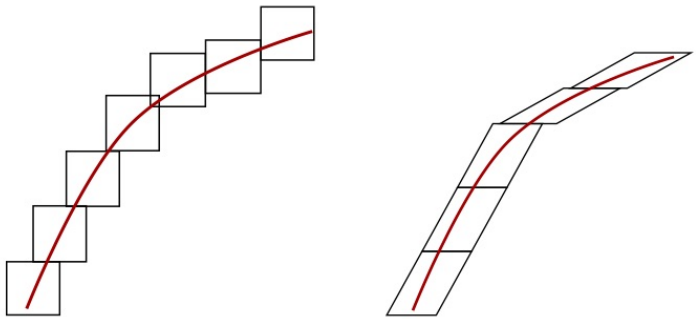

How to solve this? Scaling and Shearing

- *Scaling*

$$A_j := \begin{pmatrix} 2^j & 0 \\ 0 & 2^{j/2} \end{pmatrix}$$

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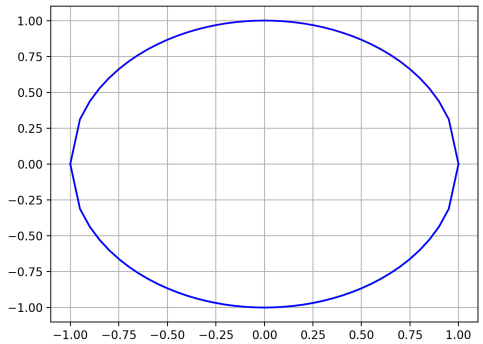

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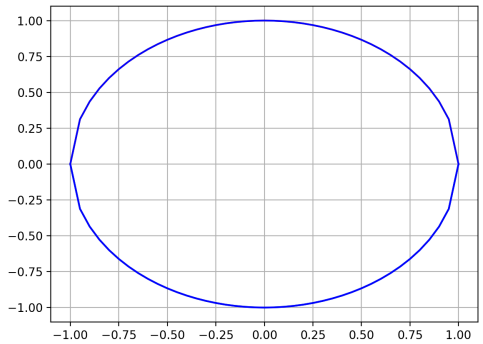

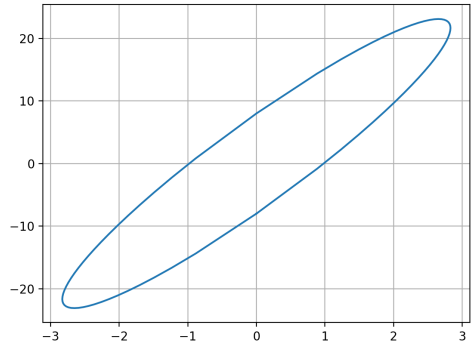

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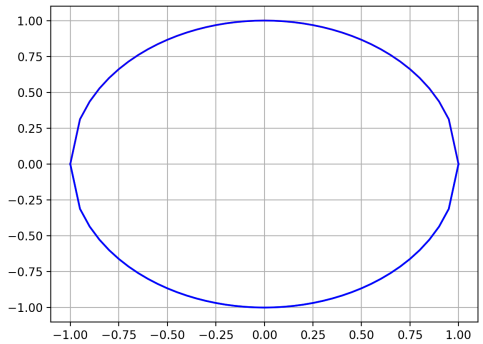

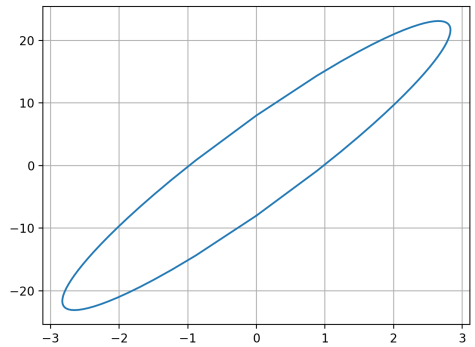

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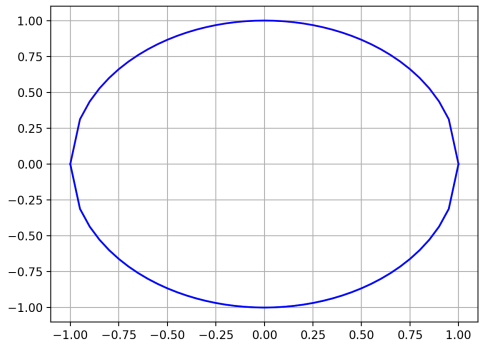

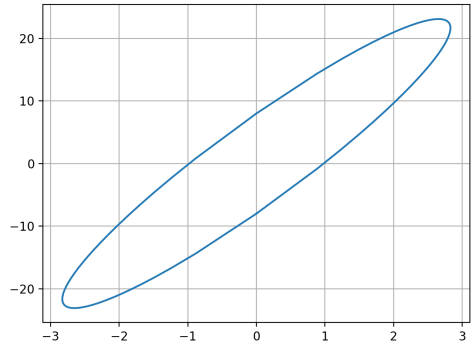

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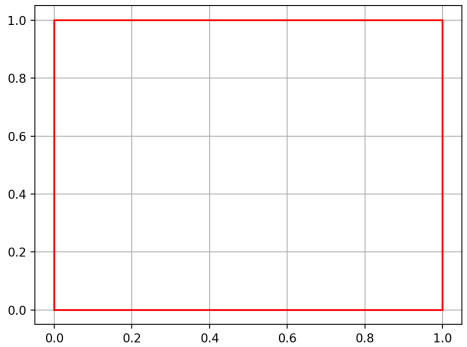

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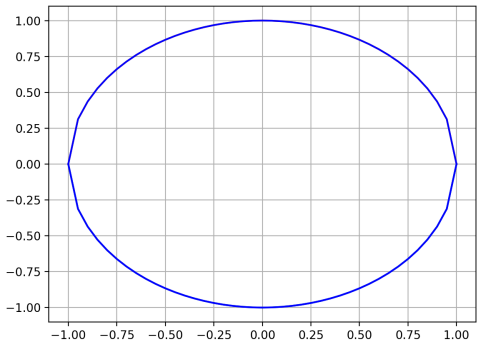

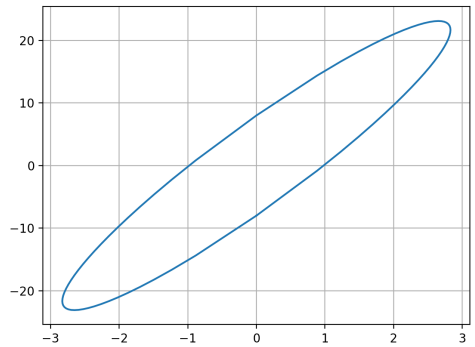

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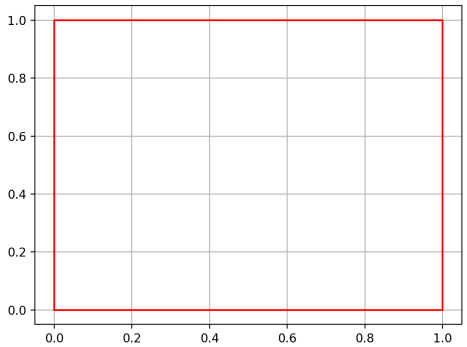

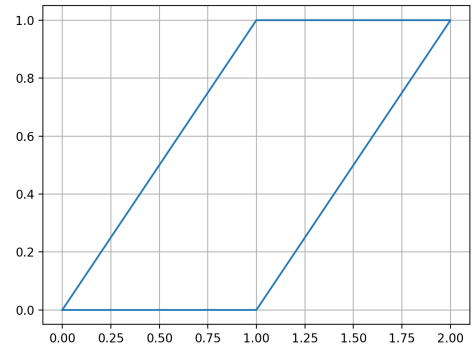

Shearlet Transform (Kutyniok, Guo, Labate, 2005)

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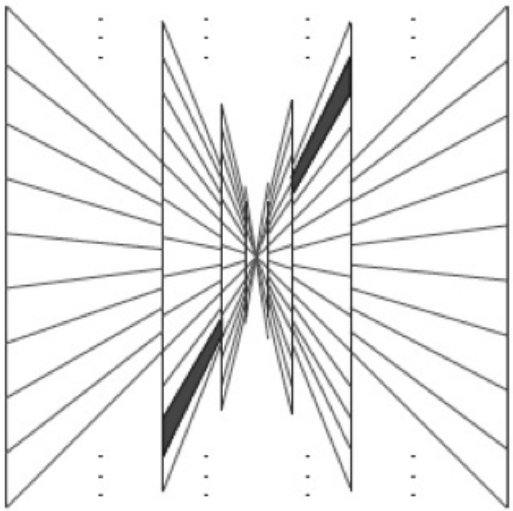

Cone-based shearlet transform

$$\mathcal{SH}(\phi, \psi, \tilde{\psi}, c) := \mathcal{P}_{\mathcal{R}} \Phi(\phi, c1) \cup \mathcal{P}_{\mathcal{C}_1} \Psi(\psi, c) \cup \mathcal{P}_{\mathcal{C}_2} \tilde{\Psi}(\tilde{\psi}, c)$$

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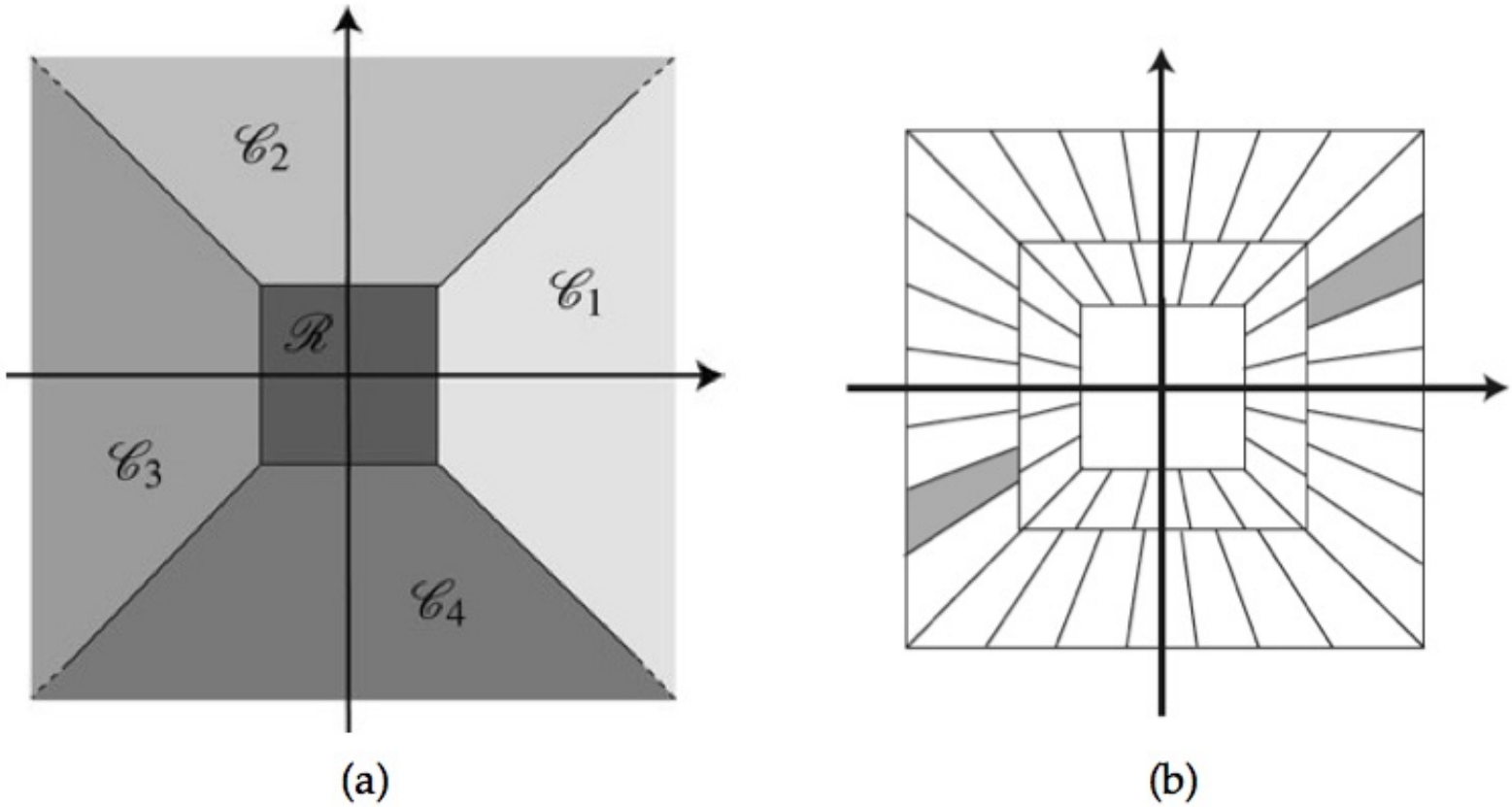

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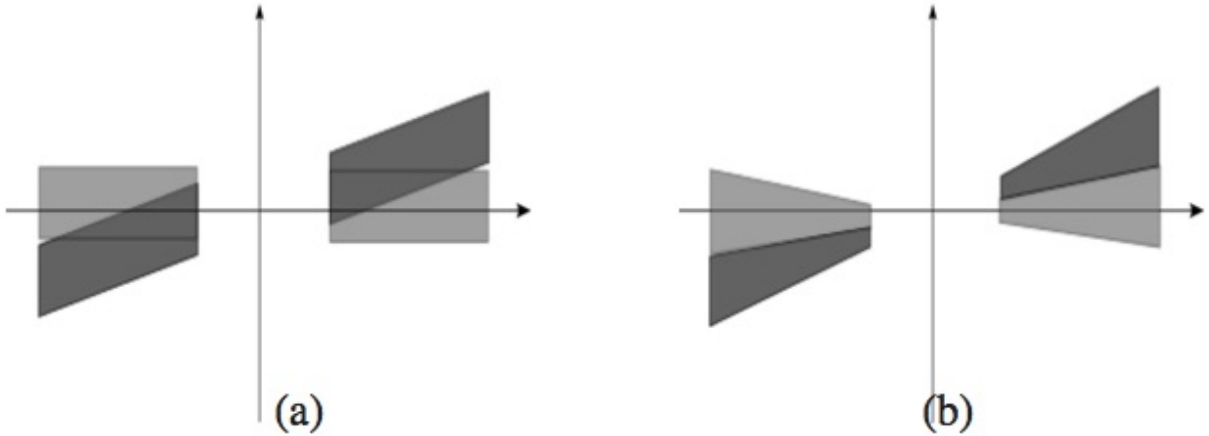

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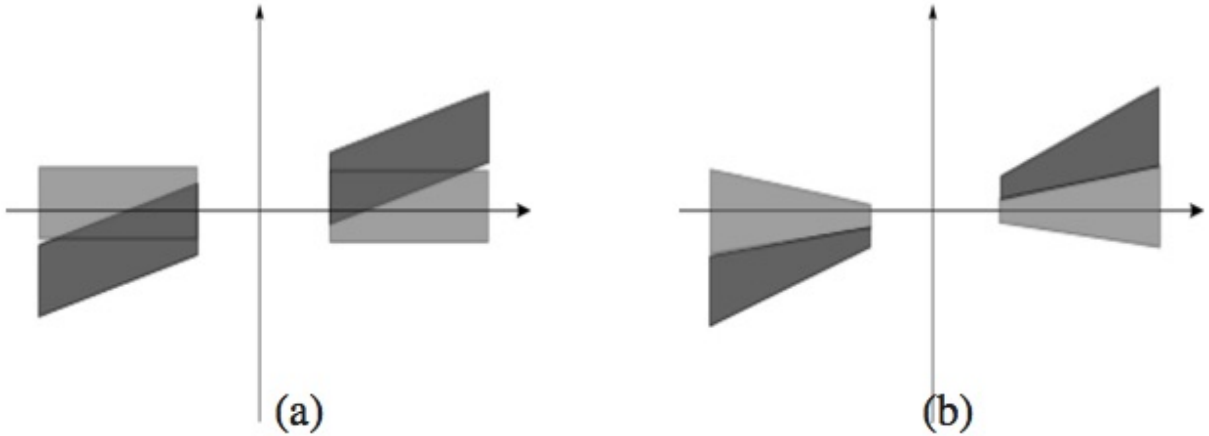

- Best N-term approximation error

$$\sigma_N(f, \{\psi_{j,k,m}\}_{j,k,m}) \sim N^{-1}(\log(N))^{3/2}$$

Current software

- Matlab
  - FFST- Fast Finite Shearlet Transform (Häuser, Steidl,TU Keiserlautern)
    http://www.mathematik.uni-kl.de/imagepro/software/ffst/
  - 2D/3D Shearlet Toolbox (D. Labate, University of Houston)
    https://www.math.uh.edu/~dlabate/software.html
  - **Shearlab3D** (G. Kutyniok, W.-Q.Lim, R. Reisenhoffer, TU Berlin)
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- **Julia**
  - **Shearlab.jl** (H. Andrade, TU Berlin)
    https://github.com/arsenal9971/Shearlab.jl

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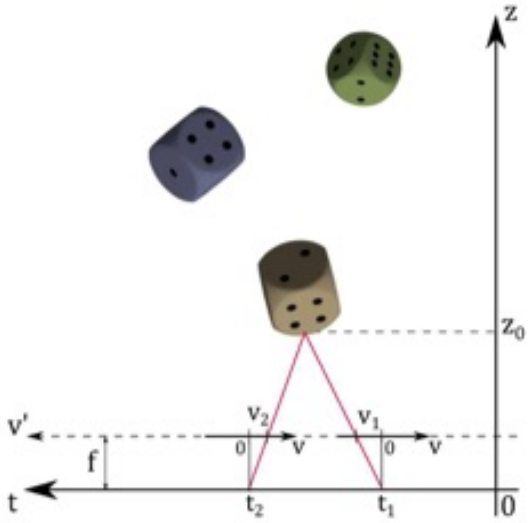
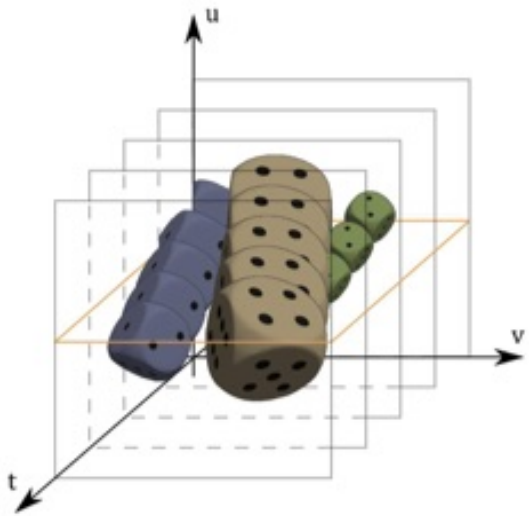
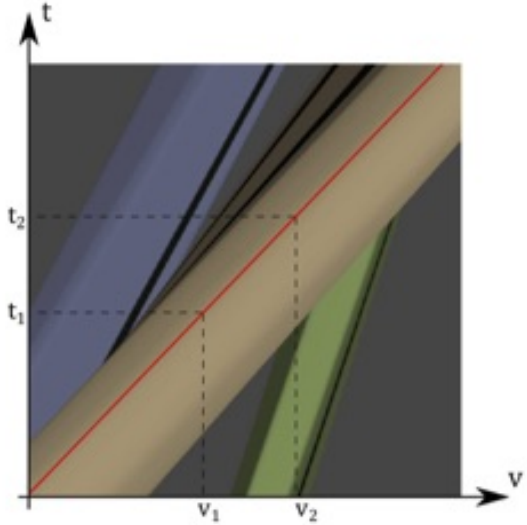
- Plenty of image filtering, import and rescaling functions with `Images.jl`, `Wavelets.jl`.
- Support of multithreading and painless GPU processing with `ArrayFire.jl`.

Current application: Light Field Recovery







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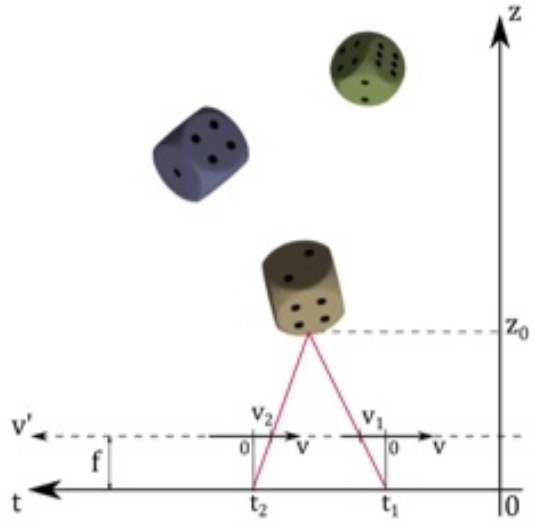
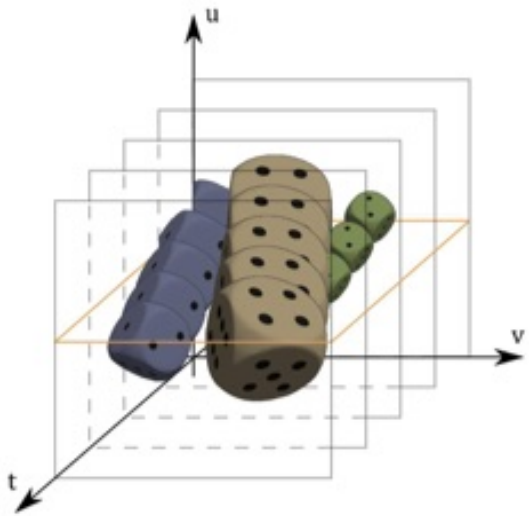
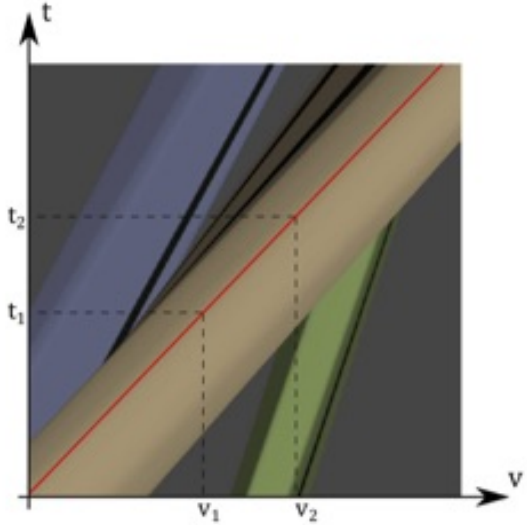
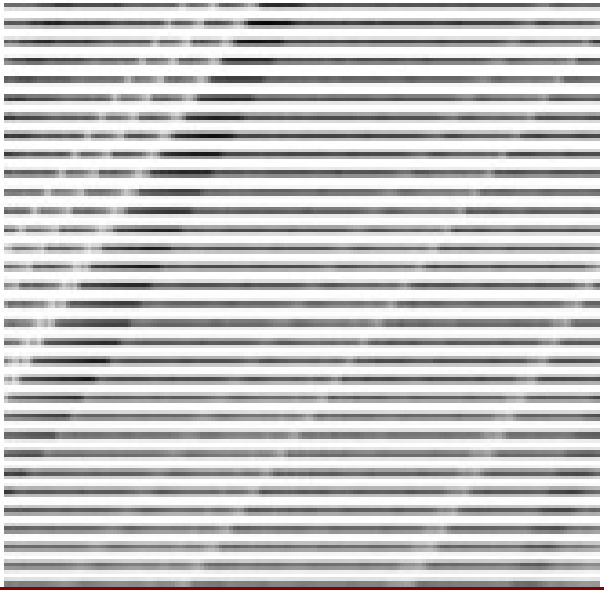
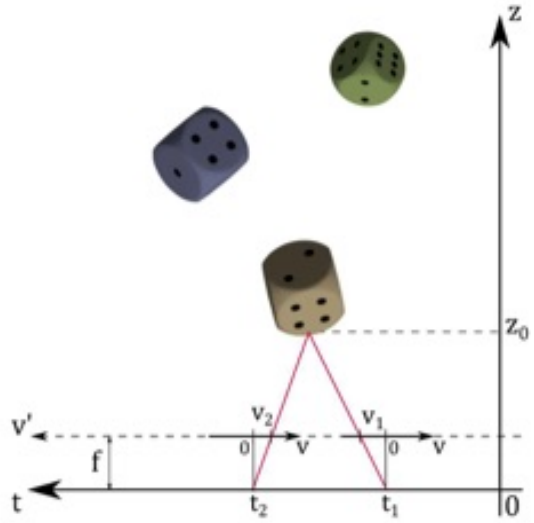


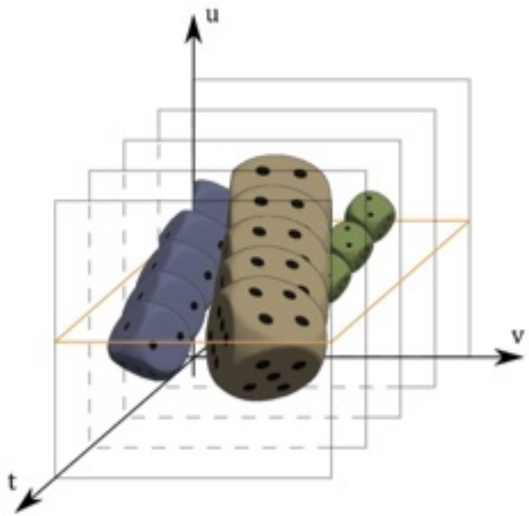


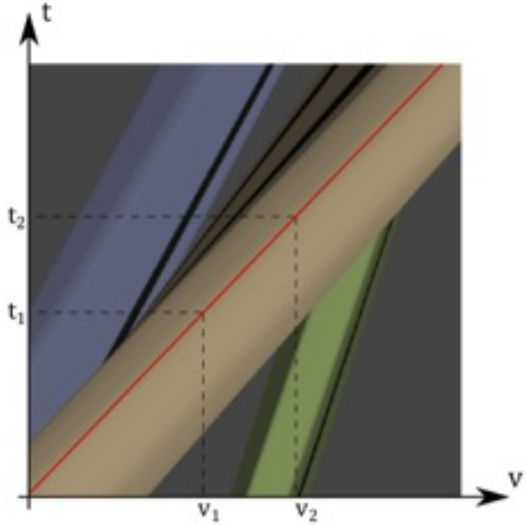


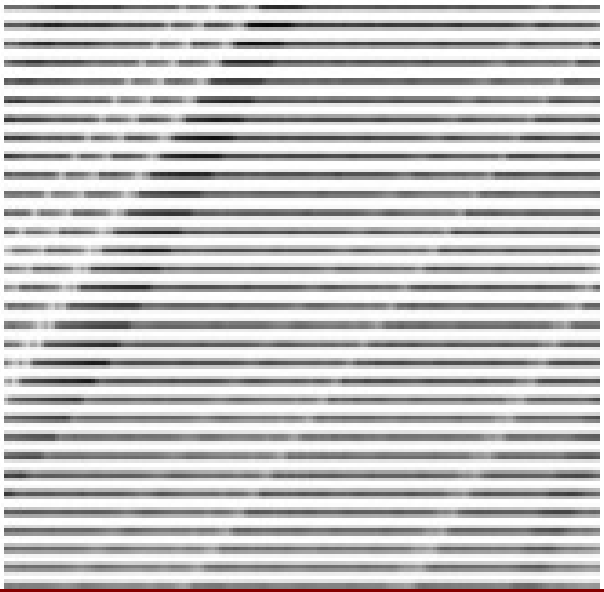


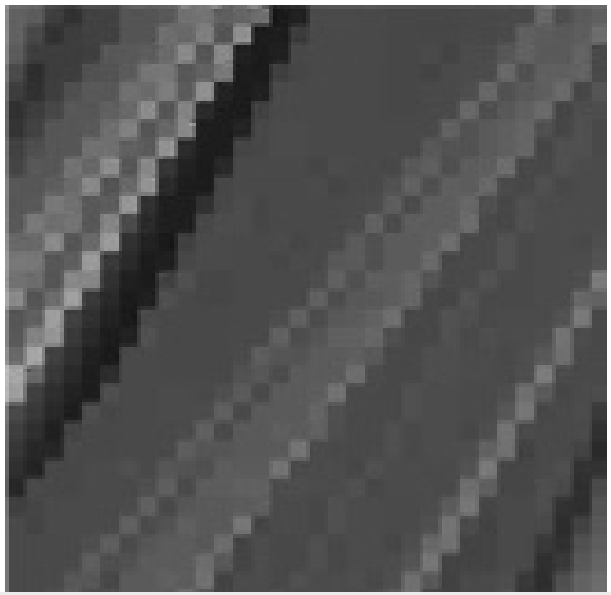
Current application: Light Field Recovery



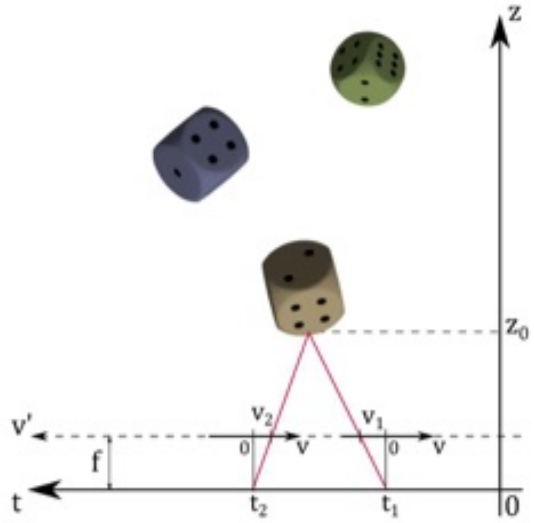


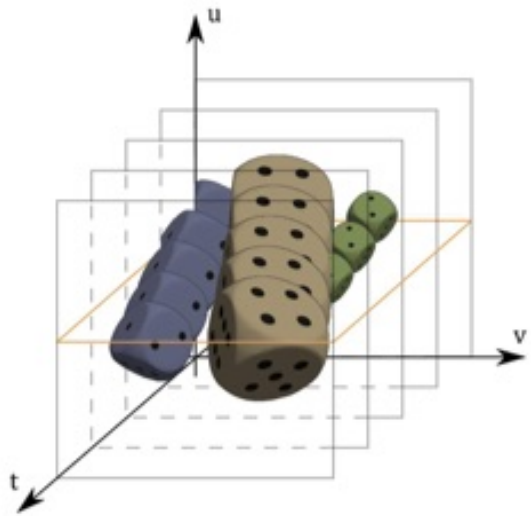


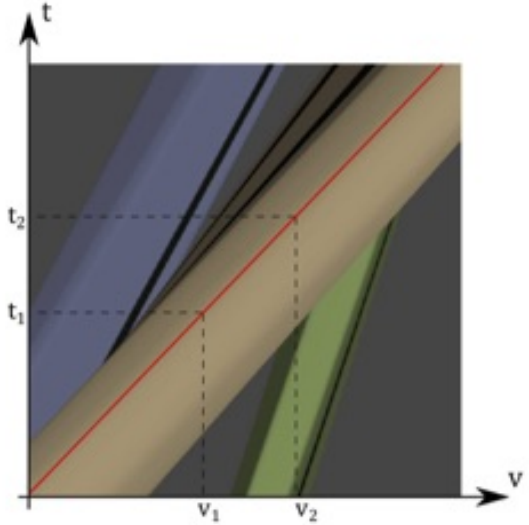


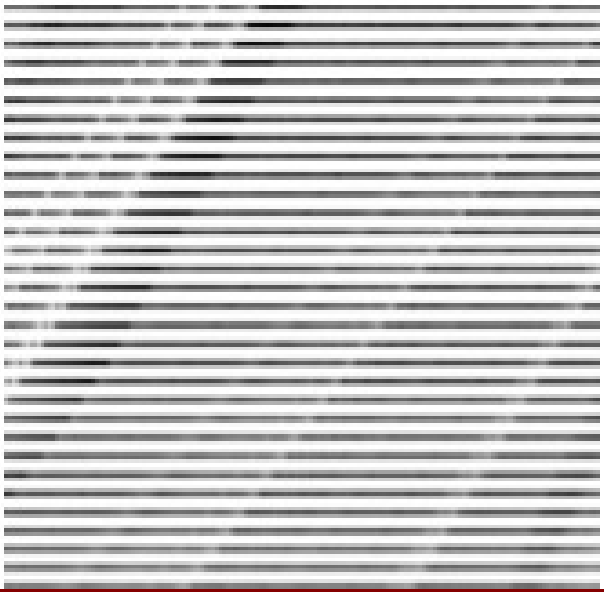


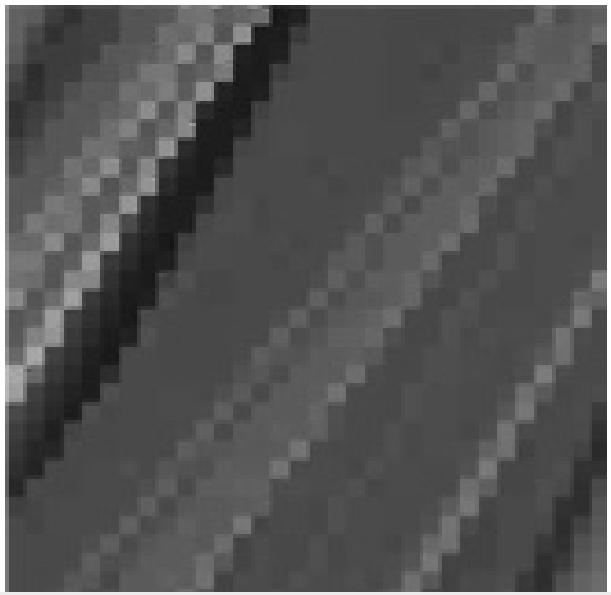
Current application: Light Field Recovery

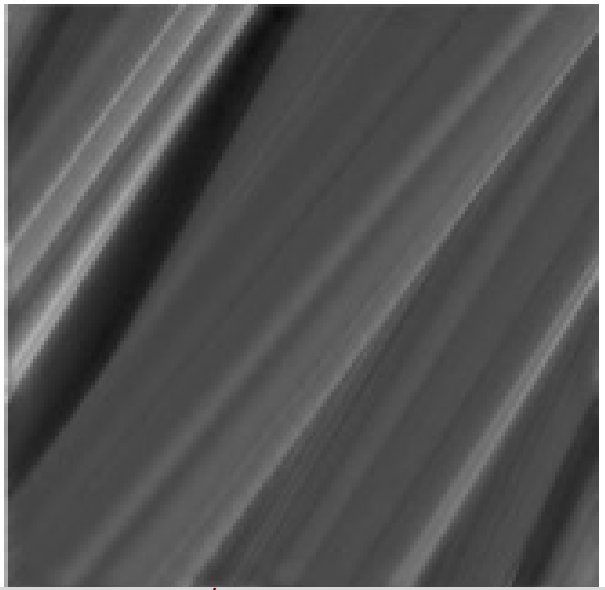












Thanks!

Questions?