Solving inverse problems in imaging with Shearlab.jl

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Daedalus Introductory Course

TU Berlin

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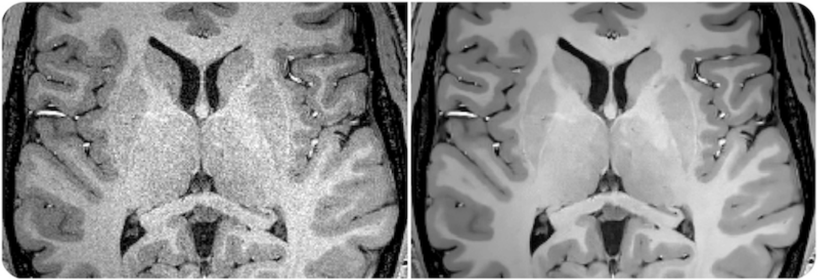

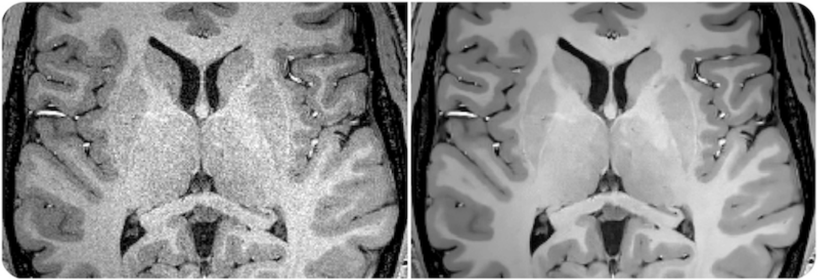

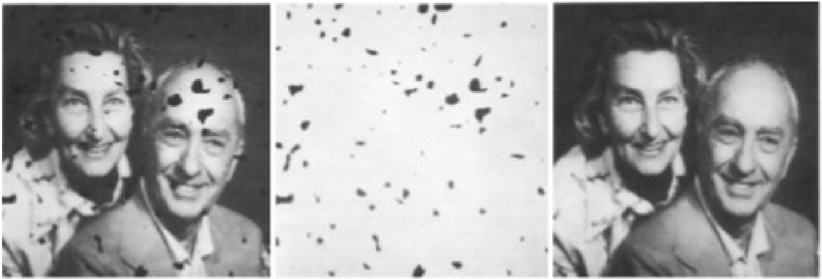

Inverse problems in Imaging

Goal

Recover parameters characterizing a system under investigation from measurements (e.g. recover image from data).

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Mathematical formulation

Recover $f_{\text{true}} \in X$ from data

$$g = \mathcal{T}(f_{\text{true}}) + \delta g$$

where $g \in Y$,$\mathcal{T} : X \longrightarrow Y$ and $\delta g \in Y$.

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- **Classical solution:** Minimization of the miss-fit against data:

$$\min_{f \in X} \mathcal{L}(\mathcal{T}(f), g)$$

$\mathcal{L} : Y \times Y \longrightarrow \mathbb{R}$ is a transformation of the negative data log-likelihood $(-\log P(f|g))$, e.g. $\mathcal{L}(f) = ||\mathcal{T}(f) - g||_2^2$.

Ill-posedness and regularization

Hadamard well-posedness

Existence and uniqueness of solution for all data and continuous dependence of solution on the data.

Ill-posed problems tend to produce overfitting when minimizing the data miss-fit, but they are the most common in applications (CT, EEG, MRI,. . .).

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- **Regularization:** Set of methods to avoid overfitting by slightly modify the original problem to increase its regularity.
- **Variational regularization:** Uses a functional $\mathcal{S} : X \longrightarrow \mathbb{R}$ (e.g. $||\cdot||_1$) to encode a priori information about f_{true}, obtaining:
$$\min_{f \in X} [\mathcal{L}(\mathcal{T}(f), g) + \lambda \mathcal{S}(f)] \quad \text{for a fixed } \lambda \geq 0$$

Image denoising

Goal

Recover an image $f \in X$ from noisy data:

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- The worst behaviour of the estimator is the supremum

$$\sup_{f \in X} \mathbb{E}||f - \tilde{f}||_2^2$$

  the *Minimax* MSE will be

$$\inf_{\tilde{f}} \sup_{f \in X} \mathbb{E}||f - \tilde{f}||_2^2$$

Minimax MSE

Frame

A frame for a Hilbert space X is a collection $\Psi = \{\psi_i\}_{i \in \mathcal{I}} \subset X$ satisfying

$$A||f||_2 \leq ||\{\langle f, \psi_i \rangle\}_{i \in \mathcal{I}}||_{\ell^2(\mathcal{I})} \leq B||f||_2 \quad \forall f \in X$$

for some $0 < A \leq B < \infty$.

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Theorem (Labate et al., 2012)

"If an image is sparse within a frame $\{\psi_i\}_{i\in\mathcal{I}}$, one can obtain a Minimax MSE estimator by thresholding the coefficients in the expansion of the noisy data:

$$g = \sum_{i\in\mathcal{I}} \langle g, \psi_i\rangle \psi_i \quad "$$

Image inpainting

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Recover an image $f \in X$ from known data:

$$g = P_K(f)$$

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Sparse Regularization/CS approach (Genzel, Kutyniok, 2014):

" If a signal (image) is sparse within a frame Ψ, it can be recovered from highly underdetermined, non-adaptive linear measurements by ℓ^1-regularization, i.e.

$$\min_{\tilde{f} \in X} ||\{\langle \tilde{f}, \psi_i \rangle\}_{i \in \mathcal{I}}||_{\ell^1(\mathcal{I})} \quad \text{s.t.} \ P_K(\tilde{f}) = g = P_K(f) \quad "$$

Theorem (Genzel, Kutyniok; 2014)

Let $\delta > 0$ *and* $\Lambda \subset \mathcal{I}$ *be a* δ-**cluster** *for* f *with respect to a frame* Ψ *(i.e.* $||\mathbb{1}_{\Lambda^c} T_\Psi f||_{\ell^1} \leq \delta$*). If* $\mu_c(\Lambda, P_M \Psi) < 1/2$ *and* f^* *is the minimizer of the problem, then*

$$||\{\langle f^* - f, \psi_i \rangle\}_{i\in\mathcal{I}}||_{\ell^1(\mathcal{I})} \leq \frac{2\delta}{1 - \mu_c(\Lambda, P_M \Psi)}$$

Cluster coherence

$$\mu_c(\Lambda, P_M \Psi) := \max_{j\in\mathcal{I}} \sum_{i\in\Lambda} |\langle P_M \psi_i, P_M \psi_j \rangle|$$

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- **Conclusion:** One can use sparsifying frames on images to perform denoising and inpainting. The quality depends on the level of sparsity.
- **Problem:** Pick a good frame for the image space.

Image space: Cartoon-like functions

Definition

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{C}$, $f \in \mathcal{E}^2(\mathbb{R}^2)$ if $f = f_0 + \chi_B f_1$, with $B \subset [0,1]^2$, $\partial B \in C^2$ and with bounded curvature. Moreover, $f_i \in C^2(\mathbb{R}^2)$ with $||f_i||_{C^2} \leq 1$ and $\text{supp} f_i \subset [0,1]^2$ for $i = 0, 1$.

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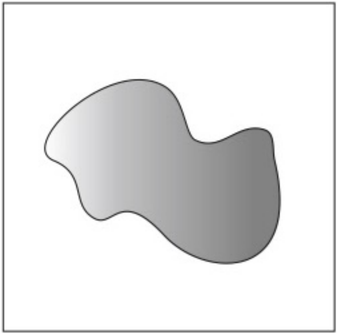
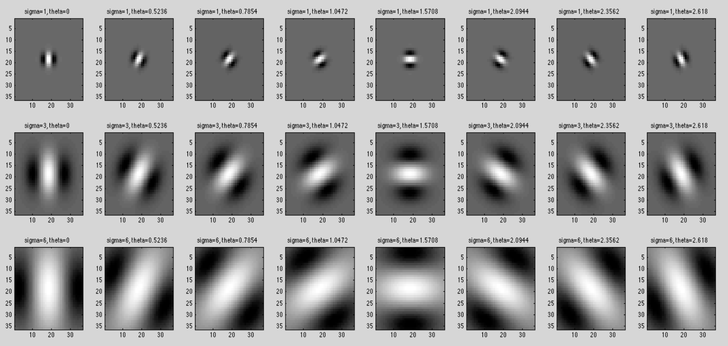


Examples of frames for images

- Gabor frames (Gabor, 1946).
- Wavelet frames (Morlet et al., 1984).
- Curvelet frames (Candès et al., 1999).
- Shearlet frames (Kutyniok et al., 2005).



Optimal approximation error for images

Best N-term approx. error (Donoho, 2001)

Let $\{\psi_\lambda\}_{\lambda\in\Lambda} \subset L^2(\mathbb{R}^2)$ a frame. The optimal best N-Term approximation error for any $f \in \mathcal{E}^2(\mathbb{R}^2)$ is

$$\sigma_N(f, \{\psi_\lambda\}_{\lambda\in\Lambda}) = O(N^{-1})$$

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Error of 2D-wavelets

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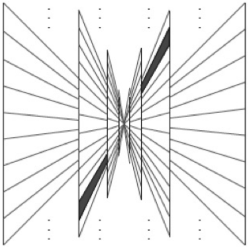
Shearlet Transform (Kutyniok, Guo, Labate, 2005)

Classical Shearlet Transform

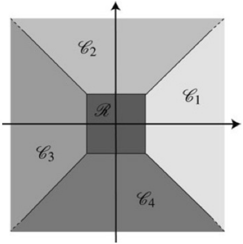
$$\langle f, \psi_{j,k,m} \rangle = \int_{\mathbb{R}^2} f(x)\overline{\psi_{j,k,m}(x)}dx$$

where

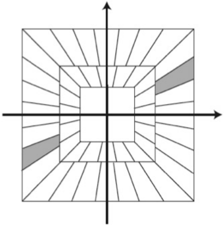
$$\mathcal{SH}(\psi) = \{\psi_{j,k,m}(x) = 2^{3j/4}\psi(S_k A_j x - m) : (j,k) \in \mathbb{Z}^2, m \in \mathbb{Z}^2\}$$



$$\mathcal{SH}(\phi, \psi, \tilde{\psi}, c) := \mathcal{P}_{\mathcal{R}}\Phi(\phi, c1) \cup \mathcal{P}_{\mathcal{C}_1}\Psi(\psi, c) \cup \mathcal{P}_{\mathcal{C}_2}\tilde{\Psi}(\tilde{\psi}, c)$$



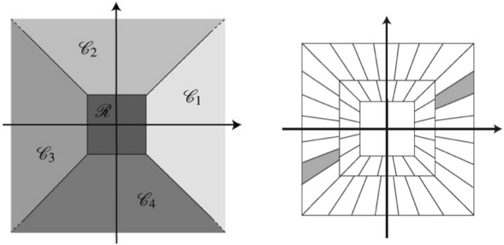
(a)



(b)

Cone-adapted shearlet transform and optimal sparsity

$$\mathcal{SH}(\phi, \psi, \tilde{\psi}, c) := \mathcal{P}_{\mathcal{R}} \Phi(\phi, c1) \cup \mathcal{P}_{\mathcal{C}_1} \Psi(\psi, c) \cup \mathcal{P}_{\mathcal{C}_2} \tilde{\Psi}(\tilde{\psi}, c)$$



(a) (b)

Cone shearlets sparsity (Band limited case: Lim, Labate; 2006), (Compactly supported case: Kutyniok, Lim, 2011)

Best N-term approximation error

$$\sigma_N(f, \{\psi_{j,k,m}\}_{j,k,m}) \sim N^{-1}(\log(N))^{3/2}$$

Current software

- Matlab
  - FFST- Fast Finite Shearlet Transform (Häuser, Steidl,TU Keiserslautern)
    http://www.mathematik.uni-kl.de/imagepro/software/ffst/
  - 2D/3D Shearlet Toolbox (D. Labate, University of Houston)
    https://www.math.uh.edu/~dlabate/software.html
  - Shearlab3D (G. Kutyniok, W.-Q.Lim, R. Reisenhofer, TU Berlin)
    http://www.shearlab.org/
- Python
  - pyShearLab (Stefan Loock, U Göttingen)
    http://na.math.uni-goettingen.de/pyshearlab/
  - alpha-Transform (Felix Voigtländer, TU Berlin, KU Eichstätt)
    https://github.com/dedale-fet/alpha-transform
- **Julia**
  - Shearlab.jl (H. Andrade, TU Berlin)
    https://github.com/arsenal9971/Shearlab.jl

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    https://www.math.uh.edu/~dlabate/software.html
  - Shearlab3D (G. Kutyniok, W.-Q.Lim, R. Reisenhofer, TU Berlin)
    http://www.shearlab.org/
- Python
  - pyShearLab (Stefan Loock, U Göttingen)
    http://na.math.uni-goettingen.de/pyshearlab/
  - alpha-Transform (Felix Voigtländer, TU Berlin, KU Eichstätt)
    https://github.com/dedale-fet/alpha-transform
- **Julia**
  - Shearlab.jl (H. Andrade, TU Berlin)
    https://github.com/arsenal9971/Shearlab.jl
- **Lets code!**