2

RIGID MOTIONS AND HOMOGENEOUS TRANSFORMATIONS

A large part of robot kinematics is concerned with the establishment of various coordinate systems to represent the positions and orientations of rigid objects, and with transformations among these coordinate systems. Indeed, the geometry of three-dimensional space and of rigid motions plays a central role in all aspects of robotic manipulation. In this chapter we study the operations of rotation and translation, and introduce the notion of homogeneous transformations.[1] Homogeneous transformations combine the operations of rotation and translation into a single matrix multiplication, and are used in Chapter 3 to derive the so-called forward kinematic equations of rigid manipulators.

We begin by examining representations of points and vectors in a Euclidean space equipped with multiple coordinate frames. Following this, we introduce the concept of a rotation matrix to represent relative orientations among coordinate frames. Then we combine these two concepts to build homogeneous transformation matrices, which can be used to simultaneously represent the position and orientation of one coordinate frame relative to another. Furthermore, homogeneous transformation matrices can be used to perform coordinate transformations. Such transformations allow us to represent various quantities in different coordinate frames, a facility that we will often exploit in subsequent chapters.

[1]Since we make extensive use of elementary matrix theory, the reader may wish to review Appendix B before beginning this chapter.

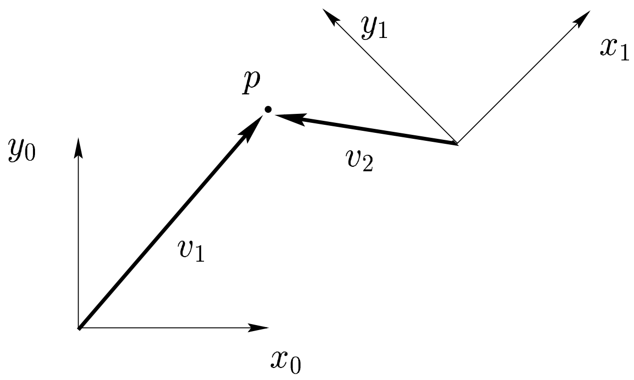

Fig. 2.1 Two coordinate frames, a point p, and two vectors v_1 and v_2.

2.1 REPRESENTING POSITIONS

Before developing representation schemes for points and vectors, it is instructive to distinguish between the two fundamental approaches to geometric reasoning: the *synthetic* approach and the *analytic* approach. In the former, one reasons directly about geometric entities (e.g., points or lines), while in the latter, one represents these entities using coordinates or equations, and reasoning is performed via algebraic manipulations.

Consider Figure 2.1. This figure shows two coordinate frames that differ in orientation by an angle of 45°. Using the synthetic approach, without ever assigning coordinates to points or vectors, one can say that x_0 is perpendicular to y_0, or that $v_1 \times v_2$ defines a vector that is perpendicular to the plane containing v_1 and v_2, in this case pointing out of the page.

In robotics, one typically uses analytic reasoning, since robot tasks are often defined using Cartesian coordinates. Of course, in order to assign coordinates it is necessary to specify a coordinate frame. Consider again Figure 2.1. We could specify the coordinates of the point p with respect to either frame $o_0x_0y_0$ or frame $o_1x_1y_1$. In the former case, we might assign to p the coordinate vector $(5, 6)^T$, and in the latter case $(-2.8, 4.2)^T$. So that the reference frame will always be clear, we will adopt a notation in which a superscript is used to denote the reference frame. Thus, we would write

$$p^0 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \qquad p^1 = \begin{bmatrix} -2.8 \\ 4.2 \end{bmatrix}$$

Geometrically, a point corresponds to a specific location in space. We stress here that p is a geometric entity, a point in space, while both p^0 and p^1 are coordinate vectors that represent the location of this point in space with respect to coordinate frames $o_0x_0y_0$ and $o_1x_1y_1$, respectively.

Since the origin of a coordinate system is just a point in space, we can assign coordinates that represent the position of the origin of one coordinate system with respect to another. In Figure 2.1, for example, we have

$$o_1^0 = \begin{bmatrix} 10 \\ 5 \end{bmatrix}, \qquad o_0^1 = \begin{bmatrix} -10.6 \\ 3.5 \end{bmatrix}$$

In cases where there is only a single coordinate frame, or in which the reference frame is obvious, we will often omit the superscript. This is a slight abuse of notation, and the reader is advised to bear in mind the difference between the geometric entity called p and any particular coordinate vector that is assigned to represent p. The former is independent of the choice of coordinate systems, while the latter obviously depends on the choice of coordinate frames.

While a point corresponds to a specific location in space, a *vector* specifies a direction and a magnitude. Vectors can be used, for example, to represent displacements or forces. Therefore, while the point p is not equivalent to the vector v_1, the displacement from the origin o_0 to the point p is given by the vector v_1. In this text, we will use the term *vector* to refer to what are sometimes called *free vectors*, i.e., vectors that are not constrained to be located at a particular point in space. Under this convention, it is clear that points and vectors are not equivalent, since points refer to specific locations in space, but a vector can be moved to any location in space. Under this convention, two vectors are equal if they have the same direction and the same magnitude.

When assigning coordinates to vectors, we use the same notational convention that we used when assigning coordinates to points. Thus, v_1 and v_2 are geometric entities that are invariant with respect to the choice of coordinate systems, but the representation by coordinates of these vectors depends directly on the choice of reference coordinate frame. In the example of Figure 2.1, we would obtain

$$v_1^0 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}, \qquad v_1^1 = \begin{bmatrix} 7.77 \\ 0.8 \end{bmatrix}, \qquad v_2^0 = \begin{bmatrix} -5.1 \\ 1 \end{bmatrix}, \qquad v_2^1 = \begin{bmatrix} -2.89 \\ 4.2 \end{bmatrix}$$

Coordinate Convention
In order to perform algebraic manipulations using coordinates, it is essential that all coordinate vectors be defined with respect to the same coordinate frame. In the case of free vectors, it is enough that they be defined with respect to "parallel" coordinate frames, i.e. frames whose respective coordinate axes are parallel, since only their magnitude and direction are specified and not their absolute locations in space.

Using this convention, an expression of the form $v_1^1 + v_2^2$, where v_1^1 and v_2^2 are as in Figure 2.1, is not defined since the frames $o_0x_0y_0$ and $o_1x_1y_1$ are not parallel. Thus, we see a clear need, not only for a representation system that allows points to be expressed with respect to various coordinate systems, but also for a mechanism that allows us to transform the coordinates of points that are expressed in one coordinate system into the appropriate coordinates with

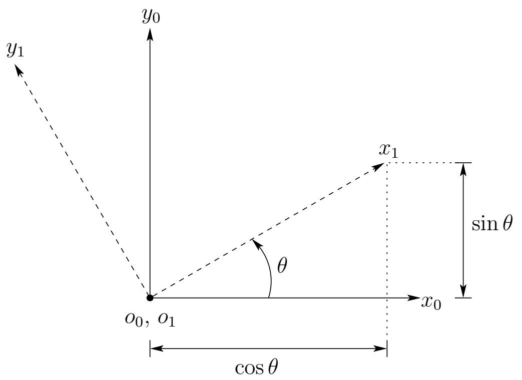

Fig. 2.2 Coordinate frame $o_1x_1y_1$ is oriented at an angle θ with respect to $o_0x_0y_0$.

respect to some other coordinate frame. Such coordinate transformations and their derivations are the topic for much of the remainder of this chapter.

2.2 REPRESENTING ROTATIONS

In order to represent the relative position and orientation of one rigid body with respect to another, we will rigidly attach coordinate frames to each body, and then specify the geometric relationships between these coordinate frames. In Section 2.1 we saw how one can represent the position of the origin of one frame with respect to another frame. In this section, we address the problem of describing the orientation of one coordinate frame relative to another frame. We begin with the case of rotations in the plane, and then generalize our results to the case of orientations in a three dimensional space.

2.2.1 Rotation in the plane

Figure 2.2 shows two coordinate frames, with frame $o_1x_1y_1$ being obtained by rotating frame $o_0x_0y_0$ by an angle θ. Perhaps the most obvious way to represent the relative orientation of these two frames is to merely specify the angle of rotation, θ. There are two immediate disadvantages to such a representation. First, there is a discontinuity in the mapping from relative orientation to the value of θ in a neighborhood of $\theta = 0$. In particular, for $\theta = 2\pi - \epsilon$, small changes in orientation can produce large changes in the value of θ (i.e., a rotation by ϵ causes θ to “wrap around” to zero). Second, this choice of representation does not scale well to the three dimensional case.

A slightly less obvious way to specify the orientation is to specify the coordinate vectors for the axes of frame $o_1x_1y_1$ with respect to coordinate frame

$o_0x_0y_0$[2]:

$$R_1^0 = \left[x_1^0 | y_1^0\right]$$

where x_1^0 and y_1^0 are the coordinates in frame $o_0x_0y_0$ of unit vectors x_1 and y_1, respectively. A matrix in this form is called a **rotation matrix**. Rotation matrices have a number of special properties that we will discuss below.

In the two dimensional case, it is straightforward to compute the entries of this matrix. As illustrated in Figure 2.2,

$$x_1^0 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \qquad y_1^0 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

which gives

$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{2.1}$$

Note that we have continued to use the notational convention of allowing the superscript to denote the reference frame. Thus, R_1^0 is a matrix whose column vectors are the coordinates of the (unit vectors along the) axes of frame $o_1x_1y_1$ expressed relative to frame $o_0x_0y_0$.

Although we have derived the entries for R_1^0 in terms of the angle θ, it is not necessary that we do so. An alternative approach, and one that scales nicely to the three dimensional case, is to build the rotation matrix by projecting the axes of frame $o_1x_1y_1$ onto the coordinate axes of frame $o_0x_0y_0$. Recalling that the dot product of two unit vectors gives the projection of one onto the other, we obtain

$$x_1^0 = \begin{bmatrix} x_1 \cdot x_0 \\ x_1 \cdot y_0 \end{bmatrix}, \qquad y_1^0 = \begin{bmatrix} y_1 \cdot x_0 \\ y_1 \cdot y_0 \end{bmatrix}$$

which can be combined to obtain the rotation matrix

$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 \end{bmatrix}$$

Thus the columns of R_1^0 specify the direction cosines of the coordinate axes of $o_1x_1y_1$ relative to the coordinate axes of $o_0x_0y_0$. For example, the first column $(x_1 \cdot x_0, x_1 \cdot y_0)^T$ of R_1^0 specifies the direction of x_1 relative to the frame $o_0x_0y_0$. Note that the right hand sides of these equations are defined in terms of geometric entities, and not in terms of their coordinates. Examining Figure 2.2 it can be seen that this method of defining the rotation matrix by projection gives the same result as was obtained in Equation (2.1).

If we desired instead to describe the orientation of frame $o_0x_0y_0$ with respect to the frame $o_1x_1y_1$ (i.e., if we desired to use the frame $o_1x_1y_1$ as the reference frame), we would construct a rotation matrix of the form

$$R_0^1 = \begin{bmatrix} x_0 \cdot x_1 & y_0 \cdot x_1 \\ x_0 \cdot y_1 & y_0 \cdot y_1 \end{bmatrix}$$

[2]We will use x_i, y_i to denote both coordinate axes and unit vectors along the coordinate axes depending on the context.

Table 2.2.1: Properties of the Matrix Group $SO(n)$

- $R \in SO(n)$
- $R^{-1} \in SO(n)$
- $R^{-1} = R^T$
- The columns (and therefore the rows) of R are mutually orthogonal
- Each column (and therefore each row) of R is a unit vector
- $\det R = 1$

Since the inner product is commutative, (i.e. $x_i \cdot y_j = y_j \cdot x_i$), we see that

$$R_0^1 = (R_1^0)^T$$

In a geometric sense, the orientation of $o_0x_0y_0$ with respect to the frame $o_1x_1y_1$ is the inverse of the orientation of $o_1x_1y_1$ with respect to the frame $o_0x_0y_0$. Algebraically, using the fact that coordinate axes are always mutually orthogonal, it can readily be seen that

$$(R_1^0)^T = (R_1^0)^{-1}$$

The column vectors of R_1^0 are of unit length and mutually orthogonal (Problem 2-4). Such a matrix is said to be **orthogonal**. It can also be shown (Problem 2-5) that $\det R_1^0 = \pm 1$. If we restrict ourselves to right-handed coordinate systems, as defined in Appendix B, then $\det R_1^0 = +1$ (Problem 2-5). It is customary to refer to the set of all such $n \times n$ matrices by the symbol $SO(n)$, which denotes the **Special Orthogonal group of order** n. The properties of such matrices are summarized in Table 2.2.1.

To provide further geometric intuition for the notion of the inverse of a rotation matrix, note that in the two dimensional case, the inverse of the rotation matrix corresponding to a rotation by angle θ can also be easily computed simply by constructing the rotation matrix for a rotation by the angle $-\theta$:

$$\begin{aligned}
\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^T
\end{aligned}$$

2.2.2 Rotations in three dimensions

The projection technique described above scales nicely to the three dimensional case. In three dimensions, each axis of the frame $o_1x_1y_1z_1$ is projected onto coordinate frame $o_0x_0y_0z_0$. The resulting rotation matrix is given by

$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

As was the case for rotation matrices in two dimensions, matrices in this form are orthogonal, with determinant equal to 1. In this case, 3×3 rotation matrices belong to the group $SO(3)$. The properties listed in Table 2.2.1 also apply to rotation matrices in $SO(3)$.

Example 2.1

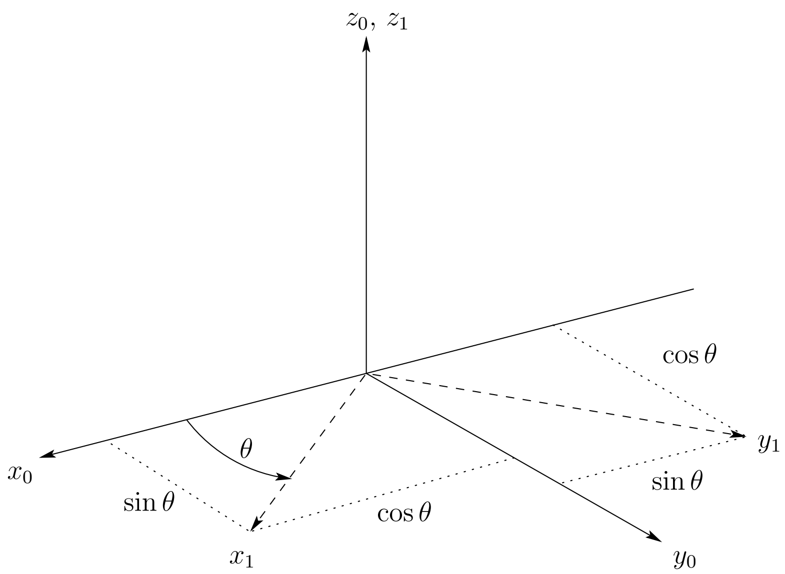

Fig. 2.3 Rotation about z_0 by an angle θ.

Suppose the frame $o_1x_1y_1z_1$ is rotated through an angle θ about the z_0-axis, and it is desired to find the resulting transformation matrix R_1^0. Note that by convention the positive sense for the angle θ is given by the right hand rule; that is, a positive rotation by angle θ about the z-axis would advance a right-hand *threaded screw along the positive z-axis*[3]. *From Figure 2.3 we see that*

$$x_1 \cdot x_0 = \cos\theta, \qquad y_1 \cdot x_0 = -\sin\theta,$$
$$x_1 \cdot y_0 = \sin\theta, \qquad y_1 \cdot y_0 = \cos\theta$$

and

$$z_0 \cdot z_1 = 1$$

while all other dot products are zero. Thus the rotation matrix R_1^0 *has a particularly simple form in this case, namely*

$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.2}$$

◇

The Basic Rotation Matrices

The rotation matrix given in Equation (2.2) is called a **basic rotation matrix** (about the z-axis). In this case we find it useful to use the more descriptive notation $R_{z,\theta}$ instead of R_1^0 to denote the matrix. It is easy to verify that the basic rotation matrix $R_{z,\theta}$ has the properties

$$R_{z,0} = I \tag{2.3}$$
$$R_{z,\theta}R_{z,\phi} = R_{z,\theta+\phi} \tag{2.4}$$

which together imply

$$\left(R_{z,\theta}\right)^{-1} = R_{z,-\theta} \tag{2.5}$$

Similarly the basic rotation matrices representing rotations about the x and y-axes are given as (Problem 2-8)

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{2.6}$$

$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \tag{2.7}$$

which also satisfy properties analogous to Equations (2.3)-(2.5).

Example 2.2

[3]See also Appendix B.

Consider the frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$ shown in Figure 2.4. Projecting the unit vectors x_1, y_1, z_1 onto x_0, y_0, z_0 gives the coordinates of x_1, y_1, z_1 in the $o_0x_0y_0z_0$ frame. We see that the coordinates of x_1 are $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)^T$, the coordinates of y_1 are $\left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)^T$ and the coordinates of z_1 are $(0,1,0)^T$. The rotation matrix R_1^0 specifying the orientation of $o_1x_1y_1z_1$ relative to $o_0x_0y_0z_0$ has these as its column vectors, that is,

$$R_1^0 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix} \tag{2.8}$$

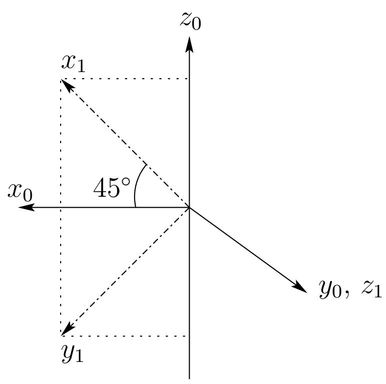

Fig. 2.4 Defining the relative orientation of two frames.

◇

2.3 ROTATIONAL TRANSFORMATIONS

Figure 2.5 shows a rigid object S to which a coordinate frame $o_1x_1y_1z_1$ is attached. Given the coordinates p^1 of the point p (i.e., given the coordinates of p with respect to the frame $o_1x_1y_1z_1$), we wish to determine the coordinates of p relative to a fixed reference frame $o_0x_0y_0z_0$. The coordinates $p^1 = (u, v, w)^T$ satisfy the equation

$$p = ux_1 + vy_1 + wz_1$$

In a similar way, we can obtain an expression for the coordinates p^0 by projecting the point p onto the coordinate axes of the frame $o_0x_0y_0z_0$, giving

$$p^0 = \begin{bmatrix} p \cdot x_0 \\ p \cdot y_0 \\ p \cdot z_0 \end{bmatrix}$$