Data 100 & 200A Principles and Techniques of Data Science

Spring 2019

Midterm 2

INSTRUCTIONS

- You have 70 minutes to complete the exam.
- The exam is closed book, closed notes, closed computer, closed calculator, except for two 8.5" × 11" crib sheets of your own creation.
- Mark your answers **on the exam itself**. We will *not* grade answers written on scratch paper.

Last name	
First name	
Student ID number	
CalCentral email (`_@berkeley.edu`)	
Exam room	
Name of the person to your left	
Name of the person to your right	
All the work on this exam is my own. **(please sign)**	

Terminology and Notation Reference:

$\exp(x)$	e^x
$\log(x)$	$\log_e x$
Linear regression model	$E[Y\|X] = X^T\beta$
Logistic (or sigmoid) function	$\sigma(t) = \frac{1}{1+\exp(-t)}$
Logistic regression model	$P(Y=1\|X) = \sigma(X^T\beta)$
Squared error loss	$L(y,\theta) = (y-\theta)^2$
Absolute error loss	$L(y,\theta) = \|y-\theta\|$
Cross-entropy loss	$L(y,\theta) = -y\log\theta - (1-y)\log(1-\theta)$
Bias	$\text{Bias}[\hat\theta,\theta] = E[\hat\theta] - \theta$
Variance	$\text{Var}[\hat\theta] = E[(\hat\theta - E[\hat\theta])^2]$
Mean squared error	$\text{MSE}[\hat\theta,\theta] = E[(\hat\theta-\theta)^2]$

1. (8 points) Feature Engineering

For each dataset depicted below in a scatterplot, fill in the squares next to **all** of the letters for the vector-valued functions f that would make it possible to choose a column vector β such that $y_i = f(x_i)^T \beta$ for all (x_i, y_i) pairs in the dataset. The input to each f is a scalar x shown on the horizontal axis, and the corresponding y value is shown on the vertical axis.

(A) $f(x) = [1 \quad x]^T$

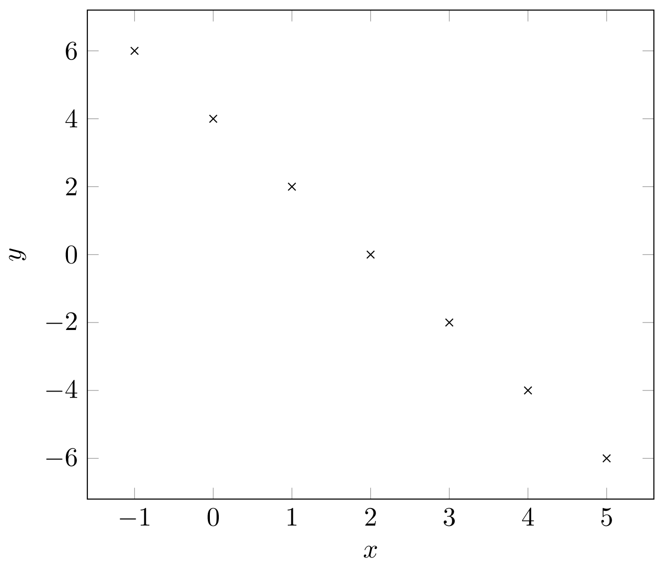
(B) $f(x) = [x \quad 2x]^T$

(C) $f(x) = [1 \quad x \quad x^2]^T$

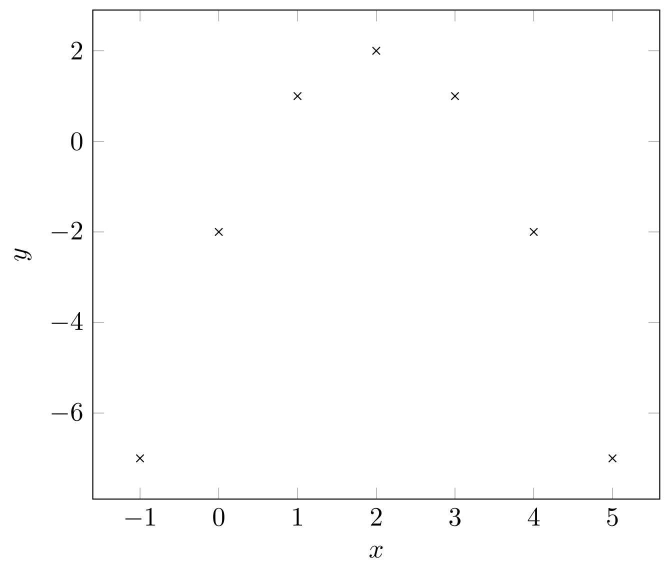
(D) $f(x) = [1 \quad |x|]^T$

(E) None of the above

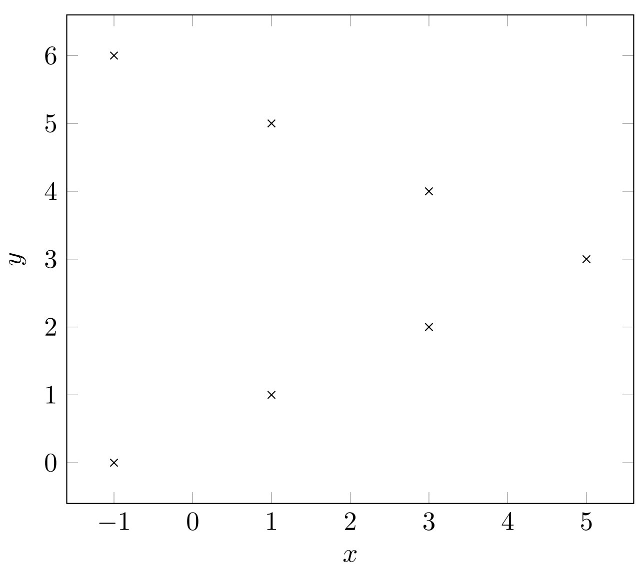
(i) (2 pt) ☐ A ☐ B ☐ C ☐ D ☐ E



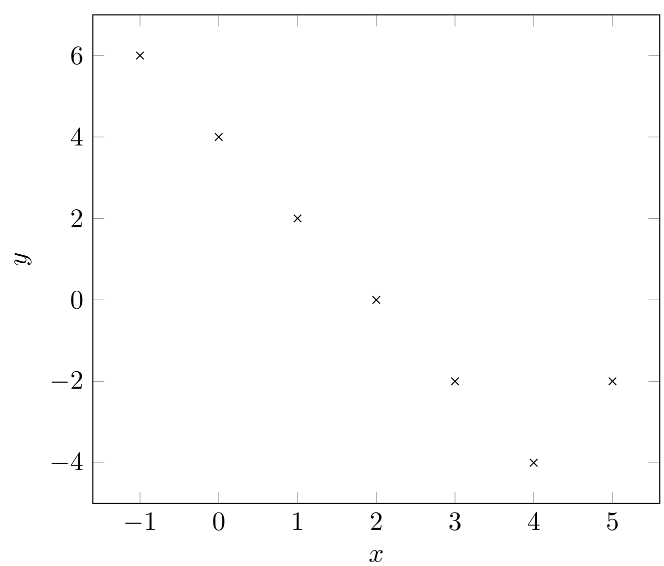
(ii) (2 pt) ☐ A ☐ B ☐ C ☐ D ☐ E



(iii) (2 pt) ☐ A ☐ B ☐ C ☐ D ☐ E



(iv) (2 pt) ☐ A ☐ B ☐ C ☐ D ☐ E



Cross entropy loss

2. (6 points) Estimation

A learning set $(x_1, y_1), \ldots, (x_{10}, y_{10})$ is sampled from a population where X and Y are both binary.

The learning set data are summarized by the following table of row counts:

x	y	Count
0	0	2
0	1	3
1	0	1
1	1	4

$L(y, \theta) = -y \log \theta - (1-y)\log(1-\theta)$

(a) (4 pt) You decide to fit a constant model $P(Y = 1|X = 0) = P(Y = 1|X = 1) = \alpha$ using the cross-entropy loss function and no regularization. What is the formula for the empirical risk on this learning set for this model and loss function? What estimate of the model parameter α minimizes empirical risk? **You must show your work for finding the estimate $\hat{\alpha}$ to receive full credit.**

Recall: Since Y is binary, $P(Y = 0|X) + P(Y = 1|X) = 1$ for any X.

$2 \times (0,0): \; L(y,\alpha) = -\log(1-\alpha)$ $\quad 1 \times (1,0): -\log(1-\alpha)$

Empirical Risk: $3 \times (0,1) \; L(y,\alpha) = -\log \alpha$ $\quad 4 \times (1,1): -\log \alpha$

Estimate $\hat{\alpha}$ (show your work):

$$R(\alpha) = \frac{-1}{10}\left(7 \log \alpha + 3 \log(1-\alpha)\right)$$

$$\frac{\partial R}{\partial \alpha} = -\frac{1}{10}\left(\frac{7}{\alpha} + \frac{3}{1-\alpha}(-1)\right) = 0$$

$$\frac{7}{\alpha} = \frac{3}{1-\alpha}$$

$$7 - 7\alpha = 3\alpha \quad 10\alpha = 7$$

$$\boxed{\hat{\alpha} = \frac{7}{10}}$$

(b) (2 pt) The true population probability $P(Y = 0|X = 0)$ is $\frac{1}{3}$. Provide an expression in terms of $\hat{\alpha}$ for the **bias** of the estimator of $P(Y = 0|X = 0)$ described in part (a) for the constant model. **You may use $E[\ldots]$ in your answer to denote an expectation under the data generating distribution of the learning set, but do not write $P(\ldots)$ in your answer.**

$\hat{P}(y=0|X=0) = 1 - \alpha$

Bias$[\hat{P}(Y = 0|X = 0), P(Y = 0|X = 0)] =$ $E[\hat{P}(y=0|x=0)] - P(y=0|X=0)$

$= E[1-\alpha] - \frac{1}{3} = \frac{2}{3} - E[\alpha]$

3. (6 points) Linear Regression

A learning set of size four is sampled from a population where X and Y are both quantitative:

$$(x_1, y_1) = (2.5, 3)$$
$$(x_2, y_2) = (2, 5)$$
$$(x_3, y_3) = (1, 3)$$
$$(x_4, y_4) = (3, 5).$$

You fit a linear regression model $E[Y|X] = \beta_0 + X\beta_1$, where β_0 and β_1 are scalar parameters, by ridge regression, minimizing the following objective function:

$$\|y - X\beta\|_2^2$$

$$\frac{1}{4}\left(\sum_{i=1}^{4}(y_i - (\beta_0 + x_i\beta_1))^2 + 4\,\frac{\beta_0^2 + \beta_1^2}{3}.\right)$$

(a) (4 pt) Fill in all blanks below to compute the parameter estimates that minimize this regularized empirical risk. (You do not need to compute their values; just fill in the matrices appropriately.)

$$\frac{1}{4}\sum_{i=1}^{4}(y_i - \hat{y}_i)^2 + \frac{1}{3}\|\beta\|_2^2$$
$$= \frac{1}{4}\left(\sum (y_i - \hat{y}_i)^2 + \frac{4}{3}\|\beta\|_2^2\right)$$

$$X_n^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2.5 & 2 & 1 & 3 \end{bmatrix}$$

$$Y_n^T = \begin{bmatrix} 3 & 5 & 3 & 5 \end{bmatrix}$$

$$\begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = (X_n^T X_n + \begin{bmatrix} 4/3 & 0 \\ 0 & 4/3 \end{bmatrix})^{-1} X_n^T Y_n.$$

(b) (2 pt) Without computing values for $\hat{\beta}_0$ and $\hat{\beta}_1$, write an expression for the squared error loss of the learning set observation (x_4, y_4) in terms of $\hat{\beta}_0$ and $\hat{\beta}_1$ and any relevant numbers. **Your solution should not contain any of $\hat{y}_4$, x_4, or y_4, but instead just numbers and $\hat{\beta}_0$ and $\hat{\beta}_1$.**

$L(y_4, \hat{y}_4) = (5 - (\beta_0 + 3\beta_1))^2$

$(x_4, y_4) = (3, 5)$

4. (8 points) Model Selection

(a) **(2 pt)** You have a quantitative outcome Y and two quantitative covariates (X_1, X_2). You want to fit a linear regression model for the conditional expected value $E[Y|X]$ of the outcome given the covariates, including an intercept. Bubble in the **minimum** dimension of the parameter vector β needed to express this linear regression model?

◯ 1 ◯ 2 ◯ 3 ◯ 4 ◯ 5 ◯ 6 ◯ 7 ◯ None of these

(b) **(2 pt)** You have a quantitative outcome Y and two qualitative covariates (X_1, X_2). $X_1 \in \{a, b, c, d\}$, $X_2 \in \{e, f, g\}$, and there is no ordering to the values for either variable. You want to fit a linear regression model for the conditional expected value $E[Y|X]$ of the outcome given the covariates, including an intercept. Bubble in the **minimum** dimension of the parameter vector β needed to express this linear regression model?

◯ 2 ◯ 3 ◯ 4 ◯ 5 ◯ 6 ◯ 7 ◯ 8 ◯ 9 ◯ 10 ◯ 11 ◯ 12 ◯ 13

(c) **(2 pt)** Bubble all true statements: In ridge regression, when the assumptions of the linear model are satisfied, the larger the shrinkage/penalty parameter,

- ☐ the larger the magnitude of the bias of the estimator of the regression coefficients β.
- ☐ the smaller the magnitude of the bias of the estimator of the regression coefficients β.
- ☐ the larger the variance of the estimator of the regression coefficients β.
- ☐ the smaller variance of the estimator of the regression coefficients β.
- ☐ the smaller the true mean squared error of the estimator of the regression coefficients β.

(d) **(2 pt)** Bubble all true statements: A good approach for selecting the shrinkage/penalty parameter in LASSO is to:

- ☐ minimize the learning set risk for the squared error (L_2) loss function.
- ☐ minimize the learning set risk for the absolute error (L_1) loss function.
- ☐ minimize the cross-validated regularized risk for the squared error (L_2) loss function.
- ☐ minimize the cross-validated risk for the squared error (L_2) loss function.
- ☐ minimize the variance of the estimator of the regression coefficients.

$P(y=1|X)=\frac{1}{1+e^{-X^T\beta}}$ $\quad$ $P(y=0|X)=\frac{e^{-X^T\beta}}{1+e^{-X^T\beta}}$

$\Rightarrow \frac{P(y=1|X)}{P(y=0|X)} = \frac{\frac{1}{\cancel{1+e^{-X^T\beta}}}}{\frac{e^{-X^T\beta}}{\cancel{1+e^{-X^T\beta}}}} = \frac{1}{e^{-X^T\beta}} = e^{X^T\beta}$

5. (12 points) Logistic Regression

(a) **(2 pt)** Bubble the expression that describes the odds ratio $\frac{P(Y=1|X)}{P(Y=0|X)}$ of a logistic regression model.
Recall: $P(Y=0|X)+P(Y=1|X)=1$ for any X.

○ $X^T\beta$ ○ $-X^T\beta$ ● $\exp(X^T\beta)$ ○ $\sigma(X^T\beta)$ ○ None of these

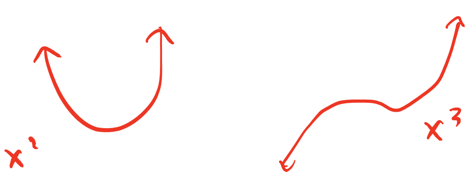
(b) **(2 pt)** Bubble the expression that describes $P(Y=0|X)$ for a logistic regression model.

● $\sigma(-X^T\beta)$ ○ $1-\log(1+\exp(X^T\beta))$ ○ $1+\log(1+\exp(-X^T\beta))$ ○ None of these

(c) **(2 pt)** Bubble **all** of the following that are typical effects of adding an L_1 regularization penalty to the loss function when fitting a logistic regression model with parameter vector β.

- ☐ The magnitude of the elements of the estimator of β are increased.
- ☑ The magnitude of the elements of the estimator of β are decreased.
- ☐ All elements of the estimator of β are non-negative.
- ☑ Some elements of the estimator of β are zero.
- ☐ None of the above.

(d) **(3 pt)** What would be the primary disadvantage of a regularization term of the form $\sum_{j=1}^{J}\beta_j^3$ rather than the more typical ridge penalty $\sum_{j=1}^{J}\beta_j^2$ for logistic regression? Answer in one sentence.



$\Sigma \beta_j^3$ has no global min,
so $\beta_i \to \infty$

(e) **(3 pt)** For a logistic regression model $P(Y=1|X)=\sigma(-2-3X)$, where X is a scalar random variable, what values of x would give $P(Y=0|X=x)\geq\frac{3}{4}$? **You must show your work for full credit.**

$P(y=0|x) = 1-P(y=1|x) = 1-\sigma(-2-3X)$

$1-\sigma(-2-3x) \geq \frac{3}{4}$

$\sigma(-2-3X) \leq \frac{1}{4}$

$\frac{1}{1+e^{-(-2-3X)}} = \frac{1}{1+e^{2+3X}} \leq \frac{1}{4}$

$1+e^{2+3X} \geq 4$

$$e^{2+3x} \geq 3$$

$$2+3x \geq \log 3$$

$$x \geq \frac{\log 3 - 2}{3}$$