Lecture 19

Recursion

Announcement

- Lab this week
- Midterm1 grade adjustment not done yet
- Concerned with your grades?

Recursion

- **Recursive Definition**:

  A definition that is defined in terms of itself

- **Recursive Function**:

  A function that calls itself (directly or indirectly)

A Mathematical Example: Factorial

- Non-recursive definition:

$$n! = n \times n\text{-}1 \times \ldots \times 2 \times 1$$
$$= n\,(n\text{-}1 \times \ldots \times 2 \times 1)$$

- Recursive definition:

$n! = n\,(n-1)!$ for $n > 0$ **Recursive case**

$0! = 1$ **Base case**

What happens if there is no base case?

Factorial as a Recursive Function

```
def factorial(n):
    """Returns: factorial of n.
    Pre: n ≥0 an int"""
    if n == 0:
        return 1

    return n*factorial(n-1)
```

- n! = n (n-1)!
- 0! = 1

Base case(s)

Recursive case

What happens if there is no base case?

Example: Fibonnaci Sequence

- Sequence of numbers: 1, 1, 2, 3, 5, 8, 13, ...

  a_0 a_1 a_2 a_3 a_4 a_5 a_6

  - Get the next number by adding previous two
  - What is a_8?

A: $a_8 = 21$

B: $a_8 = 29$

C: $a_8 = 34$

D: None of these.

Example: Fibonnaci Sequence

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A: $a_8 = 21$

B: $a_8 = 29$

C: $a_8 = 34$ **correct**

D: None of these.

Example: Fibonnaci Sequence

- Sequence of numbers: 1, 1, 2, 3, 5, 8, 13, ...

  a_0 a_1 a_2 a_3 a_4 a_5 a_6

  - Get the next number by adding previous two
  - What is a_8?
- Recursive definition:
  - $a_n = a_{n\text{-}1} + a_{n\text{-}2}$ **Recursive Case**
  - $a_0 = 1$ **Base Case**
  - $a_1 = 1$ **(another) Base Case**

Why did we need two base cases this time?

Fibonacci as a Recursive Function

```
def fibonacci(n):
    """Returns: Fibonacci no. a_n
     Precondition: n ≥ 0 an int"""
    if n <= 1:
        return 1

    return (fibonacci(n-1)+
                fibonacci(n-2))
```

Base case(s)

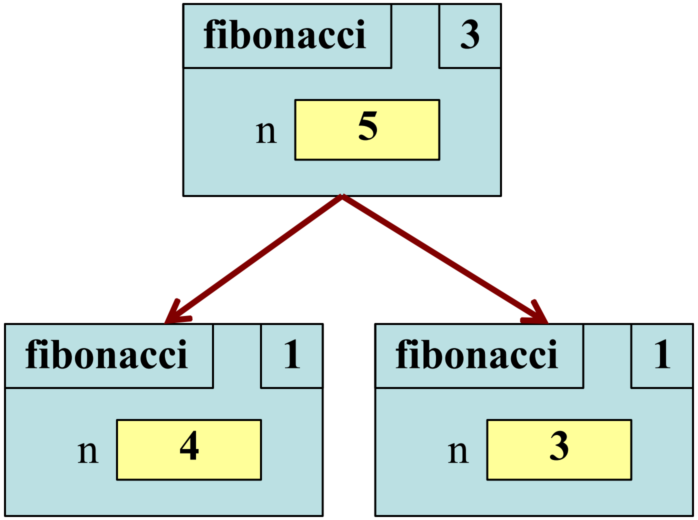
Recursive case

Note difference with base case conditional.

Fibonacci as a Recursive Function

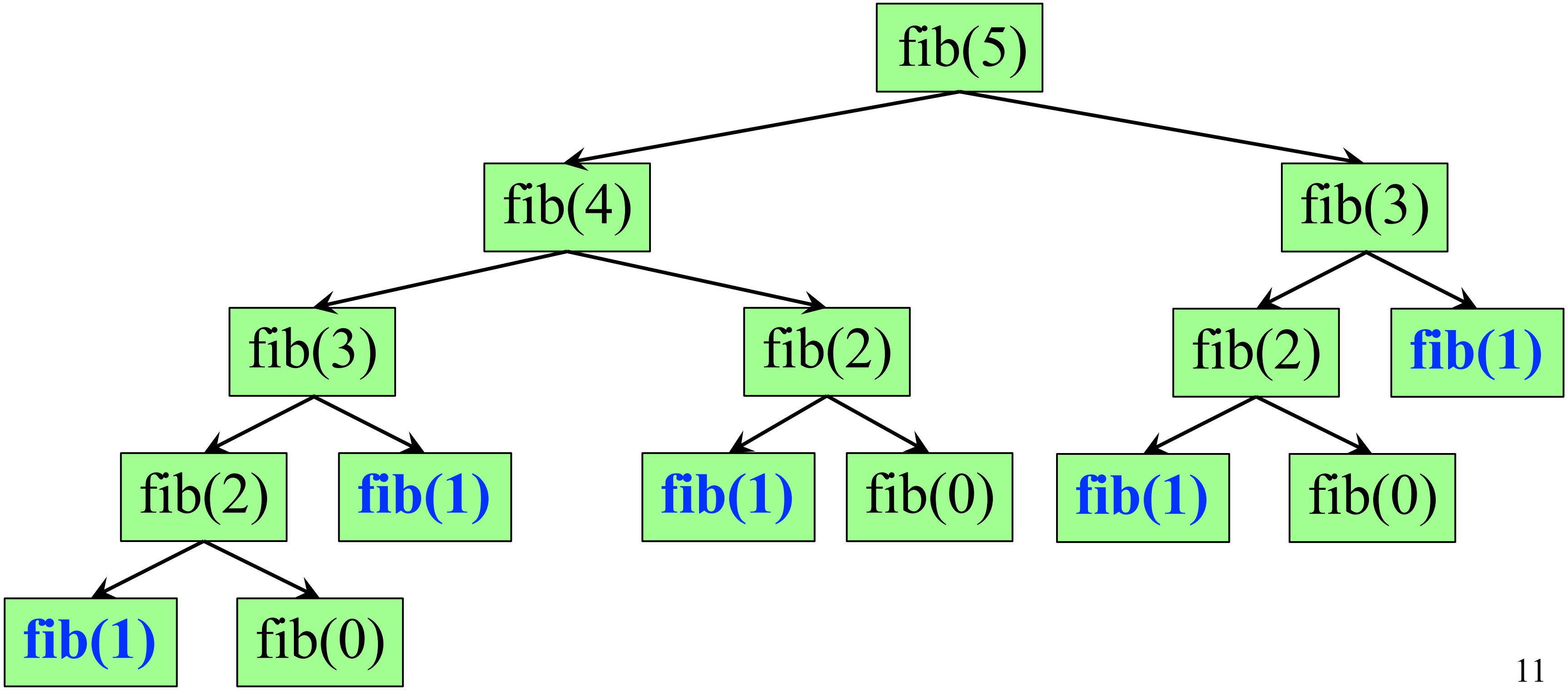
```
def fibonacci(n):
    """Returns: Fibonacci no. a_n
    Precondition: n ≥0 an int""" if
    n <= 1:
        return 1

    return (fibonacci(n-1)+
                fibonacci(n-2))
```

- Function that calls itself
  - Each call is new frame
  - Frames require memory
  - ∞ calls = ∞ memory



Fibonacci: # of Frames vs. # of Calls

- Fibonacci is very inefficient.
  - fib(*n*) has a stack that is always ≤ *n*
  - But fib(*n*) makes a lot of redundant calls



Fibonacci: # of Frames vs. # of Calls

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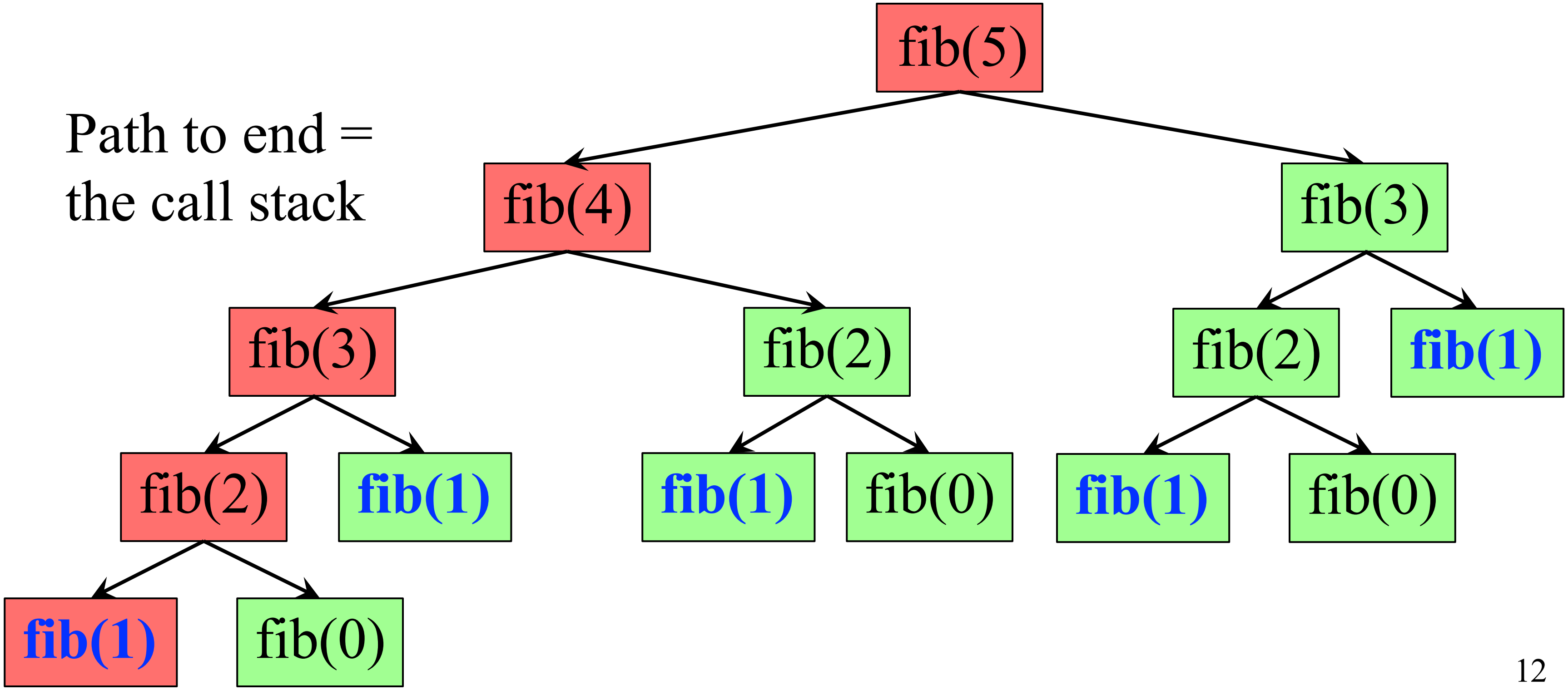
Path to end = the call stack



Recursion vs Iteration

- **Recursion** is ***provably equivalent*** to **iteration**
  - Iteration includes **for-loop** and **while-loop** (later)
  - Anything can do in one, can do in the other
- But some things are easier with recursion
  - And some things are easier with iteration
- Will **not** teach you when to choose recursion
  - This is a topic for more advanced classes
- We just want you to ***understand the technique***

Recursion is best for Divide and Conquer

Goal: Solve problem P on a piece of data

data

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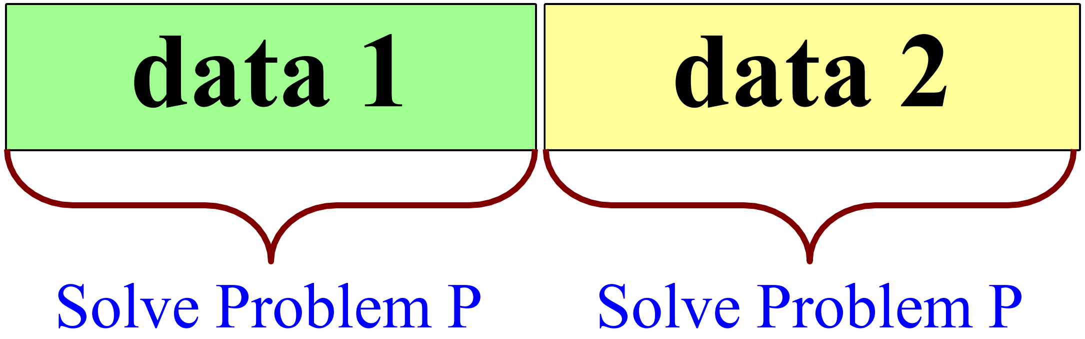
data

Idea: Split data into two parts and solve problem



data 1

data 2

Solve Problem P

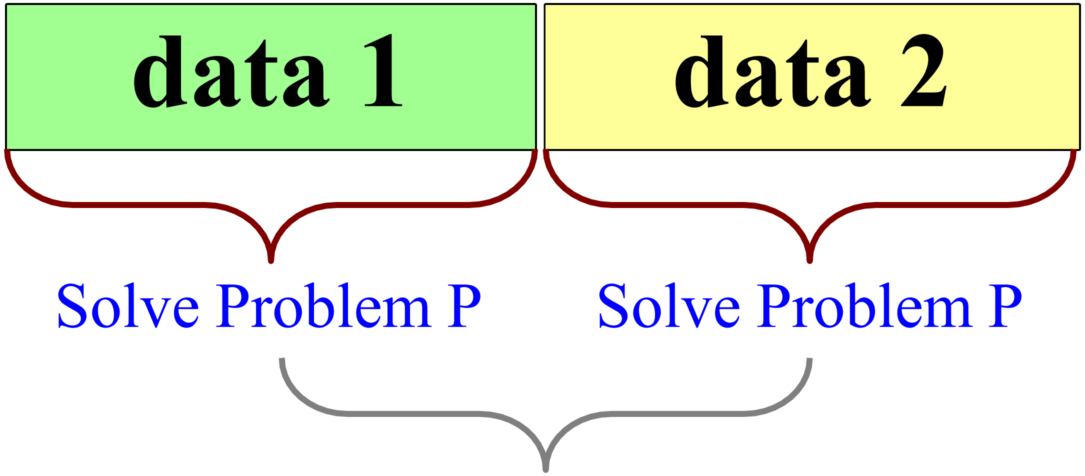
Solve Problem P

Recursion is best for Divide and Conquer

Goal: Solve problem P on a piece of data

data

Idea: Split data into two parts and solve problem



data 1

data 2

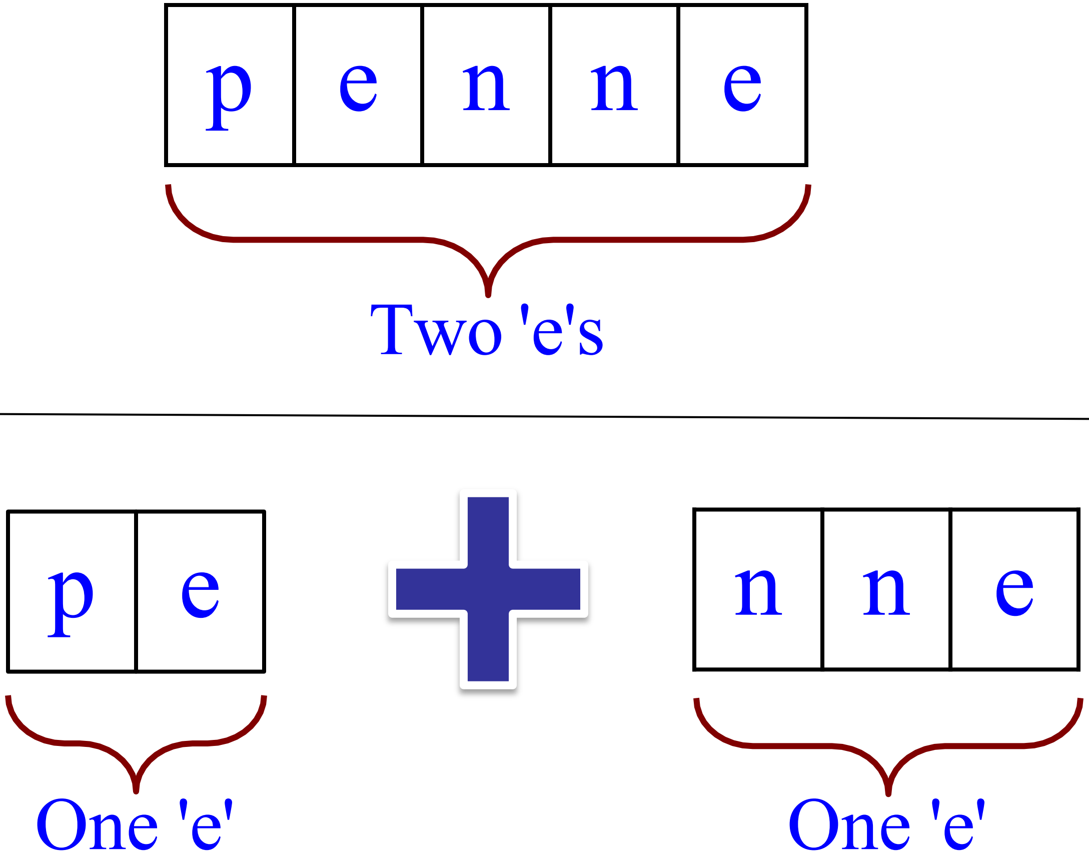
Solve Problem P

Solve Problem P

Combine Answer!

Divide and Conquer Example

Count the number of 'e's in a string:



p	e	n	n	e

Two 'e's

p	e

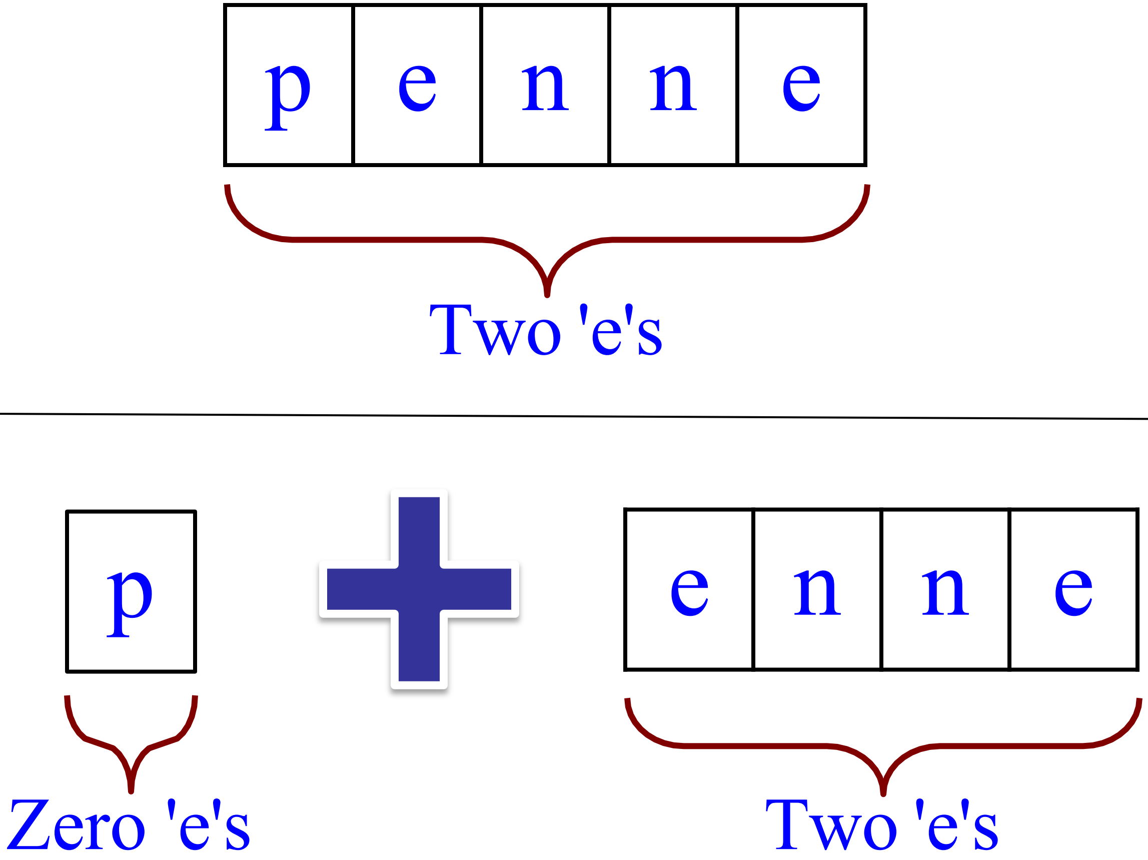
One 'e'

n	n	e

One 'e'

Divide and Conquer Example

Count the number of 'e's in a string:



p	e	n	n	e

Two 'e's

p

+

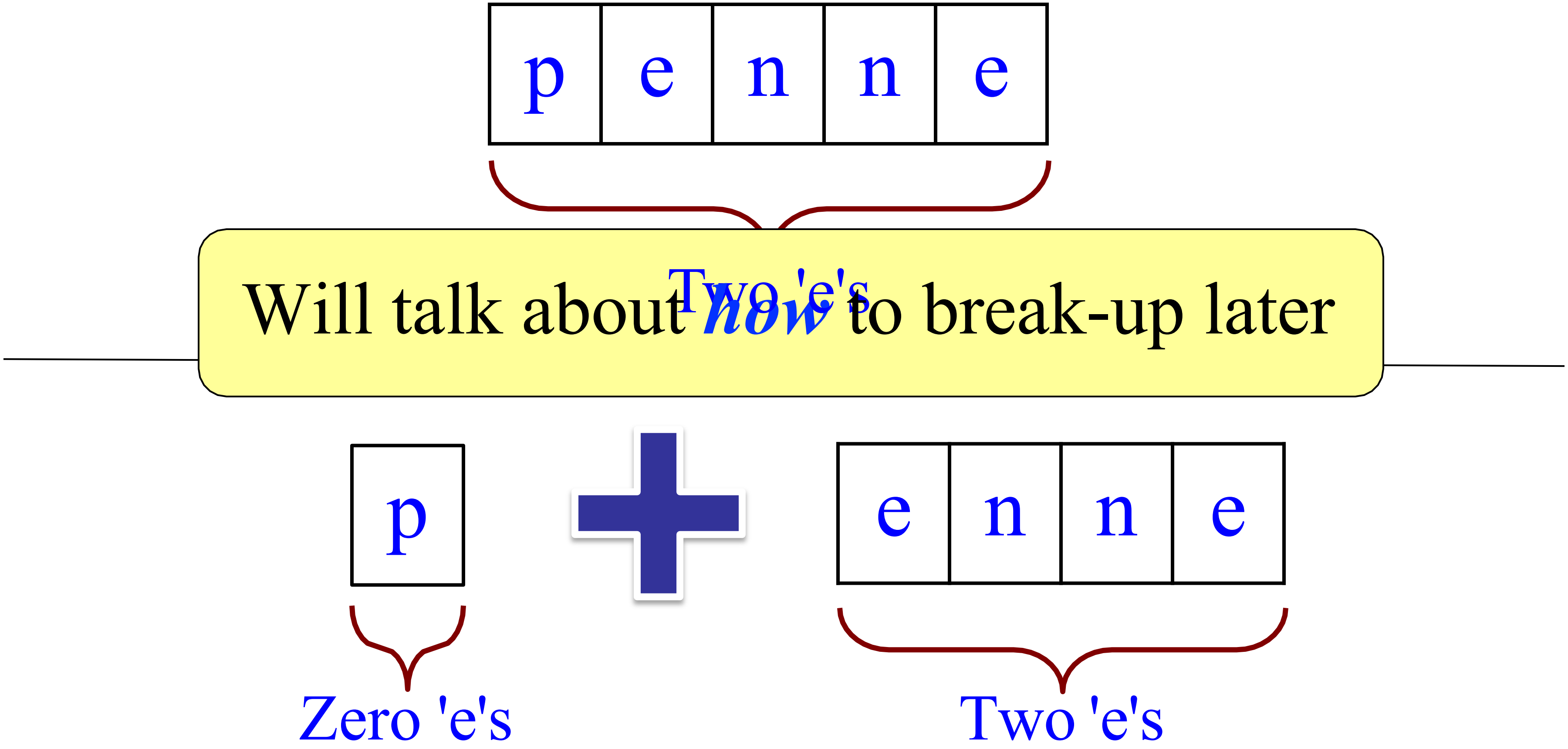
e	n	n	e

Zero 'e's

Two 'e's

Divide and Conquer Example

Count the number of 'e's in a string:



Will talk about *how* to break-up later

Three Steps for Divide and Conquer

1. Decide what to do on “small” data
   - Some data cannot be broken up
   - Have to compute this answer directly
2. Decide how to break up your data
   - Both “halves” should be smaller than whole
   - Often no wrong way to do this (next lecture)
3. Decide how to combine your answers
   - Assume the smaller answers are correct
   - Combining them should give bigger answer

Divide and Conquer Example

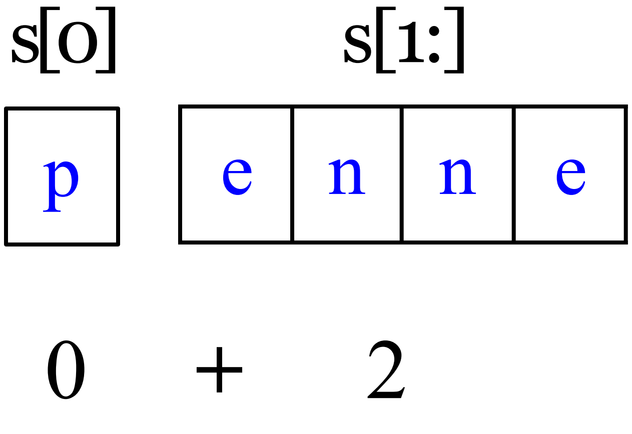
```
def num_es(s):
    """Returns: # of 'e's in s"""
    # 1. Handle small data
    if s == '':
        return 0
    elif len(s) == 1:
        return 1 if s[0] == 'e' else 0

    # 2. Break into two parts
    left = num_es(s[0])
    right = num_es(s[1:])

    # 3. Combine the result
    return left+right
```

“Short-cut” for

```
if s[0] == 'e’:
    return 1
else:
    return 0
```


s[0] s[1:]

p		e	n	n	e

0 + 2

Divide and Conquer Example

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def num_es(s):
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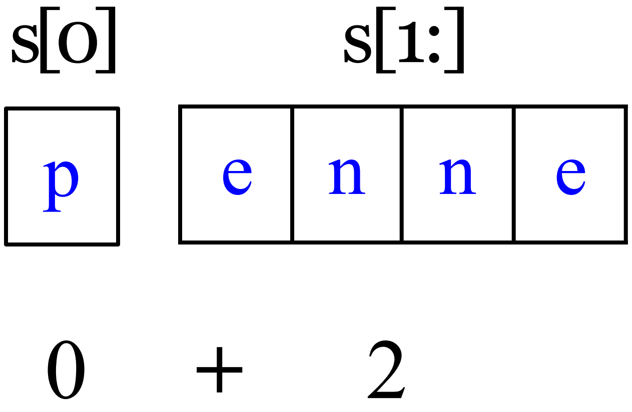

s[0]	s[1:]			
p	e	n	n	e

0 + 2

Divide and Conquer Example

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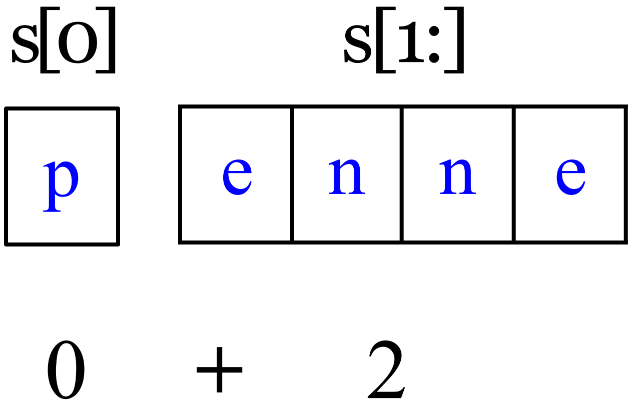

s[0] s[1:]

p		e	n	n	e

0 + 2

Divide and Conquer Example

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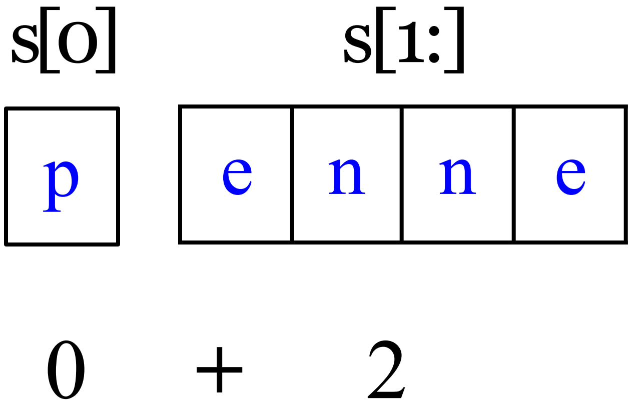

s[0] s[1:]

p		e	n	n	e

0 + 2

Divide and Conquer Example

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    # 3. Combine the result
    return left+right
```

Base Case

Recursive Case

Exercise: Remove Blanks from a String

```
def deblank(s):
    """Returns: s but with its blanks removed"""
```

1. Decide what to do on “small” data

- If it is the **empty string**, nothing to do

```
if s == '':
    return s
```

- If it is a **single character**, delete it if a blank

```
if s == ' ':     # There is a space here
    return ''  # Empty string
else:
    return s
```

Exercise: Remove Blanks from a String

```
def deblank(s):
    """Returns: s but with its blanks removed"""
```

2. Decide how to break it up

```
left = deblank(s[0])      # A string with no blanks
right = deblank(s[1:])    # A string with no blanks
```

3. Decide how to combine the answer

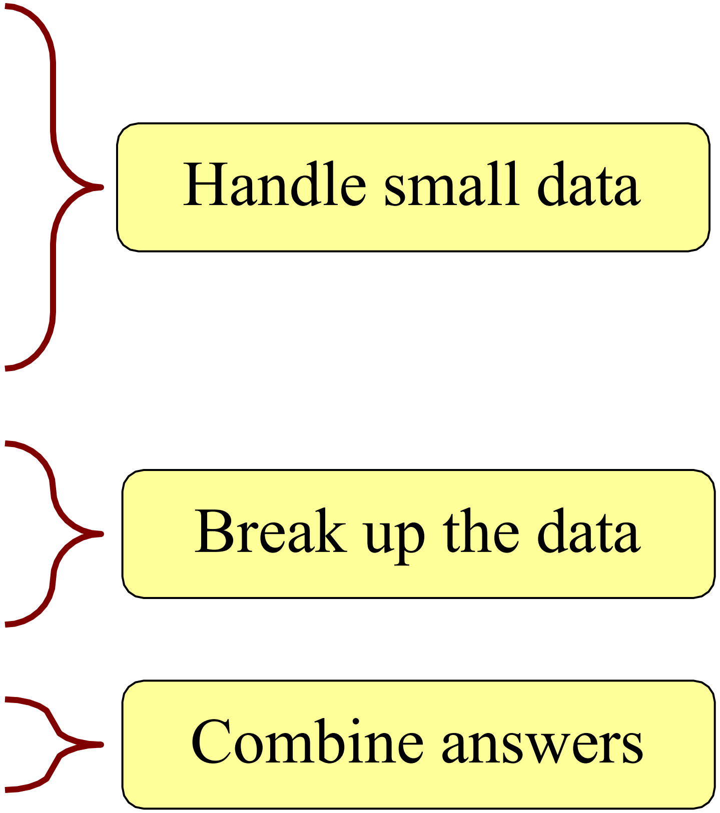
```
return left+right         # String concatenation
```

Putting it All Together

```
def deblank(s):
    """Returns: s w/o blanks"""
    if s == '':
        return s
    elif len(s) == 1:
        return '' if s[0] == ' ' else s

    left = deblank(s[0])
    right = deblank(s[1:])

    return left+right
```



Handle small data

Break up the data

Combine answers

Putting it All Together

```
def deblank(s):
    """Returns: s w/o blanks"""
    if s == '':
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    return left+right
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Base Case

Recursive Case

Minor Optimization

```
def deblank(s):
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    left = deblank(s[0])
    right = deblank(s[1:])

    return left+right
```

Needed second base case to handle s[0]

Minor Optimization

```
def deblank(s):
    """Returns: s w/o blanks"""
    if s == '':
        return s

    left = s[0]
    if s[0] == ' ':
        left = ''
    right = deblank(s[1:])

    return left+right
```

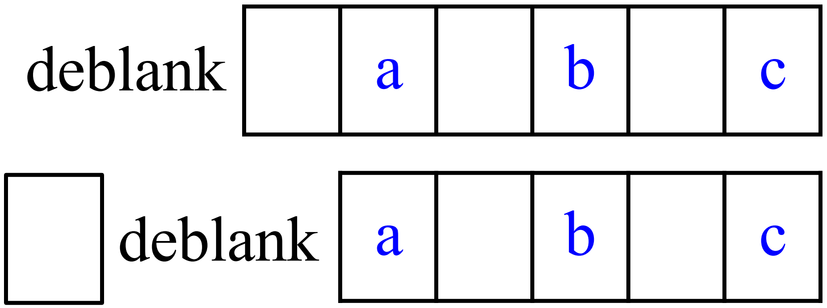
Eliminate the second base by combining

Less recursive calls

Following the Recursion

deblank		a		b		c

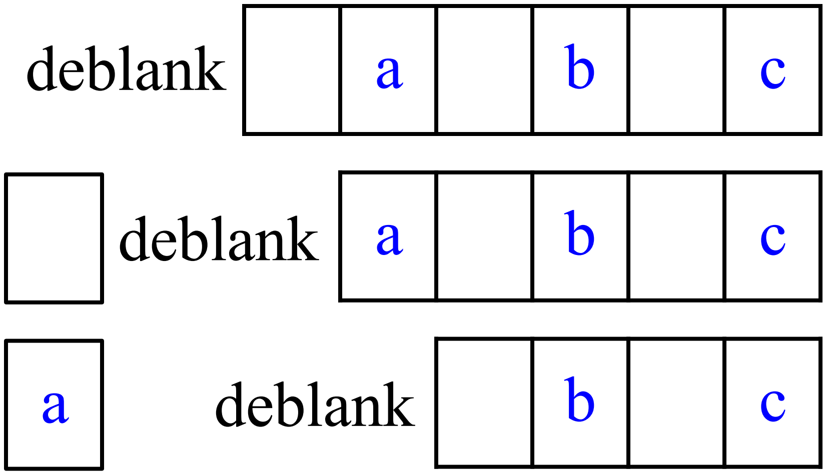
Following the Recursion



deblank a b c

deblank a b c

Following the Recursion

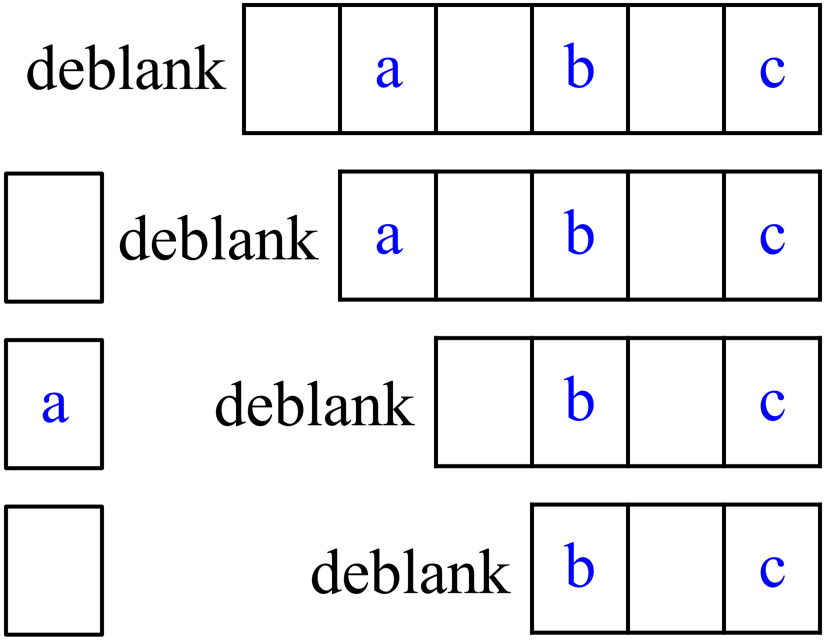


deblank a b c

deblank a b c

a deblank b c

Following the Recursion



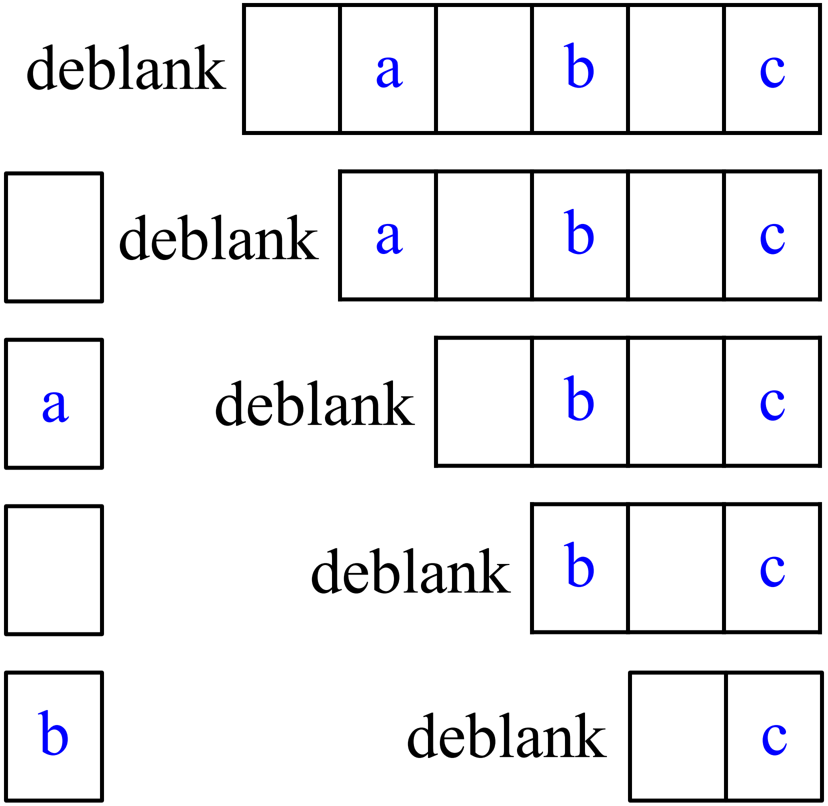
deblank a b c

deblank a b c

a deblank b c

deblank b c

Following the Recursion



deblank a b c

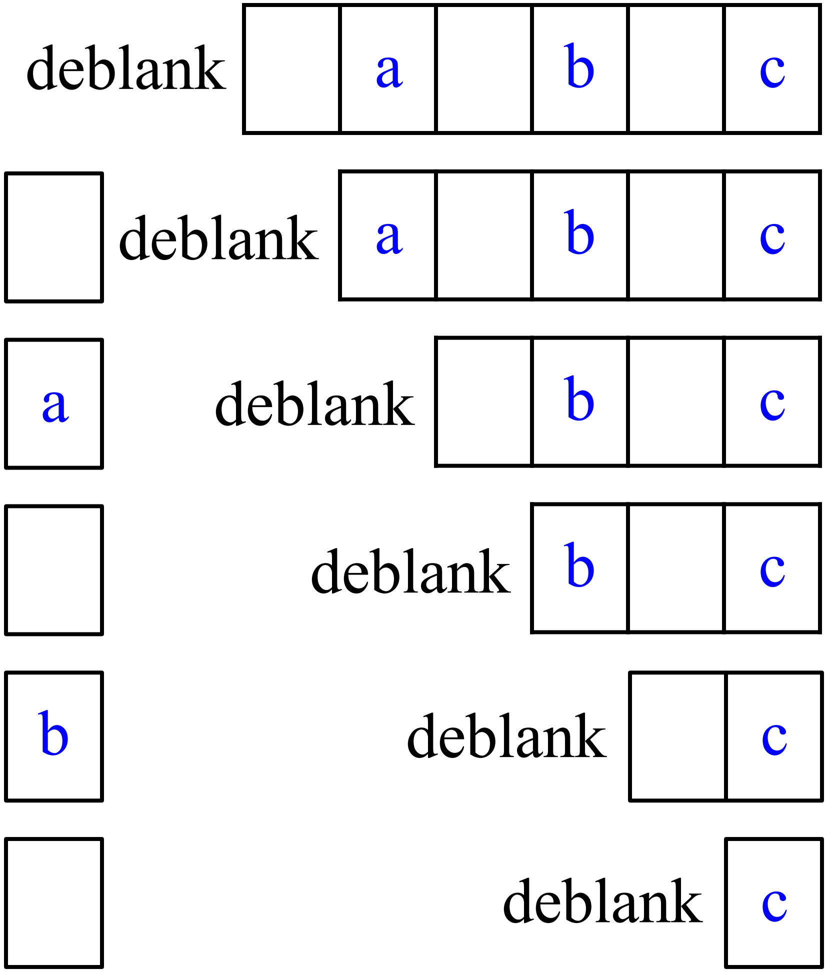
deblank a b c

a deblank b c

deblank b c

b deblank c

Following the Recursion



deblank a b c

deblank a b c

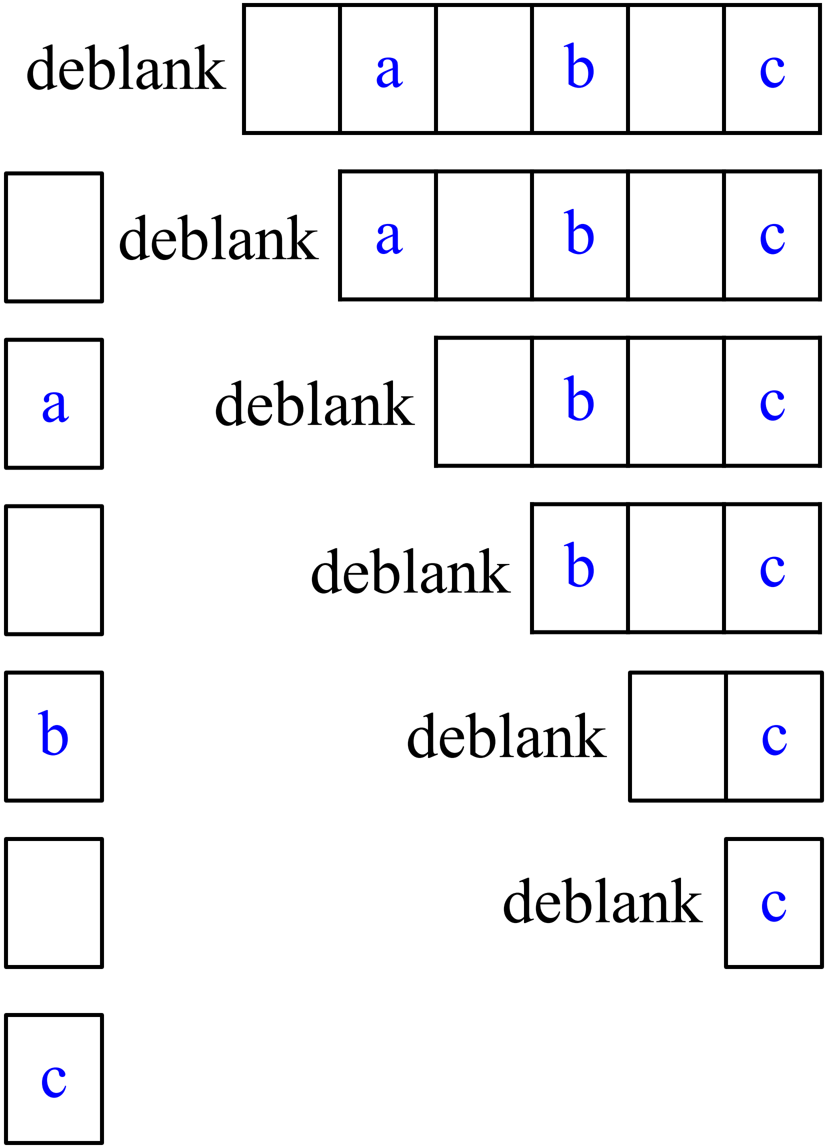
a deblank b c

deblank b c

b deblank c

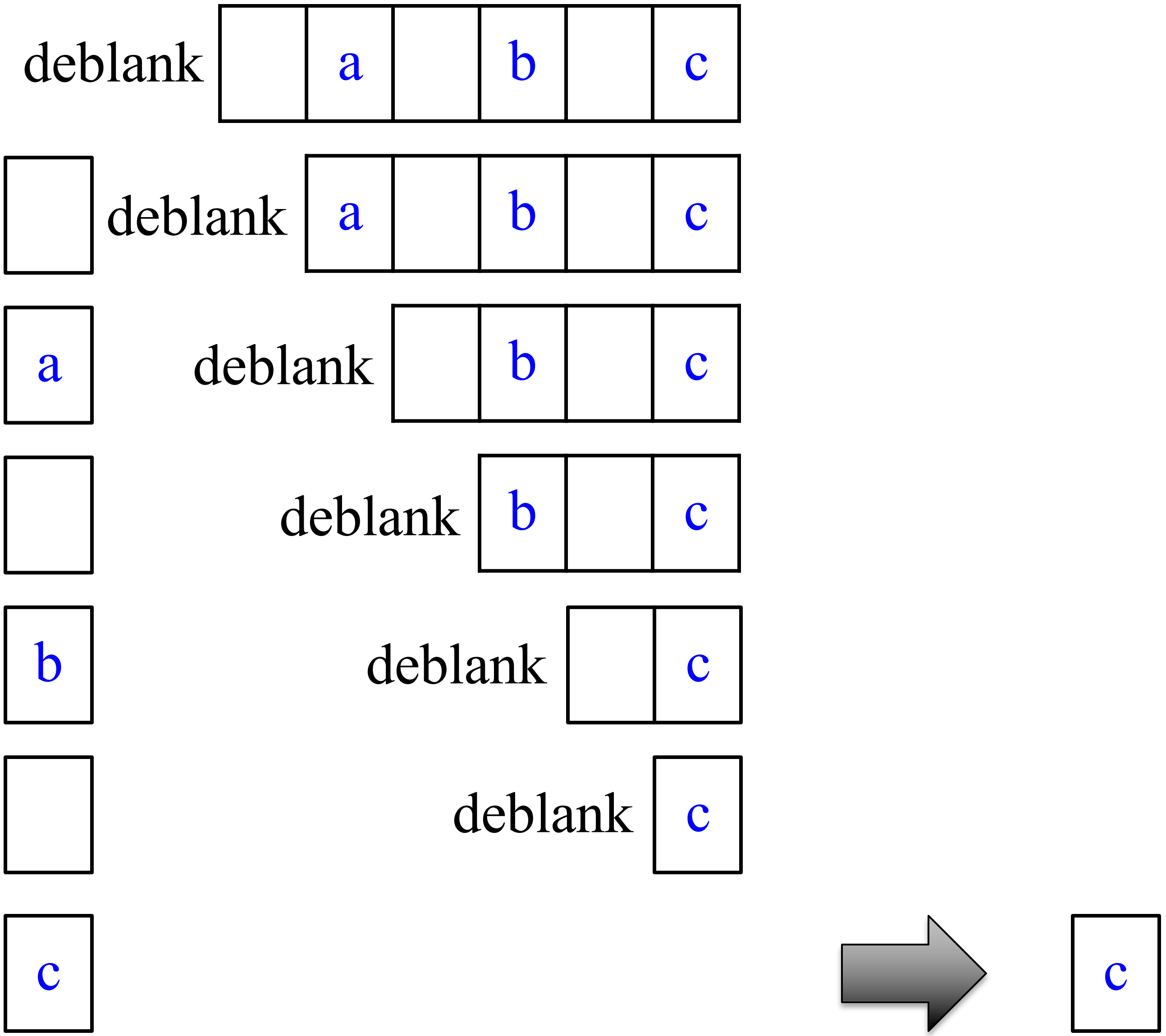
deblank c

Following the Recursion



	deblank		a		b		c
	deblank	a		b		c	
a	deblank		b		c		
	deblank	b		c			
b	deblank		c				
	deblank	c					
c							

Following the Recursion



deblank a b c

deblank a b c

a deblank b c

deblank b c

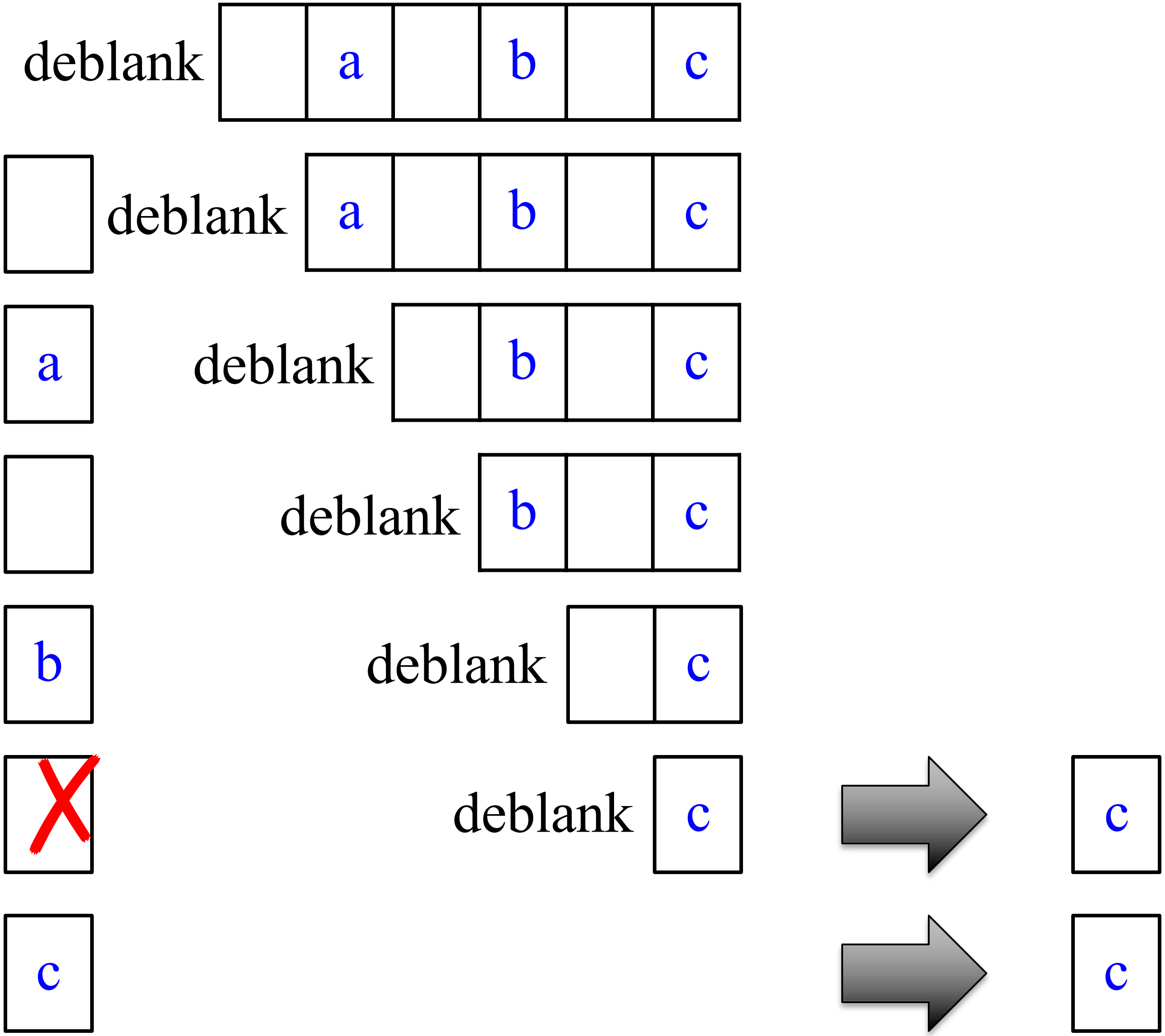
b deblank c

deblank c

c

c

Following the Recursion



deblank a b c

deblank a b c

a deblank b c

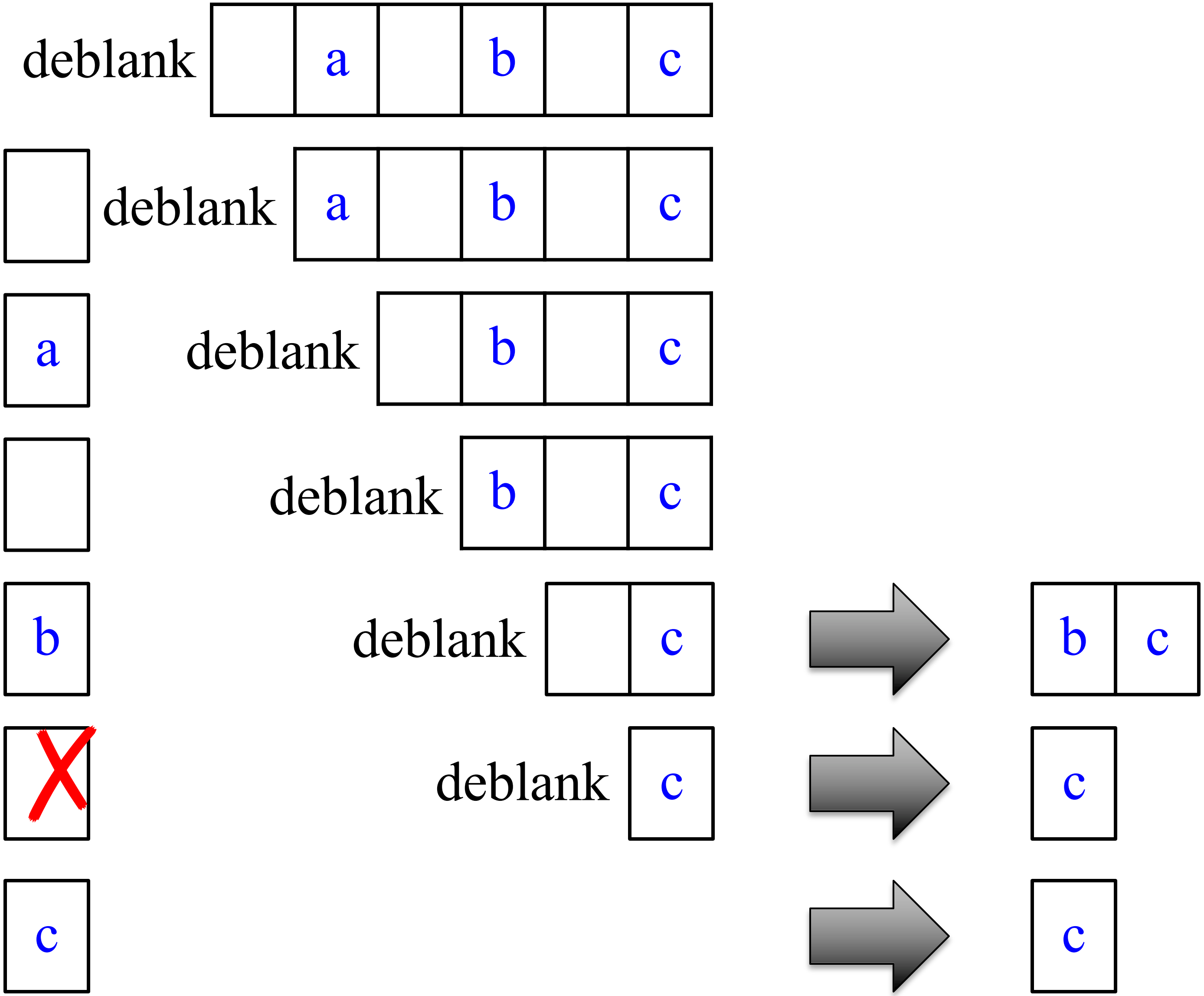
deblank b c

b deblank c

deblank c c

c c

Following the Recursion



deblank a b c

deblank a b c

a deblank b c

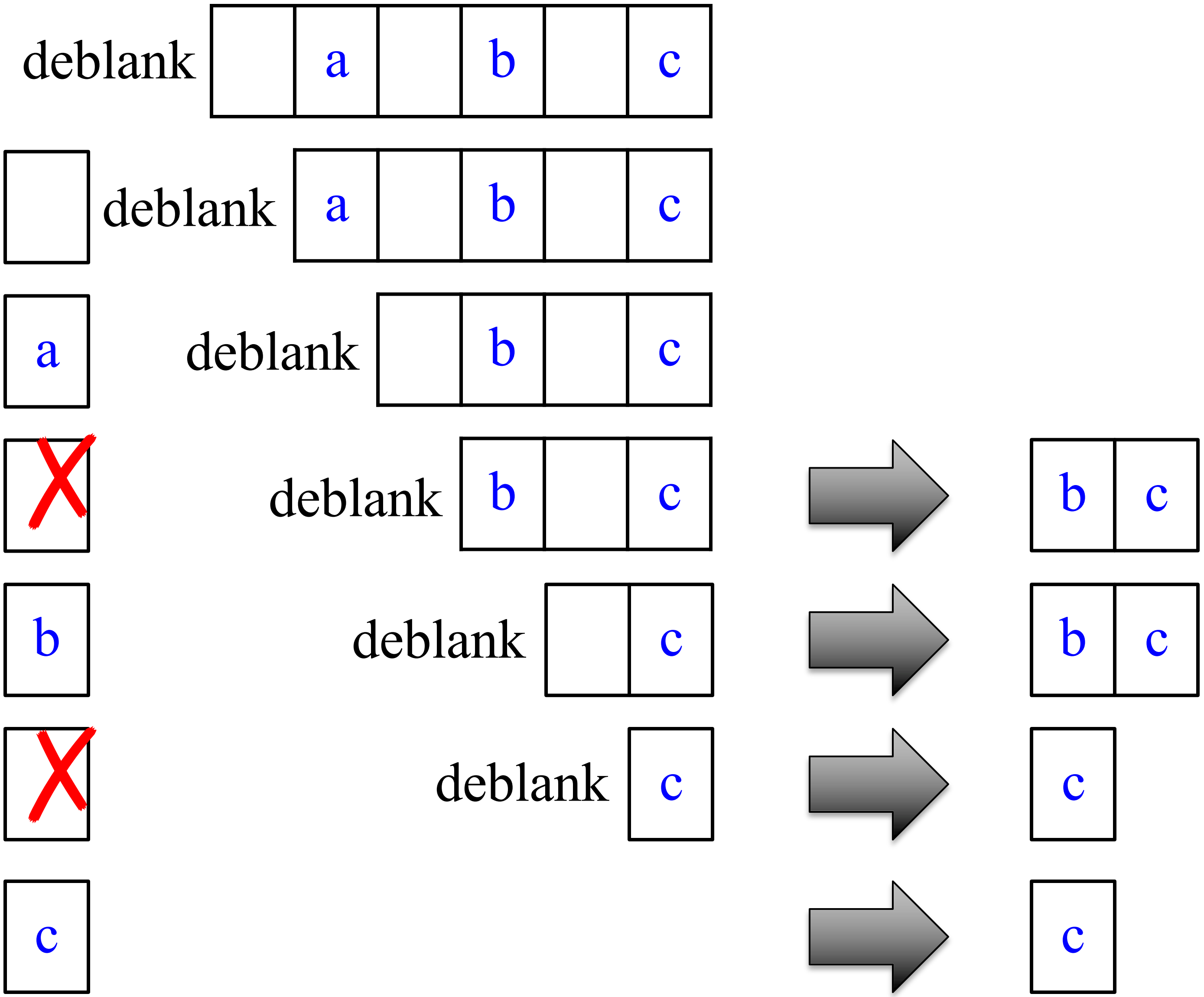
deblank b c

b deblank c → b c

deblank c → c

c → c

Following the Recursion



deblank a b c

deblank a b c

a deblank b c

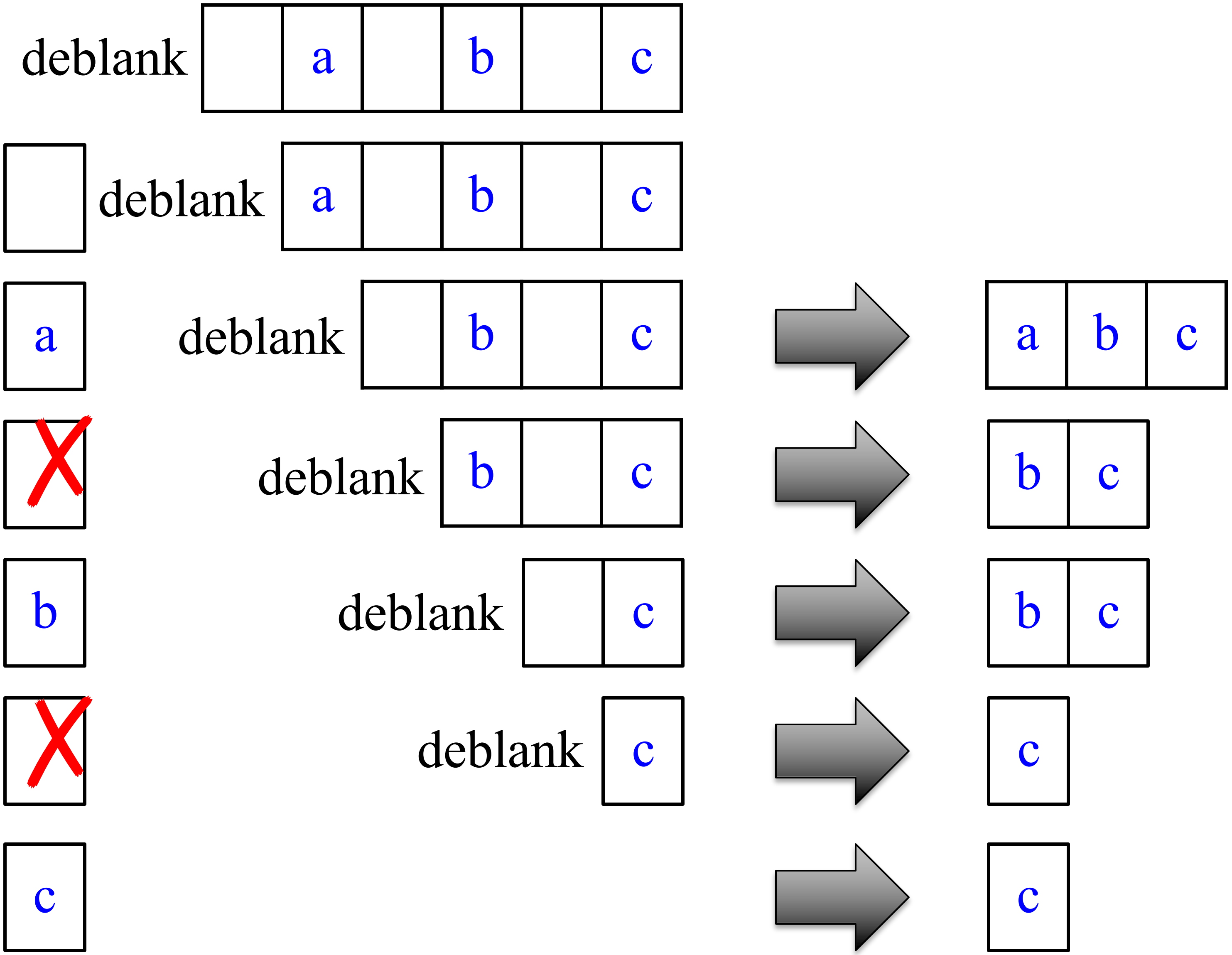
deblank b c → b c

b deblank c → b c

deblank c → c

c → c

Following the Recursion



deblank a b c

deblank a b c

a deblank b c → a b c

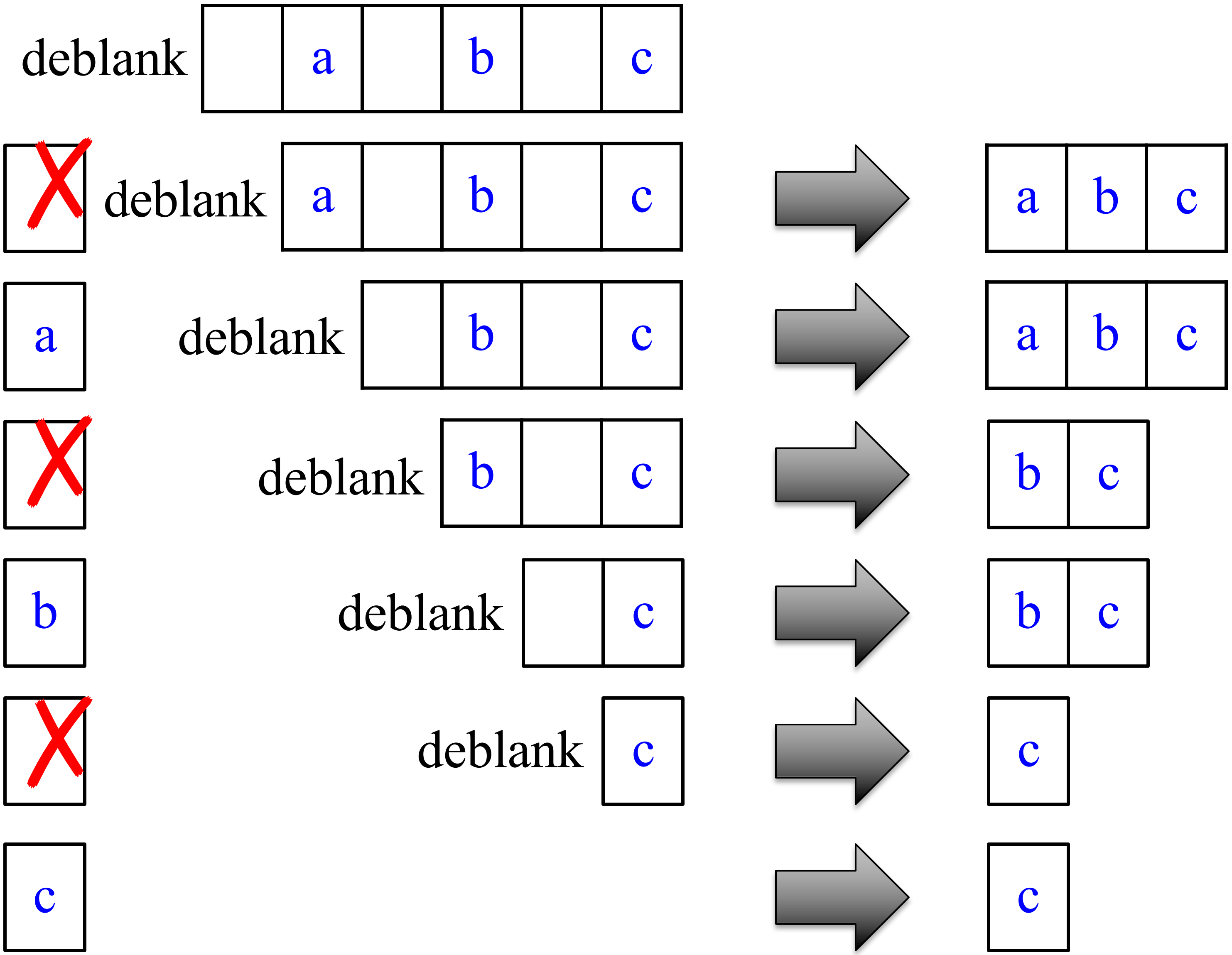
deblank b c → b c

b deblank c → b c

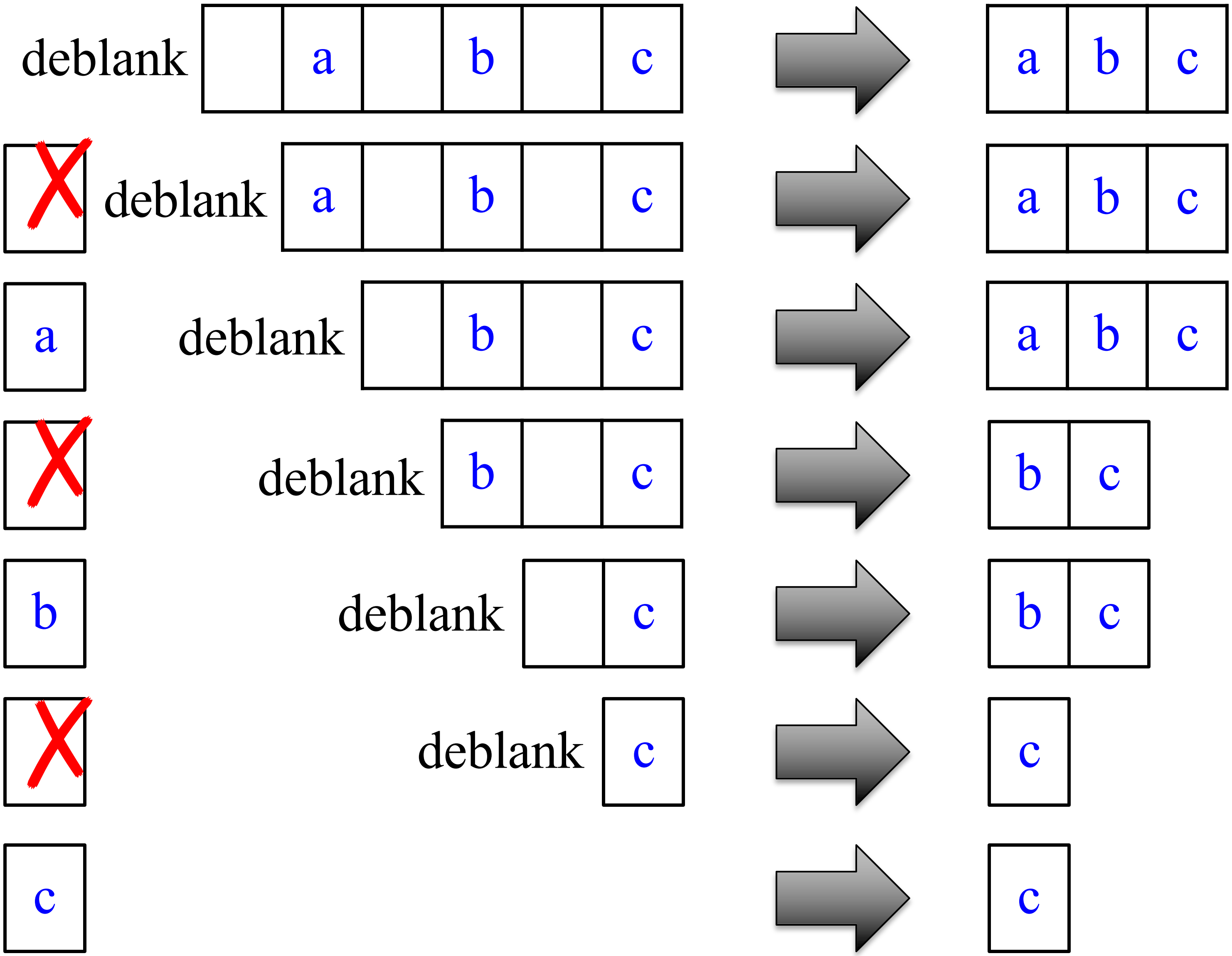
deblank c → c

c → c

Following the Recursion


deblank [ ][a][ ][b][ ][c]
✗ deblank [a][ ][b][ ][c] → [a][b][c]
a deblank [ ][b][ ][c] → [a][b][c]
✗ deblank [b][ ][c] → [b][c]
b deblank [ ][c] → [b][c]
✗ deblank [c] → [c]
c → [c]

Following the Recursion



deblank [ ][a][ ][b][ ][c] → [a][b][c]

[✗] deblank [a][ ][b][ ][c] → [a][b][c]

[a] deblank [ ][b][ ][c] → [a][b][c]

[✗] deblank [b][ ][c] → [b][c]

[b] deblank [ ][c] → [b][c]

[✗] deblank [c] → [c]

[c] → [c]

Final Modification

```
def deblank(s):
    """Returns: s w/o blanks"""
    if s == '':
        return s

    left = s[0]
    if s[0] == ' ':
        left = ''
    right = deblank(s[1:])

    return left+right
```

Real work done here

Final Modification

```
def deblank(s):
    """Returns: s w/o blanks"""
    if s == '':
        return s

    left = s
    if s[0] in string.whitespace
        left = ''
    right = deblank(s[1:])

    return left+right
```

Real work done here

Module string has special constants to simplify detection of whitespace and other characters.