SSY345 – Sensor Fusion and Non-Linear Filtering
Home Assignment 1 - Analysis

Basic information

Please upload your reports to ping-pong no later than **1 Apr 2020, 18:00**.

You are encouraged to discuss the assignments with classmates, but you must write up your own solutions (including Matlab implementations). Also, when writing up the solutions, you should write the names of people with whom you have discussed the assignments. It is important that you make an effort to present/illustrate your solutions nicely and reflect on your results!

This home assignment is related to the material in lecture 1 and 2. A large part of the assignment focuses on understanding of the basic concepts that we will rely on later in the course.

In the analysis part we want you to use the toolbox that you have developed and apply it to a practical scenario. Associated with each scenario is a set of tasks that we would like you to perform, and a set of questions on ping-pong that you should answer. The tasks are designed such that they will help you get the insights needed to answer the questions on ping-pong.

The result of the tasks should in general be visualised and compiled together in a report (pdf-file). A template for the report can also be found on course homepage. Note that, it is sufficient to write short clear captions to the figures but they should clearly explain what is seen in the figure and answer any related questions in the task. The report should also be uploaded to ping-pong before the deadline. Only properly referenced or captioned figures will result in POE.

1 Transformation of Gaussian random variables

The purpose of this scenario is to get familiar with a few common types of transformations and to practice how to calculate mean and covariance of transformed variables.

Suppose that we are interested in the position of an object in Cartesian xy-coordinates, denoted $\mathbf{x}$. However, we are not able to measure $\mathbf{x}$ directly, but rather some related entity $\mathbf{y} = h(\mathbf{x})$, where $h(\cdot)$ describes the relationship between the position of the object and what we observe. Further, assume that we have prior knowledge of the position of the object modelled as,

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 10 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.2 & 0 \\ 0 & 8 \end{bmatrix}\right), \tag{1}$$

As we will see later in the course, it is often of interest to describe what we already know (our prior on $\mathbf{x}$) in the same coordinate system as our observed entity $\mathbf{y}$. That is, we are interested in the transformed density $p(\mathbf{y}) = p(h(\mathbf{x}))$.

Task: Investigate how the properties of $p(h(\mathbf{x}))$ depend on the properties of the transforming function $h(\cdot)$ (also know as the measurement model).

a) Suppose we have a sensor which can measure the sum and the difference between the x- and y-position of the object. That is,

$$\mathbf{y} = h(\mathbf{x}) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{x}. \tag{2}$$

Use both the approxGaussianTransform and the affineGaussianTransform to calculate/approximate the mean and covariance of the transformed density $p(\mathbf{y}) = p(h(\mathbf{x}))$. Plot the transformed samples (from approxGaussianTransform) together with the mean and 3σ-ellipses from both functions. Do the ellipses match the sample points well? Why? How does the approximated ellipse's fit to the analytically calculated ellipse change with number of samples?

b) Now, suppose that we instead have a radar sensor which is capable of observing the angle, ϕ, in radians, and distance, ρ, in m to the object. In this case,

$$\mathbf{y} = \begin{bmatrix} \rho \\ \phi \end{bmatrix} = h(\mathbf{x}) = \begin{bmatrix} ||\mathbf{x}|| \\ \text{atan2}(y, x) \end{bmatrix}, \tag{3}$$

where $\text{atan2}(\cdot, \cdot)$ is the angle in radians between the object and the positive x-axis[1]. Draw samples from (**??**) and approximate the mean and covariance of the transformed density $p(h(\mathbf{x}))$ by using approxGaussianTransform. Plot the transformed samples together with the approximated mean and 3σ-ellipse. Does the ellipse match the sample points well? Why? How does the approximated ellipse's fit to true distribution change with number of samples?

[1]There is a Matlab command with the same name and the same functionality

2 Snow depth in Norway

In this problem you will practice using conditional densities, and investigate their relation to joint densities. A second purpose is that you should understand the machinery behind the Bayesian measurement updates for Gaussian densities in detail.

Anders in Lillehammer wants to ski and chooses between going to Hafjell and Kvitfjell. He remembers from the snow prognosis last night that Hafjell supposedly should have 1.1 m snow, and Kvitfjell should have 1.0 m. He also knows that these forecasts are not very reliable; in fact he considers the forecasts to be a Gaussian function, $p(x) = \mathcal{N}(x; \mu, 0.5^2)$, where x is the true snow depth and μ is what the forecast reported.

Since Anders does not like to take any chances when it comes to skiing, he calls his friends who are already in the slopes. His friend Anna in Hafjell says there is 1.0 m snow, while his friend Else in Kvitfjell says there is 2 m snow. Now he is very careful and also considers the snow measuring skills of his friends, which he models as $y = x + r$, where $r \sim \mathcal{N}(0, 0.2^2)$ for Anna, and $r \sim \mathcal{N}(0, 1^2)$ for Else.

Task:

a) Use the code you developed for `jointGaussian` and plot the 3σ-ellipsoid of $p(x, y)$ for the snow depth at Hafjell and Kvitfjell respectively. What can you say about the dependency between x and y by examining the plots? E.g. what does the slope of the major axis of the ellipse tell you?

b) Use `posteriorGaussian` and plot the two densities of the snow depth at Hafjell and Kvitfjell, $p(x_H \big| y_A)$ and $p(x_K \big| y_E)$, given the information he gained from his friends and from the forecast. How are these 1-dimensional densities related to the plot of $p(x, y)$ in task (a)?

c) Anders likes to ski where there is most snow. If he bases his decision on where to go by maximizing the expected snow depth, where should he go? Is there any information in the conditional densities calculated which speaks for the other alternative?

   Hint: You may find the command *normpdf* useful when plotting 1-D Gaussian densities, but remember to check if the functions you use take the standard deviation or the variance as input.

3 MMSE and MAP estimates for Gaussian mixture posteriors

The purpose of this scenario is to understand what the MMSE estimator is and which fundamental properties it has, specifically in comparison to the MAP estimator.

Task: Plot all three posterior densities along with the MMSE estimates obtained in each case. Also mark, with a different symbol, roughly where the MAP estimate is. How does the MMSE and MAP estimator differ? If the MAP estimate is ambiguous, it is enough to mark one MAP estimate.

a) $p(\theta|y) = 0.1\mathcal{N}(\theta;\ 1, 0.5) + 0.9\mathcal{N}(\theta;\ 1, 9)$.

b) $p(\theta|y) = 0.49\mathcal{N}(\theta;\ 5, 2) + 0.51\mathcal{N}(\theta;\ -5, 2)$.

c) $p(\theta|y) = 0.4\mathcal{N}(\theta;\ 1, 2) + 0.6\mathcal{N}(\theta;\ 2, 1)$.