Calculus 4-6

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Product Rule

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \qquad f(x+\Delta x) = f(x) + \Delta f$$

$$g'(x) = \lim_{\Delta x \to 0} \frac{\Delta g}{\Delta x} = \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \qquad g(x+\Delta x) = g(x) + \Delta g$$

$$\frac{d}{dx} f(x)g(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(f(x) + \Delta f)(g(x) + \Delta g) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x)\Delta g + g(x)\Delta f + \Delta f \Delta g}{\Delta x}$$

$$= \lim_{\Delta x \to 0} f(x)\frac{\Delta g}{\Delta x} + g(x)\frac{\Delta f}{\Delta x} + \frac{\Delta f}{\Delta x}\Delta g$$

$$= \lim_{\Delta x \to 0} f(x)\underbrace{\frac{\Delta g}{\Delta x}}_{\to g'(x)} + g(x)\underbrace{\frac{\Delta f}{\Delta x}}_{\to f'(x)} + \underbrace{\underbrace{\frac{\Delta f}{\Delta x}}_{\to f'(x)}\underbrace{\Delta g}_{\to 0}}_{\to 0}$$

$$= f(x)g'(x) + g(x)f'(x)$$

Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \lim_{\Delta x \to 0} \frac{\frac{f(x+\Delta x)}{g(x+\Delta x)} - \frac{f(x)}{g(x)}}{\Delta x}$$

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \lim_{\Delta x \to 0}\left[\frac{f(x+\Delta x)}{g(x+\Delta x)} - \frac{f(x)}{g(x)}\right]\cdot\frac{1}{\Delta x}$$

$$= \lim_{\Delta x \to 0}\left[\frac{f(x)+\Delta f}{g(x)+\Delta g} - \frac{f(x)}{g(x)}\right]\cdot\frac{1}{\Delta x}$$

$$= \lim_{\Delta x \to 0}\frac{\cancel{f(x)g(x)} + g(x)\Delta f - \cancel{f(x)g(x)} - f(x)\Delta g}{(g(x)+\Delta g)(g(x))\,\Delta x}$$

$$= \lim_{\Delta x \to 0}\left[g(x)\underbrace{\frac{\Delta f}{\Delta x}}_{\to f'(x)} - f(x)\underbrace{\frac{\Delta g}{\Delta x}}_{\to g'(x)}\right]\underbrace{\frac{1}{(g(x)+\underbrace{\Delta g}_{\to 0})\,g(x)}}_{\to g(x)^2}$$

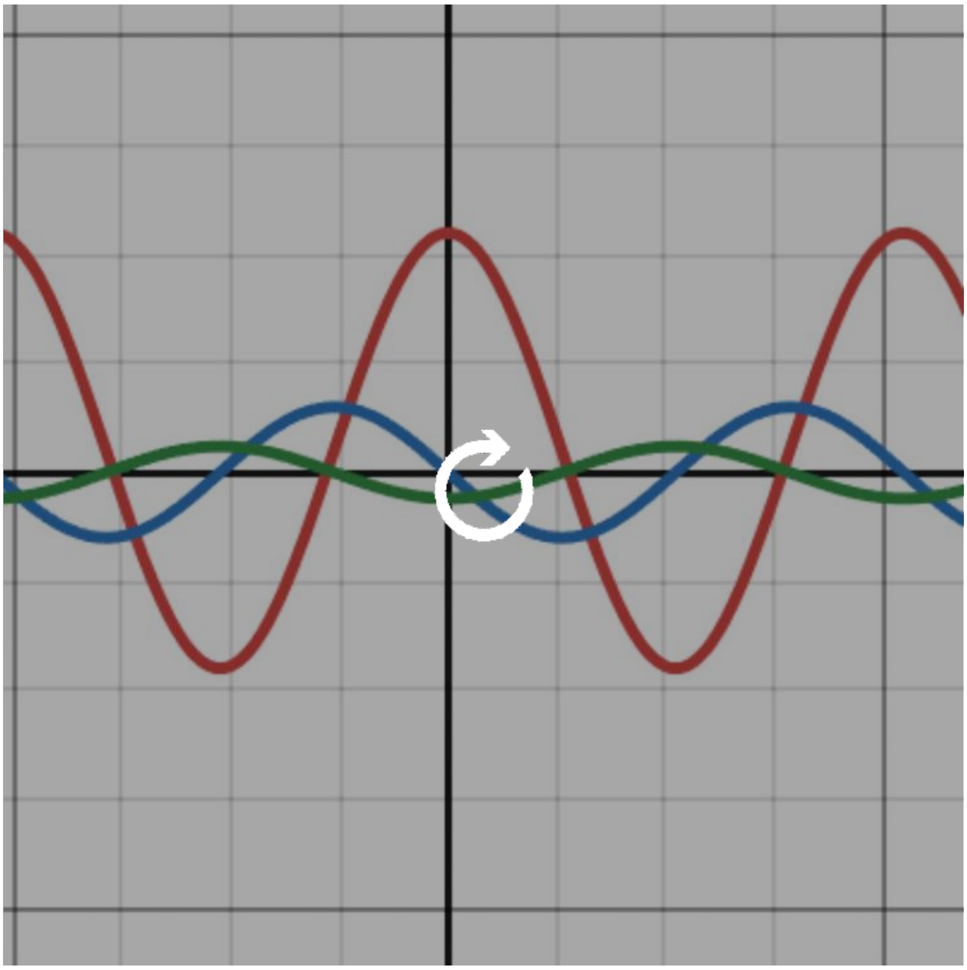
$$= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Higher order derivatives:

f'(x) is about how f(x) changes
f''(x) is about how f'(x) changes

Second order derivative tells about how change is changing.
Nth order derivative tells the rate of change of N-1th order derivative

Point of infliction: where second order derivative becomes zero, and changes its sign
[Desmos | Graphing Calculator]



Chain rule and application:

Y = f(g(x))

F and x might not be direclty influcning each other. That is Y might not be function of X.
How f changes w.r.t g
How g changes w.r.t x

Logic: rate of change of f w.r.t x = {rate of change of f w.r.t g } * {rate of change of g w.r.t x}

Why multiply ?:

Suppose $\frac{\mathrm{d}f}{\mathrm{d}g} = 10 \ \{for\ every\ 1\ unit\ increase\ in\ g, f\ increases\ by\ 10)$

$\frac{\mathrm{d}g}{\mathrm{d}x} = 20 \ \{for\ every\ 1\ unit\ increase\ in\ x, f\ increases\ by\ 20\ )$

$Then\ \frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}g} * \frac{\mathrm{d}g}{\mathrm{d}x} = 10 * 20 = 100$

Lecture 5

F(x) = x^a where a can be fractional, we lhave learned for a being integer, a = m/n

Y = $x^{m/n}$

$y^n = x^m$

Apply chain rule on left hand side and difffernaite, since its not in terms of x

$$\frac{\mathrm{d}\, y^n}{\mathrm{d}y} * \frac{\mathrm{d}y}{\mathrm{d}x} = m\, x^{m-1}$$

Implicit and explicit:

Explicit differentiation	**Implicit differentiation**
You have y = f(x)	You have an equation involving both x and y
Differentiate directly	Differentiate both sides with respect to x, apply chain rule to yyy terms

One very important application of implicit differntiation

- Finding derivatives of inverse functions:
- Eg: y = f(x)   g(y) = x
- g(f(x)) = x , therefore : g = $f^{-1}x$

 the graph of f and f^{-1} $is\ mirrored\ across\ x = y$,
Derivative of inverse function tell show input variable changes with respect to output variable