EXTRACTING PROBABILITY DENSITIES AND DETECTING FAT TAILS

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Introduction

In the volatile realm of financial markets, the precise valuation of options plays a crucial role in strategies aimed at hedging risks and optimizing speculative positions. Particularly in the context of S&P 500 Index (SPX) options, the extraction of risk-neutral densities (RND) from option prices is vital. This process not only predicts future price movements based on current market sentiments but also crucially identifies the likelihood of extreme price swings, or "fat tails," which are often underestimated in standard models.

This research focuses on the systematic extraction of these risk-neutral densities, with a particular emphasis on detecting fat tails within the distributions derived from SPX options. The aim is twofold: firstly, to enhance the accuracy and reliability of RND extraction, addressing common methodological shortcomings such as negative probabilities and computational errors; and secondly, to explore the potential of these fat tail indicators in predicting and, ultimately, mitigating substantial trading losses associated with out-of-the-money options.

By meticulously preparing, filtering, and structuring the SPX option chain data, the study lays a robust foundation for the advanced modeling techniques that follow. Utilizing the shimko.extraction function, the research adapts the volatility modeling as a quadratic function of strikes, optimized through the calibration of "Days to Expiry" (DTE) to mitigate errors and improve the fidelity of the extraction process.

Further, this paper delves into the tail behavior of the risk-neutral densities, employing sophisticated statistical tools like power law and tail index (Hill estimator) techniques to quantify and analyze the presence of fat tails. The significance of this analysis extends beyond theoretical finance; in this paper we are looking for answers to the question, by fat tail analyses can we prevent or can predict big losses from selling out money options.

The ensuing sections will detail the methodologies employed, discuss the challenges encountered and the innovative solutions applied, and evaluate the practical implications of detecting and understanding fat tails in the context of financial trading and risk management strategies.

Methodology

1.1) Data Preparation

This research involves a systematic approach to extracting risk-neutral densities from option prices, specifically focusing on S&P 500 (SPX) options. The initial phase, data preparation, is crucial as it sets the foundation for the density extraction process. The methodology used in this phase includes several detailed steps:

Data Retrieval:
A dedicated function was developed to fetch the SPX option chain data daily for 2020. This step ensures that all relevant option data, including various strike prices and associated call and put prices, are collected systematically.

Data Filtering and Sorting:
From the collected data, unique quote dates are identified. For each of these dates, the option chain is filtered to isolate the data corresponding to that specific day. Subsequently, the data for each unique date is sorted by the "Days to Expiry" (DTE). This sorting helps in selecting a specific DTE for each date, focusing on the nth smallest DTE. This methodological choice is aimed at representing a consistent time frame across different data sets, which in this study is predefined as the option's days to expiry from the quote date.

Data Structuring:
The processed data is then structured to display all strikes along with their respective call and put prices, from the quote date until the specified DTE (e.g., 4 days). This structured data serves as the basis for further analysis in the extraction of risk-neutral densities.

1.2) Extraction of Risk-Neutral Density

The extraction of the implied risk-neutral density is performed using a function known as shimko.extraction. This function models the volatility as a quadratic function of the strikes. However, this approach sometimes results in negative probabilities—a result that is theoretically implausible and undesirable in practical scenarios. Additionally, errors can occur for specific option chains.

To mitigate these issues, a calibration process was implemented for the DTE:

1.3) Calibration of DTE:

Extensive testing was conducted to identify the most reliable DTE that consistently avoids computational errors and the emergence of negative probabilities. It was found that the 8th smallest DTE almost invariably meets these criteria. This DTE was selected as it balances the need to capture immediate market dynamics without compromising the integrity and reliability of the model. A graphical representation of how probability densities change with DTE adjustments can be provided to illustrate these dynamics.

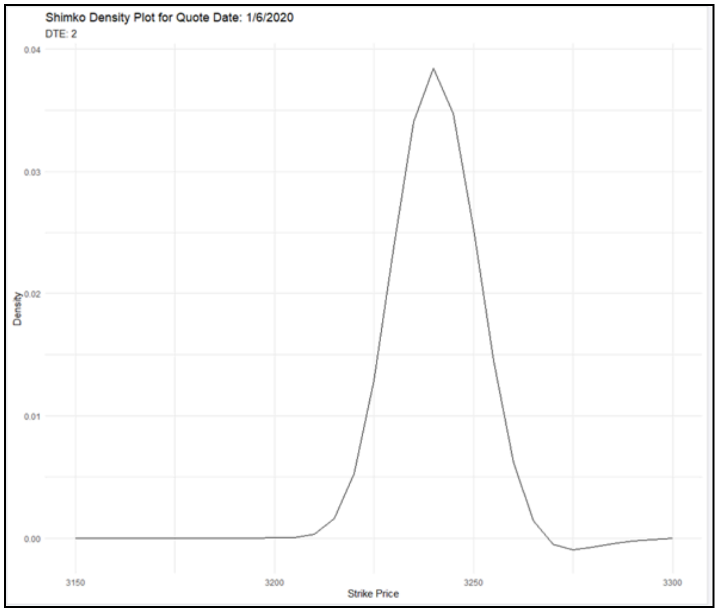

Figure 1: DTE 2 days RND extraction which implies negative probabilities.

It seems that extensive testing was conducted to identify the most reliable DTE that consistently avoids computational errors and the emergence of negative probabilities. The 8th smallest DTE was found to almost invariably meet these criteria. This DTE was selected as it balances the need to capture immediate market dynamics without compromising the integrity and reliability of the model.

Although this specific graph represents a DTE of 2, it can be inferred that similar plots were analyzed during testing to arrive at the conclusion that the 8th smallest DTE is optimal.

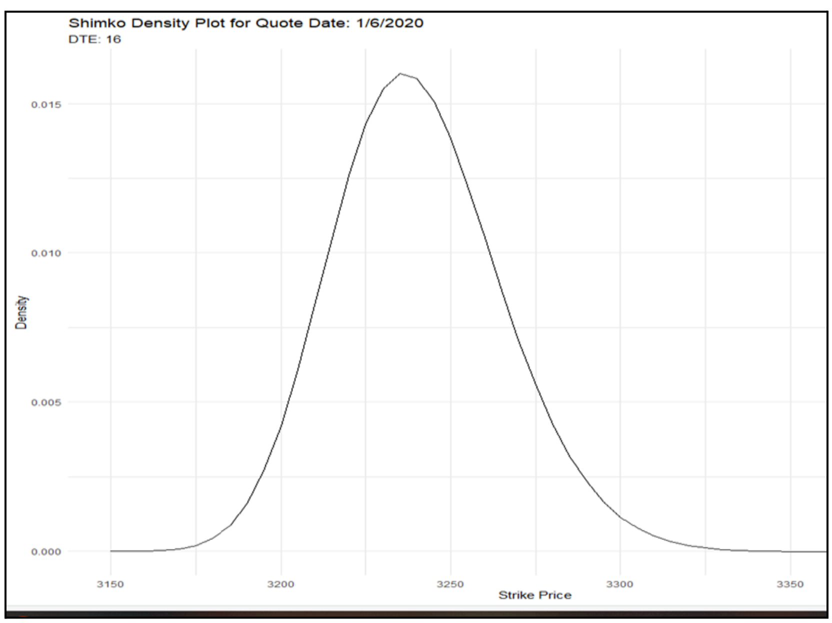

Figure 2: As DTE is set to 16 it shows a more realistic result for probability density.

This specific graph represents a DTE of 18, it can be inferred that similar plots were analyzed during testing to arrive at the conclusion that the 8th smallest DTE is optimal. This plot is a graphical representation of how probability densities change with DTE adjustments, illustrating these dynamics. The previous graph had a DTE of 2 and showed a higher peak at a specific strike price, while this graph has a DTE of 18 and shows a more evenly distributed probability density across different strike prices, forming a bell-shaped curve. This difference could be due to the change in market dynamics over the increased DTE.

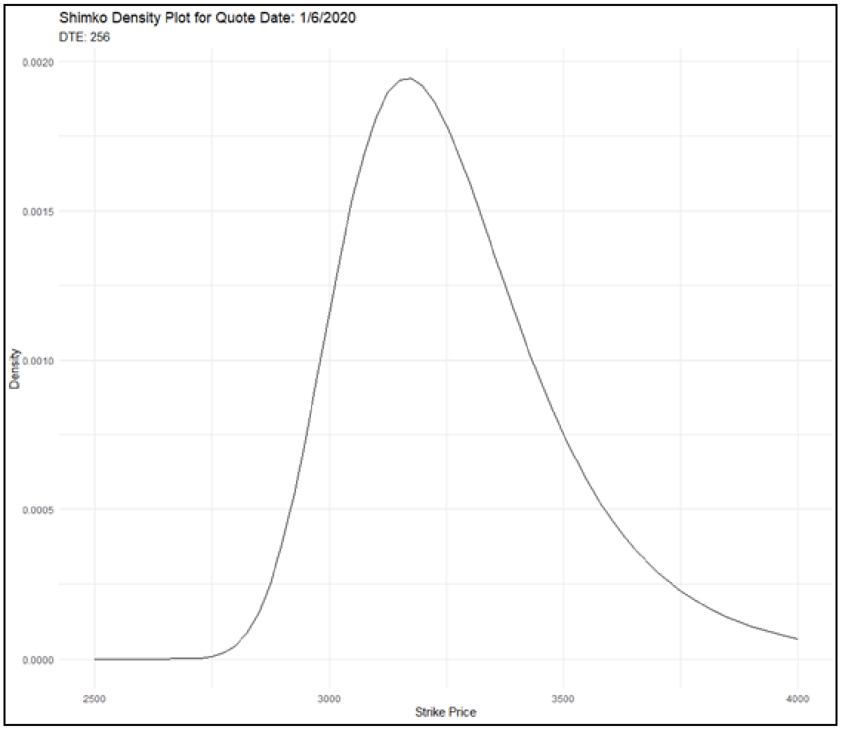

Figure 3: As DTE is set far expiry, the RND converges to log normal density.

This specific graph represents a DTE of 256, it can be inferred that similar plots were analyzed during testing to arrive at the conclusion that the 8th smallest DTE is optimal. This plot is a graphical representation of how probability densities change with DTE adjustments, illustrating these dynamics. The main differences lie in the DTE and the shape of the density curve. The first graph had a DTE of 2 and showed a higher peak at a specific strike price, while the second graph had a DTE of 18 and showed a more evenly distributed probability density across different strike prices, forming a bell-shaped curve. This third graph, with a DTE of 256, also shows a bell-shaped curve but with a wider range of strike prices. This difference could be due to the change in market dynamics over the increased DTE.

1.4) Creation of Density Data

For the actual density data creation:

Automated Extraction Loop:

An automated loop was set up to iterate over each unique date. For each iteration, the shimko.extraction function is called with parameters tailored to the specific market conditions of the day and the calibrated DTE (8th smallest). The parameters include market calls, call strikes, risk-free rate (r), dividend yield (y), time to expiry (te), initial stock price (s0), and the lower and upper bounds of the strike prices.

Data Storage:
The extracted density for each day, along with its corresponding strikes, is stored in a CSV file. This file format facilitates easy access and manipulation for further analysis and visualization.
By employing these methodological steps, the research efficiently addresses the complexities involved in extracting and analyzing the risk-neutral densities derived from SPX option prices. Each phase of the methodology is designed to ensure accuracy, reliability, and relevance of the data, thus supporting robust financial modeling and analysis.

1.5) Tail Analysis in Risk-Neutral Density Estimation

Skewness and Kurtosis Assessment

The analysis of risk-neutral densities derived from SPX option prices includes evaluating the skewness and kurtosis, two statistical measures that reveal asymmetry and tail heaviness in the probability distributions, respectively. These metrics are crucial for understanding the behavior of option prices, especially under extreme market conditions.

Detecting Fat Tails

To specifically address the presence of fat tails in the risk-neutral densities, two techniques were employed: the power law and tail index techniques.

Power Law:
A power law can be represented by the following mathematical expression:

$$P(x) \sim x^{-(alpha)}$$

P(x) represents the probability of an event of size, and alpha is a positive constant known as the power law exponent. This relationship suggests that the probability of occurrence of an event decreases polynomially with an increase in the event's size.

1. This method involves fitting a power law model to the tails of the distribution to estimate the parameter α
2. α (alpha), which characterizes the tail's thickness. A smaller alpha indicates a fatter tail, suggesting a higher probability of extreme outcomes.

1.6) Tail Index Technique:

This approach quantifies the decay rate of the tails of the distribution. A lower tail index indicates a slower decay, synonymous with fatter tails, thereby implying a greater risk of extreme events.

Data Frame Construction

For each unique date, a data frame is constructed capturing the following elements:

- Underlying Price: The price of the SPX index on the given date.

- Skewness: Measures the degree of asymmetry of the distribution around its mean.
- Kurtosis: Indicates the tail heaviness compared to a normal distribution.
- Tail Index: The decay rate of the tail of the distribution.
- Alpha: Derived from the power law fit, indicating tail thickness.
- Percentage Change of Underlying: The percentage change in the underlying SPX index price until the next quote date (usually the next day).
- Distance of Strike: This measures how many strike grids the price has moved until the next quote day.

1.7) Correlation Analysis and Regression Testing

With the data frame in place, several statistical tests are conducted:

Correlation Tests:

- Kurtosis vs. underlying price change percentage.
- Skewness vs. underlying price change percentage.
- These correlations are examined using Pearson and Spearman coefficients to understand the linear and monotonic relationships, respectively.

Filtering for Significant Fat Tails:

Data where $\alpha < 1$ are identified, indicating significant fat tails. For these specific instances, further correlation and linear regression tests are conducted to explore relationships between α and the percentage change in the underlying price.

Results

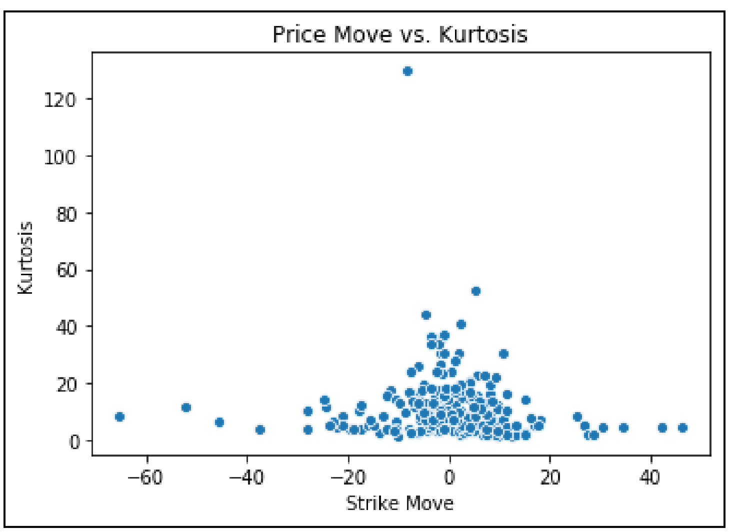

Figure 4: Price Movement with Kurtosis

Data Points: The data points, represented by blue dots, are primarily concentrated around the center of the graph, specifically around the point (0,20). This concentration suggests that most observations exhibit low values for both ‘Strike Move’ and ‘Kurtosis’.

Outliers: There is a notable outlier data point located near the coordinates (0,110). This outlier indicates an observation where ‘Strike Move’ is approximately zero, but ‘Kurtosis’ is significantly high.

Summary: Correlation between strike move and kurtosis: -0.082 and correlation between strike move and skewness: -0.112 which suggest no correlation. The scatter plot provides a visual representation of the relationship between ‘Strike Move’ and ‘Kurtosis’. The majority of data points are clustered around the center, indicating a commonality of low ‘Strike Move’ and ‘Kurtosis’ values. However, the presence of an outlier suggests potential exceptions to this trend.

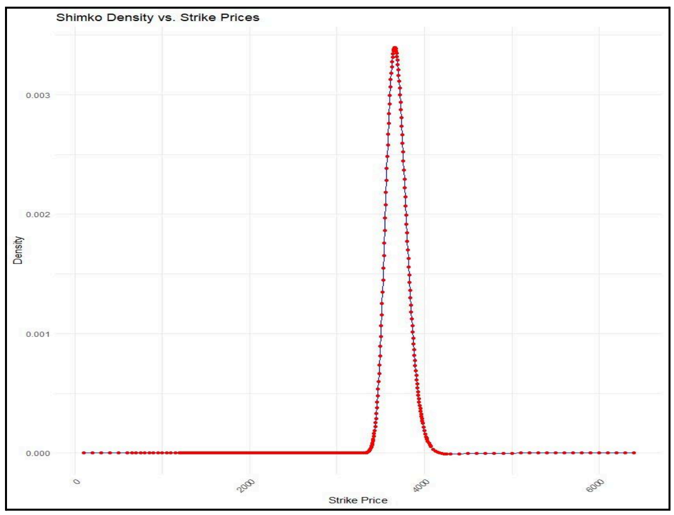

Figure 5: Shimko Density vs. Strike Prices

The data points, represented by a red line with diamond markers, show a sharp peak around a strike price of approximately 3700 (which is very close to the underlying price). The density at this peak reaches up to about 0.003 before sharply declining on both sides. This suggests that the density of strike prices is highest around 3700 and decreases for both lower and higher strike prices.

Summary: The scatter plot provides a visual representation of the relationship between ‘Strike Price’ and ‘Shimko Density’. The peak in the graph indicates a significantly higher density of strike prices around 7000. This could imply that this particular strike price is highly significant or commonly used in trading scenarios.

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Pearson correlation: (0.05309165186404286, 0.6314830478660586)
Spearman correlation: SpearmanrResult(correlation=0.025797306874557054, pvalue=0.8158140200874617)
                            OLS Regression Results
==============================================================================
Dep. Variable:                      y   R-squared:                       0.003
Model:                            OLS   Adj. R-squared:                 -0.009
Method:                 Least Squares   F-statistic:                    0.2318
Date:                Tue, 30 Apr 2024   Prob (F-statistic):              0.631
Time:                        08:59:13   Log-Likelihood:                -473.27
No. Observations:                  84   AIC:                             950.5
Df Residuals:                      82   BIC:                             955.4
Df Model:                           1
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const        -15.0554     20.494     -0.735      0.465     -55.825      25.715
x1            16.5975     34.474      0.481      0.631     -51.983      85.178
==============================================================================
Omnibus:                       30.745   Durbin-Watson:                   2.077
Prob(Omnibus):                  0.000   Jarque-Bera (JB):              164.071
Skew:                          -0.875   Prob(JB):                     2.36e-36
Kurtosis:                       9.619   Cond. No.                         6.08
==============================================================================
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Figure 6: Correlation Results

Correlation Analysis: The Pearson correlation coefficient between the variables ‘a’ and ‘Underlying_Price’ is approximately 0.0531, suggesting a negligible linear correlation between these two variables. The Spearman correlation coefficient is approximately 0.2579, indicating a weak positive monotonic relationship. This correlation analysis is conducted between fat tail incidents alpha vs underlying price change percentage.

Regression Analysis: The R-squared value of the model is 0.003, implying that the model explains a very small fraction of the variation in the dependent variable around its mean. The F-statistic probability is 0.631, suggesting that the model lacks statistical significance.

Coefficient Analysis: The coefficient for the variable ‘a’ is 16.5975. However, considering the standard error of 34.474 and a t-value of 0.481, this coefficient is not statistically significant (with a p-value of 0.631).

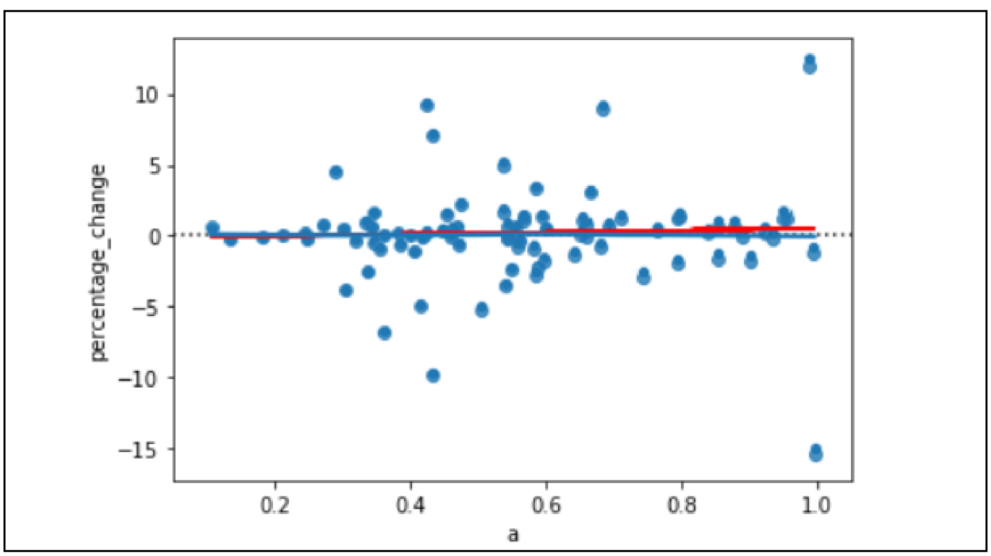

Figure 7: alpha with changes in prices

A red trend line runs horizontally through the plot, close to the 0% mark on the y-axis, indicating minimal overall percentage change across all values of ‘a’.

The scatter plot provides a visual representation of the relationship between ‘a’ and ‘percentage change’. The majority of data points are clustered around the 0% mark, indicating that for most values of ‘a’, there’s little to no percentage change. The red trend line further emphasizes this observation.

Conclusion

Shimko Method Limitations

The Shimko method, while useful in many contexts, has certain limitations when it comes to fitting data where the Days to Expiration (DTEs) are close. The method involves extending DTEs to future dates in order to calculate the Risk-Neutral Density (RND). However, this extension can lead to unreliable RNDs. The reason for this is that the further we move into the future, the more uncertain and variable the market conditions become. This increased uncertainty can introduce significant errors into the RND calculation, making it less reliable as a predictive tool.

Complications in Tail Analysis

Another limitation of the Shimko method is related to its use of a grid of strikes for calculating densities. While this approach allows for a detailed analysis of the strike price landscape, it can complicate the analysis of the tails of the distribution. The tail regions of the distribution represent extreme market conditions (i.e., very high or very low strike prices) and are often of particular interest in risk management and pricing exotic options. However, because these regions correspond to less frequent market conditions, there are typically fewer data points available for these strike prices. As a result, the grid-based approach of the Shimko method can lead to sparse and potentially unreliable density estimates in the tail regions.

References

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