8.4.13

Ellipse and hyperbola

EE25BTECH11010 - Arsh Dhoke

Question

Let a hyperbola passes through the focus of the ellipse

$$\frac{x^2}{25} + \frac{y^2}{16} = 1.$$

The transverse and conjugate axes of this hyperbola coincide with the major and minor axes of the given ellipse, also the product of eccentricities of given ellipse and hyperbola is 1, then

1. the equation of hyperbola is $\frac{x^2}{9} - \frac{y^2}{16} = 1$
2. the equation of hyperbola is $\frac{x^2}{9} - \frac{y^2}{25} = 1$
3. focus of hyperbola is $(5, 0)$
4. vertex of hyperbola is $(5\sqrt{3}, 0)$

Solution

The general equation of the conic can be written as:

$$\mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{x} + 2\mathbf{u}^{\mathbf{T}}\mathbf{x} + f = 0 \tag{1}$$

$$\mathbf{V}_E = \begin{pmatrix} \frac{1}{25} & 0 \\ 0 & \frac{1}{16} \end{pmatrix}, \quad \mathbf{u} = \mathbf{0}, \quad f = -1 \tag{2}$$

$$\lambda_1 = \frac{1}{25}, \quad \lambda_2 = \frac{1}{16} \tag{3}$$

$$e_E^2 = 1 - \frac{\lambda_1}{\lambda_2}, \quad e_E = \frac{3}{5} \tag{4}$$

$$e_H \cdot e_E = 1 \Rightarrow e_H = \frac{5}{3}, \quad e_H^2 = \frac{25}{9} \tag{5}$$

Calculating parameters for Hyperbola

$$\mathbf{V}_H = \begin{pmatrix} \lambda_1' & 0 \\ 0 & \lambda_2' \end{pmatrix}, \quad f = -1 \tag{6}$$

Hyperbola passes through (3,0), thus

$$9\lambda_1' - 1 = 0 \Rightarrow \lambda_1' = \frac{1}{9} \tag{7}$$

$$e_H^2 = 1 - \frac{\lambda_1'}{\lambda_2'} \Rightarrow \lambda_2' = \frac{\lambda_1'}{1 - e_H^2} \tag{8}$$

$$\lambda_2' = \frac{\frac{1}{9}}{1 - \frac{25}{9}} = \frac{\frac{1}{9}}{-\frac{16}{9}} = -\frac{1}{16} \tag{9}$$

$$\mathbf{V}_H = \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & -\frac{1}{16} \end{pmatrix} \tag{10}$$

Answer

$$\frac{x^2}{9} - \frac{y^2}{16} = 1 \tag{11}$$

$$b'^2 = 16, \quad c = \sqrt{\frac{|\lambda_1' - \lambda_2'|}{|\det \mathbf{V}|}} \tag{12}$$

$$c = 5 \tag{13}$$

$$\text{Foci: } \begin{pmatrix} \pm 5 \\ 0 \end{pmatrix}, \quad \text{Vertices: } \begin{pmatrix} \pm 3 \\ 0 \end{pmatrix} \tag{14}$$

$$\boxed{\text{Correct options: (1) and (3)}} \tag{15}$$

Graph

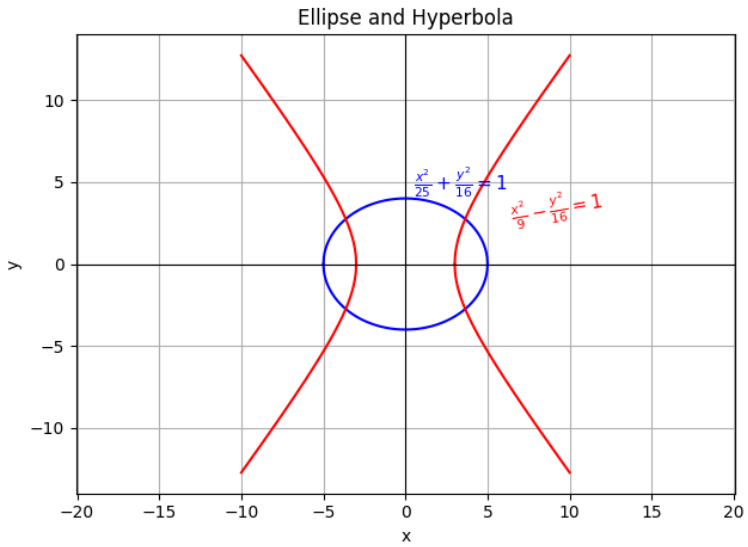

Figure: Graph

C Code

```c
#include <stdio.h>
#include <math.h>

void solve_conic() {
    double a_ellipse = 5.0, b_ellipse = 4.0;
    double e_E = sqrt(1 - (b_ellipse * b_ellipse) / (a_ellipse *
        a_ellipse));
    double e_H = 1.0 / e_E;
    double eH2 = e_H * e_H;

    double x = 3.0;
    double lambda1_H = 1.0 / (x * x);
    double lambda2_H = lambda1_H / (1 - eH2);

    double a_H = sqrt(1.0 / lambda1_H);
    double b_H = sqrt(-1.0 / lambda2_H);
    double c_H = sqrt(a_H * a_H + b_H * b_H);

}
```

Python Code

```
import numpy as np
import matplotlib.pyplot as plt

# Ellipse parameters
a_ellipse = 5
b_ellipse = 4

# Hyperbola parameters
a_hyperbola = 3
b_hyperbola = 4

# Ellipse: x âĹĹ [-a, a]
x_ellipse = np.linspace(-a_ellipse, a_ellipse, 400)
y_ellipse = b_ellipse * np.sqrt(1 - (x_ellipse / a_ellipse)**2)

# Hyperbola: valid only for |x| âĽě a
x_right = np.linspace(a_hyperbola, 10, 400)
x_left = np.linspace(-10, -a_hyperbola, 400)
```

Python Code

```
# Plot ellipse
plt.plot(x_ellipse, y_ellipse, 'b', label='Ellipse')
plt.plot(x_ellipse, -y_ellipse, 'b')

# Plot hyperbola (both branches)
plt.plot(x_right, y_right, 'r', label='Hyperbola')
plt.plot(x_right, -y_right, 'r')
plt.plot(x_left, y_left, 'r')
plt.plot(x_left, -y_left, 'r')

# Axes
plt.axhline(0, color='k', linewidth=0.8)
plt.axvline(0, color='k', linewidth=0.8)

# Labels and formatting
plt.xlabel('x')
plt.ylabel('y')
plt.title('Ellipse and Hyperbola')
```

Python Code

```
plt.axis('equal')
plt.grid(True)
plt.tight_layout()

# Annotate equations beside curves
plt.text(6.2, 2.5, r'$\frac{x^2}{9} - \frac{y^2}{16} = 1$', color
    ='r', fontsize=11, rotation=10)
plt.text(0.5, 4.5, r'$\frac{x^2}{25} + \frac{y^2}{16} = 1$',
    color='b', fontsize=11)
plt.savefig("/home/arsh-dhoke/ee1030-2025/ee25btech11010/matgeo
    /8.4.13/figs/ell.png")
plt.show()
```

Python+ C Code

```
import ctypes
import numpy as np
import matplotlib.pyplot as plt

# Load the shared library
lib = ctypes.CDLL('./code.so')

# Define the function signature
lib.solve_conic.restype = None

# Call the function
lib.solve_conic()

# Ellipse parameters
a_e, b_e = 5, 4
theta = np.linspace(0, 2*np.pi, 400)
x_ellipse = a_e * np.cos(theta)
y_ellipse = b_e * np.sin(theta)
```

Python+ C Code

```
# Hyperbola parameters
a_h, b_h = 3, 4
x_vals = np.linspace(-10, 10, 400)
y_hyperbola_pos = b_h * np.sqrt((x_vals**2 / a_h**2) - 1)
y_hyperbola_neg = -y_hyperbola_pos

# Plot
plt.figure(figsize=(6,6))
plt.plot(x_ellipse, y_ellipse, 'b', label=r'$\frac{x^2}{25} + \
    frac{y^2}{16} = 1$')
plt.plot(x_vals, y_hyperbola_pos, 'r', label=r'$\frac{x^2}{9} - \
    frac{y^2}{16} = 1$')
plt.plot(x_vals, y_hyperbola_neg, 'r')

# Annotate
plt.text(6, 0.5, r'$\frac{x^2}{9} - \frac{y^2}{16} = 1$', color='
    r')
```

Python+ C Code

```
plt.text(2, 3.5, r'$\frac{x^2}{25} + \frac{y^2}{16} = 1$', color=
    'b')

# Styling
plt.axhline(0, color='k', lw=0.8)
plt.axvline(0, color='k', lw=0.8)
plt.axis('equal')
plt.xlabel('x')
plt.ylabel('y')
plt.title('Ellipse and Hyperbola')
plt.grid(True)
plt.savefig("/home/arsh-dhoke/ee1030-2025/ee25btech11010/matgeo
    /8.4.13/figs/ell.png")
plt.show()
```