9.4.22

Quadratic with equal roots

EE25BTECH11010 - Arsh Dhoke

Question

Find the value of k such that the quadratic equation $kx(x-2)+6=0$ has equal roots. Verify your solution using graph.

Solution

$$kx^2 - 2kx + 6 = 0 \tag{1}$$

This can be represented as a conic:

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{2}$$

where

$$\mathbf{V} = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} -k \\ 0 \end{pmatrix}, \quad f = 6 \tag{3}$$

$$\mathbf{x} = \mathbf{h} + k_i \mathbf{m} \tag{4}$$

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5}$$

Solution

The value of k_i can be found out by solving the line and conic equation

$$(\mathbf{h} + k_i\mathbf{m})^\top \mathbf{V}(\mathbf{h} + k_i\mathbf{m}) + 2\mathbf{u}^\top(\mathbf{h} + k_i\mathbf{m}) + f = 0 \tag{6}$$

$$\implies k_i^2\mathbf{m}^\top\mathbf{V}\mathbf{m} + 2k_i\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u}) + \mathbf{h}^\top\mathbf{V}\mathbf{h} + 2\mathbf{u}^\top\mathbf{h} + f = 0 \tag{7}$$

$$\text{or, } k_i^2\mathbf{m}^\top\mathbf{V}\mathbf{m} + 2k_i\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u}) + g(\mathbf{h}) = 0 \tag{8}$$

Solving the above quadratic gives

$$k_i = \frac{1}{\mathbf{m}^\top\mathbf{V}\mathbf{m}}\left(-\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^\top(\mathbf{V}\mathbf{h} + \mathbf{u})]^2 - g(\mathbf{h})\left(\mathbf{m}^\top\mathbf{V}\mathbf{m}\right)}\right) \tag{9}$$

Since the tangent passes through one point of the conic, and $g\left(\mathbf{q}\right) = 0$

$$\mathbf{m}^{\mathbf{T}}\left(\mathbf{V}\mathbf{q} + \mathbf{u}\right) = 0 \tag{10}$$

$$\mathbf{m}^{\mathbf{T}}\mathbf{V}\mathbf{q} = -\mathbf{m}^{\mathbf{T}}\mathbf{u} \tag{11}$$

$$\mathbf{q} = -\frac{\left(\mathbf{m}^{\mathbf{T}}\mathbf{V}\right)^{T}\mathbf{m}^{\mathbf{T}}\mathbf{u}}{\left\|\mathbf{m}^{\mathbf{T}}\mathbf{V}\right\|^{2}} \tag{12}$$

$$\mathbf{q} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{13}$$

Solution

Since $\mathbf{q}$ lies on the conic,

$$g\left(\mathbf{q}\right) = 0 \tag{14}$$

$$\implies \mathbf{q}^{\mathbf{T}}\mathbf{V}\mathbf{q} + 2\mathbf{u}^{\mathbf{T}}\mathbf{q} + f = 0 \tag{15}$$

$\therefore k = 6$ on solving
Thus, the quadratic is

$$6x^2 - 12x + 6 = 0 \tag{16}$$

or

$$(x-1)^2 = 0 \tag{17}$$

which clearly has a double root at $x = 1$.

Verification by Graph

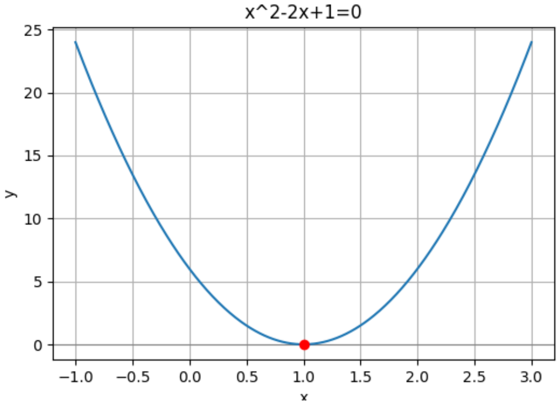

Figure: Graph of $y = 6x^2 - 12x + 6$ showing a double root at $x = 1$

C Code

```
#include <math.h>

double find_k() {
    double k1, k2;
    double a, b, c;
    double D;

    // Equation: 4k^2 - 24k = 0
    a = 4;
    b = -24;
    c = 0;
    D = b * b - 4 * a * c;

    k1 = (-b + sqrt(D)) / (2 * a);
    k2 = (-b - sqrt(D)) / (2 * a);
```

C Code

```
    // k = 0 or 6, but k = 0 makes equation invalid
    if (k1 != 0)
        return k1;
    else
        return k2;
}
```

Python Code

```
import numpy as np
import matplotlib.pyplot as plt

k = 6
x = np.linspace(-1, 3, 400)
y = k*x*(x-2) + 6 # kx^2 - 2kx + 6

plt.figure(figsize=(6,4))
plt.axhline(0, color='gray', linewidth=0.8)
plt.plot(x, y, label=f'k={k} : y=kx(x-2)+6')
plt.scatter([1], [0], color='red', zorder=5) # double root at x=1
plt.title('x^2-2x+1=0')
plt.xlabel('x')
plt.ylabel('y')
plt.grid(True)
plt.savefig("/home/arsh-dhoke/ee1030-2025/ee25btech11010/matgeo
    /9.4.22/figs/parabola.png")
plt.show()
```

Python+ C Code

```python
import ctypes
import numpy as np
import matplotlib.pyplot as plt

# Load the shared C library
lib = ctypes.CDLL("./code.so")

# Define the return type of the C function
lib.find_k.restype = ctypes.c_double

# Call the C function
k = lib.find_k()
print("Value of k for equal roots:", k)

# Define the quadratic function
def f(x, k):
    return k * x * (x - 2) + 6
```

Python+ C Code

```
# Create x values
x = np.linspace(-1, 3, 200)
y = f(x, k)

# Plot the quadratic
plt.plot(x, y, label=f"k = {k:.2f}")
plt.axhline(0, color="black", linewidth=0.8) # x-axis
plt.axvline(0, color="black", linewidth=0.8) # y-axis
plt.title("Quadratic: kx(x - 2) + 6 = 0 (Equal Roots Condition)")
plt.xlabel("x")
plt.ylabel("y")
plt.legend()
plt.grid(True)
plt.savefig("/home/arsh-dhoke/ee1030-2025/ee25btech11010/matgeo
    /9.4.22/figs/parabola.png")
plt.show()
```