9.4.22

EE25BTECH11010 - Arsh Dhoke

Question:
Find the value of k such that the quadratic equation kx(x-2) + 6 = 0 has equal roots. Verify your solution using graph.
Solution:

$$kx^2 - 2kx + 6 = 0 \tag{0.1}$$

This can be represented as a conic:

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{0.2}$$

where

$$\mathbf{V} = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} -k \\ 0 \end{pmatrix}, \quad f = 6 \tag{0.3}$$

$$\mathbf{x} = \mathbf{h} + k_i\mathbf{m} \tag{0.4}$$

$$\mathbf{m} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{0.5}$$

The value of k_i can be found out by solving the line and conic equation

$$(\mathbf{h} + k_i\mathbf{m})^T\mathbf{V}(\mathbf{h} + k_i\mathbf{m}) + 2\mathbf{u}^T(\mathbf{h} + k_i\mathbf{m}) + f = 0 \tag{0.6}$$

$$\implies k_i^2\mathbf{m}^T\mathbf{V}\mathbf{m} + 2k_i\mathbf{m}^T(\mathbf{V}\mathbf{h} + \mathbf{u}) + \mathbf{h}^T\mathbf{V}\mathbf{h} + 2\mathbf{u}^T\mathbf{h} + f = 0 \tag{0.7}$$

$$\text{or, } k_i^2\mathbf{m}^T\mathbf{V}\mathbf{m} + 2k_i\mathbf{m}^T(\mathbf{V}\mathbf{h} + \mathbf{u}) + g(\mathbf{h}) = 0 \tag{0.8}$$

Solving the above quadratic gives

$$k_i = \frac{1}{\mathbf{m}^T\mathbf{V}\mathbf{m}}\left(-\mathbf{m}^T(\mathbf{V}\mathbf{h} + \mathbf{u}) \pm \sqrt{\left[\mathbf{m}^T(\mathbf{V}\mathbf{h} + \mathbf{u})\right]^2 - g(\mathbf{h})\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)}\right) \tag{0.9}$$

Since the tangent passes through one point of the conic, and $g(\mathbf{q}) = 0$

$$\mathbf{m}^{\mathbf{T}}(\mathbf{V}\mathbf{q} + \mathbf{u}) = 0 \tag{0.10}$$

$$\mathbf{m}^{\mathbf{T}}\mathbf{V}\mathbf{q} = -\mathbf{m}^{\mathbf{T}}\mathbf{u} \tag{0.11}$$

$$\mathbf{q} = -\frac{\left(\mathbf{m^T V}\right)^T \mathbf{m^T u}}{\left\|\mathbf{m^T V}\right\|^2} \tag{0.12}$$

$$\mathbf{q} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{0.13}$$

Since $\mathbf{q}$ lies on the conic,

$$g(\mathbf{q}) = 0 \tag{0.14}$$

$$\implies \mathbf{q^T V q} + 2\mathbf{u^T q} + f = 0 \tag{0.15}$$

$\therefore k = 6$ on solving
Thus, the quadratic is

$$6x^2 - 12x + 6 = 0 \tag{0.16}$$

or

$$(x-1)^2 = 0 \tag{0.17}$$

which clearly has a double root at $x = 1$.

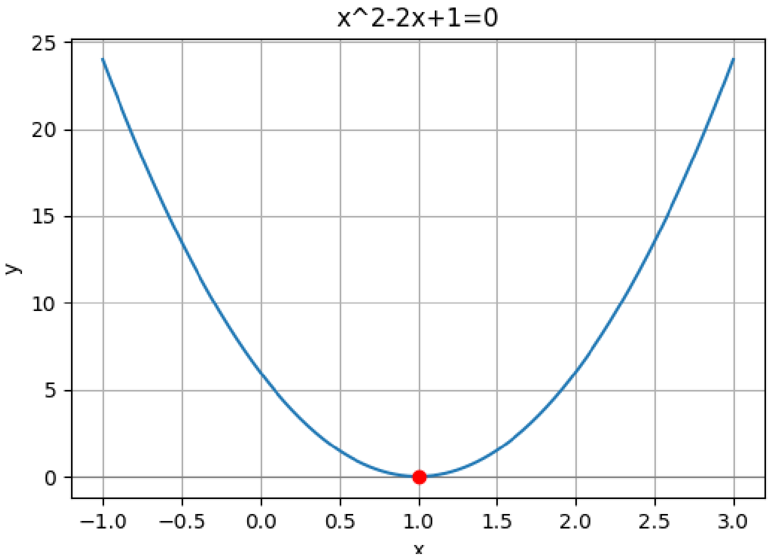

Fig. 0.1: Graph